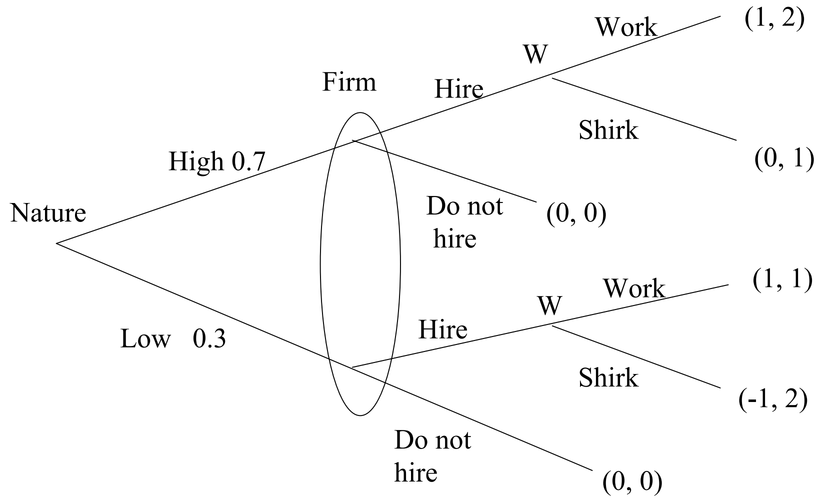
Lectures 15-18
Dynamic Games with Incomplete Information

14.12 Game Theory

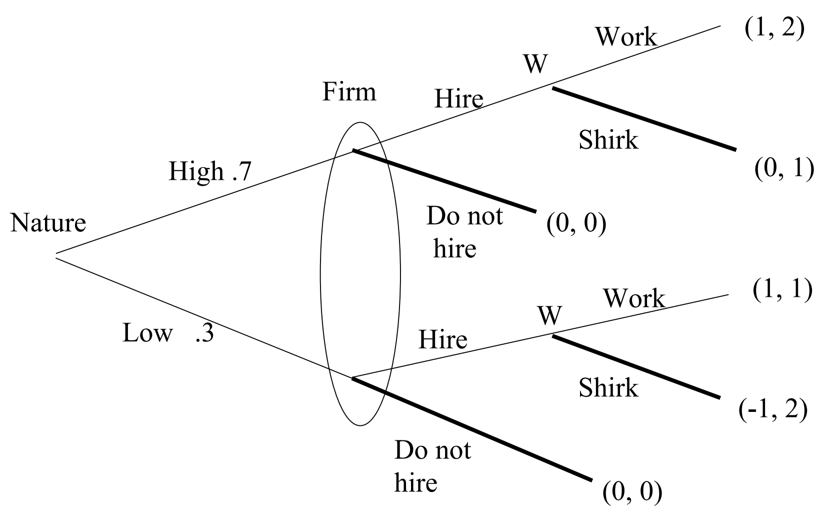
Road Map

1. Examples
2. Sequential Rationality
3. Perfect Bayesian Nash Equilibrium
4. Economic Applications
   1. Sequential Bargaining with incomplete information
   2. Reputation

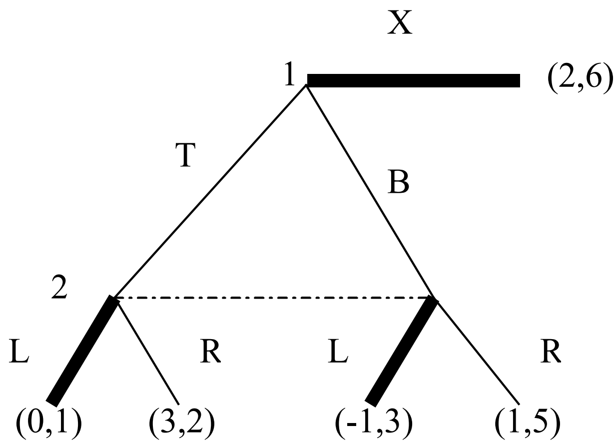
An Example



What is wrong with this equilibrium?

