P.S. – 9.1.a

The free propagator appears in the perturbative expansion of the 2-point function.

$$\langle \Omega | T \hat{\phi}(x_1) \hat{\phi}^\dagger(x_2) | \Omega \rangle =$$

$$= \lim_{T \to \infty(1-i\epsilon)} \frac{\int D\phi\, \phi(x_1) \phi^*(x_2)\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}}}{\int D\phi\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}}}$$

Choose a lattice with sites x_i and Fourier transform the field.

$$\phi(x_i) = \frac{1}{V} \sum_{k_n} e^{-ik_n \cdot x_i} \phi(k_n) \qquad k_n \in \text{Brilliouin zone}$$

$$\int D\phi = \prod_{k_n} \int_{-\infty}^{\infty} d\,\mathrm{Re}\,\phi(k_n) \int_{-\infty}^{\infty} d\,\mathrm{Im}\,\phi(k_n)$$

The quadratic portion of the action is

$$S_2[\phi] = \int d^4x \left(\partial_\mu \phi\, \partial^\mu \phi^* - m^2 \phi^* \phi \right)$$

$$= \frac{1}{V} \sum_{k_n} \frac{1}{2} \left(k_n^2 - m^2 + i\epsilon \right) \left(\mathrm{Re}\,\phi(k_n)^2 + \mathrm{Im}\,\phi(k_n)^2 \right)$$

The denominator is

$$\text{denom} = \prod_{k_n} \int_{-\infty}^{\infty} d\,\mathrm{Re}\,\phi(k_n) \int_{-\infty}^{\infty} d\,\mathrm{Im}\,\phi(k_n)\, e^{i \frac{1}{V} \sum_{k_n} (k_n^2 - m^2 + i\epsilon) \times (\mathrm{Re}\,\phi(k_n)^2 + \mathrm{Im}\,\phi(k_n)^2)}$$

$$= \prod_{k_n} \left(\int_{-\infty}^{\infty} d\,\mathrm{Re}\,\phi(k_n)\, e^{i \frac{(k_n^2 - m^2 + i\epsilon)}{V} \mathrm{Re}\,\phi(k_n)^2} \right) \times \left(\begin{matrix} \text{same with} \\ \mathrm{Re} \to \mathrm{Im} \end{matrix} \right)$$

$$= \prod_{k_n} \sqrt{\frac{i\pi V}{k_n^2 - m^2 + i\epsilon}} \sqrt{\frac{i\pi V}{k_n^2 - m^2 + i\epsilon}} = \prod_{k_n} \frac{i\pi V}{k_n^2 - m^2 + i\epsilon}$$

The numerator is

$$\text{num} = \prod_{k_n} \int_{-\infty}^{\infty} d\operatorname{Re}\phi(k_n)\, e^{i\frac{(k_n^2 - m^2 + i\epsilon)}{V}\operatorname{Re}\phi(k_n)^2} \times$$

$$\times \int_{-\infty}^{\infty} d\operatorname{Im}\phi(k_n)\, e^{i\frac{(k_n^2 - m^2 + i\epsilon)}{V}\operatorname{Im}\phi(k_n)^2} \times \underbrace{\phi(x_1)\,\phi^*(x_2)}_{\frac{1}{V}\sum_{l_n} e^{-il_n\cdot x_1}\phi(l_n)\frac{1}{V}\sum_{m_n} e^{+im_n\cdot x_2}\phi^*(m_n)}$$

$$= \frac{1}{V^2}\sum_{l_n,m_n} e^{-il_n\cdot x_1} e^{im_n\cdot x_2} \prod_{k_n}\left(\int_{-\infty}^{\infty} d\operatorname{Re}\phi(k_n)\, e^{i\frac{(k_n^2 - m^2 + i\epsilon)}{V}\operatorname{Re}\phi(k_n)^2}\right. \times$$

$$\times \int_{-\infty}^{\infty} d\operatorname{Im}\phi(k_n)\, e^{i\frac{(k_n^2 - m^2 + i\epsilon)}{V}\operatorname{Im}\phi(k_n)^2} \times \left(\operatorname{Re}\phi(l_n) + i\operatorname{Im}\phi(l_n)\right) \times$$

$$\times \left(\operatorname{Re}\phi(m_n) - i\operatorname{Im}\phi(m_n)\right)$$

$$= \frac{1}{V^2}\sum_{l_n,m_n} e^{-il_n\cdot x_1} e^{im_n\cdot x_2} \times \prod_{k_n}\left(\frac{i\pi V}{k_n^2 - m^2 + i\epsilon}\right) \times$$

$$\times \frac{1}{2}\,\frac{iV}{l_n^2 - m^2 + i\epsilon} \times \left(\delta_{l_n m_n} + 0 + 0 + \delta_{l_n m_n}\right) =$$

$$= \frac{1}{V}\sum_{l_n} e^{-il_n(x_1 - x_2)}\,\frac{i}{l_n^2 - m^2 + i\epsilon} \times \text{denom}$$

P.S. -9.1.a

$$\therefore \langle\Omega|T\hat{\phi}(x_1)\hat{\phi}^{\dagger}(x_2)|\Omega\rangle = \frac{1}{V}\sum_{l_n} e^{-il_n\cdot(x_1-x_2)} \cdot \frac{i}{l_n^2-m^2+i\epsilon}$$

$$\rightarrow \int\frac{d^4l}{(2\pi)^4}\, e^{-il\cdot(x_1-x_2)} \cdot \frac{i}{l^2-m^2+i\epsilon} = \text{position space propagator}$$

The interactions are given by

$$D_\mu\phi D^\mu\phi^* = \underbrace{\partial_\mu\phi\partial^\mu\phi}_{\text{quadratic}} + ieA_\mu(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) + -e^2 A_\mu A_\nu g^{\mu\nu}\phi\phi^*$$

The 3 point vertex appears in the perturbative expansion of the 3 point function.

