14.12 Game Theory Lecture Notes
Introduction

(Lecture 1)

Game Theory is a misnomer for Multiperson Decision Theory, analyzing the decision-making process when there are more than one decision-makers where each agent's payoff possibly depends on the actions taken by the other agents. Since an agent's preferences on his actions depend on which actions the other parties take, his action depends on his beliefs about what the others do. Of course, what the others do depends on their beliefs about what each agent does. In this way, a player's action, in principle, depends on the actions available to each agent, each agent's preferences on the outcomes, each player's beliefs about which actions are available to each player and how each player ranks the outcomes, and further his beliefs about each player's beliefs, ad infinitum.

Under perfect competition, there are also more than one (in fact, infinitely many) decision makers. Yet, their decisions are assumed to be decentralized. A consumer tries to choose the best consumption bundle that he can afford, given the prices – without paying attention what the other consumers do. In reality, the future prices are not known. Consumers' decisions depend on their expectations about the future prices. And the future prices depend on consumers' decisions today. Once again, even in perfectly competitive environments, a consumer's decisions are affected by their beliefs about what other consumers do – in an aggregate level.

When agents think through what the other players will do, taking what the other players think about them into accout, they may find a clear way to play the game. Consider the following "game":

1 \ 2	L	m	R
T	$(1,1)$	$(0,2)$	$(2,1)$
M	$(2,2)$	$(1,1)$	$(0,0)$
B	$(1,0)$	$(0,0)$	$(-1,1)$

Here, Player 1 has strategies, T, M, B and Player 2 has strategies L, m, R. (They pick their strategies simultaneously.) The payoffs for players 1 and 2 are indicated by the numbers in parentheses, the first one for player 1 and the second one for player 2. For instance, if Player 1 plays T and Player 2 plays R, then Player 1 gets a payoff of 2 and Player 2 gets 1. Let's assume that each player knows that these are the strategies and the payoffs, each player knows that each player knows this, each player knows that each player knows that each player knows this,... ad infinitum. [In that case, we formally say that the strategies and the payoffs are *common knowledge*.]

Now, player 1 looks at his payoffs, and realizes that, no matter what the other player plays, it is better for him to play M rather than B. That is, if 2 plays L, M gives 2 and B gives 1; if 2 plays m, M gives 1, B gives 0; and if 2 plays R, M gives 0, B gives -1. Therefore, he realizes that he should not play B.[1] Now he compares T and M. He realizes that, if Player 2 plays L or m, M is better than T, but if she plays R, T is definitely better than M. Would Player 2 play R? What would she play? To find an answer to these questions, Player 1 looks at the game from Player 2's point of view. He realizes that, for Player 2, there is no strategy that is outright better than any other strategy. For instance, R is the best strategy if 1 plays B, but otherwise it is strictly worse than m. Would Player 2 think that Player 1 would play B? Well, she knows that Player 1 is trying to maximize his expected payoff, given by the first entries as everyone knows. She must then deduce that Player 1 will not play B. Therefore, Player 1 concludes, she will not play R (as it is worse than m in this case). Ruling out the possibility that Player 2 plays R, Player 1 looks at his payoffs, and sees that M is now better than T, no matter what. On the other side, Player 2 goes through similar reasoning, and concludes that 1 must play M, and therefore plays L.

This kind of reasoning does not always yield such a clear prediction. Imagine that you want to meet with a friend in one of two places, about which you both are indifferent.

---

[1] After all, he cannot have any belief about what Player 2 plays that would lead him to play B when M is available.

Unfortunately, you cannot communicate with each other until you meet. This situation is formalized in the following game, which is called *pure coordination game*:

1 \ 2	Left	Right
Top	(1,1)	(0,0)
Bottom	(0,0)	(1,1)

Here, Player 1 chooses between Top and Bottom rows, while Player 2 chooses between Left and Right columns. In each box, the first and the second numbers denote the von Neumann-Morgenstern utilities of players 1 and 2, respectively. Note that Player 1 prefers Top to Bottom if he knows that Player 2 plays Left; he prefers Bottom if he knows that Player 2 plays Right. He is indifferent if he knows thinks that the other player is likely to play either strategy with equal probabilities. Similarly, Player 2 prefers Left if she knows that player 1 plays Top. There is no clear prediction about the outcome of this game.