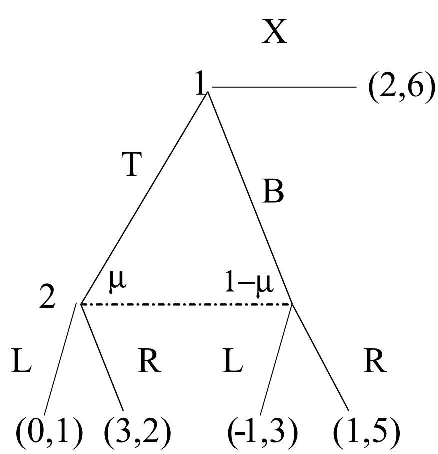


What is wrong with this equilibrium?



Beliefs

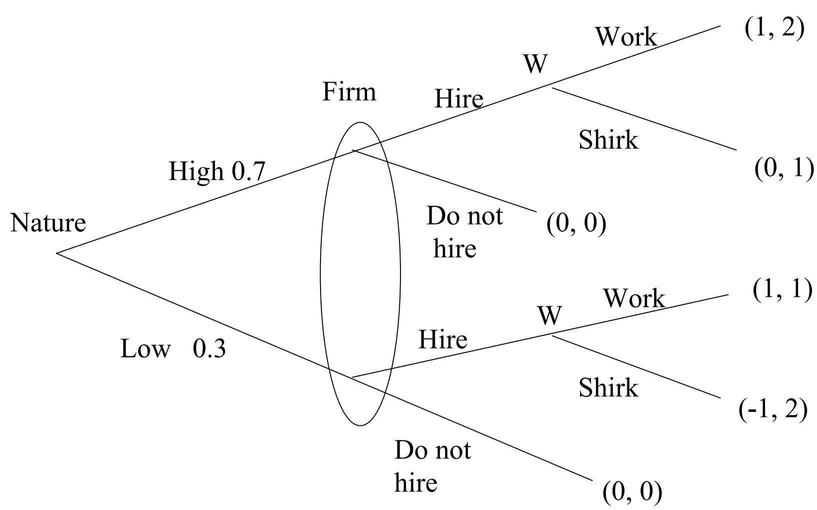
- Beliefs of an agent at a given information set is a probability distribution on the information set.
- For each information set, we must specify the beliefs of the agent who moves at that information set.



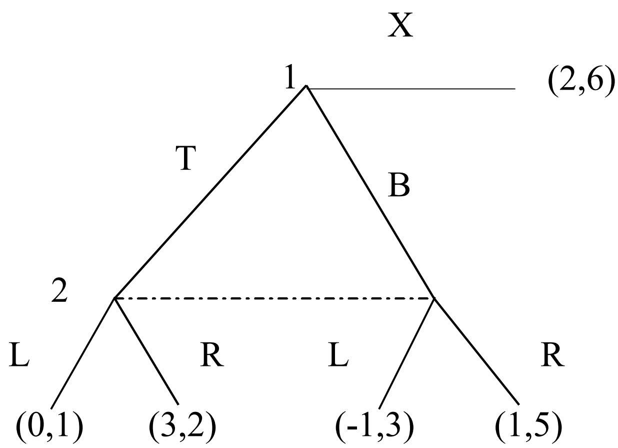
Sequential Rationality

A player is said to be **sequentially rational** iff, at each information set he is to move, he maximizes his expected utility given his beliefs at the information set (and given that he is at the information set) – even if this information set is precluded by his own strategy.

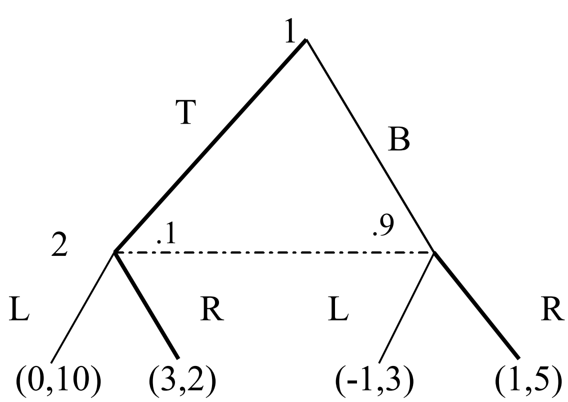
An Example



Another example



Example

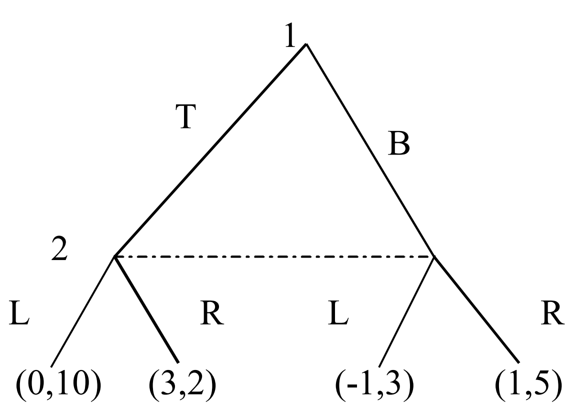


"Consistency"

