Lectures 10 -11

Repeated Games

14.12 Game Theory

Road Map

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2. Finitely Repeated Games with observable actions
   1. Entry-Deterrence/Chain-store paradox
   2. Repeated Prisoners' Dilemma
   3. A general result
   4. When there are multiple equilibria
3. Infinitely repeated games with observable actions
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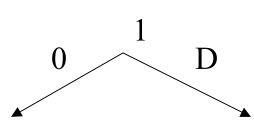
   2. Examples
   3. The Folk Theorem
   4. Repeated Prisoners' Dilemma, revisited –tit for tat
   5. Repeated Cournot oligopoly

Forward Induction

Strong belief in rationality: At any history of the game, each agent is assumed to be rational if possible. (That is, if there are two strategies s and s' of a player i that are consistent with a history of play, and if s is strictly dominated but s' is not, at this history no player j believes that i plays s.)

Burning Money



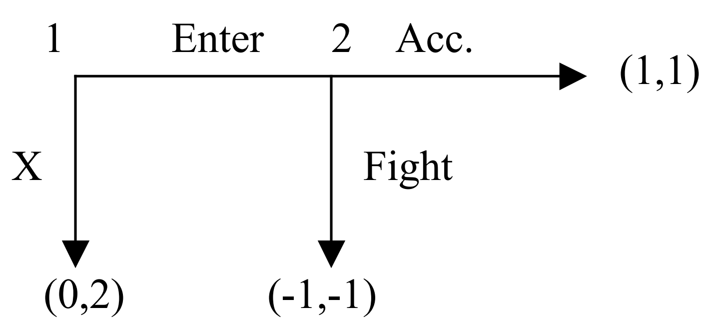
	B	S
B	3,1	.1,.1
S	.1,.1	1,3

	B	S
B	2,1	-.9,.1
S	-.9,.1	0,3

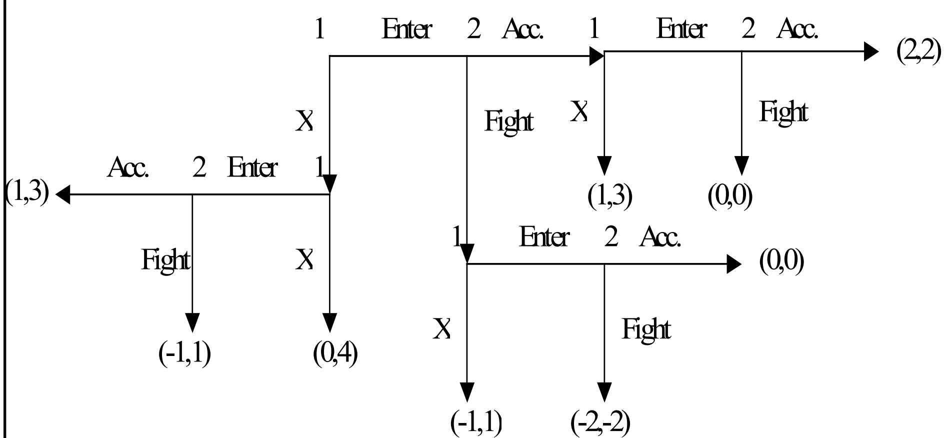
	BB	BS	SB	SS
0B	3,1	3,1	.1,.1	.1,.1
0S	.1,.1	.1,.1	1,3	1,3
DB	2,1	-.9,.1	2,1	-.9,.1
DS	-.9,.1	0,3	-.9,.1	0,3
O	T	H	E	R

Repeated Games

Entry deterrence



Entry deterrence, repeated twice, many times



What would happen if repeated n times?

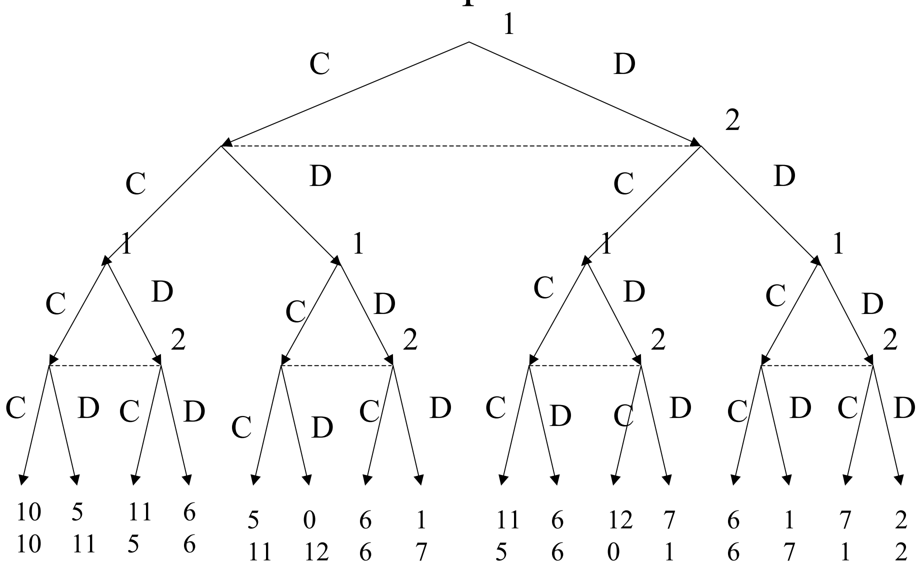
Prisoners' Dilemma, repeated twice, many times

