14.472 Problem Set 1 Suggested Solutions

Question 1a:

Because each individual's discount rate is equal to the interest rate (both are zero), there are no saving or borrowing constraints, and the utility of consumption function is the same every period, the individual will consume the same amount each period. So, $x_z^i = c_z^i = x^i \; \forall z$.

Individual of type i solves:

$$\max_{x^i, R^i} \quad T^i \log(x^i) - R^i a^i$$

$$s.t. \qquad T^i x^i = R^i w^i + I^i$$

Which has the solution:

$$\begin{aligned} R^i &= \frac{T^i}{a^i} - \frac{I^i}{w^i} \\ x^i &= \frac{w^i}{a^i} \\ V^i &= T^i \log(w^i) - T^i \log(a^i) - T^i + \frac{I^i a^i}{w^i} \end{aligned}$$

Individual of type i's marginal utility of consumption is $\frac{a^i}{w^i}$. Since $a^H < a^L$ and $w^H > w^L$, the marginal utility of the high type is lower than the marginal utility of the low type, which implies that the high type consumes more each period than the low type.

Question 1b:

NO SAVINGS!

Indiviual i's budget constraint (recall there is now no lump sum income):

$$\begin{aligned} x_z^i &= w^i \;\; \forall z < R^i \\ c_z^i &= 0 \quad \forall z \geq R^i \end{aligned}$$

Since lifetime utility would go to negative infinity if the individual has zero consumption in any period, the individual works until death:

$$\begin{aligned} R^i &= T^i \\ x_z^i &= w^i \quad \forall z \\ V^i &= T^i \log(w^i) - T^i a^i \end{aligned}$$

Again, the high type individual has a lower marginal utility of consumption than the low type. Note also that lifetime utilities of both individuals are lower than the lifetime utilities in question 1a.

Question 2a:

As in question 1a each individual will consume the same amount each period. Individual i solves:

$$\max_{R^i, x^i} \quad T^i \log(x^i) - R^i a^i$$

$$s.t. \qquad T^i x^i = R^i w^i (1 - \tau) + T^i \beta + I^i$$

The key thing to notice in this problem is that each individual takes τ and β to be constants in his budget constraint (i.e. he takes the consumption and retirement of others as given and he is too small to affect the level of τ or β by changing his consumption or retirement age). However, the sum of all the individual decisions will ultimately affect τ and β through the government's budget constraint.

The solution to individual i's problem is:

$$\begin{aligned}
R^i &= \frac{T^i}{a^i} - \frac{T^i\beta + I^i}{w^i(1-\tau)} \\
x^i &= \frac{w^i(1-\tau)}{a^i} \\
V^i &= T^i \log(w^i(1-\tau)) - T^i \log(a^i) - T^i + \frac{a^i(T^i\beta + I^i)}{w^i(1-\tau)}
\end{aligned}$$

The government's budget constraint is:

$$\beta = \frac{\tau(R^H w^H + R^L w^L)}{T^H + T^L}$$

We could use the government's budget constraint to find R^H, R^L, x^H, x^L, V^H, V^L in terms of τ, w^H, w^L, a^H, a^L, T^H,T^L, I^H, and I^L. However, the equations stated above are enough to give us the intuition that the earnings tax will lower consumption and decrease the retirment age of both types. Because each individual takes τ and β as given, the tax system has the same effect on consumption and retirement age as a decrease in the individual's wage and an increase in his lump sum income.

Question 2b:

NO SAVINGS!

Individual of type i has budget constraint (recall lump sum income is now zero):

$$\begin{aligned}
x_z^i &= w^i(1-\tau) + \beta \quad \forall z < R^i \\
c_z^i &= \beta \quad \forall z \geq R^i
\end{aligned}$$

The individual of type i solves:

$$\max_{R^i} \quad R^i \log(w^i(1-\tau)+\beta) + (T^i - R^i)\log(\beta) - R^i a^i$$

The optimal retirement age (taking τ and β as constants) is:

$$R^i = \left\{ \begin{array}{c} T^i \quad \text{if } \log(w^i(1-\tau)+\beta) - a^i > \log(\beta) \\ 0 \quad \text{if } \log(w^i(1-\tau)+\beta) - a^i < \log(\beta) \\ \in [0, T^i] \text{ if } \log(w^i(1-\tau)+\beta) - a^i = \log(\beta) \end{array} \right\}$$

$$V^i = \left\{ \begin{array}{ll} T^i \log(w^i(1-\tau)+\beta) - T^i a^i & \text{if } R^i = T^i \\ T^i \log(\beta) & \text{if } R^i = 0 \\ R^i \log(w^i(1-\tau)+\beta) + (T^i - R^i)\log(\beta) - R^i a^i & \text{if } R^i \in (0, T^i) \end{array} \right\}$$

