Lecture 6
Applications of Nash equilibrium

14.12 Game Theory

Road Map

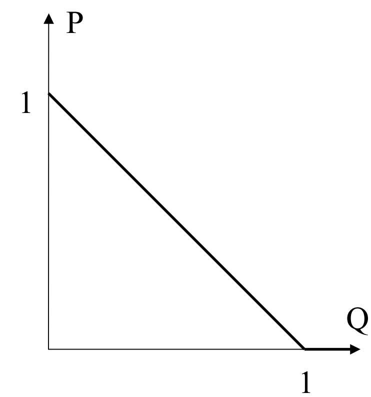
1. Cournot (quantity) Competition
   1. Nash Equilibrium in Cournot oligopoly
2. Bertrand (price) Competition
3. Commons Problem
4. Quiz
5. Mixed-strategy Nash equilibrium

Cournot Oligopoly

- $N = \{1,2,\ldots,n\}$ firms;
- Simultaneously, each firm i produces q_i units of a good at marginal cost c,
- and sells the good at price

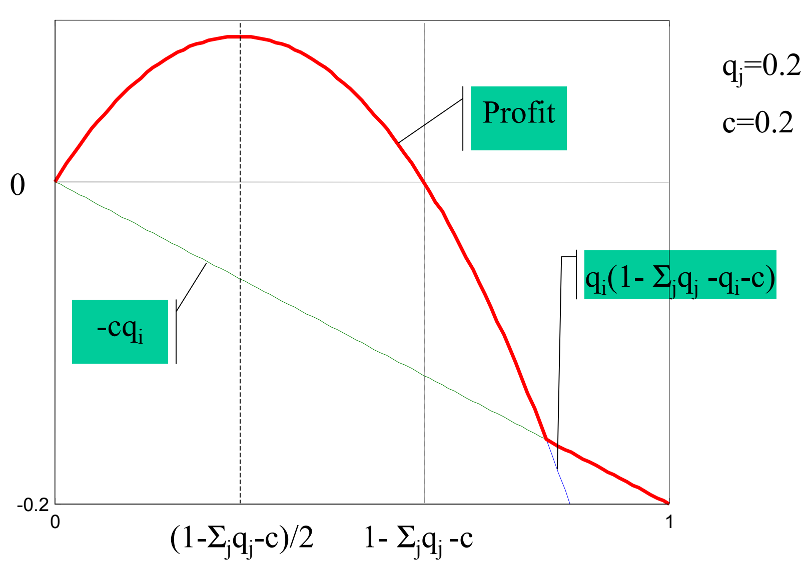
$$P = \max\{0, 1-Q\}$$

where $Q = q_1+\ldots+q_n$.
- Game $= (S_1,\ldots,S_n; \pi_1,\ldots,\pi_n)$ where $S_i = [0,\infty)$,



$$\pi_i(q_1,\ldots,q_n) = \begin{cases} q_i[1-(q_1+\ldots+q_n)-c] & \text{if } q_1+\ldots+qn < 1, \\ -q_i c & \text{otherwise.} \end{cases}$$

