14.472 Problem Set 2 Suggested Solutions

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Question 1:

First find the change in the capital stock, k, that will occur when the OLG economy moves to the new steady state after the government imposes a one time lump sum tax on the current young generation to permanently lower public per capita debt, g. The two equations we need for this exercise are the government budget constraint relating the lump sum tax on the young, T, to the per capita debt level and the market clearance equation (which states that the capital stock in a given period equals the savng done by the current old generation). These equations are:

$$T = (r-n)g \tag{1}$$

$$(1+n)(k+g) = w(1-t) - T - c^*[w(1-t) - T + \frac{b}{1+r}, r] + f \tag{2}$$

To find $\frac{dk}{dg}$, we can just implicitly differentiate the market clearance equation (2) (we assume first period marginal propensity to consume,c_Y^* , is positive and less than one).

$$\begin{aligned}
(1+n)(dk+dg) &= -\frac{dT}{dg}dg + c_Y^*\frac{dT}{dg}dg \\
\frac{dk}{dg} &= -1 - \frac{1}{1+n}\frac{dT}{dg}(1-c_Y^*) \\
\frac{dk}{dg} &= -\left\{1 + \frac{r-n}{1+n}(1-c_Y^*)\right\} < -1
\end{aligned}$$

This expression says that the amount of capital per young worker in the new steady state increases in two possible ways. First, since the government is offering fewer bonds, young workers will put their savings into private capital instead of government bonds. This effect alone increases private capital one-for-one with a decrease in the public debt (the thought experiment keeps savings by the young the same as it was in the old steady state). Second, lowering the lump sum tax gives young workers a higher income, of which they save $(1 - c_Y^*)$ (i.e. one minus the marginal propensity to consume). This second effect causes the new steady state capital per young worker to increase by more than the decrease in the public debt per young worker, since it increases the level of saving by the young.

To find the effects of lowering the public debt on utility, we can just differentiate the indirect lifetime utility functions of people in our economy. Call V=u(c*, $(w(1-t) - T - c^*)(1+r) + b$) the lifetime indirect utility function of people in our economy. There are two groups of people we need to consider when examining the effect of the change in government policy on utility - (1) the current young generation who must finance the one time decrease in the debt and (2) all the future young generations. (The current old generation is unaffected by the policy change.) The change in lifetime utility of the current young is (for all derivatives of utility wrt g, I am actually taking the negative of these derivatives because we are decreasing public debt):

$$\frac{dV^{cy}}{dg} = -\frac{du}{dc_1}(r-n)c_Y^* - \frac{du}{dc_2}(1+r)(r-n)(1-c_Y^*) < 0$$

The change in lifetime utility of the future young is

$$\frac{dV^{fy}}{dg} = \frac{du}{dc_1}(r-n)c_Y^* + \frac{du}{dc_2}(1+r)(r-n)(1-c_Y^*) = -\frac{dV^{cy}}{dg} > 0$$

To answer the normative question of should the government implement the policy, we'd need to know how the future sum of young generations' utility compared with the loss in utility of the current young generation in the social welfare function.

$$\begin{aligned} SWF &= \sum \delta^i V^i \\ \frac{dSWF}{dg} &= -\frac{dV^{fy}}{dg} + \sum \delta^i \frac{dV^{fy}}{dg} = -\frac{dV^{fy}}{dg} + \frac{\delta}{1-\delta}\frac{dV^{fy}}{dg} = \frac{2\delta - 1}{1-\delta}\frac{dV^{fy}}{dg} \end{aligned}$$

If $\frac{dSWF}{dg} > 0$ then the government should implement the policy if the government's objective is to maximize the social welfare function.

Now let's look at what happens when the government temporarily increases the lump sum tax on the current young to permanently increase funding of social security. Again, to find the change in the steady state capital stock we can differentiate the market clearance equation (2).

$$\begin{aligned}(1+n)dk &= -\frac{1}{1+r}\frac{db}{df}df \cdot c_Y^* + df \\ \frac{dk}{df} &= \frac{1}{(1+n)}\left\{1 - \frac{1}{1+r}\frac{db}{df}c_Y^*\right\}\end{aligned}$$