$$\langle\Omega|T\hat{A}_\mu(x_1)\hat{\phi}(x_2)\hat{\phi}^{\dagger}(x_3)|\Omega\rangle =$$

$$= \frac{\int DAD\phi\; A_\mu(x_1)\phi(x_2)\phi^*(x_3)\, e^{iS[A,\phi]}}{\int DAD\phi\; e^{iS[A,\phi]}}$$

$$= \int DAD\phi\; A_\mu(x_1)\phi(x_2)\phi^*(x_3)\, i\int d^4x\cdot ieA_\nu(\phi\partial^\nu\phi^* - \phi^*\partial^\nu\phi)\times$$

$$\times e^{iS_2[A]}e^{iS_2[\phi]} \Big/ \int DAD\phi\, e^{iS_2[A]}e^{iS_2[\phi]}$$

where I have expanded to first order and assumed cancellation of vacuum graphs.

$$= -e\int d^4x \underbrace{\left(\frac{\int DA\; A_\mu(x_1)A_\nu(x)\, e^{iS_2[A]}}{\int DA\, e^{iS_2[A]}}\right)}_{\text{is photon propagator } D_{\mu\nu}(x_1-x)}\times$$

$$\times \frac{\int D\phi \; \phi(x_2)\,\phi^*(x_3)\,\left(\phi(x)\partial^\nu\phi^*(x) - \phi^*(x)\partial^\nu\phi(x)\right)}{\int D\phi \; e^{iS_2[\phi]}}$$

$$= -e\int d^4x \; D_{\mu\nu}(x_1-x) \times \prod_k \int d\mathrm{Re}\,\phi(k)\, e^{(\,)\mathrm{Re}\,\phi(k)^2} \int d\mathrm{Im}\,\phi(k)\, e^{(\,)\mathrm{Im}\,\phi(k)^2}$$

$$\times \frac{1}{V}\sum_{k_2} e^{-ik_2\cdot x_2}\phi(k_2)\,\frac{1}{V}\sum_{k_3} e^{ik_3\cdot x_3}\phi^*(k_3) \times$$

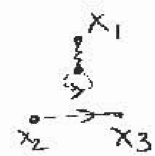
$$\frac{1}{V}\sum_{l}\frac{1}{V}\sum_{m} e^{-il\cdot x}\, e^{im\cdot x}\,\phi(l)\phi^*(m)\left(im^\nu - -il^\nu\right)\Big/\mathrm{denom}$$

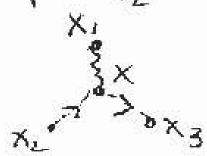
$$= -e\int d^4x\, D_{\mu\nu}(x_1-x)\,\frac{1}{V^4}\sum_{\substack{k_2,k_3\\ l,m}} i(m+l)^\nu e^{-ik_2\cdot x_2}\, e^{ik_3\cdot x_3}$$

$$e^{-il\cdot x}\, e^{im\cdot x}\prod_k \int d\mathrm{Re}\,\phi(k)\, e^{(\,)}\int d\mathrm{Im}\,\phi(k)\, e^{(\,)}\,\phi(k_2)\phi^*(k_3)\times$$

$$\times\,\phi(l)\phi^*(m)\Big/\mathrm{denom}$$

the non-zero contributions are

(1) $k_2 = k_3$ $l = m$ disconnected graph 

(2) $k_2 = m$ $k_3 = l$ connected graph 

$$= -e\int d^4x\, D_{\mu\nu}(x_1-x)\,\frac{1}{V^4}\sum_{k_2,k_3,l,m} i(m+l)^\nu e^{-ik_2\cdot x_2}e^{ik_3\cdot x_3}e^{-il\cdot x}e^{im\cdot x}$$

$$\times \prod_k \frac{i\pi V}{k^2-m^2+i\epsilon} \times 2\delta_{k_2,m}\frac{1}{2}\frac{iV}{k_2^2-m^2+i\epsilon} \times 2\delta_{k_3,l}\frac{1}{2}\frac{iV}{k_3^2-m^2+i\epsilon}\, i(m+l)^\nu\Big/\mathrm{denom}$$

P.S. - 9.1.a

$$= -e\int d^4x\, D_{\mu\nu}(x_1-x)\left(\frac{1}{V}\sum_{k_2} e^{-ik_2\cdot(x_2-x)}\frac{i}{k_2^2-m^2+i\epsilon}\right)\times$$

propagator

$$\times\left(\frac{1}{V}\sum_{k_3} e^{ik_3(x_3-x)}\frac{i}{k_3^2-m^2+i\epsilon}\right)\times\; i(k_2+k_3)^{\nu}$$

)

$$\rightarrow \int d^4x\, D_{\mu\nu}(x_1-x)\int\frac{d^4k_2}{(2\pi)^4} e^{-ik_2\cdot(x_2-x)}\frac{i}{k_2^2-m^2+i\epsilon}\times$$

$$\times\int\frac{d^4k_3}{(2\pi)^4} e^{-ik_3\cdot(x_3-x)}\frac{i}{k_3^2-m^2+i\epsilon}\times\underbrace{-ie(k_2+k_3)^{\nu}}_{\text{vertex}}$$

$$= -ie(k+p)^{\mu}$$

The 4 point vertex appears in the perturbative expansion of the 4 point function.

$$\langle\Omega|T\hat{A}^{\rho}(x_1)\hat{A}^{\sigma}(x_2)\hat{\phi}(x_3)\hat{\phi}^{\dagger}(x_4)|\Omega\rangle =$$

$$= \frac{\int DA\,D\phi\; A^{\rho}(x_1)A^{\sigma}(x_2)\phi(x_3)\phi^{*}(x_4)\, e^{iS[A,\phi]}}{\int DA\,D\phi\; e^{iS[A,\phi]}}$$

$$= \frac{\int DA\,D\phi\; A^{\rho}(x_1)A^{\sigma}(x_2)\phi(x_3)\phi^{*}(x_4)\cdot i\int d^4x\, e^2 A^{\mu}(x)A^{\nu}(x)g_{\mu\nu}\phi(x)\phi^{*}(x)e^{iS_2[A]}e^{iS_2[\phi]}}{\int DA\,D\phi\; e^{iS_2[A]}e^{iS_2[\phi]}}$$

where I have expanded to first-order and assumed cancellation of vacuum graphs.

