Supermodular games

Sergei Izmalkov and Muhamet Yildiz

(thanks to Jonathan Levin)

1 Monotone comparative statics

Suppose $X \subset \mathbb{R}$, and T is partially ordered.

Definition: A function $f : X \times T \to \mathbb{R}$ has *increasing differences* in (x, t) if for all $x' \geq x$ and $t' \geq t$,

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t).$$

Thus, $f(x', t) - f(x, t)$ is nondecreasing in t.

Symmetry: $f(x, t') - f(x, t)$ is nondecreasing in x.

Lemma: If $f \in C^2$, then f has increasing differences $\iff$ $t' \geq t$ implies $f_x(x, t') \geq f_x(x, t)$ for all x, that is,

$$f_{xt}(x, t) \geq 0 \text{ for all } x, t.$$

Define

$$x(t) = \arg\max_{x \in X} f(x,t).$$

Theorem 1: (Topkis) Suppose $X \subset \mathbb{R}$ is a compact and T is partially ordered. Suppose $f : X \times T \to \mathbb{R}$ has *ID* and is upper semi-continuous in x. Then,

(i) for all t, $x(t)$ exists, has a greatest and least elements $\bar{x}(t)$ and $\underline{x}(t)$;

(ii) for $t' \geq t$, $x(t') \geq x(t)$ in a sence $\bar{x}(t') \geq \bar{x}(t)$ and $\underline{x}(t') \geq \underline{x}(t)$.

2 Lattices

Suppose X is a partially ordered set with order $\geq$.

(think as $X \subset \mathbb{R}^n$ and $x \geq y \iff x_i \geq y_i$ for all $i = 1, .., n$.)

Define

$$\begin{aligned} \text{“join”} &: \quad x \vee y = \inf\{z \in X : z \geq x, z \geq y\}, \\ \text{“meet”} &: \quad x \wedge y = \sup\{z \in X : z \leq x, z \leq y\}. \end{aligned}$$

In $\mathbb{R}^n$,

$$\begin{aligned} (x \vee y)_i &= \max(x_i, y_i), \\ (x \wedge y)_i &= \min(x_i, y_i). \end{aligned}$$

Definition: $(X, \geq)$ is a *sub-lattice* if it is closed under $\vee$ and $\wedge$.

3 Supermodular functions

Definition: Payoff function u_i is *supermodular* in x_i if, for each $x_{-i} \in X_{-i}$ and $x_i, x_i' \in X_i$

$$u(x_i, x_{-i}) + u(x_i', x_{-i}) \leq u(x_i \vee x_i', x_{-i}) + u(x_i \wedge x_i', x_{-i}).$$

Note: If $x_i \geq x_i'$ (comparable) supermodularity is trivially satisfied.

Definition: Payoff function u_i is *supermodular* if for all $x, x' \in X$

$$u_i(x \vee x') + u_i(x \wedge x') \geq u_i(x) + u(x').$$

Theorem: Supermodularity $\Rightarrow$ supermodularity in x_i and increasing differences.

4 Supermodular games

Games with "strategic complementarities."

Definition: The game $(S_1, \ldots, S_I, u_1, \ldots, u_I)$ is a *supermodular game* if for all i: (general definition is in brackets)

- S_i is a compact subset of $\mathbb{R}$ (S_i is sub-lattice);
- u_i is upper semi-continuous in s_i, s_{-i} (u_i is supermodular in s_i);
- u_i has increasing differences in (s_i, s_{-i}).

Theorem 2: Suppose (S, u) is a supermodular game, let

$$BR_i(s_{-i}) = \text{arg} \max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

Then,

(i) $BR_i(s_{-i})$ has a greatest and least elements $\overline{BR}_i(s_{-i})$ and $\underline{BR}_i(s_{-i})$;

(ii) If $s'_{-i} \geq s_{-i}$, then $\overline{BR}_i(s'_{-i}) \geq \overline{BR}_i(s_{-i})$ and $\underline{BR}_i(s'_{-i}) \geq \underline{BR}_i(s_{-i})$.

5 Examples

5.1 Investment game

Firms $1, \ldots, I$ make simultaneous investments $s_i \in \{0, 1\}$ and payoffs are:

$$u_i(s_i, s_{-i}) = \begin{cases} \pi\left(\sum_{j=1}^{I} s_j\right) - k, & \text{if } s_i = 1, \\ 0, & \text{if } s_i = 0, \end{cases}$$

where π is increasing.

5.2 Bertrand Competition

Firms $1, \ldots, I$ simultaneously choose prices, and

$$D_i(p_i, p_{-i}) = a_i - b_i p_i + \sum_{j \neq i} d_{ij} p_j,$$

where $b_i, d_{ij} \geq 0$. Then $S_i = \mathbb{R}_+$ and

$$\begin{aligned} \pi_i(p_i, p_{-i}) &= (p_i - c_i)\, D_i(p_i, p_{-i}), \\ \frac{\partial^2 \pi_i}{\partial p_i \partial p_j} &= d_{ij} \geq 0. \end{aligned}$$

5.3 Cournot Competition

Cournot oligopoly is supermodular only if $N = 2$ and $s_1 = q_1$, $s_2 = -q_2$.