- Two dates T = {0,1};
- At each date the prisoners' dilemma is played:

	C	D
C	5,5	0,6
D	6,0	1,1

- At the beginning of 1 players observe the strategies at 0. Payoffs= sum of stage payoffs.

Twice-repeated PD



What would happen if T = {0,1,2,…,n}?

A general result

- G = "stage game" = a finite game
- T = {0,1,…,n}
- At each t in T, G is played, and players remember which actions taken before t;
- Payoffs = Sum of payoffs in the stage game.
- Call this game G(T).

Theorem: If G has a unique subgame-perfect equilibrium s*, G(T) has a unique subgame-perfect equilibrium, in which s* is played at each stage.

With multiple equilibria

T = {0,1}

1 \ 2	L	M2	R
T	1,1	5,0	0,0
M1	0,5	4,4	0,0
B	0,0	0,0	3,3

Infinitely repeated Games with observable actions

- T = {0,1,2,…,t,…}
- G = "stage game" = a finite game
- At each t in T, G is played, and players remember which actions taken before t;
- Payoffs = Discounted sum of payoffs in the stage game.
- Call this game G(T).

Definitions

The *Present Value* of a given payoff stream $\pi =$ $(\pi_0,\pi_1,\ldots,\pi_t,\ldots)$ is

$$PV(\pi;\delta) = \Sigma^{\infty}_{t=1}\, \delta^t\pi_t = \pi_0 + \delta\pi_1 + \ldots + \delta^t\pi_t + \ldots$$

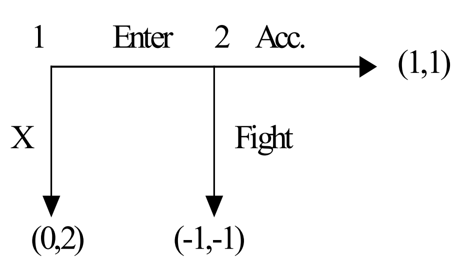
The *Average Value* of a given payoff stream π is

$$(1-\delta)PV(\pi;\delta) = (1-\delta)\Sigma^{\infty}_{t=1}\, \delta^t\pi_t$$

The *Present Value* of a given payoff stream π at t is

$$PV_t(\pi;\delta) = \Sigma^{\infty}_{s=t}\, \delta^{s-t}\, \pi_s = \pi_t + \delta\pi_{t+1} + \ldots + \delta^s\pi_{t+s} + \ldots$$

Infinite-period entry deterrence



Strategy of Entrant:
Enter iff
Accomodated before.

Strategy of Incumbent:
Accommodate iff
accomodated before.

Incumbent:

- V(Acc.) = V_A =
- V(Fight) = V_F =
- Case 1: Accommodated before.
  - Fight =>
  - Acc. =>
- Case 2: Not Accommodated
  - Fight =>
  - Acc. =>
  - Fight ⇔

Entrant:

- Accommodated
  - Enter =>
  - X =>
- Not Acc.
  - Enter =>
  - X =>

Infinitely-repeated PD

	C	D
C	5,5	0,6
D	6,0	1,1

A Grimm Strategy: Defect iff someone defected before.

- $V_D = 1/(1-\delta)$;
- $V_C = 5/(1-\delta) = 5V_D$;
- Defected before (easy)
- Not defected
  - D =>
  - C =>
  - C ⇔

Tit for Tat

- Start with C; thereafter, play what the other player played in the previous round.
- Is (Tit-for-tat,Tit-for-tat) a SPE?

- **Modified:** Start with C; if any player plays D when the previous play is (C,C), play D in the next period, then switch back to C.

