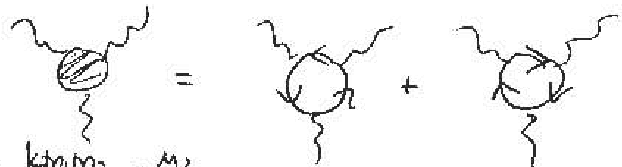
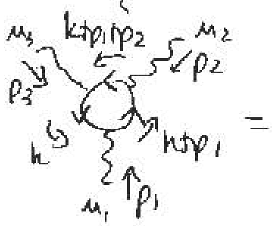
P.S.-10.1.a

$$= (-1)\int \frac{d^dk}{(2\pi)^d} \, \frac{tr\left((-ie\gamma^\mu)\, i(\not k + m)\right)}{k^2-m^2+i\epsilon} =$$

$$= -e \int \frac{d^dk}{(2\pi)^d} \, \frac{4k^\mu}{k^2-m^2+i\epsilon} = 0$$





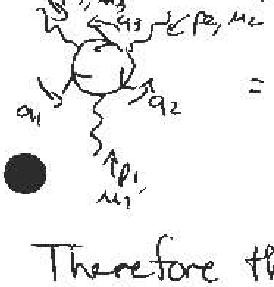
$$= (-1)\int \frac{d^dk}{(2\pi)^d} \, tr\left(\frac{i(\not q_1+m)}{q_1^2-m^2+i\epsilon}(-ie\gamma^{\mu_1})\frac{i(\not q_2+m)}{q_2^2-m^2+i\epsilon}(-ie\gamma^{\mu_2})\frac{i(\not q_3+m)}{q_3^2-m^2+i\epsilon}(-ie\gamma^{\mu_3})\right)$$

$q_1 = k$
$q_2 = k+p_1$
$q_3 = k+p_1+p_2$

$$= -e^3 \int \frac{d^dk}{(2\pi)^d} \, \frac{tr\left((\not q_1+m)\gamma^{\mu_1}(\not q_2+m)\gamma^{\mu_2}(\not q_3+m)\gamma^{\mu_3}\right)}{(q_1^2-m^2+i\epsilon)(q_2^2-m^2+i\epsilon)(q_3^2-m^2+i\epsilon)}$$

$$tr\left((\not q_1+m)\gamma^{\mu_1}(\not q_2+m)\gamma^{\mu_2}(\not q_3+m)\gamma^{\mu_3}\right) = tr\left(\not q_1\gamma^{\mu_1}\not q_2\gamma^{\mu_2}\not q_3\gamma^{\mu_3}\right) +$$
$$+ tr\left(\not q_1\gamma^{\mu_1} m\gamma^{\mu_2} m\gamma^{\mu_3}\right) + tr\left(m\gamma^{\mu_1}\not q_2\gamma^{\mu_2} m\gamma^{\mu_3}\right) +$$
$$+ tr\left(m\gamma^{\mu_1} m\gamma^{\mu_2}\not q_3\gamma^{\mu_3}\right)$$

↑ note only odd powers of q_i



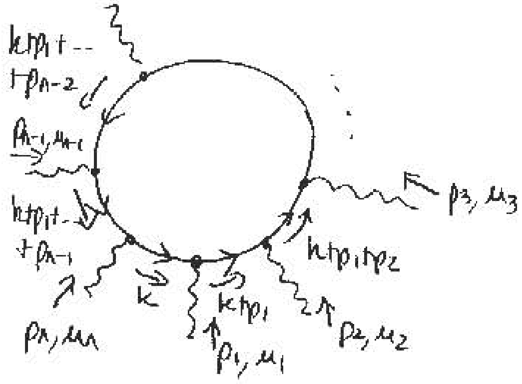
= previous diagram with $q_i \to -q_i$ because

$$\xrightarrow[\vec{q}]{} = \frac{i(\not q+m)}{q^2-m^2+i\epsilon} \qquad \xrightarrow[\overleftarrow{q}]{} = \frac{i(-\not q+m)}{q^2-m^2+i\epsilon}$$

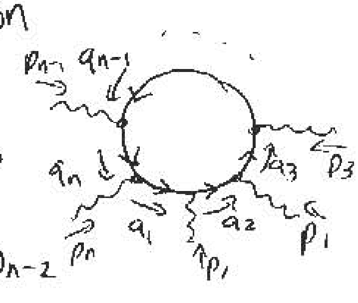
Therefore the sum of both  and  vanishes.

P.S.-10.1.a

To prove (perturbatively) that this behaviour persists for all odd point correlation functions, first consider the single loop contribution



$$\begin{aligned} q_1 &= k_1 \\ q_2 &= k_1 + p_1 \\ &\vdots \\ q_{n-1} &= k_1 + p_1 + \cdots p_{n-2} \\ q_n &= k_1 + p_1 + \cdots p_n \end{aligned} \Rightarrow$$



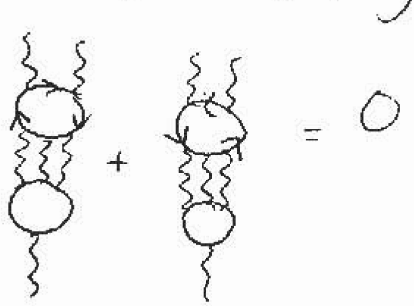
$$= (-1)\int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{tr}\left(i(\not{q}_1+m)(-ie\gamma^{\mu_1})\, i(\not{q}_2+m)(-ie\gamma^{\mu_2}) \cdots i(\not{q}_n+m)(-ie\gamma^{\mu_n})\right)}{(q_1^2-m^2+i\epsilon)(q_2^2-m^2+i\epsilon)\cdots(q_n^2-m^2+i\epsilon)}$$