One may look for the *stable* outcomes (strategy profiles) in the sense that no player has incentive to deviate if he knows that the other players play the prescribed strategies. Here, Top-Left and Bottom-Right are such outcomes. But Bottom-Left and Top-Right are not stable in this sense. For instance, if Bottom-Left is known to be played, each player would like to deviate – as it is shown in the following figure:

1 \ 2	Left	Right
Top	(1,1)	⇐⇓(0,0)
Bottom	(0,0)⇑⟹	(1,1)

(Here, ⇑ means player 1 deviates to Top, etc.)

Unlike in this game, mostly players have different preferences on the outcomes, inducing *conflict*. In the following game, which is known as the *Battle of the Sexes*, conflict and the need for coordination are present together.

1 \ 2	Left	Right
Top	(2,1)	(0,0)
Bottom	(0,0)	(1,2)

Here, once again players would like to coordinate on Top-Left or Bottom-Right, but now Player 1 prefers to coordinate on Top-Left, while Player 2 prefers to coordinate on Bottom-Right. The stable outcomes are again Top-Left and Bottom- Right.

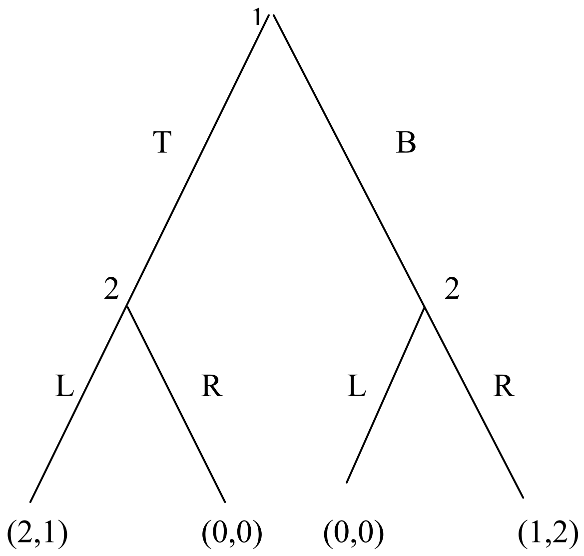


Figure 1:

Now, in the Battle of the Sexes, imagine that Player 2 knows what Player 1 does when she takes her action. This can be formalized via the following tree:

Here, Player 1 chooses between Top and Bottom, then (knowing what Player 1 has chosen) Player 2 chooses between Left and Right. Clearly, now Player 2 would choose Left if Player 1 plays Top, and choose Right if Player 1 plays Bottom. Knowing this, Player 1 would play Top. Therefore, one can argue that the only reasonable outcome of this game is Top-Left. (This kind of reasoning is called *backward induction.*)

When Player 2 is to check what the other player does, he gets only 1, while Player 1 gets 2. (In the previous game, two outcomes were stable, in which Player 2 would get 1 or 2.) That is, Player 2 prefers that Player 1 has information about what Player 2 does, rather than she herself has information about what player 1 does. When it is common knowledge that a player has some information or not, the player may prefer not to have that information – a robust fact that we will see in various contexts.

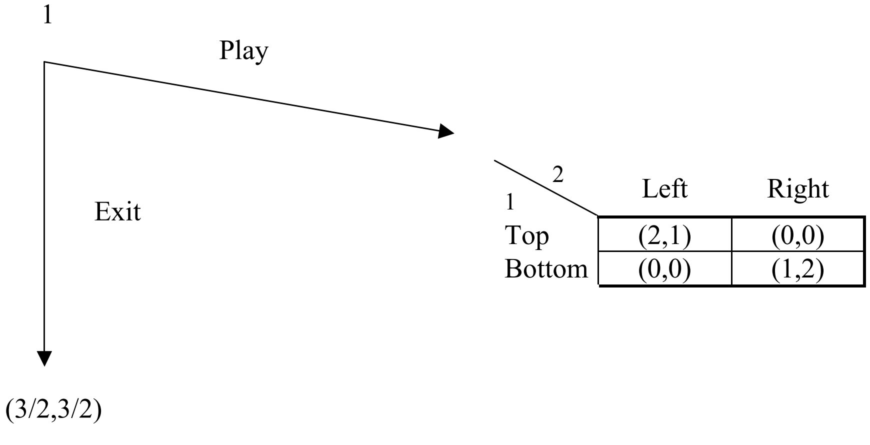
Exercise 1 *Clearly, this is generated by the fact that Player 1 knows that Player 2 will know what Player 1 does when she moves. Consider the situation that Player 1 thinks that Player 2 will know what Player 1 does only with probability* $\pi < 1$*, and this probability does not depend on what Player 1 does. What will happen in a "reasonable" equilibrium? [By the end of this course, hopefully, you will be able to formalize this* *situation, and compute the equilibria.]*