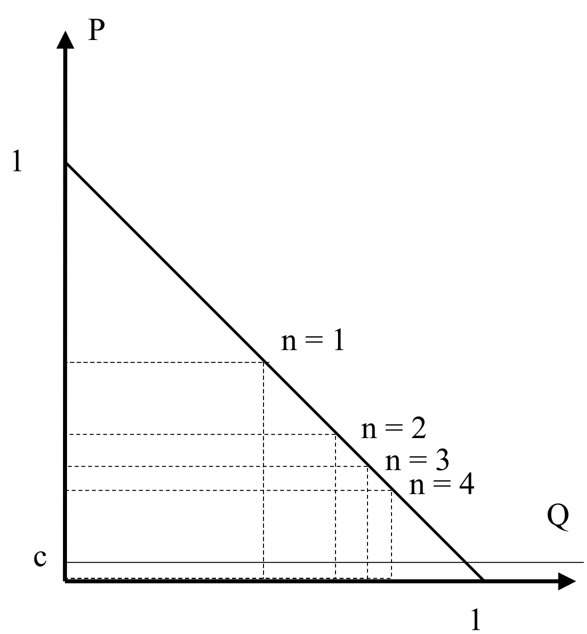
Cournot Oligopoly -- profit



Cournot Oligopoly --Equilibrium

- q>1-c is strictly dominated, so q ≤ 1-c.
- $\pi_i(q_1,\ldots,q_n) = q_i[1-(q_1+\ldots+q_n)-c]$ for each i.
- FOC: $$\left.\frac{\partial \pi_i(q_1,\ldots,q_n)}{\partial q_i}\right|_{q=q^*} = \left.\frac{\partial[q_i(1-q_1-\cdots-q_n-c)]}{\partial q_i}\right|_{q=q^*}$$
$$=(1-q_1^*-\cdots-q_n^*-c)-q_i^*=0.$$
- That is, $$\begin{aligned} 2q_1^*+q_2^*+\cdots+q_n^* &= 1-c \\ q_1^*+2q_2^*+\cdots+q_n^* &= 1-c \\ &\vdots \\ q_1^*+q_2^*+\cdots+nq_n^* &= 1-c \end{aligned}$$
- Therefore, $q_1^*=\ldots=q_n^*=(1-c)/(n+1)$.

Cournot oligopoly – comparative statics



Bertrand (price) competition

- N = {1,2} firms.
- Simultaneously, each firm i sets a price p_i;
- If $p_i < p_j$, firm i sells Q = max{1 – p_i,0} unit at price p_i; the other firm gets 0.
- If $p_1 = p_2$, each firm sells Q/2 units at price p_1, where Q = max{1 – p_1,0}.
- The marginal cost is 0.

$$\pi_1(p_1, p_2) = \begin{cases} p_1(1-p_1) & \text{if } p_1 < p_2 \\ p_1(1-p_1)/2 & \text{if } p_1 = p_2 \\ 0 & \text{otherwise.} \end{cases}$$

Bertrand duopoly -- Equilibrium

Theorem: The only Nash equilibrium in the "Bertrand game" is p* = (0,0).

Proof:

1. p*=(0,0) is an equilibrium.
2. If p = (p_1,p_2) is an equilibrium, then p = p*.
   1. If p = (p_1,p_2) is an equilibrium, then $p_1 = p_2$...
      - If $p_i > p_j = 0$, for sufficiently small ε>0, p_j' = ε is a better response to p_i for j. If $p_i > p_j > 0$, p_i' = p_j is a better response for i.
   2. Given any equilibrium p = (p_1,p_2) with $p_1 = p_2$, p = p*.
      - If $p_1 = p_2 > 0$, for sufficiently small ε>0, p_j' = p_j - ε is a better response to p_j for i.

Commons Problem

- $N = \{1,2,\ldots,n\}$ players, each with unlimited money;
- Simultaneously, each player i contributes $x_i \geq 0$ to produce $y = x_1 + \ldots x_n$ unit of some public good, yielding payoff

$$U_i(x_i,y) = y^{1/2} - x_i.$$

Quiz

Each student i is to submit a real number x_i. We will pair the students randomly. For each pair (i,j), if $x_i \neq x_j$, the student who submits the number that is closer to $(x_i+x_j)/4$ gets 100; the other student gets 20. If $x_i = x_j$, then each of i and j gets 50.

Stag Hunt

		
	(2,2)	(4,0)
	(0,4)	(5,5)

Equilibrium in Mixed Strategies

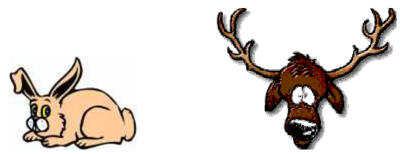
What is a strategy?

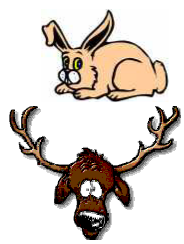
- A complete contingent-plan of a player.
- What the others think the player might do under various contingency.

What do we mean by a mixed strategy?

- The player is randomly choosing his pure strategies.
- The other players are not certain about what he will do.