Folk Theorem

Definition: A payoff vector $v = (v_1,v_2,\ldots,v_n)$ is feasible iff v is a convex combination of some pure-strategy payoff-vectors, i.e.,

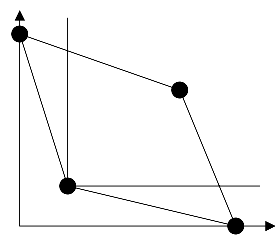
$$v = p_1u(a^1) + p_2u(a^2) +\ldots+ p_ku(a^k),$$

where $p_1 + p_2 +\ldots+ p_k = 1$, and $u(a^j)$ is the payoff vector at strategy profile a^j of the stage game.

Theorem: Let $x = (x_1,x_2,\ldots,x_n)$ be s feasible payoff vector, and $e = (e_1,e_2,\ldots,e_n)$ be a payoff vector at some equilibrium of the stage game such that $x_i > e_i$ for each i. Then, there exist $\underline{\delta} < 1$ and a strategy profile s such that s yields x as the expected average-payoff vector and is a SPE whenever $\delta \geq \underline{\delta}$.

Folk Theorem in PD

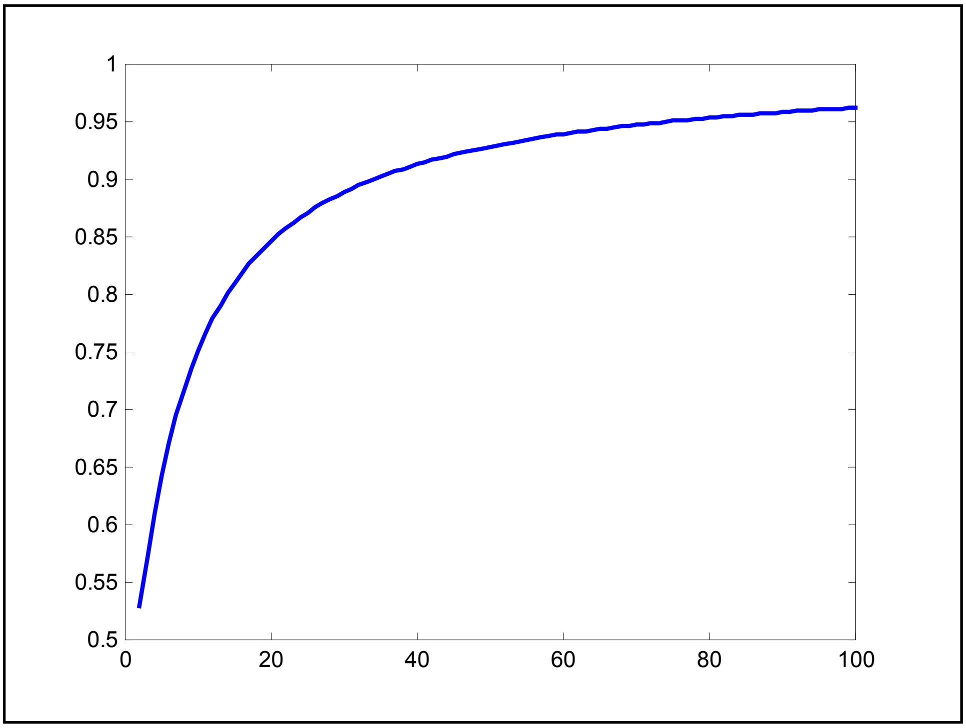
	C	D
C	5,5	0,6
D	6,0	1,1



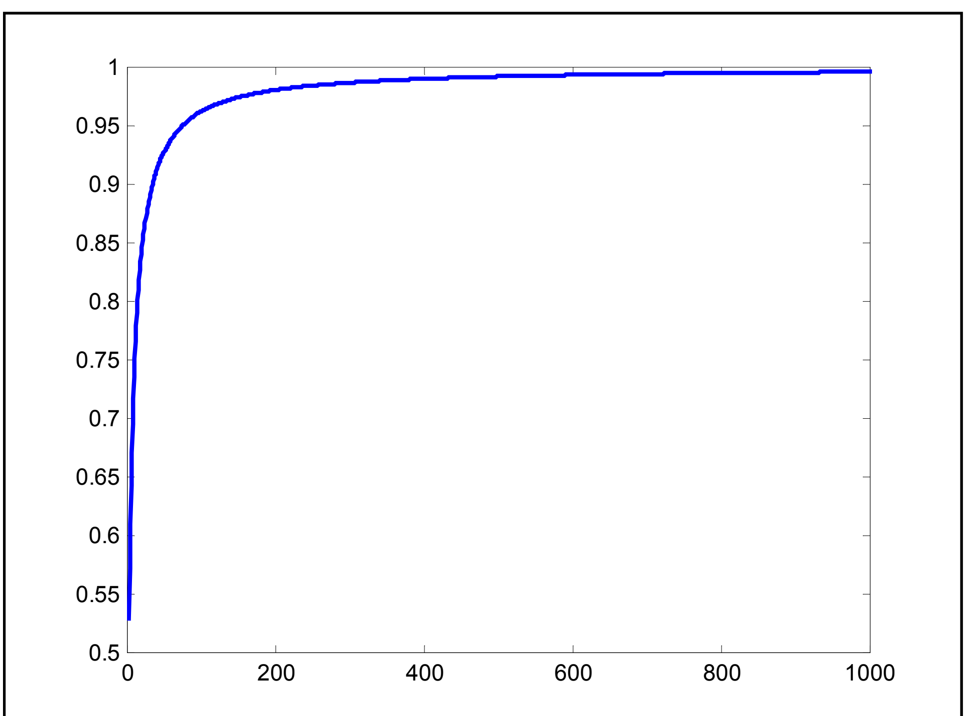
- A SPE with PV (1.1,1.1)?
  - With PV (1.1,5)?
  - With PV (6,0)?
  - With PV (5.9,0.1)?