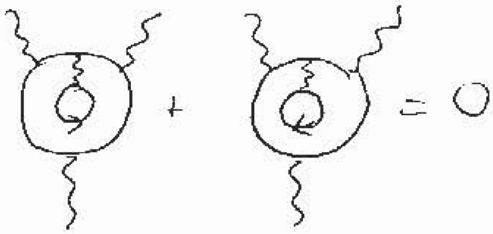
$$= -e^n \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{tr}\left((\not{q}_1+m)\gamma^{\mu_1}(\not{q}_2+m)\gamma^{\mu_2}\cdots(\not{q}_n+m)\gamma^{\mu_n}\right)}{(q_1^2-m^2+i\epsilon)(q_2^2-m^2+i\epsilon)\cdots(q_n^2-m^2+i\epsilon)}$$

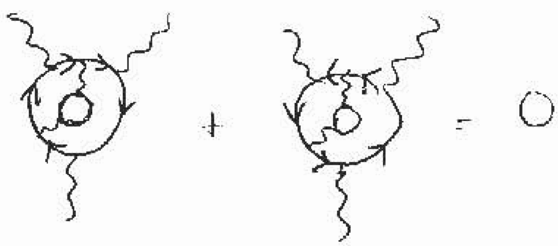
There are always an odd number of γ^{μ_i}'s in the numerator inside the trace, hence the only non-zero terms occur for an odd number of q_i's and an even number of m's. Thus this graph is odd under $q_i \to -q_i$ and hence cancels with the reversed loop, where again we use

$$\underset{q\,\rightarrow}{\longrightarrow} = \frac{i(\not{q}+m)}{q^2-m^2+i\epsilon} \qquad \underset{\leftarrow\, q}{\longrightarrow} = \frac{i(-\not{q}+m)}{q^2-m^2+i\epsilon}$$

This is sufficient to show that all diagrams contributing to an odd point function cancel in pairs as the following examples illustrate.







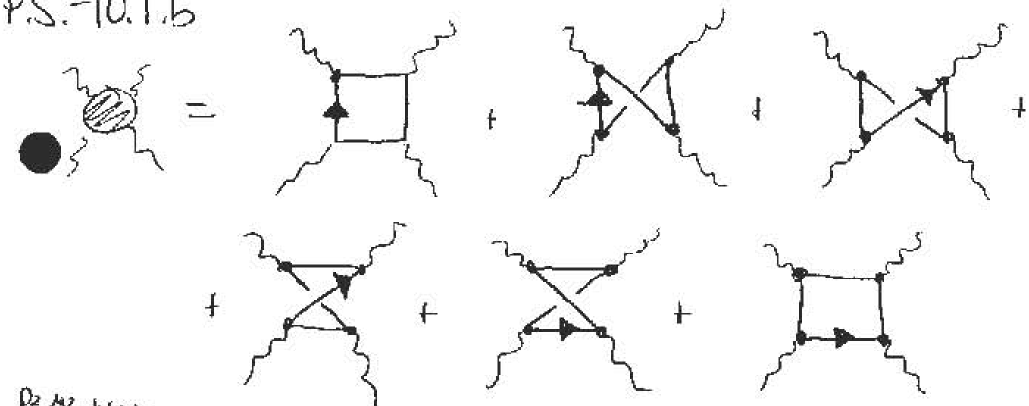
where I have drawn arrows on the loops which caused the pairwise cancellation. To turn these examples into a proof, consider the following. Suppose there are N fermion loops, $i=1,\dots,N$. Let N_γ and P_γ be the number of external and internal photon lines respectively. Let n_i be the number of photons connected to the i^{th} fermion loop. Then

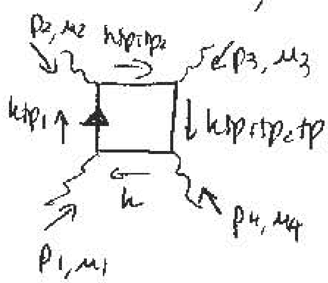
$$2\cdot P_\gamma + N_\gamma = \sum_i n_i$$

$N_\gamma = \text{odd} \Rightarrow 2P_\gamma + N_\gamma = \text{odd} \Rightarrow \exists$ some n_i which is odd

Then the graph will cancel with the corresponding graph with the i^{th} loop reversed.

P.S.-10.1.b





$$= (-1)\int \frac{d^dk}{(2\pi)^d}\, \frac{\mathrm{tr}\Big(i(\not k + m)(-ie\gamma^{\mu_1})\, i(\not k+\not p_1+m) \cdot (-ie\gamma^{\mu_2})\, i(\not k+\not p_1+\not p_2+m)(-ie\gamma^{\mu_3})\, i(\not k+\not p_1+\not p_2+\not p_3+m)(-ie\gamma^{\mu_4})\Big)}{(k^2-m^2+i\epsilon)((k+p_1)^2-m^2+i\epsilon)((k+p_1+p_2)^2-m^2+i\epsilon)((k+p_1+p_2+p_3)^2-m^2+i\epsilon)}$$

$$= -e^4\int \frac{d^dk}{(2\pi)^d}\, \frac{\mathrm{tr}(\not k\gamma^{\mu_1}\not k\gamma^{\mu_2}\not k\gamma^{\mu_3}\not k\gamma^{\mu_4})}{(k^2-m^2+i\epsilon)^4} + \text{finite}$$

$$= -e^4\int \frac{d^dk}{(2\pi)^d}\, \frac{k_{\nu_1}k_{\nu_2}k_{\nu_3}k_{\nu_4}}{(k^2-m^2+i\epsilon)^4} \cdot \mathrm{tr}(\gamma^{\nu_1}\gamma^{\mu_1}\gamma^{\nu_2}\gamma^{\mu_2}\gamma^{\nu_3}\gamma^{\mu_3}\gamma^{\nu_4}\gamma^{\mu_4})$$