$$= \int d^4x \cdot ie^2 \cdot \frac{\int \mathcal{D}A\, e^{iS_2[A]} A^\rho(x_1) A^\sigma(x_2) A^\mu(x) A^\nu(x)}{\int \mathcal{D}A\, e^{iS_2[A]}} \times g_{\mu\nu}$$

$$\times \int \mathcal{D}\phi\, \phi(x_3)\phi^*(x_4)\phi(x)\phi^*(x)\, e^{iS_2[\phi]} \Big/ \int \mathcal{D}\phi\, e^{iS_2[\phi]}$$

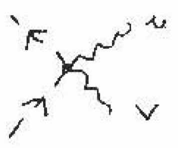
$$= \int d^4x\, ie^2 \left(D^{\rho\mu}(x_1 - x) D^{\sigma\nu}(x_2 - x) + D^{\rho\nu}(x_1-x) D^{\sigma\mu}(x_2-x) \right)$$

$$\times D(x_3 - x) D(x_4 - x) \cdot g_{\mu\nu}$$

where I have ignored disconnected contributions

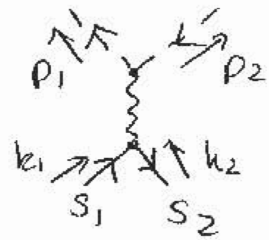
$$= \int d^4x\, D^{\rho\mu}(x_1 - x) D^{\sigma\nu}(x_2 - x) \cdot D(x_3 - x) D(x_4 - x) \times$$

$$\times \underbrace{2ie^2 g_{\mu\nu}}_{\text{vertex}}$$

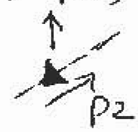


$$= 2ie^2 g_{\mu\nu}$$

P.S. -9.1.b

$i\mathcal{M} =$  $= \bar{v}(k_2,s_2)(-ie\gamma^\mu)u(k_1,s_1)\times$

$\frac{-ig_{\mu\nu}}{(k_1+k_2)^2} \times -ie(p_1-p_2)^\mu$



$$\frac{1}{2}\sum_{s_1}\frac{1}{2}\sum_{s_2}|\mathcal{M}|^2 = \frac{e^4}{(k_1+k_2)^4}(p_1-p_2)_\mu(p_1-p_2)_\nu \times$$

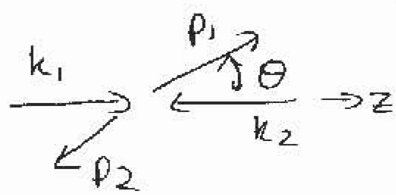
$$\times\frac{1}{4}\sum_{s_1 s_2}\bar{v}(k_2,s_2)\gamma^\mu u(k_1,s_1)\bar{u}(k_1,s_1)\gamma^\nu v(k_2,s_2)$$

$$= \frac{e^4}{(k_1+k_2)^4}(p_1-p_2)_\mu(p_1-p_2)_\nu \cdot \frac{1}{4}\underbrace{\mathrm{tr}(\not{k}_2\gamma^\mu\not{k}_1\gamma^\nu)}_{4(k_2^\mu k_1^\nu + k_2^\nu k_1^\mu - g^{\mu\nu}k_2\cdot k_1)}$$

$$= \frac{e^4}{(k_1+k_2)^4}(p_1-p_2)_\mu(p_1-p_2)_\nu\left(2k_1^\mu k_2^\nu - g^{\mu\nu}k_1\cdot k_2\right)$$

$$= \frac{e^4}{(k_1+k_2)^4}\left(2k_1\cdot(p_1-p_2)\,k_2\cdot(p_1-p_2) - k_1\cdot k_2\,(p_1-p_2)\cdot(p_1-p_2)\right)$$

In the C.M. frame, choose the following coordinates.



$k_1 = (E, E\hat{z}) \quad p_1 = (E, \vec{p})$

$k_2 = (E, -E\hat{z}) \quad p_2 = (E, -\vec{p})$

$k_1\cdot k_2 = 2E^2 \qquad k_2\cdot p_1 = E^2 + Ep\cos\theta \qquad (k_1+k_2)^2 = 4E^2$

$k_1\cdot p_1 = E^2 - Ep\cos\theta \qquad k_2\cdot p_2 = E^2 - Ep\cos\theta$

$k_1\cdot p_2 = E^2 + Ep\cos\theta \qquad p_1\cdot p_2 = E^2 + p^2$

P.S. - 9.1.b

$$2k_1\cdot(p_1-p_2)\,k_2\cdot(p_1-p_2) - k_1\cdot k_2\,(p_1-p_2)\cdot(p_1-p_2) =$$

$$= 2\left(\cancel{E^2} - Ep\cos\theta - (\cancel{E^2} + Ep\cos\theta)\right)\left(\cancel{E^2} + Ep\cos\theta - (\cancel{E^2} - Ep\cos\theta)\right)$$

$$- 2E^2\cdot\left(2m^2 - 2(E^2+p^2)\right)$$

$$p^2 = E^2 - m^2$$

$$= -8E^2 \underset{\uparrow(E^2-m^2)}{p^2}\cos^2\theta + 8E^2(E^2-m^2)$$

$$= 8E^4\left(1-\frac{m^2}{E^2}\right)\left(1-\cos^2\theta\right) = 8E^4\left(1-\frac{m^2}{E^2}\right)\sin^2\theta$$

$$\therefore \frac{1}{4}\sum_{\text{spins}}|M|^2 = \frac{e^4}{(4E^2)^2}\cdot 8E^4\left(1-\frac{m^2}{E^2}\right)\sin^2\theta =$$