Infinitely-repeated Cournot oligopoly

- N firms, MC = 0; P = max{1-Q,0};
- Strategy: Each is to produce q = 1/(2n); if any firm defects produce q = 1/(1+n) forever.
- V_C =
- V_D =
- V(D|C) =
- Equilibrium ⇔



1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0
20
40
60
80
100



1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0
200
400
600
800
1000

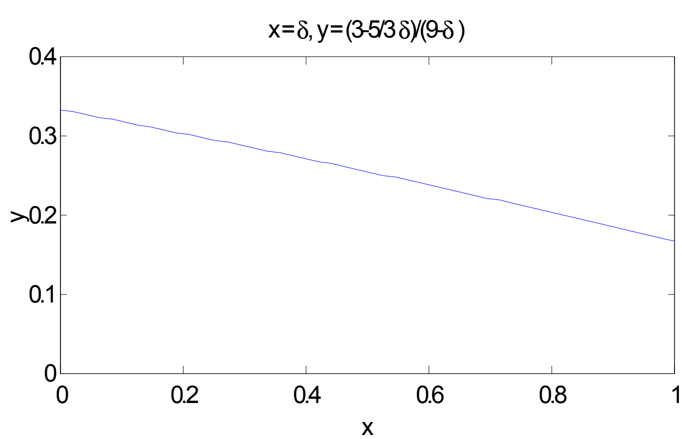
IRCD (n=2)

- Strategy: Each firm is to produce q*; if any one deviates, each produce 1/(n+1) thereafter.
- $V_C = q^*(1-2q^*)/(1-\delta)$;
- $V_D = 1/(9(1-\delta))$;
- $V_{D|C} = \max q(1-q^*-q) + \delta V_D = (1-q^*)^2/4 + \frac{\delta}{9(1-\delta)}$
- Equilibrium iff

$$q^*(1-2q^*) \geq (1-\delta)(1-q^*)^2/4 + \delta/9$$

- $\Leftrightarrow$

$$q^* \geq \frac{9-5\delta}{3(9-\delta)}$$



Carrot and Stick

Produce ¼ at the beginning; at ant $t > 0$, produce ¼ if both produced ¼ or both produced x at t-1; otherwise, produce x.

Two Phase: Cartel & Punishment

$V_C = 1/8(1-\delta)$. $V_x = x(1-2x) + \delta V_C$.

$V_{D|C} = \max q(1-1/4-q) + \delta V_X = (3/8)^2 + \delta V_X$

$V_{D|x} = \max q(1-x-q) + \delta V_X = (1-x)^2/4 + \delta V_X$

$V_C \geq V_{D|C} \Leftrightarrow V_C \geq (3/8)^2 + \delta^2 V_C + \delta\, x(1-2x)$

$\Leftrightarrow (1-\delta^2)\, V_C - (3/8)^2 \geq \delta\, x(1-2x) \Leftrightarrow (1+\delta)/8 - (3/8)^2 \geq \delta\, x(1-2x)$

$V_X \geq V_{D|C} \Leftrightarrow (1-\delta)V_x \geq (1-x)^2/4 \Leftrightarrow (1-\delta)(x(1-2x) + \delta/8(1-\delta)) \geq (1-x)^2/4$
$\Leftrightarrow (1-\delta)x(1-2x) + \delta/8 \geq (1-x)^2/4$

$$2x^2 - x + 1/8 - 9/64\delta \geq 0$$

$$(9/4-2\delta)x^2 - (3-2\delta)x + \delta/8(1-\delta) \leq 0$$