divergent integral } used P.S. Eq. A.48

$$= -e^4\, I(m^2,d)\cdot(g_{\nu_1\nu_2}g_{\nu_3\nu_4} + g_{\nu_1\nu_3}g_{\nu_2\nu_4} + g_{\nu_1\nu_4}g_{\nu_2\nu_3})\times$$
$$\times\, \mathrm{tr}(\gamma^{\nu_1}\gamma^{\mu_1}\gamma^{\nu_2}\gamma^{\mu_2}\gamma^{\nu_3}\gamma^{\mu_3}\gamma^{\nu_4}\gamma^{\mu_4})$$

$$= -e^4\, I(m^2,d)\Big\{ \mathrm{tr}(\gamma\gamma^{\mu_1}\gamma\gamma^{\mu_2}\gamma\gamma^{\mu_3}\gamma\gamma^{\mu_4}) + \mathrm{tr}(\gamma\gamma^{\mu_1}\gamma\gamma^{\mu_2}\gamma\gamma^{\mu_3}\gamma\gamma^{\mu_4})$$
$$+ \mathrm{tr}(\gamma\gamma^{\mu_1}\gamma\gamma^{\mu_2}\gamma\gamma^{\mu_3}\gamma\gamma^{\mu_4}) \Big\}$$

$= -e^4 I(m^2,d)\{ tr((-2\gamma^{\mu_1})\gamma^{\mu_2}(-2\gamma^{\mu_3})\gamma^{\mu_4}) +$

$+ tr(-2\gamma^{\mu_2}\gamma\gamma^{\mu_1}\gamma^{\mu_3}\gamma\gamma^{\mu_4}) + tr(\gamma\gamma^{\mu_1}(-2\gamma^{\mu_2})\gamma^{\mu_3}\gamma\gamma^{\mu_4})\}$

$= -e^4 I(m^2,d)\{4\, tr(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) - 2\, tr(\gamma^{\mu_2} 4g^{\mu_1\mu_3}\gamma^{\mu_4}) +$

$- 2\, tr(-2\gamma^{\mu_3}\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_4})\} =$

$= -e^4 I(m^2,d)\{4\cdot 4(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3})$

$-8g^{\mu_1\mu_3}\cdot 4g^{\mu_2\mu_4} + 4\cdot 4(g^{\mu_3\mu_2}g^{\mu_1\mu_4} - g^{\mu_3\mu_1}g^{\mu_2\mu_4} + g^{\mu_3\mu_4}g^{\mu_2\mu_1})\}$

$= -e^4 I(m^2,d)\{g^{\mu_1\mu_2}g^{\mu_3\mu_4}(16+16) + g^{\mu_1\mu_3}g^{\mu_2\mu_4}(-16-32-16)$

$+ g^{\mu_1\mu_4}g^{\mu_2\mu_3}(16+16)\} =$

$= -32e^4 I(m^2,d)\{g^{\mu_1\mu_2}g^{\mu_3\mu_4} - 2g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}\}$

Thus the divergent piece is indepedent of p_1, p_2, p_3, p_4 and only depends $\mu_1, \mu_2, \mu_3, \mu_4$. Thus the divergent piece of all 6 terms is proportional to

$\overset{(1,2,3,4)}{g^{12}g^{34} - 2g^{13}g^{24} + g^{14}g^{23}} + \overset{(1,2,4,3)}{g^{12}g^{43} - 2g^{14}g^{23} + g^{13}g^{24}}$

$\overset{(1,3,2,4)}{g^{13}g^{24} - 2g^{12}g^{34} + g^{14}g^{32}} + \overset{(1,3,4,2)}{g^{13}g^{42} - 2g^{14}g^{32} + g^{12}g^{34}}$

$\overset{(1,4,3,2)}{g^{14}g^{32} - 2g^{13}g^{42} + g^{12}g^{43}} + \overset{(1,4,2,3)}{g^{14}g^{23} - 2g^{12}g^{43} + g^{13}g^{42}}$

$= g^{\mu_1\mu_2}g^{\mu_3\mu_4}(1+1-2+1+1-2) + g^{\mu_1\mu_3}g^{\mu_2\mu_4}(-2+1+1+1-2+1)$

$$+ g^{\mu_1\mu_4} g^{\mu_2\mu_3} (1-2+1-2+1+1) = 0$$

Therefore the divergent part of the 4 point function vanishes at 1 loop.

P.S. - 10.2.a

let N_f = # external fermion lines

N_s = # " scalar "

P_f = # internal fermion lines

P_s = # " scalar lines

V_3 - # $\phi\bar{\psi}\gamma_5\psi$ vertices

V_4 = # ϕ^4 "

V = # vertices

L = # loops

D = superficial degree of divergence

then

$V = V_3 + V_4$

$D = dL - 2P_s - P_f$

$L = P_s + P_f - (V-1)$

$1 \cdot V_3 + 4 \cdot V_4 = N_s + 2P_s$

$2 \cdot V_3 + 0 \cdot V_4 = N_f + 2P_f$

we can write D as

$$D = d + \left(\frac{d-4}{2}\right) \cdot V_3 + (d-4) \cdot V_4 - \left(\frac{d-2}{2}\right) N_s - \left(\frac{d-1}{2}\right) N_f$$

$$\overset{d \to 4}{=} 4 - N_s - \frac{3}{2} N_f$$

P.S. – 10.2.a

The divergent amplitudes are:

 irrelevant

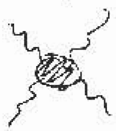
 = scalar

 = fermion

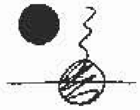
 $= 0$ due to parity

 $\sim A\Lambda^2 + Bp^2 \ln\Lambda$

 $= 0$ due to parity

 $\sim A \ln\Lambda$

 $\sim AM\ln\Lambda + B\not{p}\ln\Lambda$

 $\sim A\ln\Lambda$

There are 6 divergent numbers which can be absorbed into 2 field normalizations, 2 masses, and 2 couplings.

