Problem Set 5: Solutions

Econ 14.05

1. (a) Let's define as s the amount saved/borrowed by the consumer in the first period. If $s > 0$ the individual is a net saver, and she is a net borrower otherwise. The intertemporal budget constraints are:

$$c_1 + s = y_1$$

$$c_2 = \begin{cases} y_2 + s(1+r_s) & s \geq 0 \\ y_2 + s(1+r_b) & s \leq 0 \end{cases}$$

Note that the condition $s \geq 0$ is equivalent to $c \leq y_1$. If this is the case the consumer will be a net saver, otherwise she will be a net borrower. Considering our previous equations, we have to distinguish between the case in which $c_1 \leq y_1$, and $c_1 > y_1$. Therefore the intertemporal budget constraint is going to be the following:

$$c_1 + \frac{c_2}{1+r_s} = y_1 + \frac{y_2}{1+r_s} \qquad c_1 \leq y_1$$
$$c_1 + \frac{c_2}{1+r_b} = y_1 + \frac{y_2}{1+r_b} \qquad c_1 > y_1$$

(b) We are using a logarithmic utility, therefore the individual consumes a fixed fraction of her intertemporal income at each period. The difference lies in the expression for the intertemporal income in the case in which the individual is a net saverborrower. The fraction of income consumed is given by the ratio of the coefficients, therefore:

$$c_1 = \frac{1}{1+\beta}\left\{y_1 + \frac{y_2}{1+r_s}\right\} \qquad c_1 \leq y_1 \tag{1}$$
$$c_1 = \frac{1}{1+\beta}\left\{y_1 + \frac{y_2}{1+r_b}\right\} \qquad c_1 \geq y_1 \tag{2}$$

(c) We only have to verify the condition under which each solution lies in the desired range. That is, under what conditions on β the solution obtained using (1) verifies $c_1 \leq y_1$. The analogous procedure is required in the case of (2).
Solving from (1) we require:

$$\frac{1}{1+\beta}y_1 + \frac{y_2}{(1+\beta)(1+r_s)} < y_1,$$

solving we obtain:

$$\beta > \frac{y_2}{y_1(1+r_s)}.$$

Proceeding analogously with (2) we get:

$$\beta < \frac{y_2}{y_1(1+r_d)}.$$

As we know $y_1 > y_2$, and $r_d > r_s$, the set of conditions is:

Net borrower	$0 < \beta < \frac{y_2}{y_1(1+r_d)}$
Neither borrower nor lender	$\frac{y_2}{y_1(1+r_d)} \leq \beta \leq \frac{y_2}{y_1(1+r_s)}$
Net saver	$\frac{y_2}{y_1(1+r_s)} < \beta < 1$

(d) The analysis is straightforward from the previous conditions.

The effect of this policy is to increase the minimum discount rate required to be a net saver from $y_2/(y_1(1+r_s))$ to $y_2/(y_1(1+r_s)(1-t_s))$ Therefore, a consumer with a discount factor equal or slightly above $y_2/(y_1(1+r_s))$, she will become an "exact consumer", that is she will consume her income in every period. A consumer with a discount factor well above $y_2/(y_1(1+r_s))$ will still be a net saver but she will reduce her savings with respect to the previous case. Those consumers with a discount factor below $y_2/(y_1(1+r_s))$ are left unaffected.

2. (a) i. The intertemporal budget constraint of each individual is

$$c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = w_t - T_t$$

ii. Under logarithmic utility with no discount we know that:

$$\begin{aligned} c_{1t} &= \frac{1}{2}(w_t - T_t) \\ s_{1t} &= \frac{1}{2}(w_t - T_t) \end{aligned}$$

i.

$$\begin{aligned} K_{t+1} &= s_{1t} L_t \\ k_{t+1} &= \frac{1}{2(1+n)}(w_t - T_t) \\ k_{t+1} &= \frac{1}{2(1+n)}((1-\alpha)k_t^{\alpha} - T_t) \end{aligned}$$

ii.