Given the social security budget constraint:

$$b = tw(1+n) + (r-n)f \tag{3}$$

We can get

$$\begin{aligned}\frac{dk}{df} &= \frac{1}{1+n}\left\{1 - \frac{r-n}{1+r}c_Y^*\right\} = \frac{1}{1+n} - \frac{r-n}{(1+n)(1+r)} + \frac{(r-n)(1-c_Y^*)}{(1+n)(1+r)} \\ &= \frac{1}{1+r} + \frac{(r-n)(1-c_Y^*)}{(1+n)(1+r)} = \frac{1}{1+r}\left\{1 + \frac{(r-n)(1-c_Y^*)}{1+n}\right\} \\ &= -\frac{1}{1+r}\frac{dk}{dg}\end{aligned}$$

We can see that both policies (decreasing the public debt and increasing social security funding) have similar effects on the new steady state capital stock. Increasing funding of social security The $\frac{1}{1+r}$ just picks up a timing difference. To fund an increase in the funding of social security, the lump sum tax must increase by $\frac{1}{1+r}$, not 1, since the lump sum tax will be invested in private capital and accrue interest for one period, giving an increase in funding of 1. By increasing the benefits workers receive, future workers now have more lifetime income. This causes them to consume more in the first period; they will also consume more in the second period as well. Now consider the change in lifetime utility of the current young generation and future young generations.

$$\frac{dV^{cy}}{df} = -\frac{du}{dc_1}\frac{r-n}{1+r}c_Y^* - \frac{du}{dc_2}(r-n)(1-c_Y^*) + \frac{du}{dc_2}(r-n) = \frac{1}{1+r}\frac{dV^{cy}}{dg} + \frac{du}{dc_2}(r-n) < 0$$

$$\frac{dV^{fy}}{df} = \frac{du}{dc_1}\frac{r-n}{1+r}c_Y^* + \frac{du}{dc_2}(r-n)(1-c_Y^*) = \frac{1}{1+r}\frac{dV^{fy}}{dg} > 0$$

Notice that the utility loss to the current young generation is less than when the government decreases the public debt. This is because the current young generation gets extra consumption in retirement as a result of the higher social security benefits from the increase in funding. The future young generations are unequivocally better off.

Question 2:

Now only a fraction α of each young generation saves in a lifetime consistent manner. The rest consume their after-tax income in the first period and consume only social security benefits when old. The market clearance equation is now given by:

$$(1+n)(k+g) = \alpha\left\{w(1-t) - T - c^*[w(1-t) - T + \frac{b}{1+r}, r]\right\} + f \tag{4}$$

The government budget constraints remain the same as in equations (1) and (3). To find the effect on steady state capital of a decrease in the public debt we differentiate the new market clearance constraint:

$$\begin{aligned}(1+n)(dk+dg) &= \alpha\left\{-\frac{dT}{dg}dg + c_Y^*\frac{dT}{dg}dg\right\} \\ \frac{dk}{dg} &= -1 - \frac{\alpha(r-n)(1-c_Y^*)}{1+n}\end{aligned}$$

The capital stock increases with a decrease in public debt, but by less than in question 1 because there are fewer savers (i.e. the second effect of a decrease in public debt causing an increase in the level of private saving is mitigated by a factor of 1-α). In calculating utility effects we now need to consider four groups of people - (1) current young savers; (2) current young non-savers; (3) future young savers; (4) future young non-savers. The lifetime indirect utility function of savers is V^{save}=u(c*, $(w(1-t)-T-c^*)(1+r)+b)$. The lifetime indirect utility function of non-savers is $V^{non-save}$=u($w(1-t)-T$, b). The effect on current young savers is:

$$\frac{dV^{cy}_{save}}{dg} = -\frac{du}{dc_1}(r-n)c^*_Y - \frac{du}{dc_2}(1+r)(r-n)(1-c^*_Y) < 0$$

The effect on current young non-savers is:

$$\frac{dV^{cy}_{non-save}}{dg} = -\frac{du}{dc_1}(r-n) < 0$$

The effect on future young savers is:

$$\frac{dV^{fy}_{save}}{dg} = -\frac{dV^{cy}_{save}}{dg}$$

The effect on future young non-savers is:

$$\frac{dV^{fy}_{non-save}}{dg} = -\frac{dV^{cy}_{non-save}}{dg}$$

Now let's consider a permanent increase in social security funding:

$$
\begin{aligned}
(1+n)dk &= -\alpha c_Y^* \frac{db}{df} df + df \\
(1+n)\frac{dk}{df} &= 1 - \alpha c_Y^* \frac{r-n}{1+r} = 1 - \frac{\alpha(r-n)}{1+r} + \frac{\alpha(r-n)(1-c_Y^*)}{1+r} = \frac{1+\alpha n + (1-\alpha)r}{1+r} + \frac{\alpha(r-n)(1-c_Y^*)}{1+r} \\
\frac{dk}{df} &= \frac{1}{(1+r)(1+n)} \{1 + \alpha n + (1-\alpha)r + \alpha(r-n)(1-c_Y^*)\} > -\frac{1}{1+r}\frac{dk}{dg} \qquad (r > n)
\end{aligned}
$$