The bare Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_B)^2 - \frac{1}{2}m_B^2\phi_B^2 - \frac{\lambda_B}{4!}\phi_B^4 + \bar{\psi}_B(i\not{\partial} - M_B)\psi_B - ig_B\phi_B\bar{\psi}_B\gamma_5\psi_B$$

rescale the fields, $\phi = 1/\sqrt{Z}\,\phi_B$ $\quad \psi = 1/\sqrt{Z_2}\,\psi_B$

$$\mathcal{L} = Z\,\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}Zm_B^2\phi^2 - \frac{Z^2\lambda_B}{4!}\phi^4 + Z_2\bar{\psi}i\not{\partial}\psi - Z_2M_B\bar{\psi}\psi$$
$$- ig_B\sqrt{Z_B}\,Z_2\,\phi\bar{\psi}\gamma_5\psi$$

define counter terms as: $Z = 1+\delta_Z$, $Zm_B^2 = m^2 + \delta_m$

$Z^2\lambda_B = \lambda + \delta_\lambda$, $Z_2 = 1+\delta_2$, $Z_2M_B = M + \delta_M$, $g_B\sqrt{Z_B}\,Z_2 = gZ_1$

$Z_1 = 1+\delta_1$

P.S.-10.2-a

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \bar\Psi(i\not\partial - M)\Psi - ig\phi\bar\Psi\gamma_5\Psi +$$

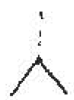
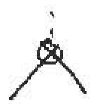
$$+ \frac{\delta_Z}{2}(\partial_\mu\phi)^2 - \frac{\delta_m}{2}\phi^2 - \frac{\delta_\lambda}{4!}\phi^4 + \bar\Psi(i\not\partial\delta_2 - \delta_M)\Psi - ig\delta_1\bar\Psi\gamma_5\Psi\phi$$

The Feynman rules are

 $= \dfrac{i}{p^2 - m^2 + i\epsilon}$  $= i(p^2\delta_Z - \delta_m)$

 $= \dfrac{i}{\not p - m + i\epsilon}$  $= i(\not p\delta_2 - \delta_M)$

 $= -i\lambda$  $= -i\delta_\lambda$

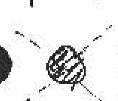
 $= g\gamma^5$  $= g\delta_1\gamma_5$

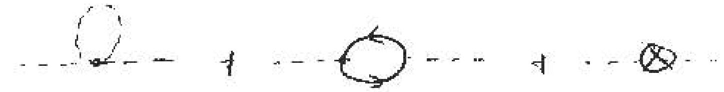
The renormalization conditions are:

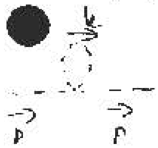
$$\left.\begin{aligned} &\Sigma(M) = 0 \\ &\frac{d}{d\not p}\Sigma(\not p)\Big|_{\not p = M} = 0 \end{aligned}\right\} \quad -i\Sigma(\not p) =$$

$$\left.\begin{aligned} &M^2(m^2) = 0 \\ &\frac{d}{dp^2}M^2(p^2)\Big|_{p^2 = m^2} = 0 \end{aligned}\right\} \quad -iM^2(p^2) =$$

 $\Big|_{q\to 0} = g\gamma^5$ (tree)

 $\Big|_{s=4m^2,\, t=0,\, u=0} = -i\lambda$ (tree)

$-iM^2(p^2) =$  (to 1 loop)

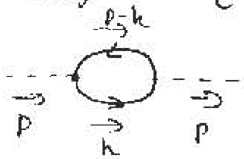
 $= \left(\frac{1}{2}\right)(-i\lambda)\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2+i\epsilon} = \frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2+i\epsilon}$

$= \frac{\lambda}{2}\frac{(-1)^1\cdot i}{(4\pi)^{d/2}}\frac{\Gamma(1-\frac{d}{2})}{\Gamma(1)}\left(\frac{1}{m^2}\right)^{1-\frac{d}{2}} = -\frac{i}{2}\frac{\lambda}{(4\pi)^2}\Gamma(-1+\epsilon/2)(4\pi)^{\frac{\epsilon}{2}}(m^2)^{1-\epsilon/2}$

$= -\frac{i}{2}\frac{\lambda}{(4\pi)^2}\cdot m^2\cdot\frac{(-1)^1}{1!}\frac{1}{-1+\epsilon/2+1}$ + finite

$= i\frac{\lambda}{(4\pi)^2}\cdot m^2\cdot\frac{1}{\epsilon}$

fermion loop

 $= (-1)\cdot g\cdot g\int\frac{d^dk}{(2\pi)^d}\frac{tr(\gamma^5 i(\not k+M)\gamma^5 i(\not k-\not p+M))}{(k^2-M^2+i\epsilon)((k-p)^2-M^2+i\epsilon)}$

$= g^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{tr((-\not k+M)(\not k-\not p+M))}{\underbrace{(x(k^2-M^2+i\epsilon)+(1-x)((k-p)^2-M^2+i\epsilon))}^2}$

$D = \ell^2-\Delta+i\epsilon \quad \ell = k-(1-x)p \quad \Delta = M^2-x(1-x)p^2$

$= g^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{tr((-\not\ell-(1-x)\not p+M)(\not\ell-x\not p+M))}{(\ell^2-\Delta+i\epsilon)^2}$

← ignore ϵ contribution from traces, only gives finite pieces

$= g^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}4\frac{(-\ell^2+x(1-x)p^2+M^2)}{(\ell^2-\Delta+i\epsilon)^2}$

