Economic Applications with Incomplete Information

14.12 Game Theory

Road Map

1. Cournot duopoly with Incomplete Information
2. A first price auction
3. A double auction
4. Quiz

1 Cournot duopoly with Incomplete Information

- Demand:
$$P(Q) = a - Q$$
where $Q = q_1 + q_2$.
- The marginal cost of Firm 1 = c; common knowledge.
- Firm 2's marginal cost:
$$\begin{array}{ll} c_H & \text{with probaility } \theta, \\ c_L & \text{with probaility } 1-\theta; \end{array}$$
its private information.
- Each firm maximizes its expected profit.

1.1 Bayesian Nash Equilibrium

Firm 2 of high type:

$$\max_{q_2}(P - c_H)q_2 = \max_{q_2}\left[a - q_1^* - q_2 - c_H\right]q_2.$$

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2} \qquad \text{(*)}$$

Firm 2 of low type:

$$\max_{q_2}\left[a - q_1^* - q_2 - c_L\right]q_2,$$

$$q_2^*(c_L) = \frac{a - q_1^* - c_L}{2}. \qquad \text{(**)}$$

Firm 1:

$$\max_{q_1}\theta\left[a - q_1 - q_2^*(c_H) - c\right]q_1 + (1-\theta)\left[a - q_1 - q_2^*(c_L) - c\right]q_1$$

$$q_1^* = \frac{\theta\left[a - q_2^*(c_H) - c\right] + (1-\theta)\left[a - q_2^*(c_L) - c\right]}{2} \qquad \text{(***)}$$

Solve *, **, and *** for $q_1^*, q_2^*(c_L), q_2^*(c_H)$.

$$\begin{pmatrix} q_1^* \\ q_2^*(c_H) \\ q_2^*(c_L) \end{pmatrix} = \begin{bmatrix} 2 & \theta & 1-\theta \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1} \begin{pmatrix} a-c \\ a-c_H \\ a-c_L \end{pmatrix};$$

$$\begin{aligned} q_2^*(c_H) &= \frac{a-2c_H+c}{3}+\frac{(1-\theta)(c_H-c_L)}{6} \\ q_2^*(c_L) &= \frac{a-2c_L+c}{3}-\frac{\theta(c_H-c_L)}{6} \\ q_1^* &= \frac{a-2c+\theta c_H+(1-\theta)c_L}{3} \end{aligned}$$

2 A First-price Auction

- One object, two bidders
- $v_i =$ Valuation of bidder i; iid with uniform distribution over $[0, 1]$.
- Simultaneously, each bidder i submits a bid b_i, then the highest bidder wins the object and pays her bid.
- The payoffs:

$$u_i(b_1, b_2, v_1, v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j, \\ \frac{v_i - b_i}{2} & \text{if } b_i = b_j, \\ 0 & \text{if } b_i < b_j. \end{cases}$$

- Objective of i:

$$\max_{b_i} E\left[u_i(b_1, b_2, v_1, v_2)\right]$$

where

$$E\left[u_i\right] = (v_i - b_i)\Pr\{b_i > b_j(v_j)\} + \frac{1}{2}(v_i - b_i)\Pr\{b_i = b_j(v_j)\}$$

2.1 Symmetric, linear equilibrium

$$b_j = a + cv_j.$$

Then,

$$\Pr\{b_i = b_j(v_j)\} = 0.$$

Equilibrium condition: $a \leq b_i \leq v_i$.

Hence,

$$\begin{aligned} E[u_i] &= (v_i - b_i)\Pr\{b_i \geq a + cv_j\} \\ &= (v_i - b_i)\Pr\{v_j \leq \frac{b_i - a}{c}\} \\ &= (v_i - b_i) \cdot \frac{b_i - a}{c}. \end{aligned}$$

FOC:

$$b_i = \begin{cases} \frac{v_i + a}{2} & \text{if } v_i \geq a \\ a & \text{if } v_i < a. \end{cases} \tag{1}$$

Therefore,

$$b_i = \frac{1}{2} v_i.$$

2.2 Any symmetric equilibrium

$$b_i = b(v_i)$$

Hence,

$$\begin{aligned} E[u_i] &= (v_i - b_i)\Pr\{b_i \geq b(v_j)\} \\ &= (v_i - b_i)\Pr\{v_j \leq b^{-1}(b_i)\} \\ &= (v_i - b_i)b^{-1}(b_i). \end{aligned}$$