5.4 Diamond search model

I agents exerting effort searching for trading partners:

e_i and $c(e_i)$ – effort and cost of effort for agent i,

$$u_i(e_i, e_{-i}) = e_i \cdot \sum_{j \neq i} e_j - c(e_i)$$

has increasing differences in e_i, e_{-i}.

6 Solving Bertrand game.

Suppose there are 2 firms, $D_i(p_i, p_j) = 1 - 2p_i + p_j$, and $c = 0$. Suppose $S_i^0 = [0, 1]$.

$$\begin{aligned} \pi_i(p_i, p_{-i}) &= p_i(1 - 2p_i + p_j), \\ \frac{\partial \pi_i(p_i, p_{-i})}{\partial p_i} &= 1 - 4p_i + p_j. \end{aligned}$$

Iterated elimination of strictly dominated strategies gives:

- Any $p_i < \frac{1}{4}$ is strictly dominated by $p_i = \frac{1}{4}$; any $p_i > \frac{1}{2}$ is strictly dominated by $p_i = \frac{1}{2}$.

Thus, $S_i^1 = \left[\frac{1}{4}, \frac{1}{2}\right]$. Note that $S_i^1 = BR_i(S_j^0)$.

- Repeating the procedure we have $S_i^k = BR_i(S_j^{k-1})$.

- Converges to the point $\left(\frac{1}{3}, \frac{1}{3}\right)$.

7 Main result

Theorem 3: Let (S, u) be a supermodular game. Then the set of strategies surviving iterated strict dominance has greatest and least elements $\bar{s}$ and $\underline{s}$; and $\bar{s}$, $\underline{s}$ are both Nash equilibria.

Corollary:

1. Pure strategy Nash equilibrium exist in supermodular games.

2. The largest and the smallest strategies compatible with iterated strict dominance, rationalizability, correlated equilibrium, and Nash equilibrium are the same.

3. If a supermodular game has a unique NE, then it is dominance solvable (and so a lot of learning or adjustment rules will converge to it (e.g. best-response dynamics)).

7.1 Proof of Theorem 3

- Iterate best-response mapping.

- $S^0 = S$; $s^0 = (s_1^0, \ldots, s_I^0)$ – largest element in S^0.

  $s_i^1 = \overline{BR}_i(s_{-i}^0)$; $S_i^1 = \left\{ s_i \in S_i^0 : s_i \leq s_i^1 \right\}$.

- Any $s_i \notin S_i^1$ is dominated by s_i^1 because

$$\begin{aligned} & u_i(s_i, s_{-i}) - u_i(s_i^1, s_{-i}) \\ \leq \; & u_i(s_i, s_{-i}^0) - u_i(s_i^1, s_{-i}^0) < 0. \end{aligned}$$

- $s_i^k = \overline{BR}_i(s_{-i}^{k-1})$; $S_i^k = \left\{ s_i \in S_i^{k-1} : s_i \leq s_i^{k-1} \right\}$.

$$\begin{aligned} s^k &\leq s^{k-1} \Longrightarrow \\ s_i^{k+1} &= \overline{BR}_i(s_{-i}^k) \geq \overline{BR}_i(s_{-i}^{k-1}) = s_i^k. \end{aligned}$$

- Define
$$\bar{s}_i = \lim_{k\to\infty} s_i^k.$$
Only strategies $s_i \leq \bar{s}_i$ are undominated.

- $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_I)$ – Nash equilibrium, indeed
$$\begin{aligned} u_i(s_i^{k+1}, s_{-i}^k) &\geq u_i(s_i, s_{-i}^k), \\ u_i(\bar{s}_i, \bar{s}_{-i}) &\geq u_i(s_i, \bar{s}_{-i}). \end{aligned}$$

- Similarly define $s^0 = (s_1^0, \ldots, s_I^0)$ – smallest element in S^0;

  $s_i^1 = \underline{BR}(s_{-i}^0)$; $S_i^1 = \left\{s_i \in S_i^0 : s_i \geq s_i^1\right\}$ and so on...

- Obtain $\underline{s} = (\underline{s}_1, \ldots, \underline{s}_I)$, prove that it is Nash Equilibrium.

8 Properties of supermodular games

Idea: Use monotonicity to obtain comparative statics results.

- *A supermodular game* (S, u) *is indexed by* t if each players payoff function is indexed by $t \in T$, some ordered set, and for all i, $u_i(s_i, s_{-i}, t)$ has increasing differences in (s_i, t).

Proposition: Suppose (S, u) is a supermodular game is indexed by t. The largest and smallest Nash equilibria are increasing in t.

- *A supermodular game* (S, u) *has positive spillovers* if for all i, $u(s_i, s_{-i})$ is increasing in s_{-i}.

Proposition: Suppose (S, u) is a supermodular game with positive spillovers. Then the largest Nash equilibrium is Pareto-preferred.