P.S. - 10.2.b

$$= 4g^2 \int_0^1 dx \left\{ \{-1\} \cdot \frac{(-1)^{2-1} \cdot i}{(4\pi)^{d/2}} \cdot \frac{d}{2} \cdot \frac{\Gamma(2-\frac{d}{2}-1)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}-1} + \right.$$

$$\left. + \{x(1-x)p^2 + M^2\} \cdot \frac{(-1)^2 \cdot i}{(4\pi)^{d/2}} \cdot \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \right\}$$

$$= \frac{4ig^2}{(4\pi)^2} \int_0^1 dx \, (4\pi)^{\epsilon/2} \left\{ \left(2 - \frac{\epsilon}{2}\right) \Gamma\left(\frac{\epsilon}{2}-1\right) \left(\frac{1}{\Delta}\right)^{\frac{\epsilon}{2}-1} + (x(1-x)p^2 + M^2)\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{1}{\Delta}\right)^{\frac{\epsilon}{2}} \right\}$$

$$= \frac{4ig^2}{(4\pi)^2} \int_0^1 dx \, (4\pi)^{\epsilon/2} \left\{ \left(2 - \frac{\epsilon}{2}\right) \Gamma\left(\frac{\epsilon}{2}-1\right) \left(M^2 - x(1-x)p^2\right) + \left(\frac{\epsilon}{2}-1\right)\Gamma\left(\frac{\epsilon}{2}-1\right) \times \right.$$

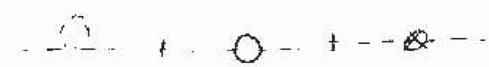
$$\left. \times \left(x(1-x)p^2 + M^2\right) \right\} \left(\frac{1}{\Delta}\right)^{\frac{\epsilon}{2}}$$

$$= \frac{4ig^2}{(4\pi)^2} \int_0^1 dx \left(2(M^2 - x(1-x)p^2) - (x(1-x)p^2 + M^2) \right) \cdot \Gamma\left(\frac{\epsilon}{2}-1\right) + \text{finite}$$

$$= \frac{4ig^2}{(4\pi)^2} \Gamma\left(\frac{\epsilon}{2}-1\right) \int_0^1 dx \left(M^2 - 3x(1-x)p^2\right) + \text{finite}$$

$$= \frac{4ig^2}{(4\pi)^2} \Gamma\left(\frac{\epsilon}{2}-1\right) \left(M^2 - \frac{1}{2}p^2\right) = \frac{4ig^2}{(4\pi)^2} \frac{(-1)^1}{1!} \cdot \frac{2}{\epsilon} \left(M^2 - \frac{1}{2}p^2\right)$$

$$= -8\frac{ig^2}{(4\pi)^2} \left(M^2 - \frac{1}{2}p^2\right) \cdot \frac{1}{\epsilon} + \text{finite}$$

$-iM^2(p^2) =$ 

$$= \frac{i\lambda}{(4\pi)^2} m^2 \cdot \frac{1}{\epsilon} - 8\frac{ig^2}{(4\pi)^2} \left(M^2 - \frac{1}{2}p^2\right) \frac{1}{\epsilon} + i(p^2 \delta_Z - \delta_m)$$

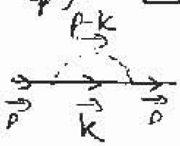
$$M^2(p^2) = -\frac{\lambda}{(4\pi)^2} m^2 \frac{1}{\epsilon} + \frac{8g^2}{(4\pi)^2}\left(\mathcal{M}^2 - \tfrac{1}{2}p^2\right)\frac{1}{\epsilon} - p^2\delta_Z + \delta_m$$

$$\frac{d}{dp^2} M^2(p^2) = -\frac{4g^2}{(4\pi)^2}\frac{1}{\epsilon} - \delta_Z = 0 \quad {}_{p^2\to m^2} \Rightarrow \boxed{\delta_Z = -\frac{4g^2}{(4\pi)^2}\cdot\frac{1}{\epsilon}}$$

$$M^2(p^2=m^2) = -\frac{\lambda}{(4\pi)^2} m^2 \cdot \frac{1}{\epsilon} + \frac{8g^2\left(\mathcal{M}^2 - \frac{1}{2}m^2\right)}{(4\pi)^2}\frac{1}{\epsilon} - m^2\cdot\frac{(-4g^2)}{(4\pi)^2}\frac{1}{\epsilon} + \delta_m$$

$$\boxed{\delta_m = \frac{\lambda}{(4\pi)^2} m^2 \frac{1}{\epsilon} - \frac{8g^2\mathcal{M}^2}{(4\pi)^2}\frac{1}{\epsilon}}$$

$-i\Sigma(\not p) =$  $+$ 



$$= \int \frac{d^dk}{(2\pi)^d}\, g\gamma_5 \frac{i(\not k + \mathcal{M})}{k^2 - \mathcal{M}^2 + i\epsilon} \cdot g\gamma_5 \frac{i}{(p-k)^2 - m^2 + i\epsilon}$$

$$= -g^2\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(-\not k + \mathcal{M})}{\underbrace{\left(x(k^2 - \mathcal{M}^2 + i\epsilon) + (1-x)((p-k)^2 - m^2 + i\epsilon)\right)}^2}$$

$$D = \ell^2 - \Delta + i\epsilon \quad \ell = k - (1-x)p \quad \Delta = x\mathcal{M}^2 + (1-x)m^2 - x(1-x)p^2$$

$$= -g^2\int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(-\not\ell - (1-x)\not p + \mathcal{M})}{(\ell^2 - \Delta + i\epsilon)^2} = -g^2\int_0^1 dx\, ((x-1)\not p + \mathcal{M})\, \frac{(-1)^2\, i\, \Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}\,\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

$$= \frac{-ig^2}{(4\pi)^2}\left(-\tfrac{1}{2}\not p + \mathcal{M}\right)\frac{2}{\epsilon} + \text{finite} = -2i\frac{g^2}{(4\pi)^2}\left(\mathcal{M} - \tfrac{1}{2}\not p\right)\cdot\frac{1}{\epsilon} + \text{finite}$$

$-i\Sigma(\not p) =$  $+$  $=$

$$= -\frac{2ig^2}{(4\pi)^2}\left(\mathcal{M} - \tfrac{1}{2}\not p\right)\frac{1}{\epsilon} + i(\not p\,\delta_Z - \delta_{\mathcal{M}})$$