The government budget constraint is the same as it was in 2a.

$$\beta = \frac{\tau(R^H w^H + R^L w^L)}{T^H + T^L}$$

Note that given the government's budget constraint $R^i = 0$ for $i = H, L$ is not a possible solution since $\log(\beta) \to -\infty$ in this case and this clearly cannot be greater than $\log(w^i(1-\tau)+\beta) - a^i$. Under certain conditions on the wages and disutilities of labor, it might be possible to have the low type never work and the high type work for some time (perhaps his entire life). Under other conditions, both individuals may work their entire lives ($a^i s$ small enough and $w^i s$ big enough), or the high type may work his entire life and the low type may work but retire before he dies.

Question 3a:

The government solves:

$$\max_{\tau,\beta} \quad V^H + V^L$$

$$\max_{\tau,\beta} \quad T^H \log(w^H(1-\tau)) + \frac{a^H(T^H\beta + I^H)}{w^H(1-\tau)} + T^L \log(w^L(1-\tau)) + \frac{a^L(T^L\beta + I^L)}{w^L(1-\tau)}$$

$$
\begin{aligned}
s.t.\ \ (T^H+T^L)\beta &= \tau(R^H w^H + R^L w^L) \\
&= \tau\left\{ w^H\left(\frac{T^H}{a^H} - \frac{T^H\beta + I^H}{w^H(1-\tau)}\right) + w^L\left(\frac{T^L}{a^L} - \frac{T^L\beta + I^L}{w^L(1-\tau)}\right)\right\}
\end{aligned}
$$

Question 3b:

NO SAVINGS!

The government again wants to find the optimal τ and β. Now it must find the optimal τ and β for two cases: (1) Both types work and (2) Only the high type works. The government will choose the optimal tax system for the case which yields the higher sum of lifetime utilities.

Case 1 (Both types work):

$$
\max_{\tau,\beta} \quad R^H \log(w^H(1-\tau)+\beta) + R^L \log(w^L(1-\tau)+\beta) + (T^H - R^H + T^L - R^L)\log(\beta) - a^H R^H - a^L R^L
$$

$$
\begin{aligned}
s.t. \qquad\qquad \beta &= \frac{\tau(R^H w^H + R^L w^L)}{T^H + T^L} \\
(1) \qquad \log(\beta) &\leq \log(w^H(1-\tau)+\beta) - a^H \\
(2) \qquad \log(\beta) &\leq \log(w^L(1-\tau)+\beta) - a^L
\end{aligned}
$$

Constraint (2) will bind before constraint (1). Here, the optimal solution will have (2) bind; constraint (1) will be slack. So, the high type works his entire life, $R^H = T^H$. The low type will work for the amount of time necessary to make constraint (2) bind. Note that if constraint (2) binds with $R^L = 0$, we are technically in case 2 below.

Case 2 (Only high type works):

$$
\max_{\tau,\beta} \quad R^H \log(w^H(1-\tau)+\beta) + (T^H - R^H + T^L)\log(\beta) - a^H R^H
$$

$$
\begin{array}{llcl}
s.t. & \beta & = & \dfrac{\tau(R^H w^H)}{T^H + T^L} \\
(3) & \log(\beta) & \leq & \log(w^H(1-\tau)+\beta) - a^H \\
(4) & \log(\beta) & > & \log(w^L(1-\tau)+\beta) - a^L
\end{array}
$$

Now we set $R^L = 0$, and pick R^H to make constraint (3) bind.

Notice that β will be higher in case 2 than in case 1. Notice also that the high type must always have higher lifetime utility than the low type in both cases. The low type has higher lifetime utility in case 2 than in case 1. It's not clear whether lifetime utility of the high type is higher or lower in case 1. To decide which case it wants to implement, the government will compare total lifetime utility in the two cases given specific parameter values.

Question 4a:

Individual of type i solves:

$$\max_{x^i, R^i} \; T^i \log(x^i) - R^i a^i$$

$$s.t. \quad T^i x^i = R^i(w^i(1-(1-\alpha)t)+b) + I^i$$

As in question 2a, the individual takes b as a constant in his budget constraint. The solution to the problem is:

$$
\begin{aligned}
R^i &= \frac{T^i}{a^i} - \frac{I^i}{w^i(1-(1-\alpha)t)+b} \\
x^i &= \frac{w^i(1-(1-\alpha)t)+b}{a^i}
\end{aligned}
$$

The social security budget constraint is:

$$(1-\alpha)t(R^H w^H + R^L w^L) = b(R^H + R^L)$$

What are the differences in incentives compared to question 2a? The individual is still facing an earnings tax - now $(1-\alpha)t$ instead of τ, but this does not substantially alter incentives. The main difference is that the flow benefit b now is only available when the individual is working. (β was available every period in question 2a.) The fact that receiving b now is contingent upon working changes retirement incentives for the individual - i.e. it provides an incentive for the individual to delay retirement vis-a-vis his retirement decision in question 2a. Note that $\frac{\partial R^i}{\partial \beta} < 0$ and $\frac{\partial R^i}{\partial b} > 0$.