$$k^* = \frac{1}{2(1+n)}((1-\alpha)k^{*\alpha} - T)$$

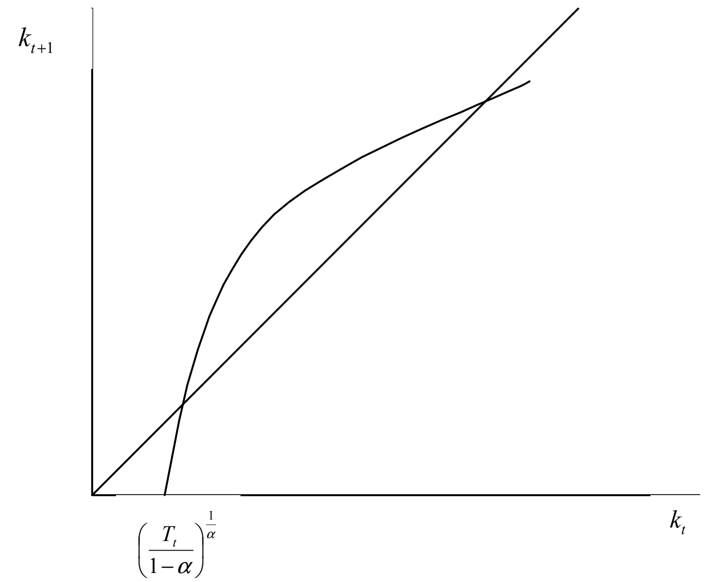

Figure 1:

iii. Economy has 3 equilibria: $k^* = 0$, k_0^*, k_1^*, only $k^* = 0$, and k_1^* are stable.

i.

$$\begin{aligned} k_{t+1} &= \frac{1}{2(1+n)} w_t (1 - \tau_{wt}) \\ k_{t+1} &= \frac{1}{2(1+n)} (1-\alpha) k_t^{\alpha} (1 - \tau_{wt}) \end{aligned}$$

ii.

$$k^* = \frac{1}{2(1+n)} (1-\alpha) k^{*\alpha} (1 - \tau_w) \tag{3}$$

Equation 3 can be written as:

$$\begin{aligned} k^* &= \frac{1}{2(1+n)} ((1-\alpha) k^{*\alpha} - (1-\alpha) k^{*\alpha} \tau_w) \\ &= \frac{1}{2(1+n)} ((1-\alpha) k^{*\alpha} - w^* \tau_w) \\ &= \frac{1}{2(1+n)} ((1-\alpha) k^{*\alpha} - T) \end{aligned}$$

Therefore the level of capital per capita is the same. The interesting question (this is not a required part of the answer) is why a lump-sum tax and a proportional tax have the same effect. Intuitively, the labor tax is a distiortionant tax, so under general conditions it should affect the decisions of the individual. The reason why it acts as a lump sum tax is that the individual has no disutility of labor (or utility for leisure), therefore she supplies her labor inelastically, and changes in the price of labor don't distort the decisions of the individual.

i. The new IBC is:

$$c_{1t} + \frac{c_{2t+1}}{1 + r_{t+1}(1 - \tau_{rt+1})} = w_t$$

Under logarithmic utility with no discount we know that:

$$\begin{aligned} c_{1t} &= \frac{1}{2} w_t \\ s_{1t} &= \frac{1}{2} w_t \\ c_{2t+1} &= \frac{1}{2} w_t (1 + r_{t+1}(1 - \tau_{rt+1})) \end{aligned}$$

Therefore, the introduction of this tax does not affect neither period 1 consupmtion nor savings, and only affects period 2 consumption.
Proceeding as before:

$$\begin{aligned} K_{t+1} &= s_{1t} L_t \\ k_{t+1} &= \frac{1}{2(1+n)} w_t \\ k_{t+1} &= \frac{1}{2(1+n)} (1-\alpha) k_t^{\alpha} \end{aligned}$$

ii.

