Lecture 5-6
Applications of Nash equilibrium
Rationalizablity & Backwards Induction

14.12 Game Theory

Road Map

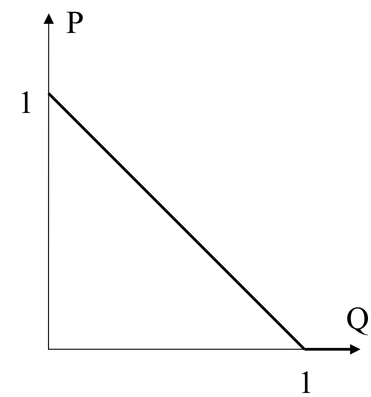
1. Cournot (quantity) Competition
   1. Nash Equilibrium in Cournot duopoly
   2. Nash Equilibrium in Cournot oligopoly
   3. Rationalizability in Cournot duopoly
2. Bertrand (price) Competition
3. Commons Problem
4. Quiz
5. Mixed-strategy Nash equilibrium
6. Backwards induction

Cournot Oligopoly

- $N = \{1,2,\ldots,n\}$ firms;
- Simultaneously, each firm i produces q_i units of a good at marginal cost c,
- and sells the good at price

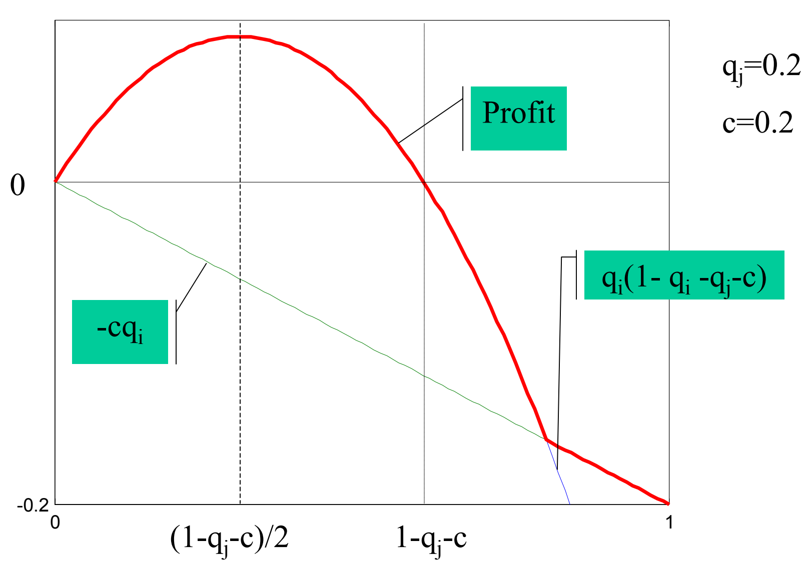
  $$P = \max\{0,1-Q\}$$

  where $Q = q_1+\ldots+q_n$.
- Game = $(S_1,\ldots,S_n; \pi_1,\ldots,\pi_n)$ where $S_i = [0,\infty)$,



$$\pi_i(q_1,\ldots,q_n) = \begin{cases} q_i[1-(q_1+\ldots+q_n)-c] & \text{if } q_1+\ldots+qn < 1, \\ -q_i c & \text{otherwise.} \end{cases}$$

Cournot Duopoly -- profit

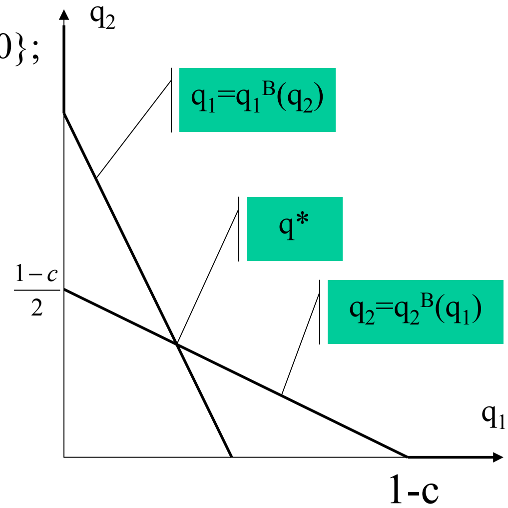


C-D – best responses

- $q_i^B(q_j) = \max\{(1-q_j-c)/2, 0\}$;
- Nash Equilibrium q*:

  $q_1^* = (1-q_2^*-c)/2$;

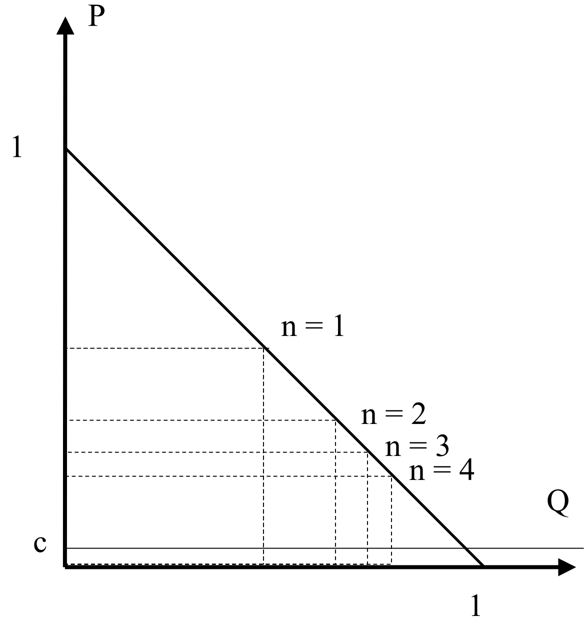
  $q_2^* = (1-q_1^*-c)/2$;
- $q_1^* = q_2^* = (1-c)/3$



Cournot Oligopoly --Equilibrium

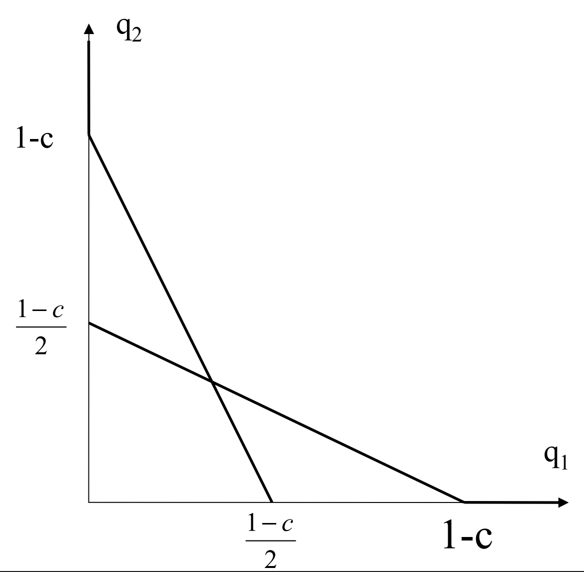
- $q > 1-c$ is strictly dominated, so $q \le 1-c$.
- $\pi_i(q_1,\ldots,q_n) = q_i[1-(q_1+\ldots+q_n)-c]$ for each i.
- FOC: $$\left.\frac{\partial \pi_i(q_1,\ldots,q_n)}{\partial q_i}\right|_{q=q^*} = \left.\frac{\partial[q_i(1-q_1-\cdots-q_n-c)]}{\partial q_i}\right|_{q=q^*}$$ $$= (1-q_1^*-\cdots-q_n^*-c)-q_i^* = 0.$$
- That is, $$\begin{aligned} 2q_1^* + q_2^* + \cdots + q_n^* &= 1-c \\ q_1^* + 2q_2^* + \cdots + q_n^* &= 1-c \\ &\vdots \\ q_1^* + q_2^* + \cdots + nq_n^* &= 1-c \end{aligned}$$
- Therefore, $q_1^* = \ldots = q_n^* = (1-c)/(n+1)$.