Now increasing social security funding increases the capital stock more than decreasing the public debt because increasing funding effectively forces non-savers to save more. The utility effect on the four groups:

$$
\begin{aligned}
\frac{dV_{save}^{cy}}{df} &= -\frac{du}{dc_1}\frac{r-n}{1+r}c_Y^* - \frac{du}{dc_2}(r-n)(1-c_Y^*) + \frac{du}{dc_2}(r-n) = \frac{dV_{save}^{cy}}{dg} + \frac{du}{dc_2}(r-n) < 0 \\
\frac{dV_{non-save}^{cy}}{df} &= -\frac{du}{dc_1}\frac{r-n}{1+r} + \frac{du}{dc_2}(r-n) ><= 0 \\
\frac{dV_{save}^{fy}}{df} &= \frac{du}{dc_1}\frac{r-n}{1+r}c_Y^* + \frac{du}{dc_2}(r-n)(1-c_Y^*) > 0 \\
\frac{dV_{non-save}^{fy}}{df} &= \frac{du}{dc_2}(r-n) > 0
\end{aligned}
$$

The welfare effect on current young non-savers is ambiguous. Forcing them to save more may actually make them better off; however, it may making an already bad situation worse, it these people by not saving are maximizing lifetime utility (this would require that non-savers had a different u$(\cdot,\cdot)$ than savers). Savers have the same utility changes as in question 1. The current savers are worse off, while all future savers are better off.

Question 3:

Now we have a Leontief production function, so the per capital capital stock is fixed in equilibrium at $\frac{y}{a}$, and output per capita is q = min{y, $a \cdot \frac{y}{a}\} = y$. The equilibrium equations are given by (assume perfect competition and price of output is 1):

$$
\begin{aligned}
w &= q(k) - rk = y - \frac{ry}{a} && (5)\\
T &= (r-n)g && (6)\\
b &= t(y - \frac{ry}{a})(1+n) + (r-n)f && (7)\\
(1+n)(k+g) &= (y - \frac{ry}{a})(1-t) - T - c^*[(y - \frac{ry}{a})(1-t) - T + \frac{b}{1+r}] + f && (8)
\end{aligned}
$$

We want to find the effect a decrease in the public debt will have on the steady state interest rate (which is endogenous since capital is fixed by the production technology). We can again implicitly differentiate the market clearance equation (8).

$$
\begin{aligned}
(1+n)dg &= -(1-t)\frac{y}{a}dr - gdr - (r-n)dg - c_Y^*\{-(1-t)\frac{y}{a}dr - gdr - (r-n)dg + \frac{1}{1+r}\frac{db}{dr}\\
&\quad -\frac{b}{(1+r)^2}dr\} - c_r^* dr\\
\frac{db}{dr} &= -t(1+n)\frac{y}{a}dr + fdr\\
\{1+n+(1-c_Y^*)(r-n)\}\,dg &= \left\{(1-c_Y^*)\left[-(1-t)\frac{y}{a} - g\right] + c_Y^*\left[\frac{1}{1+r}(t(1+n)\frac{y}{a} - f) + \frac{b}{(1+r)^2}\right] - c_r^*\right\} dr\\
\frac{dr}{dg} &= \frac{1+n+(1-c_Y^*)(r-n)}{-(1-c_Y^*)\left[(1-t)\frac{y}{a} + g\right] + c_Y^*\left[\frac{1}{1+r}(t(1+n)\frac{y}{a} - f) + \frac{b}{(1+r)^2}\right] - c_r^*} >\!<\!= 0 \ \ \text{(I think)}
\end{aligned}
$$

The lifetime indirect utility function is $V = u(c^*, (y - \frac{ry}{a})(1-t) - T - c^* + b)$. Welfare effects are ambiguous. There are now fewer government bonds and the private capital stock is fixed, so there are fewer places to put savings. This may cause the interest rate to increase and the wage to decrease (wage decrease only for future workers). The current generation who is taxed may benefit in the second period from the higher interest rate. The future generation will be happier if the interest rate is higher, but a lower wage will mitigate this happiness.