$$k^* = \frac{1}{2(1+n)} (1-\alpha) k^{*\alpha} \tag{4}$$

The level of taxation does not affect the equilibrium level of capital. Moreover, from equation 4, we have that the equilibrium level of capital is the same that would be obtained without taxes, and comparing with any of the two cases (lump sum tax or proportinal tax), the equilibrium level of capital is clearly higher.

i. Taxation on interest income.

ii. This result relies completelly on the use of logarithmic utility. As discussed in the book, logarithmic utility has the property that the income effect and substitution effects associated with a larger interest rate cancel out, hence the fraction of the intertemporal income saved is independent of the interest rate.

3. **Social Security in the Diamond model (based on Problem 2.16 in Romer's textbook). Part I: Pay as you go social security:**

(a) The utility function is given by:

$$\log c_{1,t} + \frac{1}{1+\rho}\log c_{2,t+1}$$

with the social security tax of T per person, the individual faces the following constraints (with g, the growth rate of technology, equal to 0, A is simply a constant throughout):

$$c_{1,t} + s_t = Aw_t - T$$

$$c_{2,t+1} = (1 + r_{t+1})\, s_t + (1+n)\, T$$

where s_t represents the individual's saving in the first period. As far as the individual is concerned, the rate of return on social security is $(1+n)$, which in general will not be equal to the return on private saving which is $(1+r_{t+1})$. From the budget constraint, $(1 + r_{t+1})\, s_t = c_{2,t+1} - (1+n)\, T$. Solving for s_t yields:

$$s_t = \frac{c_{2,t+1}}{1+r_{t+1}} - \frac{1+n}{1+r_{t+1}}T.$$

Now solve for the intertemporal budget constraint by substitution:

$$c_{1,t} + \frac{c_{2,t+1}}{1+r_{t+1}} = Aw_t - \frac{(r_{t+1}-n)}{1+r_{t+1}}T.$$

(b) We know that with logarithmic utility, the individual will consume fraction $\frac{1+\rho}{2+\rho}$ of her wealth in the first period. Thus:

$$c_{1,t} = \left(\frac{1+\rho}{2+\rho}\right)\left(Aw_t - \left(\frac{r_{t+1}-n}{1+r_{t+1}}\right)T\right).$$

To solve for saving per person, substitute:

$$s_t = \frac{Aw_t}{2+\rho} - \left(\frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})}\right)T$$

Note that if $r_{t+1} = n$ then saving is reduced one-for-one by the social security tax. If $r_{t+1} > n$, saving falls less than one-for one. Finally, if $r_{t+1} < n$, saving falls more than one-for-one.
Denote $Z_t = \left(\frac{(2+\rho)(1+r_{t+1})-(1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})}\right)$ and rewrite:

$$s_t = \frac{Aw_t}{2+\rho} - Z_t T.$$

(c) It is still true that the capital stock in period $t+1$ will be equal to the total saving of the young in period t, hence

$$K_{t+1} = S_t L_t. \tag{5}$$

Converting this into units of effective labor by dividing both sides by AL_{t+1} and using equation the expression for the savings:

$$k_{t+1} = \frac{1}{1+n}\left(\frac{w_t}{2+\rho} - Z_t\frac{T}{A}\right) \tag{6}$$

With Cobb-Douglas production function, the real wage is given by:

$$w_t = (1-\alpha)\,k_t^{\alpha} \tag{7}$$

Substituting gives the new relationship between capital in period $t+1$ and capital in period t, all in units of effective labor:

$$k_{t+1} = \frac{1}{1+n}\left(\frac{1}{2+\rho}(1-\alpha)\,k_t^{\alpha} - Z_t\frac{T}{A}\right).$$

(d) To see what effect the introduction of the social security system has on the balanced-growth-path value of k, we must determine the sign of Z_t. If positive, the introduction of the tax, T, shifts down the k_{t+1} curve and reduces the balanced growth path value of k.

$$Z_t = \frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})} = \frac{(1+1+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})}$$

simplifying further allows us to sign Z_t:

$$Z_t = \frac{(1+r_{t+1}) + (1+\rho)(1+n)}{(2+\rho)(1+r_{t+1})} > 0.$$

Thus, the k_{t+1} curve shifts down, relative to the case without social security, and k^* is reduced.