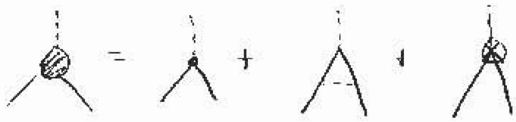
P.S. - 10.2.b

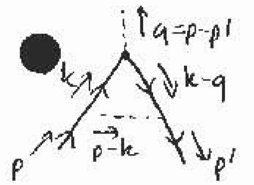
$$\Sigma(\not p) = \frac{2g^2}{(4\pi)^2}\left(M - \tfrac{1}{2}\not p\right)\frac{1}{\epsilon} - \not p\,\delta_Z + \delta_M$$

$$\frac{d\Sigma}{d\not p} = \frac{-g^2}{(4\pi)^2}\cdot\frac{1}{\epsilon} - \delta_Z \underset{\not p \to M}{=} 0 \quad \Rightarrow \quad \boxed{\delta_Z = \frac{-g^2}{(4\pi)^2}\frac{1}{\epsilon}}$$

$$\Sigma(\not p = M) = 0 = \frac{2g^2}{(4\pi)^2}\left(M - \tfrac{1}{2}M\right)\frac{1}{\epsilon} - M\cdot\left(\frac{-g^2}{(4\pi)^2}\frac{1}{\epsilon}\right) + \delta_M$$

$$\boxed{\delta_M = \frac{-2g^2 M}{(4\pi)^2}\frac{1}{\epsilon}}$$





$$= \int \frac{d^dk}{(2\pi)^d}\, g\gamma^5 \cdot \frac{i(\not k + M)}{k^2 - M^2 + i\epsilon} \cdot g\gamma^5 \cdot \frac{i(\not k - \not q + M)}{(k-q)^2 - M^2 + i\epsilon} \cdot g\gamma^5 \frac{i}{(p-k)^2 - m^2 + i\epsilon}$$

$$= -ig^3 \int \frac{d^dk}{(2\pi)^d} \frac{(-\not k + M)(\not k - \not q + M)\cdot\gamma^5}{(k^2 - M^2 + i\epsilon)((k-q)^2 - M^2 + i\epsilon)((p-k)^2 - m^2 + i\epsilon)}$$

examine the case $q=0$

$$= -ig^3 \int \frac{d^dk}{(2\pi)^d} \frac{(-\not k + M)(\not k + M)\gamma^5}{(k^2 - M^2 + i\epsilon)(k^2 - M^2 + i\epsilon)((p-k)^2 - m^2 + i\epsilon)}$$

$$= -ig^3 \int \frac{d^dk}{(2\pi)^d} \frac{\cancel{(-k^2 + M^2)}\,\gamma^5}{\cancel{(k^2 - M^2)}\,(k^2 - M^2 + i\epsilon)((p-k)^2 - m^2 + i\epsilon)}$$

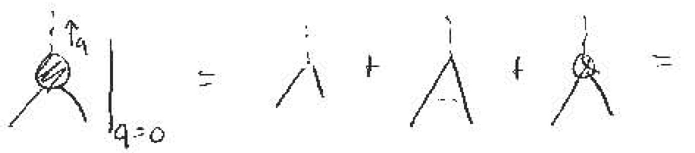
$$= -ig^3 \int \frac{d^dk}{(2\pi)^d} \frac{(-1)\gamma^5}{(k^2 - M^2 + i\epsilon)((p-k)^2 - m^2 + i\epsilon)}$$

$$= ig^3\gamma^5\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{\big(\underbrace{x(k^2-M^2+i\epsilon)+((p-k)^2+m^2+i\epsilon)}_{D=\ell^2-\Delta+i\epsilon}\big)^2}$$

$l = k-(1-x)p \qquad \Delta = xM^2+(1-x)m^2 - x(1-x)p^2$

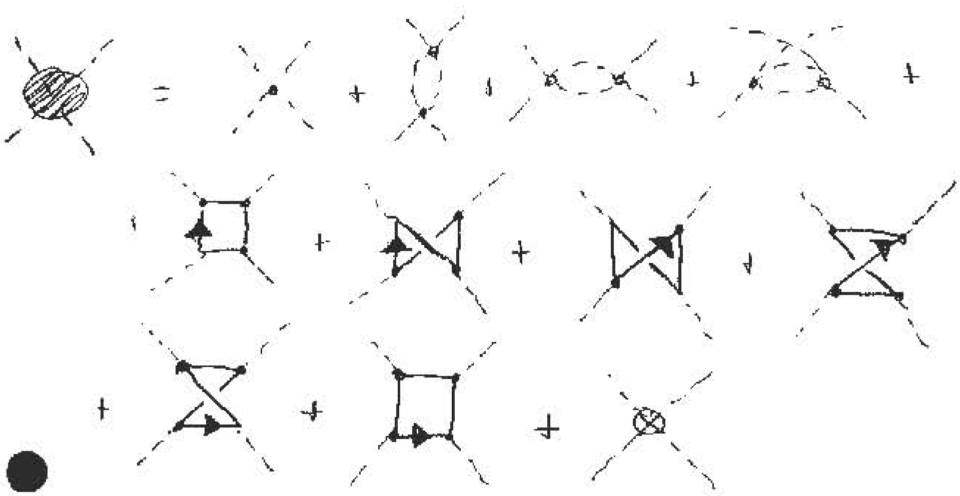
$$= ig^3\gamma^5\int_0^1 dx\,\frac{(-1)^2\, i}{(4\pi)^{d/2}}\cdot\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

$$= -\frac{g^2}{(4\pi)^2}\cdot g\gamma^5\frac{2}{\epsilon} + \text{finite} = -\frac{2g^2}{(4\pi)^2}\cdot g\gamma^5\cdot\frac{1}{\epsilon} + \text{finite}$$