Cournot oligopoly – comparative statics



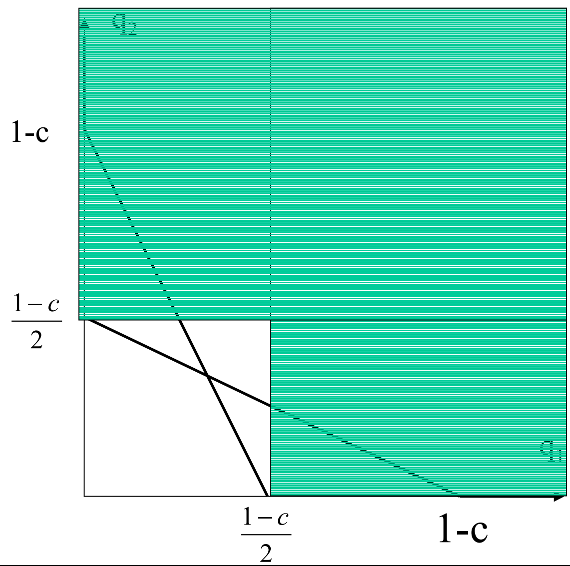
Rationalizability in Cournot Duopoly

Assume that players are rational.



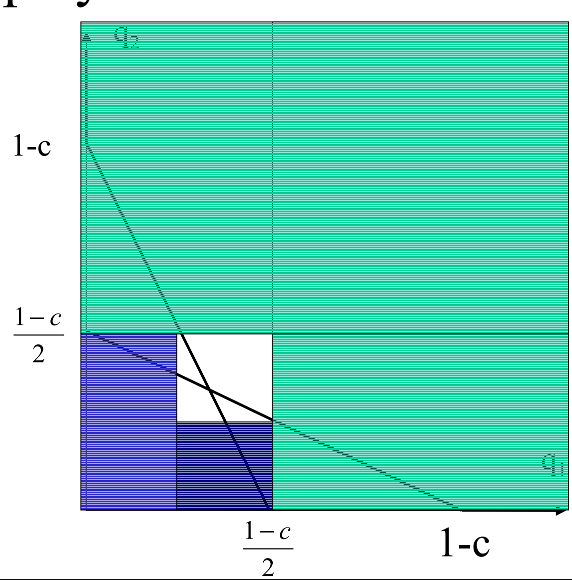
Players are rational:

Assume that players know this.



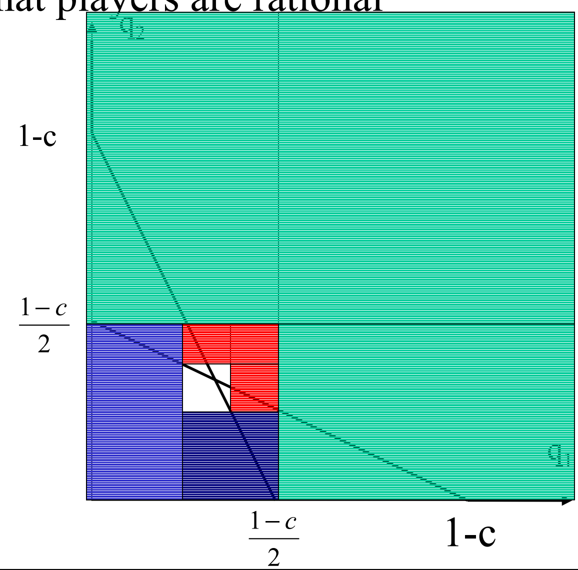
Players are rational and know that players are rational

Assume that players know this.



Players are rational; players know that players are rational; players know that players know that players are rational

Assume that players know this.



Rationalizability in Cournot duopoly

- If i knows that $q_j \le q$, then $q_i \ge (1\text{-}c\text{-}q)/2$.
- If i knows that $q_j \ge q$, then $q_i \le (1\text{-}c\text{-}q)/2$.
- We know that $q_j \ge q^0 = 0$.
- Then, $q_i \le q^1 = (1\text{-}c\text{-}q^0)/2 = (1\text{-}c)/2$ for each i;
- Then, $q_i \ge q^2 = (1\text{-}c\text{-}q^1)/2 = (1\text{-}c)(1\text{-}1/2)/2$ for each i;
- …
- Then, $q^n \le q_i \le q^{n+1}$ or $q^{n+1} \le q_i \le q^n$ where
  $q^{n+1} = (1\text{-}c\text{-}q^n)/2 = (1\text{-}c)(1\text{-}1/2\text{+}1/4\text{-}\ldots\text{+}(\text{-}1/2)^n)/2$.
- As $n \to \infty$, $q^n \to (1\text{-}c)/3$.