Another interpretation is that Player 1 can communicate to Player 2, who cannot communicate to player 1. This enables player 1 to commit to his actions, providing a strong position in the relation.

Exercise 2 *Consider the following version of the last game: after knowing what Player 2 does, Player 1 gets a chance to change his action; then, the game ends. In other words, Player 1 chooses between Top and Bottom; knowing Player 1's choice, Player 2 chooses between Left and Right; knowing 2's choice, Player 1 decides whether to stay where he is or to change his position. What is the "reasonable" outcome? What would happen if changing his action would cost player 1 c utiles?*

Imagine that, before playing the Battle of the Sexes, Player 1 has the option of exiting, in which case each player will get 3/2, or playing the Battle of the Sexes. When asked to play, Player 2 will know that Player 1 chose to play the Battle of the Sexes.

There are two "reasonable" equilibria (or stable outcomes). One is that Player 1 exits, thinking that, if he plays the Battle of the Sexes, they will play the Bottom-Right equilibrium of the Battle of the Sexes, yielding only 1 for player 1. The second one is that Player 1 chooses to Play the Battle of the Sexes, and in the Battle of the Sexes they play Top-Left equilibrium.



Some would argue that the first outcome is not really reasonable. Because, when asked to play, Player 2 will know that Player 1 has chosen to play the Battle of the Sexes, forgoing the payoff of 3/2. She must therefore realize that Player 1 cannot possibly be planning to play Bottom, which yields the payoff of 1 max. That is, when asked to play, Player 2 should understand that Player 1 is planning to play Top, and thus she should play Left. Anticipating this, Player 1 should choose to play the Battle of the Sexes game, in which they play Top-Left. Therefore, the second outcome is the only reasonable one. (This kind of reasoning is called *Forward Induction.*)

Here are some more exmples of games:

1. Prisoners' Dilemma:

1 \ 2	Confess	Not Confess
Confess	(-1, -1)	(1, -10)
Not Confess	(-10, 1)	(2, 2)

This is a well known game that most of you know. [It is also discussed in Gibbons.] In this game no matter what the other player does, each player would like to confess, yielding (-1,-1), which is dominated by (2,2).

2. Hawk-Dove game

1 \ 2	Hawk	Dove
Hawk	$\left(\frac{V-C}{2}, \frac{V-C}{2}\right)$	$(V, 0)$
Dove	$(0,V)$	$(\frac{V}{2},\frac{V}{2})$

This is a generic biological game, but is also quite similar to many games in economics and political science. V is the value of a resource that one of the players will enjoy. If they shared the resource, their values are $V/2$. Hawk stands for a "tough" strategy, whereby the player does not give up the resource. However, if the other player is also playing hawk, they end up fighting, and incur the cost $C/2$ each. On the other hand, a Hawk player gets the whole resource for itself when playing a Dove. When $V > C$, we have a Prisoners' Dilemma game, where we would observe fight.

When we have $V < C$, so that fighting is costly, this game is similar to another well-known game, inspired by the movie Rebel Without a Cause, named "Chicken", where two players driving towards a cliff have to decide whether to stop or continue. The one who stops first loses face, but may save his life. More generally, as class of games called "wars of attrition" are used to model this type of situations. In this case, a player would like to play Hawk if his opponent plays Dove, and play Dove if his opponent playes Hawk.