$$= \frac{e^4}{2}\left(1-\frac{m^2}{E^2}\right)\sin^2\theta$$

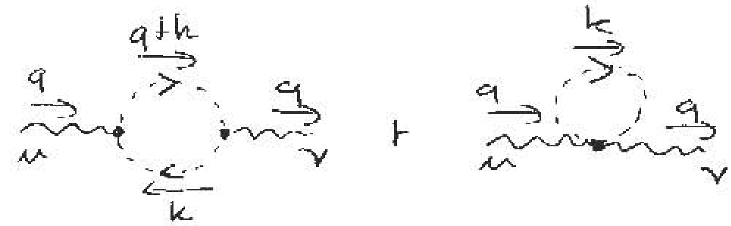
$$\frac{d\sigma}{d\Omega} = \frac{1}{2\frac{E_{cm}}{2}}\,\frac{1}{2\frac{E_{cm}}{2}}\,\frac{1}{2}\,\frac{\overset{\frac{E_{cm}}{2}\sqrt{1-\frac{m^2}{E^2}}}{|\vec{p}|}}{(2\pi)^2\,4E_{cm}}\cdot\frac{e^4}{2}\left(1-\frac{m^2}{E^2}\right)\sin^2\theta$$

$$= \frac{\alpha}{8E_{cm}^2}\left(1-\frac{m^2}{E^2}\right)^{3/2}\sin^2\theta = \frac{\alpha}{32E^2}\left(1-\frac{m^2}{E^2}\right)^{3/2}\sin^2\theta$$

$$\sigma = \frac{\alpha}{8E_{cm}^2}\left(1-\frac{m^2}{E^2}\right)^{3/2}\cdot 2\pi\int_{-1}^{1}\sin^2\theta\, d\cos\theta$$

$$= \frac{\pi\alpha^2}{3E_{cm}^2}\left(1-\frac{m^2}{E^2}\right)^{3/2} = \frac{\pi\alpha^2}{12E^2}\left(1-\frac{m^2}{E^2}\right)^{3/2}$$

P.S. – 9.1.c

$i\Pi^{\mu\nu}(q) =$ 

$$= \int \frac{d^4k}{(2\pi)^4} \left\{ (-ie)(k+k+q)^\mu (-ie)(k+k+q)^\nu \frac{i}{(k+q)^2 - m^2 + i\epsilon} \frac{i}{k^2 - m^2 + i\epsilon} + \right.$$

$$\left. + 2ie^2 g^{\mu\nu} \frac{i}{k^2 - m^2 + i\epsilon} \right\} =$$

$$= e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\{(2k+q)^\mu (2k+q)^\nu - 2g^{\mu\nu}((k+q)^2 - m^2 + \cancel{i\epsilon})\}}{((k+q)^2 - m^2 + i\epsilon)\,(k^2 - m^2 + i\epsilon)}$$

$$= e^2 \underset{\substack{\uparrow \\ \text{Feynman} \\ \text{parameters}}}{\int_0^1 dx} \int \frac{d^4k}{(2\pi)^4} \frac{\{(2k+q)^\mu (2k+q)^\nu - 2g^{\mu\nu}((k+q)^2 - m^2)\}}{(\underbrace{x(k^2 - m^2 + i\epsilon) + (1-x)((k+q)^2 - m^2 + i\epsilon)}_{D})^2}$$

$$D = xk^2 + (1-x)(k^2 + q^2 + 2k\cdot q) - m^2 + i\epsilon$$

$$= (\underbrace{k + (1-x)q}_{\ell = k + (1-x)q \quad k = \ell + (x-1)q})^2 - (1-x)^2 q^2 + (1-x)q^2 - m^2 + i\epsilon$$

$$= \ell^2 + x(1-x)q^2 - m^2 + i\epsilon$$

$$= \ell^2 - \Delta + i\epsilon \qquad \Delta = m^2 - x(1-x)q^2$$

$$\Pi^{\mu\nu}(q) = -ie^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{D^2} \{ (2\ell + 2(x-1)q + q)^\mu \times$$

$$\times (2\ell + 2(x-1)q + q)^\nu - 2g^{\mu\nu}((\ell + (x-1)q + q)^2 - m^2) \}$$

P.S. - 9.1.c.

$$= -ie^2 \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \frac{1}{D^2} \{ 4 l^\mu l^\nu + (2x-1)^2 q^\mu q^\nu$$

$$-2g^{\mu\nu} (l^2 + x^2 q^2 - m^2) \}$$ (dropping odd l terms)

$$= -ie^2 \int_0^1 dx \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Gamma(2)} \{ \{ (2x-1)^2 q^\mu q^\nu - 2g^{\mu\nu}(x^2q^2 - m^2) \} \times$$

$$\times (-1)^2 \Gamma(2-\tfrac{d}{2}) \left(\tfrac{1}{\Delta}\right)^{2-\frac{d}{2}} + 4 (-1)^{2-1} \frac{g^{\mu\nu}}{2} \Gamma(2-\tfrac{d}{2}-1) \left(\tfrac{1}{\Delta}\right)^{2-\frac{d}{2}-1} +$$

$$-2g^{\mu\nu} (-1)^{2-1} \frac{d}{2} \Gamma(2-\tfrac{d}{2}-1) \left(\tfrac{1}{\Delta}\right)^{2-\frac{d}{2}-1} \}$$

$$= \frac{e^2}{(4\pi)^{\frac{d}{2}}} \Gamma(1-\tfrac{d}{2}) \int_0^1 dx \left(\tfrac{1}{\Delta}\right)^{2-\frac{d}{2}} \{ \{ (2x-1)^2 q^\mu q^\nu - 2g^{\mu\nu}(x^2q^2-m^2) \} (1-\tfrac{d}{2})$$

$$\underbrace{-2g^{\mu\nu}\Delta + d g^{\mu\nu}\Delta}_{-2(1-\frac{d}{2}) g^{\mu\nu}\Delta} \}$$

$$= \frac{e^2}{(4\pi)^{\frac{d}{2}}} \Gamma(1-\tfrac{d}{2})(1-\tfrac{d}{2}) \int_0^1 dx \left(\tfrac{1}{\Delta}\right)^{2-\frac{d}{2}} \{ (2x-1)^2 q^\mu q^\nu$$

$$-2g^{\mu\nu}(x^2q^2 - \cancel{m^2}) - 2g^{\mu\nu} (\cancel{m^2} - x(1-x)q^2) \}$$

$$= \frac{e^2}{(4\pi)^{\frac{d}{2}}} \Gamma(2-\tfrac{d}{2}) \int_0^1 dx \left(\tfrac{1}{\Delta}\right)^{2-\frac{d}{2}} \{ (2x-1)^2 q^\mu q^\nu$$