Bertrand (price) competition

- N = {1,2} firms.
- Simultaneously, each firm i sets a price p_i;
- If $p_i < p_j$, firm i sells $Q = \max\{1 - p_i, 0\}$ unit at price p_i; the other firm gets 0.
- If $p_1 = p_2$, each firm sells Q/2 units at price p_1, where $Q = \max\{1 - p_1, 0\}$.
- The marginal cost is 0.

$$\pi_1(p_1, p_2) = \begin{cases} p_1(1-p_1) & \text{if } p_1 < p_2 \\ p_1(1-p_1)/2 & \text{if } p_1 = p_2 \\ 0 & \text{otherwise.} \end{cases}$$

Bertrand duopoly -- Equilibrium

Theorem: The only Nash equilibrium in the "Bertrand game" is $p^* = (0,0)$.

Proof:

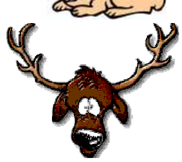
1. $p^*=(0,0)$ is an equilibrium.
2. If $p = (p_1,p_2)$ is an equilibrium, then $p = p^*$.
   1. If $p = (p_1,p_2)$ is an equilibrium, then $p_1 = p_2$...
   2. Given any equilibrium $p = (p_1,p_2)$ with $p_1 = p_2$, $p = p^*$.

Commons Problem

- N = {1,2,…,n} players, each with unlimited money;
- Simultaneously, each player i contributes $x_i \geq 0$ to produce $y = x_1 + \ldots x_n$ unit of some public good, yielding payoff

$$U_i(x_i, y) = y^{1/2} - x_i.$$

Stag Hunt

		
	(2,2)	(4,0)
	(0,4)	(5,5)

Equilibrium in Mixed Strategies

What is a strategy?

– A complete contingent-plan of a player.

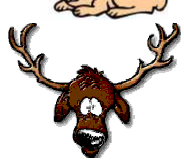
– What the others think the player might do under various contingency.

What do we mean by a mixed strategy?

– The player is randomly choosing his pure strategies.

– The other players are not certain about what he will do.

Stag Hunt

		
	(2,2)	(4,0)
	(0,4)	(5,5)

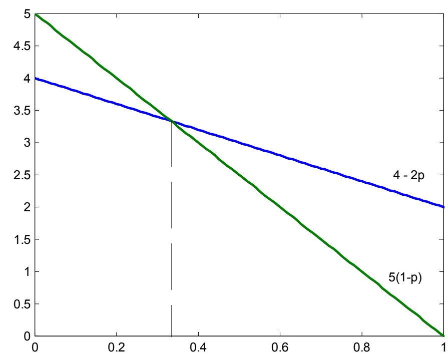
Mixed-strategy equilibrium in Stag-Hunt game

- Assume: Player 2 thinks that, with probability p, Player 1 targets for Rabbit. What is the best probability q she wants to play Rabbit?
- His payoff from targeting Rabbit:

  $U_2(R;p) = 2p + 4(1-p)$
  $= 4-2p.$
- From Stag:

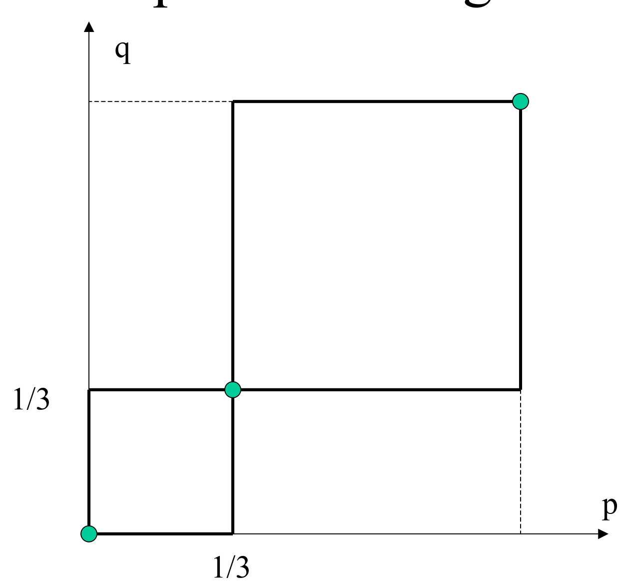
  $U_2(S;p) = 5(1-p)$
- She is indifferent iff

