14.12 Game Theory

Lecture 2: Decision Theory

Road Map

1. Basic Concepts (Alternatives, preferences,…)
2. Ordinal representation of preferences
3. Cardinal representation – Expected utility theory
4. Applications: Risk sharing and Insurance
5. **Quiz**

Basic Concepts: Alternatives

- Agent chooses between the alternatives
- X = The set of all alternatives
- Alternatives are
  - Mutually exclusive, and
  - Exhaustive
- **Example:** Options = {Tea, Coffee}
  X = {T, C, TC, NT} where
  T= Tea, C = Coffee, TC = Tea and Coffee,
  NT = Neither Tea nor Coffee

Basic Concepts: Preferences

- A relation $\succsim$ (on X) is any subset of $X \times X$,
  - e.g.,
    $\succsim^* = \{(T,C),(T,CT),(T,NT),(C,CT),(C,NT),(NT,CT)\}$
  - $T \succsim C \equiv (T,C) \in \succsim$.
- $\succsim$ is complete iff $\forall x,y \in X$, $x \succsim y$ or $y \succsim x$,
- $\succsim$ is transitive iff, $\forall$ $x,y,z \in X$,

$$[x \succsim y \text{ and } y \succsim z] \Rightarrow x \succsim z.$$

Definition: A relation is a preference relation iff it is complete and transitive.

Side

- Strict preference:

$$x \succ y \Leftrightarrow [x \succsim y \text{ and } y \not\succsim x],$$

- indifference:

$$x \sim y \Leftrightarrow [x \succsim y \text{ and } y \succsim x].$$

Examples

- Define a relation among the students in this class by
  - x T y iff x is at least as tall as y;
  - x M y iff x's final grade in 14.04 is at least as high as y's final grade;
  - x H y iff x and y went to the same high school;
  - X Y y iff x is strictly younger than y;
  - x S y iff x is as old as y;

Ordinal representation

- $\succsim$ represented by $u : X \to R$ iff
  $x \succsim y \Leftrightarrow u(x) \geq u(y) \quad \forall x,y \in X.$ (OR)

E.g.,

$\succsim^{**} = \{(T,C),(T,CT),(T,NT),(C,CT),(C,NT),$
$(NT,CT),(C,C),(T,T),(CT,CT),(NT,NT)\}$

is represented by u^{**} where

$u^{**}(C) =$

$u^{**}(T) =$

$u^{**}(CT) =$

$u^{**}(NT) =$

Exercises

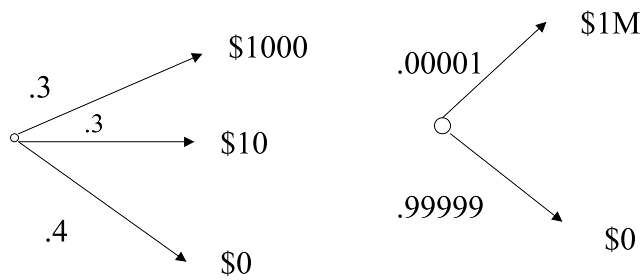
- Imagine a group of students sitting around a round table. Define a relation R, by writing x R y iff x sits to the right of y. Can you represent R by a utility function?
- Consider a relation $\succeq$ among positive real numbers represented by u with $u(x) = x^2$.
  Can this relation be represented by $u^*(x) = \sqrt{x}$?
  What about $u^{**}(x) = 1/x$?

Theorem (Ordinal representation)

Let X be finite. A relation $\succsim$ can be represented by a utility function U in the sense of (OR) iff $\succsim$ is a preference relation. If $U : X \to R$ represents $\succsim$, and if $f : R \to R$ is strictly increasing, then $f \circ U$ also represents $\succsim$.

A Lottery

Two Lotteries



Cardinal representation

- Z = a finite set of consequences or prizes.
- A lottery is a probability distribution on Z.
- P = the set of all lotteries.
- VNM-representation:

$$p \succsim q$$

$$\Leftrightarrow$$

$$U(p) \equiv \sum_{z \in Z} u(z)p(z) \geq \sum_{z \in Z} u(z)q(z) \equiv U(q)$$

Axioms

A1 $\succsim$ is complete and transitive.