$$- q^2 g^{\mu\nu} \underbrace{(2x^2 - 2x(1-x))}_{2x(2x-1) = (2x-1)^2 + (2x-1)} \}$$

$$= \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \times \frac{-e^2}{(4\pi)^{\frac{d}{2}}} \Gamma(2-\tfrac{d}{2}) \int_0^1 dx \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} (2x-1)^2$$

$$+ \frac{e^2}{(4\pi)^{\frac{d}{2}}} \Gamma(2-\tfrac{d}{2}) \int_0^1 dx \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \times -q^2 g^{\mu\nu} (2x-1)$$

proportional to $\int_0^1 dx \; (2x-1) \; (m^2 - x(1-x)q^2)^{\frac{d}{2}-2}$ $\quad y = x - \frac{1}{2}$, $\; x = u + \frac{1}{2}$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \; (2y+1-1)(m^2 - (u+\tfrac{1}{2})(\tfrac{1}{2}-u)q^2)^{\frac{d}{2}-2}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \; \underbrace{2\cdot y \,(m^2 - (\tfrac{1}{2}+u)(\tfrac{1}{2}-u)q^2)^{\frac{d}{2}-2}}_{\text{odd in } y} = 0$$

$$= \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \cdot \frac{-e^2}{(4\pi)^{\frac{d}{2}}} \Gamma(2-\tfrac{d}{2}) \int_0^1 dx \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} (2x-1)^2$$

$$\therefore \; \Pi(q^2) = \frac{-e^2}{(4\pi)^{\frac{d}{2}}} \Gamma(2-\tfrac{d}{2}) \int_0^1 dx \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} (2x-1)^2$$

$d = 4 - \epsilon$

$$\Pi(q^2) = \frac{-e^2}{(4\pi)^2} \int_0^1 dx \; (2x-1)^2 \cdot \underset{\uparrow \; \left(\frac{2}{\epsilon} - \gamma + O(\epsilon)\right)}{\Gamma(\tfrac{\epsilon}{2})} \; \underset{\uparrow \; \left(1 + \frac{\epsilon}{2}\ln(\frac{4\pi}{\Delta}) + O(\epsilon^2)\right)}{\left(\frac{4\pi}{\Delta}\right)^{\frac{\epsilon}{2}}} \quad e^{\frac{\epsilon}{2}\ln(\frac{4\pi}{\Delta})}$$

$$\xrightarrow{d\to 4} \; -\frac{\alpha}{4\pi} \int_0^1 dx \; (2x-1)^2 \left(\frac{2}{\epsilon} + \ln(4\pi) - \ln(\Delta) - \gamma\right)$$

P.S. - 9.1.c

$$\Pi(q^2) \xrightarrow{d\to 4} -\frac{\alpha}{4\pi}\int_0^1 dx\,(2x-1)^2\left(\frac{2}{\epsilon} - \ln\left(m^2 - x(1-x)q^2\right) + \ln 4\pi - \gamma\right)$$

$$\xrightarrow{q^2 \gg m^2} -\frac{\alpha}{4\pi}\int_0^1 dx\,(2x-1)^2\left(\frac{2}{\epsilon} - \ln\left(\frac{q^2}{m^2}\right) + \underbrace{\ln(4\pi) - \gamma}_{\text{Subleading}}\right)$$

$$= -\frac{\alpha}{12\pi}\left(\frac{2}{\epsilon} - \ln\left(\frac{q^2}{m^2}\right) + \ln(4\pi) - \gamma\right)$$

The QED result in the same limit is

$$-\frac{\alpha}{3\pi}\left(\frac{2}{\epsilon} - \ln\left(\frac{q^2}{m^2}\right) + \ln(4\pi) - \gamma\right) = 4 \times (\quad)$$

P.S. -9.2a

$$Z = \mathrm{tr}(e^{-\beta H}) = \int dx \langle x| e^{-\beta H} |x\rangle =$$

$$= \int dx \langle x| \underbrace{e^{-\epsilon H} \cdots e^{-\epsilon H}}_{N \text{ factors}} |x\rangle \qquad \epsilon = \beta/N$$

insert $\mathbb{1} = \int dx_i |x_i\rangle\langle x_i|$ between each factor

$$= \int dx \int dx_{N-1} \cdots \int dx_1 \langle x| e^{-\epsilon H} |x_{N-1}\rangle \cdots \langle x_1| e^{-\epsilon H} |x\rangle$$

$$\langle x| e^{-\epsilon H} |y\rangle = \langle x| (\mathbb{1} - \epsilon H + \cdots) |y\rangle = \langle x|y\rangle - \epsilon \langle x|H|y\rangle$$

$$= \int \frac{dp}{2\pi} e^{ip(x-y)} - \epsilon \int \frac{dp}{2\pi} H\left(\tfrac{x+y}{2}, p\right) \cdot e^{ip(x-y)}$$

assuming Weyl ordering

$$= \int \frac{dp}{2\pi} e^{ip(x-y) - \epsilon H\left(\frac{x+y}{2}, p\right)}$$

$$Z = \int dx \int dx_{N-1} \cdots \int dx_1 \times \int \frac{dp_{N-1}}{2\pi} \cdots \int \frac{dp_0}{2\pi} \times e^{ip_{N-1}(x_N - x_{N-1}) - \epsilon H\left(\frac{x_N + x_{N-1}}{2}, p_{N-1}\right)}$$

$$\times \cdots e^{ip_0(x_1 - x_0) - \epsilon H\left(\frac{x_1 + x_0}{2}, p_0\right)} \qquad \text{where } x_N = x = x_0$$

$$= \left(\prod_{i=0}^{N-1} \int dx_i \int \frac{dp_i}{2\pi}\right) e^{\sum_{i=0}^{N-1} ip_i(x_{i+1} - x_i) - \epsilon H\left(\frac{x_{i+1} + x_i}{2}, p_i\right)}$$

compare to P.S. Eq. 9.11

assume $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

P.S. - 9.2.a

$$Z = \left(\prod_{i=0}^{N-1} \int dx_i \int \frac{dp_i}{2\pi} \right) e^{\sum_{i=0}^{N-1} i p_i (x_{i+1} - x_i) - \epsilon \frac{p_i^2}{2m} - \epsilon V\left(\frac{x_{i+1}+x_i}{2}\right)}$$

$$= \left(\prod_{i=0}^{N-1} \int dx_i \right) e^{\sum_{i=0}^{N-1} - \epsilon V\left(\frac{x_{i+1}+x_i}{2}\right)} \times \left(\prod_{i=0}^{N-1} \int_{-\infty}^{\infty} \frac{dp_i}{2\pi} e^{i p_i (x_{i+1} - x_i) - \frac{\epsilon p_i^2}{2m}} \right)$$