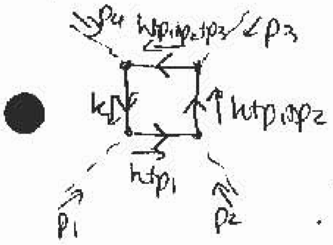


$= g\gamma_5$ (tree)

$$= g\gamma_5 - \frac{2g^2}{(4\pi)^2}\frac{1}{\epsilon}\cdot g\gamma^5 + g\,\delta_1\gamma^5 \Rightarrow \boxed{\delta_1 = \frac{2g^2}{(4\pi)^2}\frac{1}{\epsilon}}$$



P.S. – 10.2.b



$$= (-1)\int \frac{d^dk}{(2\pi)^d}\, \mathrm{tr}\left\{ \frac{i(\not k + M)}{(k^2 - M^2 + i\epsilon)}\, g\gamma_5 \cdot \frac{i(\not k + \not p + M)}{(k+p)^2 - M^2 + i\epsilon}\, g\gamma_5 \cdot \frac{i(\not k + \not p_1 + \not p_2 + M)}{(k+p_1+p_2)^2 - M^2 + i\epsilon}\, g\gamma_5\, \frac{i(\not k + \not p_1 + \not p_2 + \not p_3 + M)}{(k+p_1+p_2+p_3)^2 - M^2 + i\epsilon}\, g\gamma_5 \right\}$$

The only divergence occurs for the term in the numerator containing all 4 $\not k$s. The divergence is also independent of the p_i's, so we can drop them.

$$= -g^4 \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{tr}(\not k \gamma_5 \not k \gamma_5 \not k \gamma_5 \not k \gamma_5)}{(k^2 - M^2 + i\epsilon)^4} + \text{finite}$$

$$= -g^4 \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{tr}(\not k \not k \not k \not k)}{(k^2 - M^2 + i\epsilon)^4} = -g^4 \int \frac{d^dk}{(2\pi)^d} \cdot \frac{4(k^2)^2}{(k^2)^2 \cdot (k^2 - M^2 + i\epsilon)^2} =$$

$$= -4g^4 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - M^2 + i\epsilon)^2} = -4g^4 \cdot \frac{(-1)^2 i}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{M^2}\right)^{2-\frac{d}{2}}$$

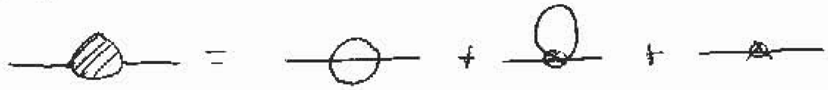
$$= -4i \frac{g^4}{(4\pi)^2} \cdot \frac{2}{\epsilon} + \text{finite} = -8i \frac{g^4}{(4\pi)^2} \frac{1}{\epsilon} + \text{finite}$$

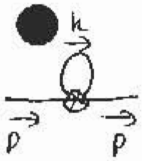
P.S. p. 326–327 divergence is independent of p_i's

$$= -i\cancel{\lambda} + 3 \cdot \frac{i\lambda^2}{(4\pi)^2} \frac{1}{\epsilon} + 6 \cdot \left(-\frac{8ig^4}{(4\pi)^2} \frac{1}{\epsilon}\right) - i\delta_\lambda$$

$$\boxed{\delta_\lambda = \frac{3\lambda^2}{(4\pi)^2} \frac{1}{\epsilon} - 48 \frac{g^4}{(4\pi)^2} \frac{1}{\epsilon}}$$

P.S. – 10.3

 at 2 loop

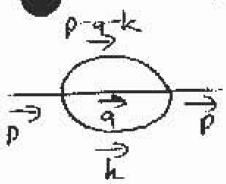
 $= \frac{1}{2} \cdot (-i\delta_\lambda) \cdot \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2 - m^2 + i\epsilon} =$

↙ 1 loop counter-term

$= -\frac{i\delta_\lambda}{2} \cdot \frac{(-1)^1 \cdot i}{(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{\Gamma(1)} \left(\frac{1}{m^2}\right)^{1-\frac{d}{2}}$

$= -\frac{i\delta_\lambda}{2} \frac{m^2}{(4\pi)^2} \cdot (4\pi)^\epsilon \cdot (m^2)^{-\frac{\epsilon}{2}} \Gamma(\frac{\epsilon}{2} - 1)$

$= -\frac{i\delta_\lambda}{2} \frac{(4\pi)^\epsilon \Gamma(\frac{\epsilon}{2}-1)}{(4\pi)^2} (m^2)^{1-\frac{\epsilon}{2}} \xrightarrow[m^2 \to 0]{} 0$ for $\epsilon < 2$

 $= \frac{1}{3!} (-i\lambda)^2 \int \frac{d^dk}{(2\pi)^d} \int \frac{d^dq}{(2\pi)^d} \frac{i}{k^2} \frac{i}{q^2} \frac{i}{(p-q-k)^2} =$

$= \frac{i\lambda^2}{6} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} \int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2 \cdot (p-q-k)^2}$

$= \frac{i\lambda^2}{6} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} \int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{1}{(\underbrace{xq^2 + (1-x)(p-q-k)^2}_{= \ell^2 - \Delta,\ \ell = q + (1-x)(k-p),\ \Delta = -x(1-x)(k-p)^2})^2}$

$= \frac{i\lambda^2}{6} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} \int_0^1 dx \frac{(-1)^2 \cdot i}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$

$= -\frac{\lambda^2}{6} \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} \int_0^1 dx \frac{1}{(-x(1-x)(k-p)^2)^{2-\frac{d}{2}}} =$