FOC ($\partial/\partial b_i = 0$):

$$\begin{aligned} -b^{-1}(b_i) + (v_i - b_i)\frac{db^{-1}(b_i)}{db_i} &= 0 \\ -v_i + (v_i - b(v_i))\frac{1}{b'(v_i)} &= 0 \\ b'(v_i)v_i + b(v_i) &= v_i \\ \frac{d[b(v_i)v_i]}{dv_i} &= v_i \end{aligned}$$

$$\begin{aligned} b(v_i)v_i &= v_i^2/2 + const. \\ b(v_i) &= v_i/2 + const/v_i. \end{aligned}$$

$b(0) = 0$, => cons = 0.

$$b(v_i) = v_i/2.$$

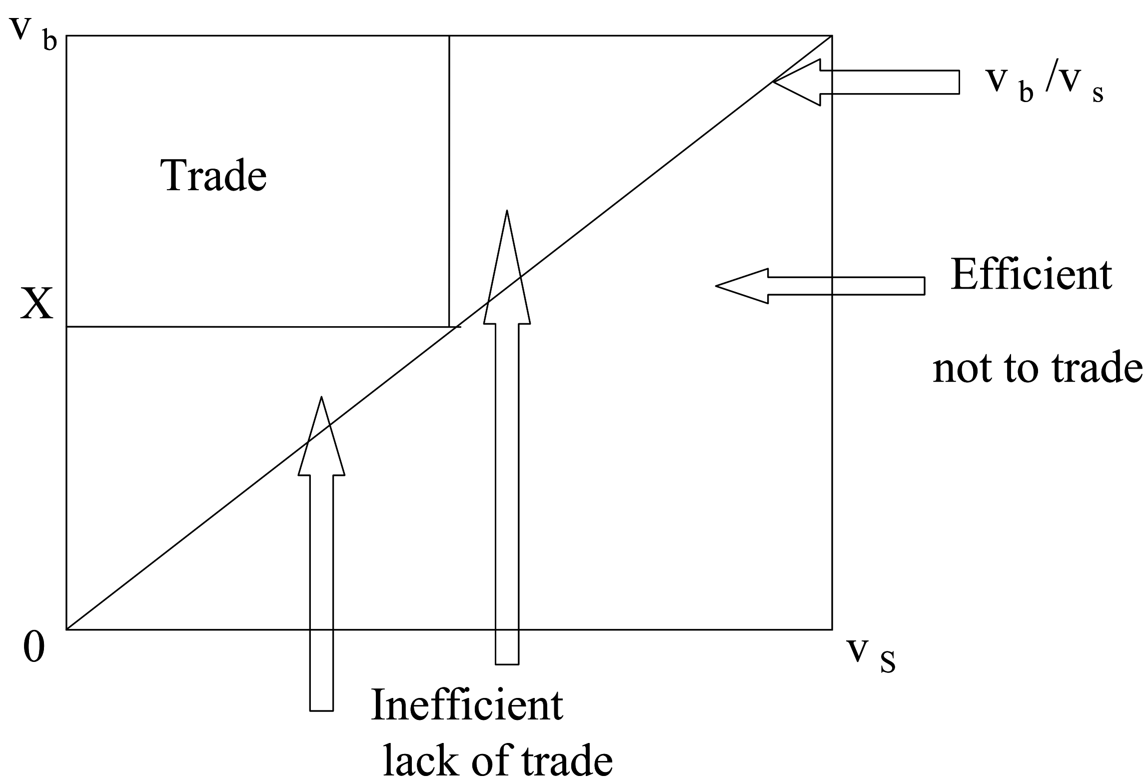
3 Double Auction

1. Simultaneously, Seller names p_s and Buyer names p_b.
   a. If $p_b < p_s$, then no trade;
   b. if $p_b \geq p_s$, trade at price $p = \frac{p_b+p_s}{2}$.
2. Valuations are private information: v_b, v_s iid w/ uniform on $[0, 1]$.
3. Payoffs:
$$u_b = \begin{cases} v_b - \frac{p_b+p_s}{2} & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases}$$
$$u_s = \begin{cases} \frac{p_b+p_s}{2} - v_s & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases}$$
4. The buyer's problem:
$$\max_{p_b} E\left[v_b - \frac{p_b + p_s(v_s)}{2} : p_b \geq p_s(v_s)\right].$$
5. The seller's problem:
$$\max_{p_s} E\left[\frac{p_s + p_b(v_b)}{2} - v_s : p_b(v_b) \geq p_s\right].$$

An Equilibrium:

$$p_b = \begin{cases} X & \text{if } v_b \geq X \\ 0 & \text{otherwise} \end{cases},$$

$$p_s = \begin{cases} X & \text{if } v_s \leq X \\ 1 & \text{otherwise} \end{cases}.$$