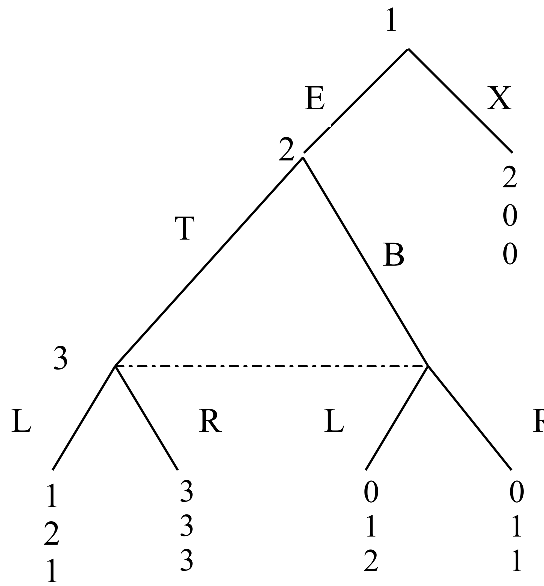
Definition: Given any (possibly mixed) strategy profile s, an information set is said to be **on the path of play** iff the information set is reached with positive probability if players stick to s.

Definition: Given any strategy profile s and any information set I on the path of play of s, a player's beliefs at I is said to be **consistent** with s iff the beliefs are derived using the Bayes' rule and s.

Example



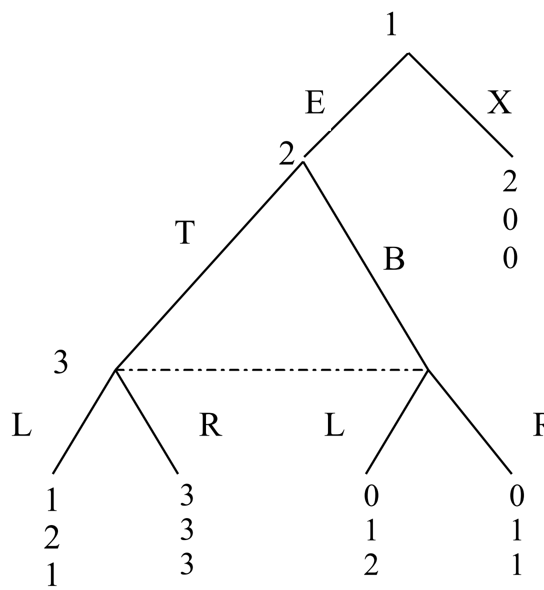
Example



"Consistency"

- Given s and an information set I, even if I is off the path of play, the beliefs must be derived using the Bayes' rule and s "whenever possible," e.g., if players tremble with very small probability so that I is on the path, the beliefs must be very close to the ones derived using the Bayes' rule and s.

Example



Sequential Rationality

A strategy profile is said to be **sequentially rational** iff, at each information set, the player who is to move maximizes his expected utility

1. given his beliefs at the information set, and
2. given that the other players play according to the strategy profile in the continuation game (and given that he is at the information set) .

Perfect Bayesian Nash Equilibrium

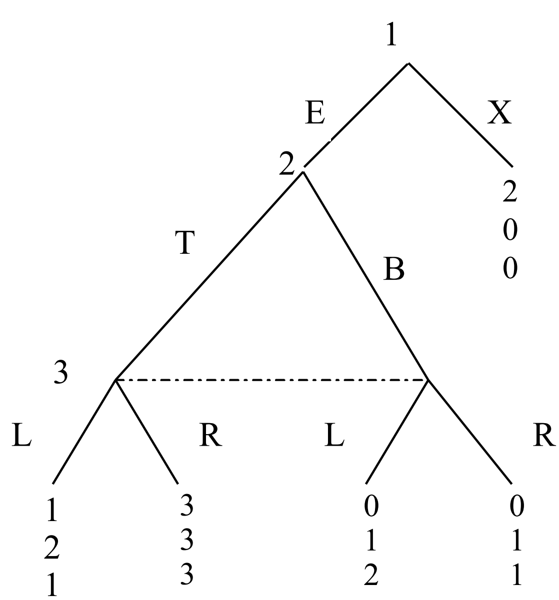
A Perfect Bayesian Nash Equilibrium is a pair (s,b) of strategy profile and a set of beliefs such that