(e) If the economy is initially dynamically efficient, a marginal increase in T results in a gain to the old generation who receive the extra benefits. However, it reduces k^* further below k_{GR} and thus leaves future generations worse off, with lower consumption possibilities. If the economy was initially dynamically inefficient, so that $k^* > k_{GR}$, the old generation again gains due to the extra benefits. Now, the reduction in k^* actually allows for higher consumption for future generations and is welfare improving. The introduction of the tax in this case reduces or may possibly eliminate the dynamic inefficiency caused by the over-accumulation of capital.

4. **Social Security in the Diamond model (based on Problem 2.16 in Romer's textbook). Part II: fully funded social security:**

(a) The period 2 budget constraint becomes

$$c_{2,t+1} = (1+r_{t+1})\,s_t + (1+r_{t+1})\,T$$

As far as the individual is concerned, the rate of return on social security is the same as that on private saving. We can derive the intertemporal budget constraint. From above:

$$s_t = \frac{c_{2,t+1}}{1+r_{t+1}} - T.$$

Substituting into period 1 budget constraint gives:

$$c_{1,t} + \frac{c_{2,t+1}}{1+r_{t+1}} = Aw_t - T + T$$

or simply:

$$c_{1,t} + \frac{c_{2,t+1}}{1+r_{t+1}} = Aw_t$$

This is just the usual intertemporal budget constraint in the Diamond model.

(b) Solving the individual's maximization problem yields the usual Euler equation:

$$\frac{c_{2,t+1}}{c_{1,t}} = \frac{1+r_{t+1}}{1+\rho} Aw_t.$$

The private saving per person is given by

$$s_t = \frac{1}{2+\rho} Aw_t - T.$$

The social security tax causes a one-for-one reduction in private saving.

(c) The capital stock in period $t+1$ will be equal to the sum of total private saving of the young plus the total amount saved by the government, hence:

$$K_{t+1} = S_t L_t + T L_t.$$

Dividing both sides by AL_{t+1} to convert this into units of effective labor, and using equation yields:

$$k_{t+1} = \frac{1}{1+n}\left(\frac{w_t}{2+\rho} - \frac{T}{A}\right) + \frac{1}{1+n}\frac{T}{A}$$

which simplifies to:

$$k_{t+1} = \frac{1}{1+n}\frac{1}{2+\rho} w_t$$

Using equation to substitute for the wage yields

$$k_{t+1} = \frac{1}{1+n}\frac{1}{2+\rho}\left(1-\alpha\right) k_t^{\alpha}$$

Thus the fully-funded social security system has no effect on the relationship between the capital stock in successive periods.

(d) Since there is no effect on the relationship between k_{t+1} and k_t, the balanced growth path value of k is the same as it was before the introductions of the fully funded social security system (Note that we have been assuming that the amount of the tax is not greater than the amount of saving each individual would have done in the absence of the tax.) The basic idea is that total investment and saving is still the same each period, the government is simply doing some of the saving for the young. Since social security pays the same rate of return as private saving, individuals are indifferent as to who does the saving. Thus individuals offset one-for-one any saving that the government does for them.

(e) The results show that the equilibrium level of capital per effective worker is lower in the pay as you go system than in the fully funded system. The intuition is straightforward: in the pay as you go system the tax levied on the young is immediately transferred to the old, while in the fully funded system it is saved. Otherwise the problems are quite similar (compare the expressions for the saving function for the case in which $n = r_{t+1}$). In other words, in the fully funded system the tax revenue is capitalized, while in the pay as you go system it is consumed.