Stag Hunt





	(2,2)	(4,0)
	(0,4)	(5,5)

Mixed-strategy equilibrium in Stag-Hunt game

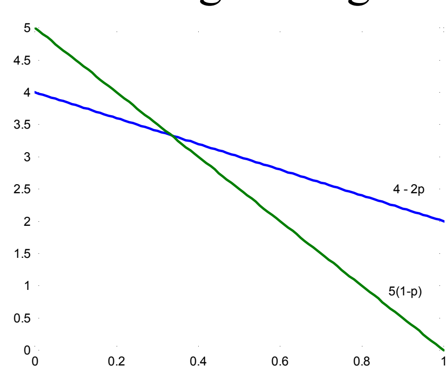
- Assume: Player 2 thinks that, with probability p, Player 1 targets for Rabbit. What is the best probability q she wants to play Rabbit?
- His payoff from targeting Rabbit:

  $U_2(R;p) = 2p + 4(1-p)$

  $= 4-2p.$
- From Stag:

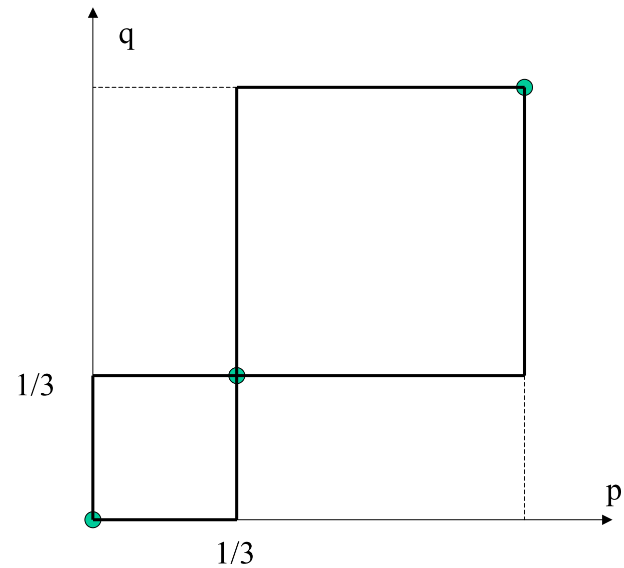
  $U_2(S;p) = 5(1-p)$
- She is indifferent iff

  $4-2p = 5(1-p)$ iff $p = 1/3$.



$$q^{BR}(p) = \begin{cases} 0 & \text{if } p < 1/3 \\ q \in [0,1] & \text{if } p = 1/3 \\ 1 & \text{if } p > 1/3 \end{cases}$$

Best responses in Stag-Hunt game



Bertrand Competition with costly search

- N = {F1,F2,B}; F1, F2 are firms; B is buyer
- B needs 1 unit of good, worth 6;
- Firms sell the good; Marginal cost = 0.
- Possible prices P = {3,5}.
- Buyer can check the prices with a small cost c > 0.

Game:

1. Each firm i chooses price p_i;
2. B decides whether to check the prices;
3. (Given) If he checks the prices, and $p_1 \neq p_2$, he buys the cheaper one; otherwise, he buys from any of the firm with probability ½.

Bertrand Competition with costly search

Check

F1 \ F2	High	Low
High	5/2 5/2 1-c	0 1 3-c
Low	3 0 3-c	3/2 3/2 3-c

Don't Check

F1 \ F2	High	Low
High	5/2 5/2 1	5/2 3/2 2
Low	3/2 5/2 2	3/2 3/2 3

Mixed-strategy equilibrium

- Symmetric equilibrium: Each firm charges "High" with probability q;
- Buyer Checks with probability r.
- U(check;q) = $q^2 1 + (1-q^2)3 - c = 3 - 2q^2 - c$;
- U(Don't;q) = $q1 + (1-q)3 = 3 - 2q$;
- Indifference: $2q(1-q) = c$; i.e.,
- U(high;q,r) = $0.5(1-r(1-q))5$;
- U(low;q,r) = $qr3 + 0.5(1-qr)3$
- Indifference: $r = 2/(5-2q)$.