1. Strategy profile s is sequentially rational given beliefs b, and
2. Beliefs b are consistent with s.

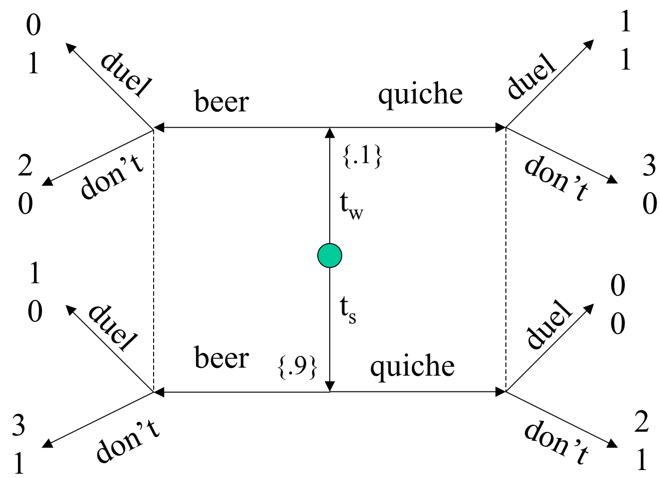
Nash → Subgame-perfect

Bayesian Nash → Perfect Bayesian

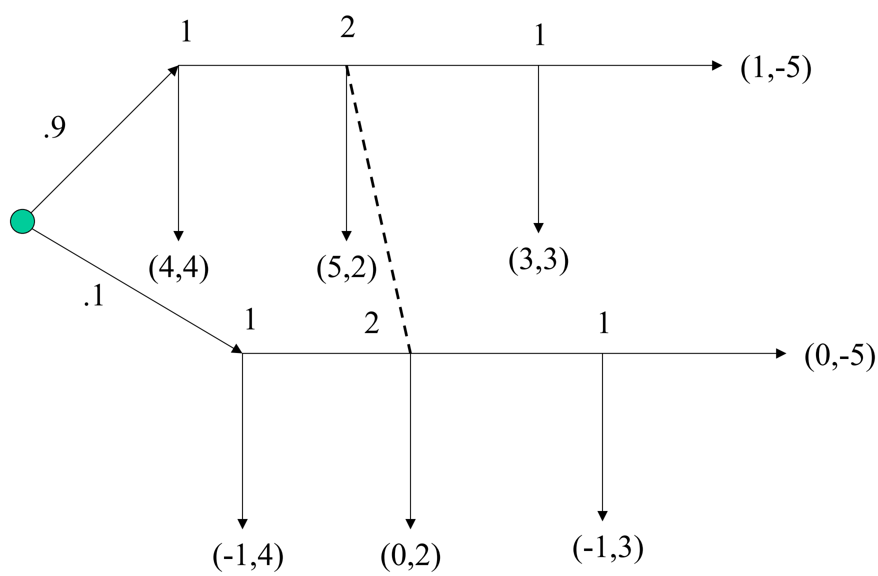
Example



Beer – Quiche



Example

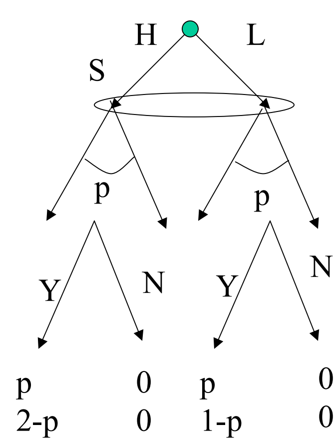


Sequential Bargaining

1. 1-period bargaining – 2 types
2. 2-period bargaining – 2 types
3. 1-period bargaining – continuum
4. 2-period bargaining – continuum

Sequential bargaining 1-p

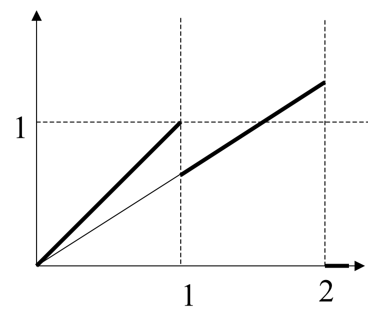
- A seller S with valuation 0
- A buyer B with valuation v;
  - B knows v, S does not
  - v = 2 with probability π
  - = 1 with probability $1-\pi$
- S sets a price $p \geq 0$;
- B either
  - buys, yielding (p,v-p)
  - or does not, yielding (0,0).