A2 (**Independence**) For any p,q,r $\in$ P, and any a $\in$(0,1],

$$ap+(1-a)r \succ aq+(1-a)r \;\Leftrightarrow\; p \succ q.$$

A3 (**Continuity**) For any p,q,r $\in$ P, if p $\succ$ q $\succ$ r, then there exist a,b $\in$(0,1) such that

$$ap+(1-a)r \succ q \succ bp+(1-r)r.$$

Theorem (Cardinal representation)

A relation $\succsim$ on P can be represented by a VNM utility function u : Z $\rightarrow$ R iff $\succsim$ satisfies Axioms A1-A3.

u and ũ represent $\succsim$ iff ũ=au+b for some a and b where a > 0.

Exercise

- Consider an agent with VNM utility function u with $u(x) = x^2$.

  Can his preferences be represented by VNM utility function $u^*(x) = \sqrt{x}$?

  What about $u^{**}(x) = 1/x$?

"Proof"

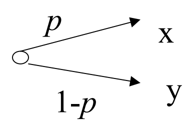
By A2 and A3,

$$p \sim q \Leftrightarrow ap+(1-a)r \sim aq+(1-a)r.$$

Hence,

1. The indifference curves on the lotteries are straight lines
2. .... and parallel to each other.

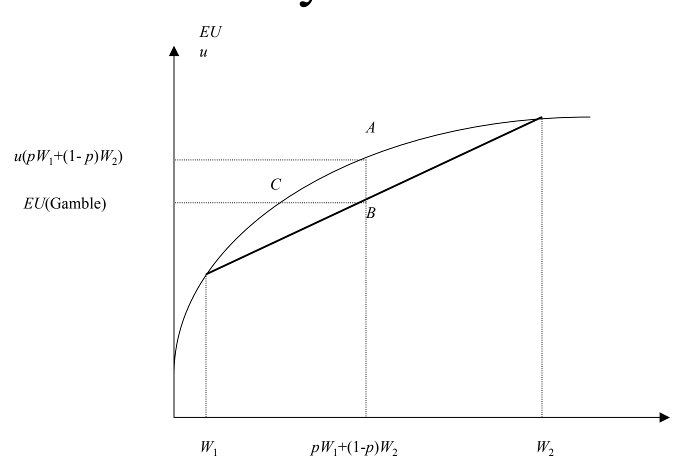
Attitudes towards Risk

- A fair gamble: 

 $px+(1-p)y = 0.$

- An agent is said to be *risk neutral* iff he is indifferent towards all fair gambles. He is said to be *(strictly) risk averse* iff he never wants to take any fair gamble, and (strictly) *risk seeking* iff he always wants to take fair gambles.

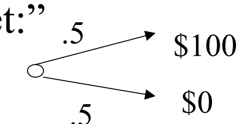
A utility function



- An agent is risk-neutral iff he has a linear utility function, i.e., $u(x) = ax + b$.
- An agent is risk-averse iff his utility function is concave.
- An agent is risk-seeking iff his utility function is convex.

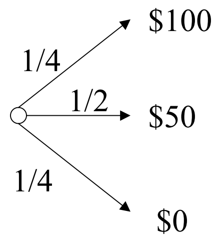
Risk Sharing

- Two agents, each having a utility function u with $u(x) = \sqrt{x}$ and an "asset:"



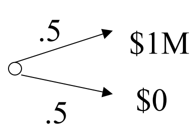
- For each agent, the value of the asset is
- Assume that the value of assets are independently distributed.

- If they form a mutual fund so that each agent owns half of each asset, each gets



Insurance

- We have an agent with $u(x) = \sqrt{x}$ and



- And a risk-neutral insurance company with lots of money, selling full insurance for "premium" P.