Equilibrium with linear strategies:

$$
\begin{aligned}
p_b &= a_b + c_b v_b \\
p_s &= a_s + c_s v_s.
\end{aligned}
$$

$$
p_b \geq p_s(v_s) = a_s + c_s v_s \iff v_s \leq \frac{p_b - a_s}{c_s}.
$$

$$
p_s \leq p_b(v_b) = a_b + c_b v_b \iff v_b \geq \frac{p_s - a_b}{c_b}.
$$

$$\begin{aligned}
E\left[u_b\right] &= E\left[v_b - \frac{p_b + p_s(v_s)}{2} : p_b \geq p_s(v_s)\right] \\
&= \int_0^{\frac{p_b - a_s}{c_s}} \left[v_b - \frac{p_b + p_s(v_s)}{2}\right] dv_s \\
&= \int_0^{\frac{p_b - a_s}{c_s}} \left[v_b - \frac{p_b + a_s + c_s v_s}{2}\right] dv_s \\
&= \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2}\right) - \frac{c_s}{2}\int_0^{\frac{p_b - a_s}{c_s}} v_s dv_s \\
&= \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2}\right) - \frac{c_s}{4}\left(\frac{p_b - a_s}{c_s}\right)^2 \\
&= \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2} - \frac{p_b - a_s}{4}\right) \\
&= \frac{p_b - a_s}{c_s}\left(v_b - \frac{3p_b + a_s}{4}\right).
\end{aligned}$$

F.O.C. ($\max_{p_b} E\left[u_b\right]$):

$$\frac{1}{c_s}\left(v_b - \frac{3p_b + a_s}{4}\right) - \frac{3(p_b - a_s)}{4c_s} = 0$$

i.e.,

$$p_b = \frac{2}{3}v_b + \frac{1}{3}a_s. \qquad (2)$$

Similarly,

$$\begin{aligned}
E\left[u_s\right] &= E\left[\frac{p_s + p_b(v_b)}{2} - v_s : p_b(v_b) \geq p_s\right] \\
&= \int_{\frac{p_s - a_b}{c_b}}^{1} \left[\frac{p_s + a_b + c_b v_b}{2} - v_s\right] dv_b \\
&= \left(1 - \frac{p_s - a_b}{c_b}\right)\left[\frac{p_s + a_b}{2} - v_s\right] + \frac{c_b}{2}\int_{\frac{p_s - a_b}{c_b}}^{1} v_b dv_b \\
&= \left(1 - \frac{p_s - a_b}{c_b}\right)\left[\frac{p_s + a_b}{2} - v_s\right] \\
&\quad + \frac{c_b}{4}\left(1 - \left(\frac{p_s - a_b}{c_b}\right)^2\right) \\
&= \left(1 - \frac{p_s - a_b}{c_b}\right)\left[\frac{p_s + a_b}{2} - v_s + \frac{c_b}{4} + \frac{p_s - a_b}{4}\right] \\
&= \left(1 - \frac{p_s - a_b}{c_b}\right)\left[\frac{3p_s + a_b}{4} - v_s + \frac{c_b}{4}\right]
\end{aligned}$$

F.O.C. $(\max_{p_s} E[u_s])$**:**

$$-\frac{1}{c_b}\left[\frac{3p_s+a_b}{4}-v_s+\frac{1}{4}\right]+\frac{3}{4}\left(1-\frac{p_s-a_b}{c_b}\right)=0$$

$$-\left[\frac{3p_s+a_b}{4}-v_s+\frac{c_b}{4}\right]+\frac{3}{4}\left(c_b-(p_s-a_b)\right)=0,$$

$$\frac{3p_s}{2}=-\frac{a_b}{4}+v_s-\frac{c_b}{4}+\frac{3}{4}\left(c_b+a_b\right)=v_s+\frac{a_b+c_b}{2}$$

i.e.,

$$p_s=\frac{2}{3}v_s+\frac{a_b+c_b}{3}. \tag{3}$$

- $a_b = a_s/3$
- $a_s = a_b/3 + 2/9$
- Hence, $9a_s = a_s + 2$,
- $a_s = 1/4$; $a_b = 1/12$.

$$p_b = \frac{2}{3}v_b+\frac{1}{12} \tag{4}$$

$$p_s = \frac{2}{3}v_s+\frac{1}{4}. \tag{5}$$

We have trade iff

$$p_b \geq p_s$$

iff

$$\frac{2}{3}v_b+\frac{1}{12}\geq\frac{2}{3}v_s+\frac{1}{4}$$

iff

$$v_b - v_s \geq \frac{3}{2}\left(\frac{1}{4} - \frac{1}{12}\right) = \frac{3}{2}\frac{1}{6} = \frac{1}{4}.$$