P.S. – 10.3

$$= -\frac{\lambda^2}{6}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx\,(-x(1-x))^{\frac{d}{2}-2}\int\frac{d^dk}{(2\pi)^d}\underbrace{\frac{1}{k^2}\frac{1}{((k-p)^2)^{2-\frac{d}{2}}}}_{\int_0^1 dy\,\frac{(2-\frac{d}{2})\cdot y^{2-\frac{d}{2}-1}}{((1-y)\cdot k^2+y(k-p)^2)^{2-\frac{d}{2}+1}}}$$

$$= -\frac{\lambda^2}{6}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx\,(-x(1-x))^{\frac{d}{2}-2}\int_0^1 dy\,(2-\tfrac{d}{2})y^{1-\frac{d}{2}}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(\underbrace{(1-y)k^2+y(k-p)^2}_{l^2-\Delta})^{3-\frac{d}{2}}}$$

$$= -\frac{\lambda^2}{6}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx\,(-x(1-x))^{\frac{d}{2}-2}\,\cdot \qquad l = k-yp \quad \Delta = y(y-1)p^2$$

$$\cdot\int_0^1 dy\,(2-\tfrac{d}{2})y^{1-\frac{d}{2}}\,\frac{(-1)^{3-\frac{d}{2}}\cdot i}{(4\pi)^{d/2}}\,\frac{\Gamma(3-\frac{d}{2}-\frac{d}{2})}{\Gamma(3-\frac{d}{2})}\left(\frac{1}{y(y-1)p^2}\right)^{3-\frac{d}{2}-\frac{d}{2}}$$

$$= \frac{-i\lambda^2}{6}\frac{1}{(4\pi)^d}(-1)^2(-1)^{3-\frac{d}{2}}\,\underbrace{\Gamma(2-\tfrac{d}{2})(2-\tfrac{d}{2})}_{\Gamma(2-\frac{d}{2}+1)}\,\frac{\Gamma(3-d)}{\Gamma(3-\frac{d}{2})}\cdot(p^2)^{d-3}$$

$$\cdot\int_0^1 dx\,(x(1-x))^{\frac{d}{2}-2}\cdot\int_0^1 dy\,y^{1-\frac{d}{2}}(y(y-1))^{d-3}$$

$$= \frac{i\lambda^2}{6}\frac{1}{(4\pi)^4}(4\pi)^{\epsilon}(-1)^{\frac{\epsilon}{2}}\,\Gamma(-1+\epsilon)\cdot p^2(p^2)^{-\epsilon}\int_0^1 dx\,(x(1-x))^{-\frac{\epsilon}{2}}$$

$$\cdot\int_0^1 dy\,y^{-\frac{\epsilon}{2}}(y-1)^{1-\epsilon}$$

We can set $\epsilon = 0$ in all finite p^2 independent terms since this only affects the p^2-constant term.

$$= \frac{i\lambda^2}{6} \frac{1}{(4\pi)^4} p^2 \underbrace{\int_0^1 dy\,(1-y)}_{-\frac{1}{2}} \cdot \underbrace{\Gamma(-1+\epsilon)}_{-\frac{1}{\epsilon}} \cdot \underbrace{(p^2)^{-\epsilon}}_{(1-\epsilon \ln p^2)}$$

$$= \frac{i\lambda^2}{12\,(4\pi)^4} p^2 \left(\frac{1}{\epsilon} - \ln p^2 + \text{constant independent of } p^2 \right)$$

W: 12.1

$\int d^dx\, \partial_\mu \varphi \partial^\mu \varphi$ is dimensionless $\Rightarrow -d + 2 + 2d_\varphi = 0$

- $d_\varphi = \frac{d-2}{2}$

For Lorentz invariance, each operator must contain an even number of derivatives, $\mathcal{O}_{n,m} \sim (\partial\partial)^n \varphi^m$, then

$d_{n,m} = 2n\, d_\partial + m d_\varphi = 2n + m\left(\frac{d-2}{2}\right)$

$\Delta_{n,m} = d - d_{n,m} = d - 2n - m\left(\frac{d-2}{2}\right)$

$\Delta_{n,m} \geq 0 \Rightarrow d - 2n - \left(\frac{d-2}{2}\right)m \geq 0$

$d = 2 \quad 2 - 2n \geq 0 \Rightarrow n \leq 1$

- The renormalizable operators are φ^p $p = 0, 1, \ldots$, $\partial_\mu \varphi \partial^\mu \varphi\, \varphi^p$ $p = 0, 1, \ldots$, and $\partial^2\varphi\, \varphi^p$ $p = 0, 1, \ldots$. (Classically one could integrate by parts to relate $\partial^2\varphi \cdot \varphi$ to $\partial_\mu \varphi \partial^\mu \varphi$, but $\pi_1(U(1)) = \pi_1(S_1) = \mathbb{Z}$ may prevent the surface term from vanishing.)

$d = 3 \quad 3 - 2n - \frac{1}{2}m \geq 0$

$n = 0 \quad m \leq 6$

$n = 1 \quad m \leq 2$

- The renormalizable interactions are φ^p $p = 0, 1, \ldots, 6$, and $\partial_\mu \varphi \partial^\mu \varphi$. (Here $\pi_2(S_1) = 0$ allows one to integrate by parts and hence ignore ∂^2 terms.

$d=6$ $\quad 6-2n-2m \geq 0$

$n=0$ $\quad m \leq 3$ $\quad \phi^p \ p=0,..,3$

$n=1$ $\quad m \leq 2$ $\quad \partial_\mu \phi \partial^\mu \phi$ $\quad$ ($\phi \partial^2 \phi$, ignored, integrate by parts)

$n=2$ $\quad m \leq 1$ $\quad \partial^4 \phi$ ignored, total derivative

$n=3$ $\quad m \leq 0$ $\quad$ nothing

using $\Pi_5(S_1)=0$

The list of terms is: $1, \phi, \phi^2, \phi^3, \partial_\mu \phi \partial^\mu \phi$

can be shifted away

most likely leads to a sick theory

In $d=6$, the scalar theory should be free.