Solution

1. B buys iff $v \geq p$;
   1. If $p \leq 1$, both types buy: S gets p.
   2. If $1 < p \leq 2$, only H-type buys: S gets πp.
   3. If $p > 2$, no one buys.
2. S offers

- 1 if $\pi < ½$,
- 2 if $\pi > ½$.



Sequential bargaining 2-period

- A seller S with valuation 0
- A buyer B with valuation v;
  - B knows v, S does not
  - v = 2 with probability π
  - = 1 with probability $1-\pi$

1. At t = 0, S sets a price $p_0 \geq 0$;
2. B either
   - buys, yielding $(p_0, v-p_0)$
   - or does not, then
3. At t = 1, S sets a price $p_0 \geq 0$;
4. B either
   - buys, yielding $(\delta p_0, \delta(v-p_0))$
   - or does not, yielding (0,0)

Solution, 2-period

1. Let $\mu = \Pr(v = 2|\text{history at } t=1)$.
2. At $t = 1$, buy iff $v \geq p$;
3. If $\mu > ½$, $p_1 = 2$
4. If $\mu < ½$, $p_1 = 1$.
5. If $\mu = ½$, mix between 1 and 2.
6. B with $v=1$ buys at $t=0$ if $p_0 \leq 1$.
7. If $p_0 > 1$, $\mu = \Pr(v = 2|p_0,t=1) \leq \pi$.

Solution, cont. $\pi < 1/2$

1. $\mu = \Pr(v = 2|p_0,t=1) \leq \pi < 1/2$.
2. At $t = 1$, buy iff $v \geq p$;
3. $p_1 = 1$.
4. B with $v=2$ buys at $t=0$ if

   $$(2-p_0) \geq \delta(2-1) = \delta \Leftrightarrow p_0 \leq 2-\delta.$$
5. $p_0 = 1$:

$$\pi(2-\delta) + (1-\pi)\delta = 2\pi(1-\delta) + \delta < 1-\delta+\delta = 1.$$

Solution, cont. $\pi > 1/2$

- If v=2 is buying at $p_0 > 2-\delta$, then
  - $\mu = \Pr(v = 2|p_0 > 2-\delta, t=1) = 0$;
  - $p_1 = 1$;
  - v = 2 should not buy at $p_0 > 2-\delta$.
- If v=2 is not buying at $2 > p_0 > 2-\delta$, then
  - $\mu = \Pr(v = 2|p_0 > 2-\delta, t=1) = \pi > 1/2$;
  - $p_1 = 2$;
  - v = 2 should buy at $2 > p_0 > 2-\delta$.
- No pure-strategy equilibrium.

Mixed-strategy equilibrium, $\pi > 1/2$

1. For $p_0 > 2-\delta$, $\mu(p_0) = ½$;
2. $\beta(p_0) = 1 - \Pr(v=2 \text{ buys at } p_0)$

$$\mu = \frac{\beta(p_0)\pi}{\beta(p_0)\pi + (1-\pi)} = \frac{1}{2} \Leftrightarrow \beta(p_0)\pi = 1-\pi \Leftrightarrow \beta(p_0) = \frac{1-\pi}{\pi}.$$

3. v = 2 is indifferent towards buying at p_0:

$$2 - p_0 = \delta\gamma(p_0) \Leftrightarrow \gamma(p_0) = (2 - p_0)/\delta$$

where $\gamma(p_0) = \Pr(p_1 = 1|p_0)$.

Sequential bargaining, v in [0,1]

- 1 period:
  - B buys at p iff v ≥ p;
  - S gets U(p) = p Pr(v ≥ p);
  - v in [0,a] => U(p) = p(a-p)/a;
  - p = a/2.