This integral is of the form

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{i p \cdot a - \frac{\epsilon p^2}{2m}} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{\epsilon}{2m}\left(p - \frac{im}{\epsilon} a\right)^2 - \frac{m}{2}\frac{a^2}{\epsilon}}$$

$$= \frac{1}{2\pi} e^{-\frac{m}{2}\frac{a^2}{\epsilon}} \sqrt{\frac{\pi\, 2m}{\epsilon}} = e^{-\frac{m}{2}\frac{a^2}{\epsilon}} \sqrt{\frac{m}{2\pi\epsilon}}$$

$$= \left(\prod_{i=0}^{N-1} \int dx_i \right) e^{\sum_{i=0}^{N-1} - \epsilon V\left(\frac{x_{i+1}+x_i}{2}\right)} \prod_{i=0}^{N-1} e^{-\frac{m}{2}\frac{(x_{i+1}-x_i)^2}{\epsilon}} \sqrt{\frac{m}{2\pi\epsilon}}$$

$$= \left(\prod_{i=0}^{N-1} \sqrt{\frac{m}{2\pi\epsilon}} \int dx_i \right) e^{-\sum_{i=0}^{N-1} \epsilon \left(\frac{m}{2} \left(\frac{x_{i+1}-x_i}{\epsilon}\right)^2 + V\left(\frac{x_{i+1}+x_i}{2}\right)\right)}$$

$$\rightarrow \int_{x(0)=x(\beta)} \mathcal{D}x \; e^{-\int_0^\beta d\tau \left(\frac{m}{2} \left(\frac{dx}{d\tau}\right)^2 + V(x(\tau))\right)} = \int_{x(0)=x(\beta)} \mathcal{D}x \; e^{-\int_0^\beta dt\, L_E(x(t))}$$

P.S. - 9.2.b

$$X(t) = \sum_n X_n \cdot \frac{1}{\sqrt{\beta}}\, e^{i\frac{2\pi n}{\beta}\cdot t} \qquad \omega_n = \frac{2\pi n}{\beta}$$

using $X = X^*$

$$\int_0^\beta dt\, L_E[X(t)] = \int_0^\beta dt \left(\frac{1}{2}\dot{X}\dot{X}^* + \frac{\omega^2}{2} X X^*\right) =$$

$$= \frac{1}{\sqrt{\beta}}\sum_n \frac{1}{\sqrt{\beta}} \sum_m \int_0^\beta dt \left(\frac{1}{2}(i\omega_n)(-i\omega_m) X_n X_m^*\, e^{i\omega_n t} e^{-i\omega_m t}\right.$$

$$\left. + \frac{\omega^2}{2} X_n X_m^*\, e^{i\omega_n t} e^{-i\omega_m t}\right) =$$

$$= \frac{1}{\beta}\sum_{n,m} \underbrace{\int_0^\beta dt\, e^{i\omega_n t} e^{-i\omega_m t}}_{\beta\,\delta_{nm}} \cdot \frac{1}{2}(\omega^2+\omega_n\omega_m) X_n X_m^*$$

$$= \sum_n \frac{\omega^2+\omega_n^2}{2}|X_n|^2 = \left(\frac{\omega^2+\omega_0^2}{2}\right)\cdot X_0^2 + 2\sum_{n>0}\left(\frac{\omega^2+\omega_n^2}{2}\right)\times$$

where $X = X^* \Rightarrow X_0 = X_0^*$ $X_n = X_n^*$ $n \neq 0$ $\times(\mathrm{Re}\,X_n^2 + \mathrm{Im}\,X_n^2)$

$$= \frac{\omega^2}{2} X_0^2 + \sum_{n>0}(\omega^2+\omega_n^2)\cdot(\mathrm{Re}\,X_n^2 + \mathrm{Im}\,X_n^2)$$

$$Z = \int_{-\infty}^{\infty} dX_0 \prod_{n>0}\int_{-\infty}^{\infty} d\mathrm{Re}\,X_n \int_{-\infty}^{\infty} d\mathrm{Im}\,X_n\, e^{\frac{\omega^2}{2}X_0^2} \prod_{n>0} e^{(\omega^2+\omega_n^2)(\mathrm{Re}\,X_n^2 + \mathrm{Im}\,X_n^2)}$$

$$= \sqrt{\frac{2\pi}{\omega^2}} \times \prod_{n>0}\sqrt{\frac{\pi}{\omega^2+\omega_n^2}}\sqrt{\frac{\pi}{\omega^2+\omega_n^2}} =$$

$$= \sqrt{2\pi}\,\frac{1}{\omega}\prod_{n>0}\frac{\pi}{\omega^2+\omega_n^2}$$

P.S. -9.2.b

$$= \sqrt{2\pi}\,\frac{1}{\omega}\prod_{n>0}\frac{\pi\left(\frac{\beta}{2\pi n}\right)^2}{1+\left(\frac{\omega\beta}{2\pi n}\right)^2} = \sqrt{2\pi}\,\frac{\beta}{2}\,\frac{1}{\frac{\omega\beta}{2}}\prod_{n>0}\frac{\pi\left(\frac{\beta}{2\pi n}\right)^2}{1+\left(\frac{\omega\beta}{2\pi n}\right)^2}$$

$$= \sqrt{2\pi}\,\frac{\beta}{2}\left(\prod_{n>0}\left(\frac{\beta}{2\pi n}\right)^2\right)\times\frac{1}{\sinh\left(\frac{\omega\beta}{2}\right)}$$

$$= \left\{\sqrt{2\pi}\cdot\beta\cdot\left(\prod_{n>0}\left(\frac{\beta}{2\pi n}\right)^2\right)\right\}\cdot\underbrace{\frac{1}{2\sinh\left(\frac{\omega\beta}{2}\right)}}_{Z(\beta)\ \text{for single harmonic oscillator}}$$

