We will consider an economy with two classes of workers: $i = H, L$, both of whom do consistent lifetime utility maximization or both of whom do no saving or borrowing, but still plan retirement in an optimal manner. Assume that workers work until some retirement age and do no work thereafter (that is, we ignore indifference to the timing of work). Assume that there are equal numbers of the two types. Assume that labor supply is a zero-one choice. Assume that there are no liquidity constraints on consistent lifetime savers. Assume that the interest rate is zero.

Notation

| | |
|---|---|
| $V^i$ | Lifetime utility, $i = H, L$, equal to the undiscounted integral of flow utility |
| $u^i\left[x_z^i\right] - a_z^i$ | flow utility at age z if working |
| $x_z^i$ | consumption when working |
| $v^i\left[c_z^i\right]$ | flow utility at age z if not working |
| $c_z^i$ | consumption when not working |
| $w_z^i$ | wage at age z |
| $R^i$ | retirement age |
| $T^i$ | age at death |
| $I^i$ | lump-sum income |
| $t, b$ | social security taxes and benefits |
| $\tau, \beta$ | flow income tax rate and benefit |

Assumptions

$u^i\left[x_z^i\right] = \log\left[x_z^i\right]$

$a_z^i = a^i$

$v^i\left[c_z^i\right] = \log\left[c_z^i\right]$

$w_z^i = w^i$

$a^L \geq a^H, T^L \leq T^H, w^L \leq w^H, I^L \leq I^H$

**Part A: consistent lifetime utility maximization**

**Question 1a.**

Assuming no social security and no income taxes, solve for the equations for individually optimal retirement age and consumption at each age in terms of the parameters assuming time-consistent optimal savings. Solve for lifetime utility. Show the comparison of marginal utilities of consumption of the two types.

**Question 2a.**

Assume that there is a linear tax on earnings at rate $\tau$. Assume that each person receives a flow benefit from the income tax authority of size $\beta$ whether working or not. Assume that the income tax authority has a PDV break-even budget constraint.

Solve for the equations for individually optimal retirement age and consumption at each age in terms of the parameters assuming time-consistent optimal savings. Solve for lifetime utility. State the government budget constraint that relates the benefit to the tax rate.

**Question 3a.**

Using the setup in Question 2a, set up the maximization for the optimal $\tau$ and $\beta$ that maximize the sum of lifetime utilities. Do not derive the first order condition.

**Question 4a.**

Now assume that there is no income tax, but there is a social security system. There is a payroll tax at rate t. Assume that workers have individual accounts. Assume that the fraction $\alpha$ of an individual's taxes goes into that individual's account. The rest of payroll tax revenue will be used to finance a flow contribution of b into the account of any worker who has not yet retired. Since there are no liquidity constraints and workers do time-consistent lifetime saving, it does not matter when benefits are paid.

Derive the first order conditions for worker savings and retirement.

In terms of the two retirement ages, write the social security budget constraint, assuming that social security must break even for the cohort.

Discuss the differences in incentives between this economy and that in questions 2a-3a.

Discuss how the analysis of incentives would be different if accounts had to be annuitized and annuitization were individually fair.

Discuss the differences between incentives if instead of benefits being based on fair annuities separately for each type, they are fair for the cohort as a whole. Note the ambiguity in the question based on possible differences in retirement ages.

**Part B: zero savings.**
**Assume** $I^L = I^H = 0$

**Question 1b.**

Assuming no social security and no income taxes, solve for the equations for individually optimal retirement age and consumption at each age in terms of the parameters assuming no savings. Solve for lifetime utility.

**Question 2b.**

Assume that there is a linear tax on earnings at rate $\tau$. Assume that each person receives a flow benefit from the income tax authority of size $\beta$ whether working or not. Assume that the income tax authority has a PDV break-even budget constraint.

Solve for the equations for individually optimal retirement age and consumption at each age in terms of the parameters assuming no savings. Solve for lifetime utility. Solve for the government budget constraint that relates the benefit to the tax rate.

**Question 3b.**

Using the setup in Question 2b, derive the first order condition for the optimal $\tau$ and $\beta$ that maximize the sum of lifetime utilities. Note there are several cases to analyze.

**Question 4b.**

Now assume that there is no income tax, but there is a social security system. There is a payroll tax at rate t. Assume that workers have individual accounts. Assume that the fraction $\alpha$ of an individual's taxes goes into that individual's account. The rest of payroll tax revenue will be used to finance a flow contribution of b into the account of any worker who has not yet retired. Assume that accounts have to be annuitized and annuitization is individually fair.

Derive the first order conditions for worker savings and retirement.

In terms of the two retirement ages, write the social security budget constraint, assuming that social security must break even for the cohort.

Discuss the differences in incentives between this economy and that in questions 2b-3b.

Discuss the differences between incentives if instead of benefits being based on fair annuities separately for each type, they are fair for the cohort as a whole. Note the ambiguity in the question based on possible differences in retirement ages.

**Questions 5a and 5b**

Sketch without details any significant differences if the disutility of labor satisfied $a_z^i = a^i z$.