  $4-2p = 5(1-p)$ iff $p = 1/3$.



$$q^{BR}(p) = \begin{cases} 0 & \text{if } p < 1/3 \\ q \in [0,1] & \text{if } p = 1/3 \\ 1 & \text{if } p > 1/3 \end{cases}$$

Best responses in Stag-Hunt game



Bertrand Competition with costly search

- N = {F1,F2,B}; F1, F2 are firms; B is buyer
- B needs 1 unit of good, worth 6;
- Firms sell the good; Marginal cost = 0.
- Possible prices P = {1,5}.
- Buyer can check the prices with a small cost $c > 0$.

Game:

1. Each firm i chooses price p_i;
2. B decides whether to check the prices;
3. (Given) If he checks the prices, and $p_1 \neq p_2$, he buys the cheaper one; otherwise, he buys from any of the firm with probability ½.

Bertrand Competition with costly search

F1 \ F2	High	Low
High		
Low		

Check

F1 \ F2	High	Low
High		
Low		

Don't Check

Mixed-strategy equilibrium

- Symmetric equilibrium: Each firm charges "High" with probability q;
- Buyer Checks with probability r.
- $U(\text{check};q) = q^2 1 + (1-q^2)5 – c = 5 - 4 q^2 – c$;
- $U(\text{Don't};q) = q1 + (1-q)5 = 5 - 4 q$;
- Indifference: $4q(1-q) = c$; i.e.,
- $U(\text{high};q,r) = 0.5(1-r(1-q))5$;
- $U(\text{low};q,r) = qr1 + 0.5(1-qr)$
- Indifference = $r = 4/(5-4q)$.

Dynamic Games of Perfect Information & Backward Induction

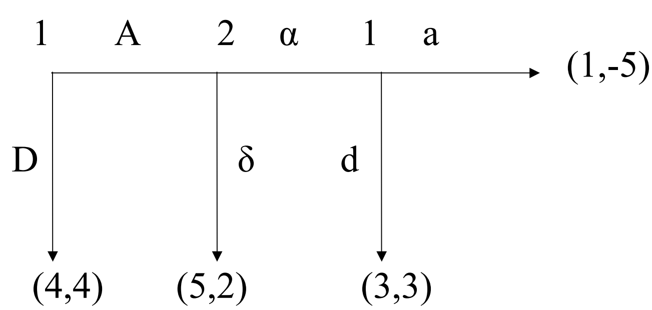
Definitions

Perfect-Information game is a game in which all the information sets are singleton.

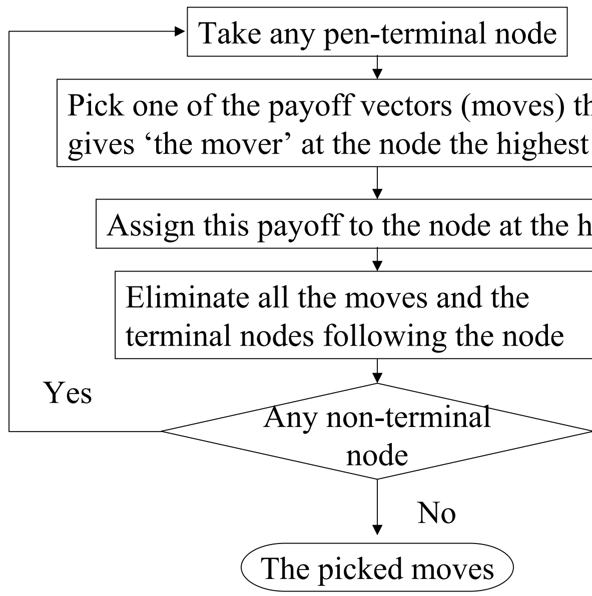
Sequential Rationality: A player is sequentially rational iff, at each node he is to move, he maximizes his expected utility conditional on that he is at the node – even if this node is precluded by his own strategy.