P.S. - 9.2.c 1

The previous result for a single degree of freedom can be generalized to N degrees of freedom, x_i $i=1 \to N$.

$$Z(\beta) = \int \prod_{i=1}^{N} Dx_i(t) \, e^{-\int_0^\beta dt \sum_{i=1}^{N} \mathcal{L}_E(x_i(t))}$$
$$x_i(0)=x_i(\beta)$$

which for a continuous field becomes

$$\int D\phi(x) \, e^{-\int_0^\beta d^4x \, \mathcal{L}_E[\phi(x)]}$$
$$\phi(\vec{x},0)=\phi(\vec{x},\beta)$$

For a free scalar field,

$$-\int_0^\beta d^4x \mathcal{L}_E = -\frac{1}{2}\int_0^\beta d^4x \left(\partial_\mu\phi\partial^\mu\phi^* + m^2\phi\phi^*\right) =$$

$$= -\frac{1}{2}\int_0^\beta d^4x \left(\partial_\mu(\phi\partial^\mu\phi^*) - \phi\partial_\mu\partial^\mu\phi^* + m^2\phi\phi^*\right) =$$

$$= -\frac{1}{2}\int_0^\beta d^4x \, \phi(-\partial^2 + m^2)\phi^* \; \underbrace{-\frac{1}{2}\int_0^\beta d^4x \, \partial_\mu(\phi\partial^\mu\phi^*)}_{=0}$$

$= 0$ because $\phi(\vec{x},0) = \phi(\vec{x},\beta)$ and $\phi(\vec{x},t) \to 0$ $|\vec{x}| \to \infty$

$$Z(\beta) = \int D\phi \, e^{-\frac{1}{2}\int_0^\beta d^4\vec{x} \, \phi(-\partial^2+m^2)\phi^*}$$
$$\phi(\vec{x},0)=\phi(\vec{x},\beta)$$

$$= (\text{constant}) \times \left(\det(-\partial^2 + m^2)\right)^{-1/2}$$ using P.S. Eq. 9.24

P.S.-9.2.c

where $-\partial^2+m^2$ acts on the space of functions periodic in time with period β.

To determine $\det(-\partial^2+m^2)$, diagonalize $-\partial^2+m^2$ by Fourier transform.

$$\phi(\vec{x},t) = \sum_n \int \frac{d^3\vec{k}}{(2\pi)^3} e^{-i(\omega_n t - \vec{k}\cdot\vec{x})} \phi_n(\vec{k}) \qquad \omega_n = \frac{2\pi n}{\beta}$$

$$= \sum_n \sum_{\vec{k}} e^{-i(\omega_n t - \vec{k}\cdot\vec{x})} \phi_n(\vec{k})$$

$\uparrow$ box normalization

$$(-\partial^2+m^2)\phi(\vec{x},t) = \sum_n \sum_{\vec{k}} \left\{ (-1)(-\omega_n^2 - \vec{k}^2) + m^2 \right\} e^{-i(\omega_n t - \vec{k}\cdot\vec{x})} \phi_n(\vec{k})$$

$\partial^2 = \partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2$ in Euclidean space

The eigenvalues are $\omega_n^2 + m^2 + \vec{k}^2 = \omega_n^2 + E_{\vec{k}}^2$. For a real scalar field, we have the same eigenvalues but with degeneracy $= \begin{Bmatrix} 1 & n=0 \\ 2 & n\neq 0 \end{Bmatrix} \times \begin{Bmatrix} 1 & \vec{k}=0 \\ 2 & \vec{k}\neq 0 \end{Bmatrix}$

Therefore,

$$\frac{1}{\det(-\partial^2+m^2)} = \Bigg(\prod_{n>0} \underbrace{\frac{1}{\omega_n^2+m^2} \; \frac{1}{\omega_n^2+m^2}}_{n\neq 0 \text{ degeneracy}} \Bigg) \cdot \frac{1}{m^2} \times \quad \leftarrow \vec{k}=0 \text{ eigenvalues}$$

$$\prod_{\vec{k}\neq 0} \Bigg(\prod_{n>0} \underbrace{\frac{1}{\omega_n^2+m^2+\vec{k}^2} \; \frac{1}{\omega_n^2+m^2+\vec{k}^2}}_{n\neq 0 \text{ degeneracy}} \Bigg) \frac{1}{m^2+\vec{k}^2} \quad \leftarrow \vec{k}\neq 0 \text{ eigenvalues}$$

$\uparrow$ $\vec{k}>0$ and $\vec{k}<0$ takes care of $\vec{k}\neq 0$ degeneracy

P.S. - 9.2.c

$$\frac{1}{\sqrt{\det(-\partial^2+m^2)}} = \left(\prod_{n>0} \frac{1}{\omega_n^2+m^2}\right) \cdot \frac{1}{m} \times \left(\prod_{\vec{k}\neq 0}\left(\prod_{n>0} \frac{1}{\omega_n^2+E_{\vec{k}}^2}\right) \cdot \frac{1}{E_{\vec{k}}}\right)$$

$$= \left(\prod_{n>0} \frac{\left(\frac{\beta}{2\pi n}\right)^2}{1+\left(\frac{m\beta}{2\pi n}\right)^2}\right) \frac{1}{\frac{m\beta}{2}} \cdot \frac{\beta}{2} \times \left(\prod_{\vec{k}\neq 0}\left(\prod_{n>0} \frac{\left(\frac{\beta}{2\pi n}\right)^2}{1+\left(\frac{E_{\vec{k}}\beta}{2\pi n}\right)^2}\right) \frac{1}{\frac{E_{\vec{k}}\beta}{2}} \cdot \frac{\beta}{2}\right)$$

