14.12 Game Theory – Midterm I Solutions

1. Consider the following game.

1\2	L	M	R
T	3,2	4,0	1,1
M	2,0	3,3	0,0
B	1,1	0,2	2,3

   (a) Iteratively eliminate all the strictly dominated strategies. [10]

   Answer: For Player 1, T strictly dominates M, hence we eliminate M of Player 1.

1\2	L	M	R
T	3,2	4,0	1,1
B	1,1	0,2	2,3

   Now, R strictly dominates M for Player 2, hence we eliminate M of Player 2, and obtain

1\2	L	R
T	3,2	1,1
B	1,1	2,3

   (b) State the rationality/knowledge assumptions corresponding to each elimination. [10]

   Answer: For the first elimination, we assume that Player 1 is rational (hence he would not play a strictly dominated strategy). For the second elimination, we assume that Player 2 is rational and that he knows that player 1 is rational. [Many of you lost point for this part; those who stated general rationality assumptions did not get any credit, in general.]

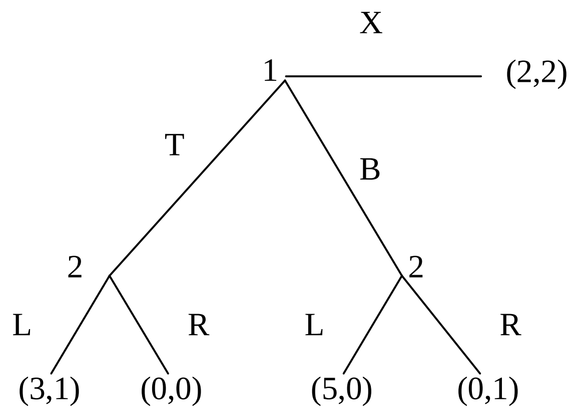
   (c) What are the rationalizable strategies? [3]

   Answer: {T,B} for Player 1, and {L,R} for player 2.

   (d) Find all the Nash equilibria. (Don't forget the mixed-strategy equilibrium!) [10]

   Answer: Pure strategy Nash Equilibria are (T,L) and (B,R). [5]. Mixed strategy Nash equilibrium is $[\sigma_1(T) = 2/3; \sigma_1(B) = 1/3; \sigma_2(L) = 1/3; \sigma_2(R) = 2/3]$. [5]

2. Consider the following extensive form game.



(a) Find the normal form representation of this game. [11]

Answer: Player 1 has three strategies: {X,T,B}. Player 2 has four strategies:

- LL = L if Player 1 chooses T; L if Player 1 chooses B.
- LR = L if Player 1 chooses T; R if Player 1 chooses B.
- RL = R if Player 1 chooses T; L if Player 1 chooses B.
- RR = R if Player 1 chooses T; R if Player 1 chooses B.

Therefore, the normal form is the following.

1\2	LL	LR	RL	RR
X	2,2	2,2	2,2	2,2
T	3,1	3,1	0,0	0,0
B	5,1	0,1	5,1	0,1

[Those who assigned only two strategies to player 2 received no credit. For this is the mistake that those who do not know any Game Theory can make. Many of you were confused; they assigned eight strategies to player 2, assigning 2 trivial moves after player 1 plays X. This is clearly wrong, for player 2 does not have any move after player 1 plays X. They received eight points.]

(b) Find all pure strategy Nash equilibria. [11]

Answer: The pure strategy Nash equilibria are (T, LR), and (X,RR).

(c) Which of these equilibria are subgame perfect? [11]

Answer: (T,LR) is subgame perfect. (X,RR) is not; for 2 would not play R if 1 played T.

3. Consider two agents $\{1,2\}$ owning one dollar which they can use only after they divide it. Each player's utility of getting x dollar at t is $\delta^t x$ for $\delta \in (0,1)$. Given any $n > 0$, consider the following n-period symmetric, random bargaining model. Given any date $t \in \{0,1,\ldots,n-1\}$, we toss a fair coin; if it comes Head (which comes with probability 1/2), we select player 1; if it comes Tail, we select player 2. The selected player makes an offer $(x,y) \in [0,1]^2$ such that $x + y \leq 1$. Knowing what has been offered, the

other player accepts or rejects the offer. If the offer (x, y) is accepted, the game ends, yielding payoff vector $(\delta^t x, \delta^t y)$. If the offer is rejected, we proceed to the next date, when the same procedure is repeated, except for $t = n - 1$, after which the game ends, yielding (0,0). The coin tosses at different dates are stochastically independent. And everything described up to here is common knowledge.

(a) Compute the subgame perfect equilibrium for $n = 1$. What is the value of playing this game for a player? (That is, compute the expected utility of each player before the coin-toss, given that they will play the subgame-perfect equilibrium.) [10]

Answer: If a player rejects an offer, he will get 0, hence he will accept any offer that gives him at least 0. (He is indifferent between accepting and rejecting an offer that gives him exactly 0; but rejecting such an offer is inconsistent with an equilibrium.) Hence, the selected player offers 0 to his opponent, taking entire dollar for himself; and his offer will be accepted. Therefore, in any subgame perfect equilibrium, the outcome is (1,0) if it comes Head, and (0, 1) if it comes Tail. The expected payoffs are

$$V = \frac{1}{2}(1,0) + \frac{1}{2}(0,1) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

(b) Compute the subgame perfect equilibrium for $n = 2$. Compute the expected utility of each player before the <u>first</u> coin-toss, given that they will play the subgame-perfect equilibrium. [10]

Answer: In equilibrium, on the last day, they will act as in part (a). Hence, on the first day, if a player rejects the offer, the expected payoff of each player will be $\delta \cdot 1/2 = \delta/2$. Thus, he will accept an offer if an only if it gives him at least $\delta/2$. Therefore, the selected player offers $\delta/2$ to his opponent, keeping $1 - \delta/2$ for himself, which is more than $\delta/2$, his expected payoff if his offer is rejected. Therefore, in any subgame perfect equilibrium, the outcome is $(1 - \delta/2, \delta/2)$ if it comes Head, and $(\delta/2, 1 - \delta/2)$ if it comes Tail. The expected payoff of each player before the first coin toss is

$$\frac{1}{2}(1 - \delta/2) + \frac{1}{2}(\delta/2) = \frac{1}{2}.$$

(c) What is the subgame perfect equilibrium for $n \geq 3$. [13]

Answer: Part (b) suggests that, if expected payoff of each player at the beginning of date $t + 1$ is $\delta^{t+1}/2$, the expected payoff of each player at the beginning of t will be $\delta^t/2$. [Note that in terms of dollars these numbers correspond to $\delta/2$ and $1/2$, respectively.] Therefore, the equilibrium is follows: At any date $t < n - 1$, the selected player offers $\delta/2$ to his opponent, keeping $1 - \delta/2$ for himself; and his opponent accepts an offer iff he gets at least $\delta/2$; and at date $n - 1$, a player accepts any offer, hence the selected player offers 0 to his opponent, keeping 1 for himself. [You should be able to prove this using mathematical induction and the argument in part (b).]