In a finite game of perfect information, the "common knowledge" of sequential rationality gives **"Backward Induction"** outcome.

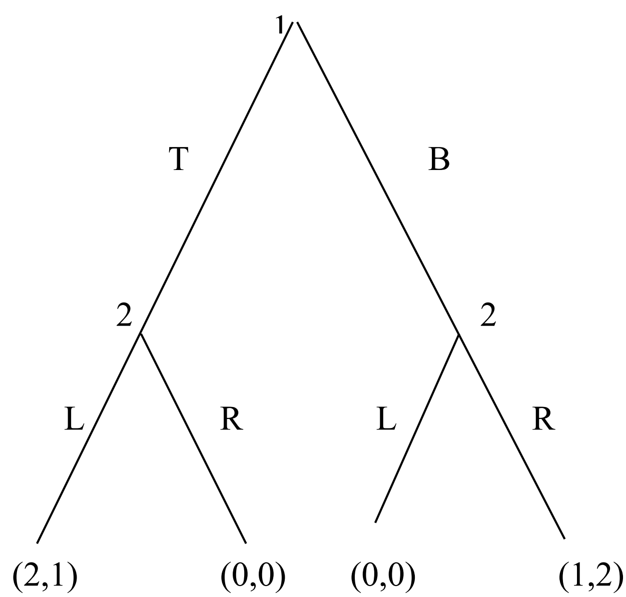
A centipede game



Backward Induction



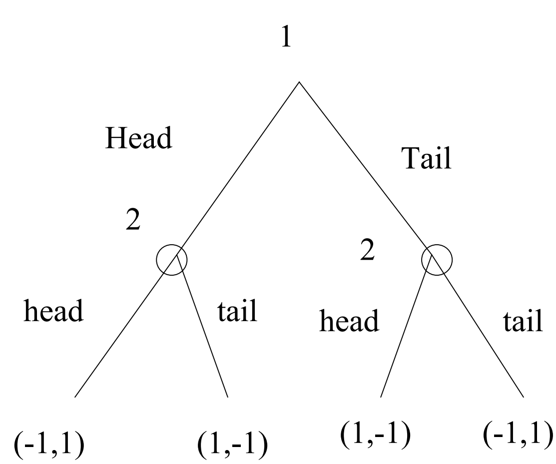
Battle of The Sexes with perfect information



Note

- There are Nash equilibria that are different from the Backward Induction outcome.
- Backward Induction always yields a Nash Equilibrium.
- That is, Sequential rationality is stronger than rationality.

Matching Pennies (wpi)



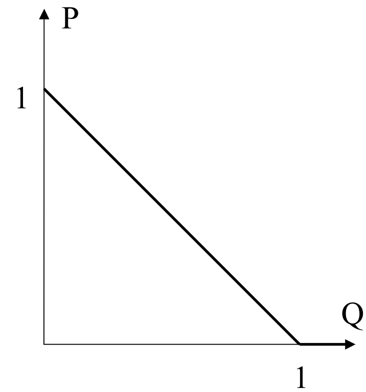
Stackelberg Duopoly

Game:

$N = \{1,2\}$ firms w $MC = 0$;

1. Firm 1 produces q_1 units
2. Observing q_1, Firm 2 produces q_2 units
3. Each sells the good at price

   $P = \max\{0,1-(q_1+q_2)\}$.



$$\pi_i(q_1, q_2) = \begin{cases} q_i[1-(q_1+q_2)] & \text{if } q_1+ q_2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

"Stackelberg equilibrium"

- If $q_1 > 1$, $q_2^*(q_1) = 0$.
- If $q_1 \leq 1$, $q_2^*(q_1) = (1-q_1)/2$.
- Given the function q_2^*, if $q_1 \leq 1$

  $\pi_1(q_1;q_2^*(q_1)) = q_1[1-(q_1+ (1-q_1)/2)]$

  $= q_1 (1-q_1)/2$;

  0 otherwise.
- $q_1^* = ½$.
- $q_2^*(q_1^*) = ¼$.