Sequential bargaining, v in [0,1]

- 2 periods: (p_0,p_1)
  - At t = 0, B buys at p_0 iff $v \geq a(p_0)$;
  - $p_1 = a(p_0)/2$;
  - Type $a(p_0)$ is indifferent:

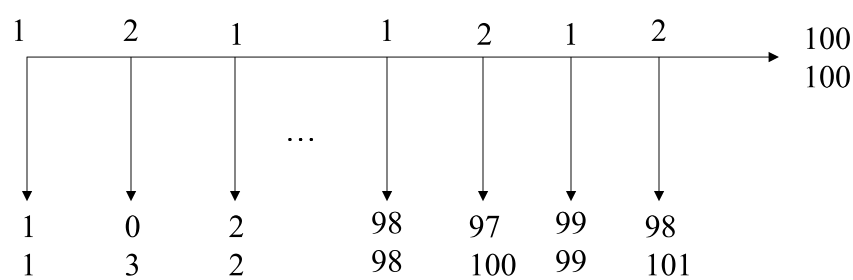
$$a(p_0) - p_0 = \delta(a(p_0) - p_1) = \delta a(p_0)/2$$

$$\Leftrightarrow a(p_0) = p_0/(1-\delta/2)$$

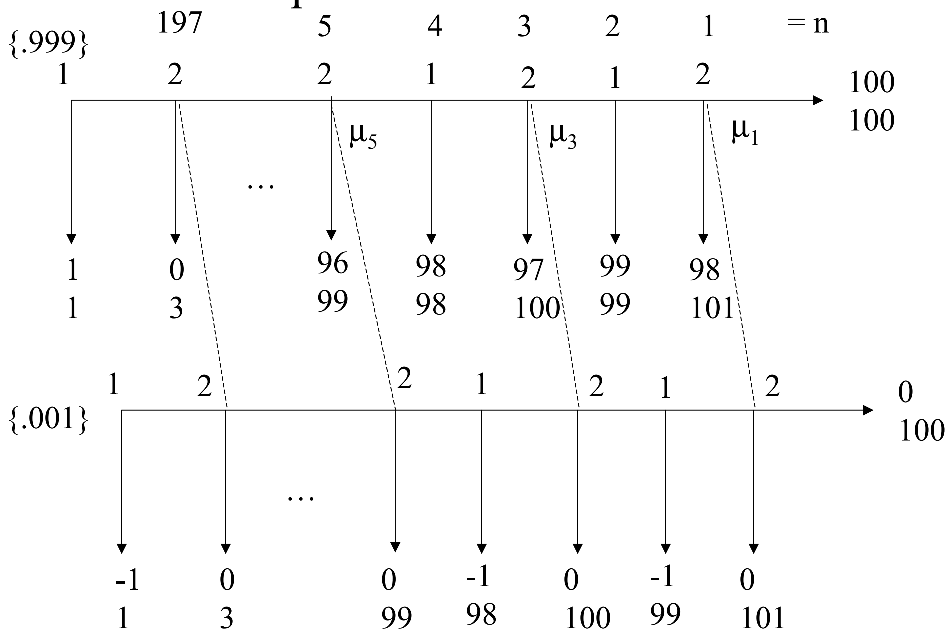
  - S gets $$\left(1 - \frac{p_0}{1-\delta/2}\right)p_0 + \left(\frac{p_0}{2-\delta}\right)^2$$
  - FOC: $$1 - \frac{2p_0}{1-\delta/2} + \frac{2p_0}{2-\delta} = 0 \Rightarrow p_0 = \frac{(1-\delta/2)^2}{2(1-3\delta/4)}$$

Reputation

Centipede Game



Centipede Game – with doubt



Facts about the Centipede

- Every information set of 2 is reached with positive probability.
- 2 always goes across with positive probability.
- If 2 strictly prefers to go across at n, then
  - she must strictly prefer to go across at n+1,
  - her posterior at n is her prior.
- For any n > 2, 1 goes across with positive probability. If 1 goes across w/p 1 at n, then 2's posterior at n-1 is her prior.

If 2's payoff at any n is x and 2 is mixing, then

$$x = \mu_n(x+1) + (1-\mu_n)[(x-1)p_n + (1-p_n)(x+1)]$$

$$= \mu_n(x+1) + (1-\mu_n)[(x+1) - 2p_n]$$

$$= x+1 - 2p_n(1-\mu_n)$$

$$\Leftrightarrow (1-\mu_n)\, p_n = 1/2$$

$$\mu_{n-1} = \frac{\mu_n}{\mu_n + (1-\mu_n)(1-p_n)} = \frac{\mu_n}{\mu_n + (1-\mu_n) - p_n(1-\mu_n)} = 2\mu_n$$

$$\mu_n = \frac{\mu_{n-1}}{2}$$