

PS 3: Solution

Econ 14.05: Intermediate Applied Macroeconomics

1. (a) A fall in the rate of depreciation means that the effective depreciation rate goes down, and consequently the break even investment line moves down. Nothing happens to the actual investment line. See Figure 1. With less depreciation, capital per unit of effective labor is going to increase in the steady state.
- (b) This is just the opposite case from before. Now the break even investment line is steeper and nothing happens to actual investment. The capital stock per unit of effective labor falls in steady state. See Figure 2.
- (c) A greater α means that marginal product diminishes more slowly. Intuitively, this is going to imply a higher capital stock per unit of effective labor in the long run. But this is not clear. On the graph, nothing happens to the break even investment. What happens to actual investment can be seen on Figure 3. For $k < 1$ the curves lies below the old one, while for $k > 1$ it is above. This is because $\frac{\partial sk^\alpha}{\partial \alpha} = sk^\alpha \ln k$. What happens to the steady state level of capital per unit of effective labor? It depends on whether $s \stackrel{\leq}{\geq} n + g + n$. When $s > n + g + n$ then k rises for sure. This is the case in Figure 3.
- (d) The actual investment curve (that includes B now), shifts up, we have more output per unit of capital. Break even investment does not change. Thus, capital per unit of effective worker raises in the steady state.

2. Problem 2: Natural Resources in the Solow model:

- (a) The text tells us that it is assumed that $\dot{A}/A = g$, $\dot{L}/L = n$ and $\dot{R}/R = 0$. We need to show that \dot{K}/K and \dot{Y}/Y converge to a constant. There are two ways of addressing this question, one direct way and the other using the hint provided. Consider first the direct way. The equation for capital accumulation is:

$$\dot{K} = sY - \delta K$$

Rewriting and defining $g_K = \dot{K}/K$ gives:

$$g_K = \dot{K}/K = sY/K - \delta$$

This equation tells us that Y and K need to grow at the same rate if the growth rate of capital is to be constant in the steady state. Therefore in the steady state $g_K = \dot{Y}/Y$. Next, take the time derivative of the production technology and replace for the growth rates:

$$\begin{aligned} \dot{Y}/Y &= \alpha \dot{K}/K + \beta (\dot{AL})/(AL) + (1 - \alpha - \beta) \dot{R}/R \\ g_K &= \alpha g_K + \beta(g + n) \\ g_K &= \frac{\beta}{1 - \alpha}(n + g). \end{aligned}$$

So, the economy has a steady state in which the growth rate of capital and output is constant and equal to $\frac{\beta}{1 - \alpha}(n + g)$, and productivity and labor grow at exogenous constant rates g and n respectively.

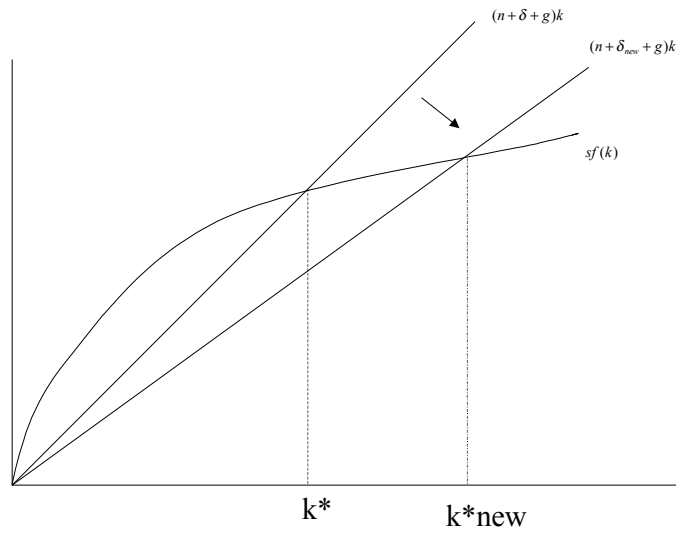


Figure 1:

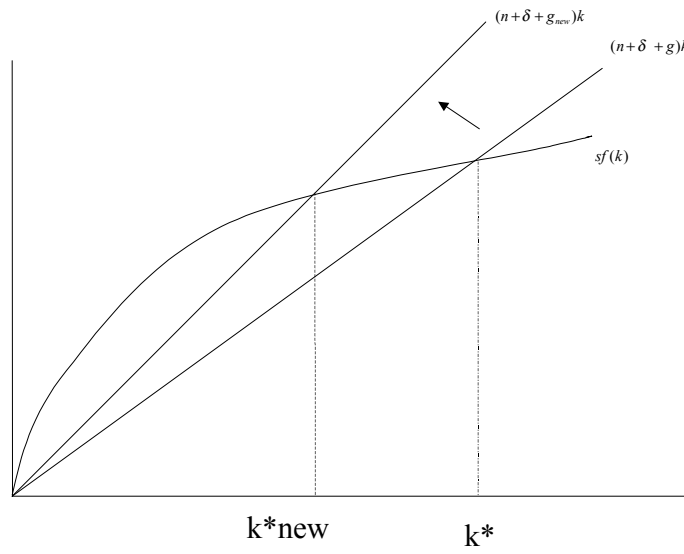


Figure 2:

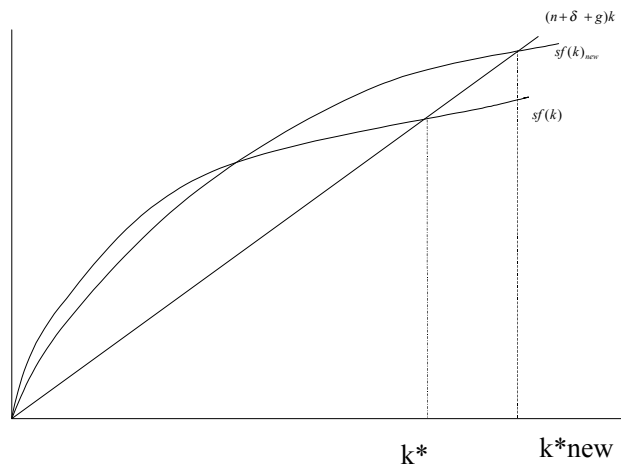


Figure 3:

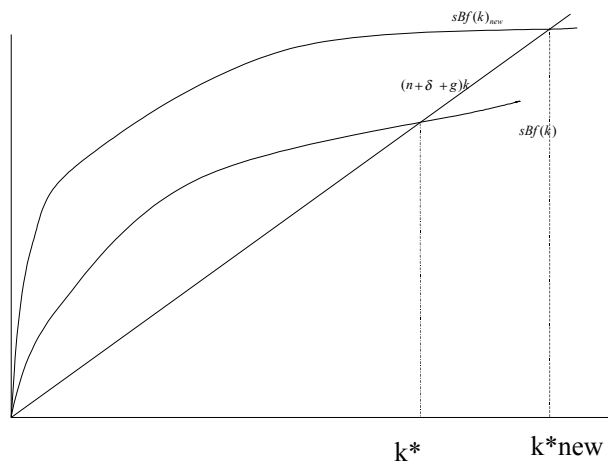


Figure 4:

The Hint provides a normalization that creates variables that are constant in the steady state: k and y . From the definition,

$$\begin{aligned} g_y &= g_Y - M(g_A + g_L) - Ng_T \\ &= g_Y - \frac{\beta}{1-\alpha}(n+g), \end{aligned}$$

so, as g_y is constant in the steady state, this implies that

$$g_Y = \frac{\beta}{1-\alpha}(n+g),$$

which is what we obtained using the direct approach. Finally, you can check that the condition that k is constant in the steady state implies that $g_K = g_Y$.

- (b) From the answer to part (a), permanent growth is still possible with a fixed stock of land. The reason is that even though land is constant, the number of effective workers is growing at an exogenous constant rate.
- (c) Consider the growth rate of output per capita:

$$(\dot{Y}/L)/(Y/L) = \dot{Y}/Y - \dot{L}/L = g_K - n = \frac{\beta}{1-\alpha}(n+g) - n = \frac{\beta g - (1-\beta-\alpha)n}{1-\alpha}$$

Clearly, the growth rate of output per capita can either be positive or negative. Two considerations are important: the bigger the rate of technological progress relative to population growth, the less likely is it that growth of output per capita is stagnant. Second, the bigger the coefficient of land in the production function $(1-\beta-\alpha)$ relative to the coefficient on technological progress β , the more likely that growth is negative.

The reason for the possibility of permanent stagnation of output per person is that the presence of land - which is not accumulable - yields decreasing returns to the other two factors K and AL .

- (d) There are two differences between this model and the mechanism Malthus had in mind. The first is that he did not consider the rate of population growth as constant, but as endogenous, being influenced by the level of output per capita. The second difference is that he did not think about technological progress, as determined in part (c), it is the latter that really makes a difference: if technical progress is fast enough, it may offset the decreasing returns effect generated by fixed stock of land.