If accounts had to be annuitized and the annuities were individually fair, then at retirement an individual would receive a flow benefit θ^i each period until his death such that the sum of these benefits $(T^i - R^i)\theta^i$ equalled the amount in his account at retirement $R^i(\alpha t w^i + b)$. Such a scheme would not alter incentives since all it does is alter when benefits are paid, but not the amount of benefits to be paid. Because there are no saving or borrowing constraints, this will not affect the individual's optimal solution.

Now suppose each individual receives benefit ϕ each period after he retires such that $(T^H - R^H + T^L - R^L)\phi = R^H(b + \alpha t w^H) + R^L(b + \alpha t w^L) = t(R^H w^H + R^L w^L)$ (i.e. the benefit is set such that it breaks even for the cohort). Given this set-up, there will be redistribution from individual's with bigger $T^i - R^i$ to individuals with smaller $T^j - R^j$. If there is no redistribution from the individuals with lower $T^j - R^j$ to the individuals with higher $T^i - R^i$ (i.e. if the account runs out of money no more benefits are paid or if the individual dies before all the account has been paid out the leftover can pay off accumulated debts), then again all this is doing is altering when benefits are paid out of the account. Since there are no liquidity constraints this won't alter incentives. However, if taxes are just paid into a common pot, then individuals will observe changes in their budget constraints which will alter incentives. Now the budget constraint can be written as a tax on earnings t and a flow benefit ϕ which is paid after retirement (i.e. we have gotten rid of the notion of an individual account).

$$T^i x^i = R^i w^i (1-t) + (T^i - R^i)\phi + I^i$$

The government's budget constraint is now

$$(T^H - R^H + T^L - R^L)\phi = t(R^H w^H + R^L w^L)$$

Since individual's don't take into account that their retirement decisions will affect the benefit level they receive when they retire, there are now incentives to retire earlier.

Question 4b:

NO SAVINGS!

Individual accounts are annuitized at retirement and are individually fair

$$\begin{aligned} x_z^i &= w^i(1-t) \quad \forall z < R^i \\ c_z^i &= \frac{R^i}{T^i - R^i}(\alpha t w^i + b) \quad \forall z \geq R^i \end{aligned}$$

Individual of type i solves:

$$\max_{R^i} \; R^i \log(w^i(1-t)) + (T^i - R^i)\log(\frac{R^i}{T^i - R^i}(\alpha t w^i + b)) - R^i a^i$$

The FOC for retirement age equates the marginal utility of working with the marginal utility of retiring:

$$\log(w^i(1-t)) - a^i = \log(\frac{R^i}{T^i - R^i}(\alpha t w^i + b)) - (T^i - R^i)\frac{1}{R^i} + 1$$

The social security budget constraint remains the same as in 4a:

$$(1-\alpha)t(R^H w^H + R^L w^L) = b(R^H + R^L)$$

What are the differences in incentives between this economy and the economy in question 2b? Now individuals realize that their future benefit payments depend on when they retire (as opposed to just being constants in the budget constraint). There is a marginal incentive for work, unlike in 2b (though retirement age could switch dramtically depending on the tax system)

Now suppose that benefits are annuitized and fair for the cohort but not the individual. Individual i's budget constraint becomes:

$$\begin{aligned} x_z^i &= w^i(1-t) \quad \forall z < R^i \\ c_z^i &= \phi \quad \forall z \geq R^i \end{aligned}$$

The government's budget constraint is the same as in 4a:

$$(T^H - R^H + T^L - R^L)\phi = t(R^H w^H + R^L w^L)$$

Again incentives has changed. The notion of individual accounts now is meaningless; each individual is atomistic. The optimal retirement ages will be described similarly to those in question 2b.

Questions 5a and 5b:

(This was the functional form of disutility of labor that Peter originally had in the problem set....so you can infer from this that the analysis becomes a lot messier!) But besides that, since disutility of labor is increasing with age there will be a general move towards earlier retirement and lower consumption levels in the part A questions. In general lifetime utilities will be lower since obtaining consumption is more onerous.

In 1b, still work until die, but lifetime utility is lower. There will be a greater chance for an interior solution in 2b. In 4b retirement age will fall, as will consumption.