$$= \frac{\beta}{2} \cdot \prod_{n>0} \left(\frac{\beta}{2\pi n}\right)^2 \times \prod_{\vec{k}\neq 0} \frac{\beta}{2} \prod_{\vec{k}\neq 0} \prod_{n>0} \left(\frac{\beta}{2\pi n}\right)^2 \times$$

$$\times \frac{1}{\sinh\left(\frac{m\beta}{2}\right)} \times \prod_{\vec{k}\neq 0} \frac{1}{\sinh\left(\frac{E_{\vec{k}}\beta}{2}\right)} = N(\beta) \times \prod_{\vec{k}} \frac{1}{\sinh\left(\frac{E_{\vec{k}}\beta}{2}\right)}$$

$$\therefore Z(\beta) = (\text{constant}) \times N(\beta) \times \prod_{\vec{k}} \frac{1}{\sinh\left(\frac{E_{\vec{k}}\beta}{2}\right)}$$

$$= (\text{constant})\, e^{\ln\left(\prod_{\vec{k}} \frac{1}{\sinh\left(\frac{E_{\vec{k}}\beta}{2}\right)}\right)}$$

$$= (\text{constant})\, e^{\sum_{\vec{k}} \ln\left(\frac{1}{\sinh\left(\frac{E_{\vec{k}}\beta}{2}\right)}\right)} = (\text{constant})\, e^{-\sum_{\vec{k}} \ln\left(\sinh\left(\frac{E_{\vec{k}}\beta}{2}\right)\right)}$$

$$\rightarrow (\text{constant}) \cdot e^{-\int \frac{d^3\vec{k}}{(2\pi)^3} \ln\left(\sinh\left(\frac{E_{\vec{k}}\beta}{2}\right)\right)}$$

P.S.-9.2.d

$$Z(\beta) = \int D\bar{\Psi} D\Psi \, e^{-\int_0^\beta dt \, \bar{\Psi}(\partial + \omega)\Psi}$$

$$= \det(\partial + \omega)$$

Diagonalize $\partial + \omega$ on the space of anti-periodic functions.

$$\varphi(t) = \sum_{n=-\infty}^{\infty} e^{i\frac{2\pi(n+\frac{1}{2})}{\beta}t} \varphi_n$$

$$(\partial + \omega)\varphi(t) = \sum_{n=-\infty}^{\infty} \left(i\frac{2\pi(n+\frac{1}{2})}{\beta} + \omega \right) e^{i\frac{2\pi(n+\frac{1}{2})}{\beta}t} \varphi_n$$

$$\det(\partial + \omega) = \prod_{n=-\infty}^{\infty} \left(i\frac{2\pi(n+\frac{1}{2})}{\beta} + \omega \right) =$$

$$= \prod_{n=0}^{\infty} \left(i\frac{2\pi(n+\frac{1}{2})}{\beta} + \omega \right) \cdot \prod_{\substack{n=1 \\ \uparrow \\ m=n-1}}^{\infty} \left(i\frac{2\pi(-n+\frac{1}{2})}{\beta} + \omega \right)$$

$$= \prod_{n=0}^{\infty} \left(i\frac{2\pi(n+\frac{1}{2})}{\beta} + \omega \right) \prod_{m=0}^{\infty} \left(i\frac{2\pi}{\beta}(-m-1+\tfrac{1}{2}) + \omega \right)$$

$$= \prod_{n=0}^{\infty} \left(i\frac{2\pi(n+\frac{1}{2})}{\beta} + \omega \right) \cdot \prod_{n=0}^{\infty} \left(-i\frac{2\pi(n+\frac{1}{2})}{\beta} + \omega \right)$$

$$= \prod_{n=0}^{\infty} \left(\left(\frac{2\pi(n+\frac{1}{2})}{\beta} \right)^2 + \omega^2 \right)$$

P.S. - 9.2.d

$$= \prod_{n=0}^{\infty} \left(\frac{2\pi(n+\frac{1}{2})}{\beta}\right)^2 \left(1 + \left(\frac{\omega\beta}{2\pi(n+\frac{1}{2})}\right)^2\right).$$

$$= \underbrace{\prod_{n=0}^{\infty} \left(\frac{2\pi(n+\frac{1}{2})}{\beta}\right)^2}_{N(\beta)} \cdot \prod_{n=0}^{\infty} \left(1 + \left(\frac{\omega\beta}{2\pi(\frac{2n+1}{2})}\right)^2\right)$$

$$= N(\beta) \cdot \prod_{\substack{n=1 \\ \text{odd}}}^{\infty} \left(1 + \left(\frac{\omega\beta}{\pi n}\right)^2\right) = N(\beta) \frac{\prod_{n=1}^{\infty} \left(1 + \left(\frac{\omega\beta}{\pi n}\right)^2\right)}{\prod_{\substack{n=2, \\ \text{even}}}^{\infty} \left(1 + \left(\frac{\omega\beta}{\pi n}\right)^2\right)}$$

$$= \frac{N(\beta)}{2} \cdot \frac{\omega\beta}{\frac{\omega\beta}{2}} \cdot \frac{\prod_{n=1}^{\infty} \left(1 + \left(\frac{\omega\beta}{\pi n}\right)^2\right)}{\prod_{n=1}^{\infty} \left(1 + \left(\frac{\omega\beta}{\pi 2n}\right)^2\right)}$$

$$= \frac{N(\beta)}{2} \cdot \frac{\sinh(\omega\beta)}{\sinh(\frac{\omega\beta}{2})} = \frac{N(\beta)}{2} \cdot 2\cosh(\omega\beta/2) =$$

$$= N(\beta)\cosh\left(\frac{\omega\beta}{2}\right)$$

partition function of a 2 state system with energies $\frac{\omega\beta}{2}$ and $-\frac{\omega\beta}{2}$

W-9.1

An acceptable solution can be found in "Quantum Field Theory" by Lowell S. Brown on page 21-29.

W.-9.4

The solution can be found in the literature.
Two possible references are

Phys. Rev. D 3, 2153-2161 (1971)

Phys. Rev. C 39, 2339-2348 (1989)