3. The Solow Model II:

- (a) Due to the assumption of constant returns to scale, the production function can be expressed as:

$$y = \frac{Y}{AL} = F(K/AL, 1) = F(k, 1) = f(k) \tag{1.1'}$$

Expressing the budget constraint in per effective worker and replacing for the production function gives:

$$\dot{k} = sf(k) - (\delta + g + n)k \tag{1.2'}$$

(b) It is straightforward to see that the growth rate of capital per effective unit of labor is:

$$g_k = s \frac{f(k)}{k} - (\delta + g + n) \quad (1.2'')$$

In the case of smooth production technology, e.g. if $f(k) = k^\alpha$, this will look like:

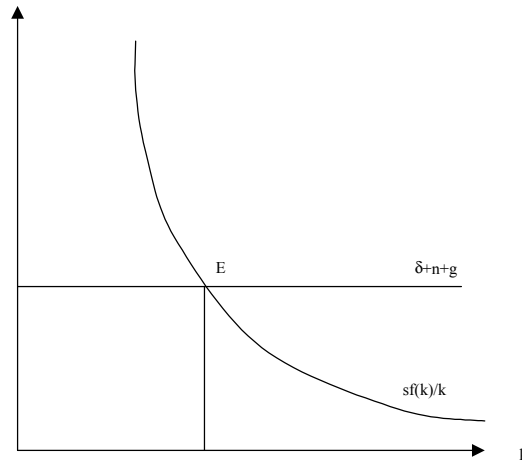


Figure 5

At point E in the figure, $g_k = 0$. Note that $sf(k)/k = k^{-(1-\alpha)}$ in the example of the Cobb-Douglas production function, which is a hyperbola.

(c) Now the production technology is in intensive form:

$$f(k) = \min(ak, b) \quad (1.5')$$

The production function is piecewise linear. Below b , each additional unit of investment in effective units of capital raises effective units of output by a . Above b , additional investment in k is useless. The situation is depicted in Figure 6.

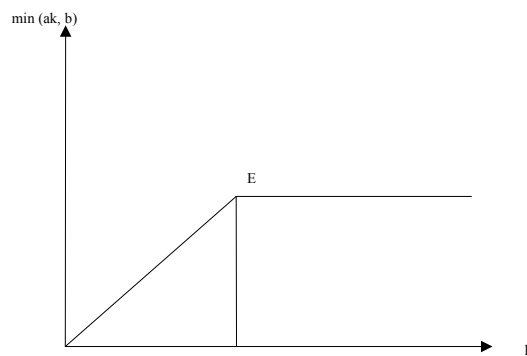


Figure 6

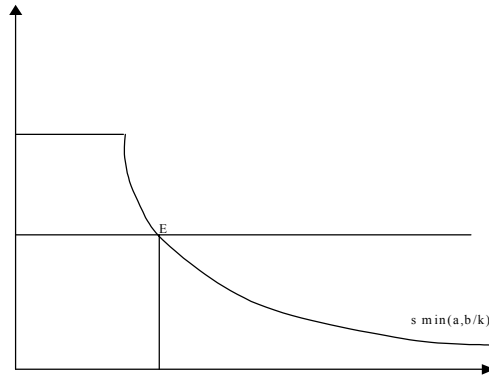
(d) In the case of this production technology, the growth rate of capital is:

$$g_k = s \cdot \min(a, b/k) - (\delta + g + n) \quad (1.5'')$$

In order to understand the steady state of this economy, two cases need to be distinguished. In the first case, when $s \cdot a > \delta + n + g$, the marginal productivity of capital is sufficiently high, and there is a steady state equilibrium with positive growth. The steady state level of effective units of capital is

$$k^* = sb/(\delta + n + g) \quad (1.6)$$

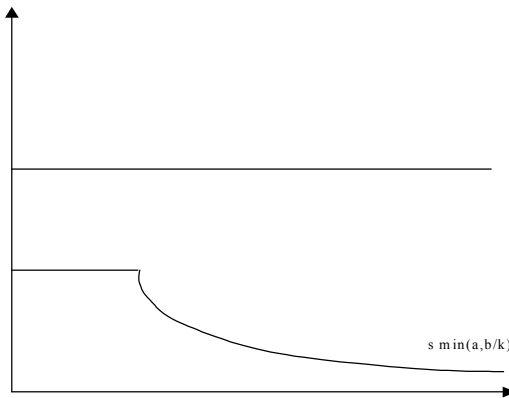
The following Figure 7 shows this first equilibrium:



The case when $sa > \delta + n + g$

Figure 7

In the second case, when $s \cdot a < \delta + n + g$, the marginal productivity of capital is too low. no capital accumulation is possible, and the only equilibrium is $k^* = 0$. This can be seen in Figure 8:



The case when $sa < \delta + n + g$

Figure 8

- (e) The determination of equilibrium is the relative size of the marginal product of capital $-a-$, the savings rate $-s-$, and the other exogenous parameters $\delta + n + g$. One interpretation of the model is that the economy can end up in a poverty trap with zero capital. As the savings rate increases, the response is highly nonlinear: at first, nothing happens, then, the economy jumps to a new equilibrium with positive income (point E in the figure 7).