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Status Report on

ESTIMATION AND SIGNAL PROCESSING FOR SPATIAL DATA: EFFICIENT ALGORITHMS, INVERSE PROBLEMS, AND COMPUTATIONAL VISION AND GEOMETRY

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SUMMARY

This status report presents a brief description of the research carried out by faculty, staff and students of the M.I.T. Department of Electrical Engineering and Computer Science under NSF Grant ECS-8700903. The principal investigator for this research is Prof. Alan S. Willsky, the co-principal investigator is Bernard C. Levy (at U.C. Davis since September 1987), and Profs. George C. Verghese and Sanjoy K. Mitter are the other major faculty contributors. The period covered by the report is August 1, 1987 to April 15, 1988.

The basic scope of the grant is to carry out fundamental research on several interrelated classes of problems involving estimation and signal processing of spatial data. Our research is divided into three broad areas:

I. Efficient Estimation and Signal Processing for Spatial Data. The major emphasis here is on using notions of recursion, spatial decomposition and symmetry to obtain recursive and highly parallel algorithms for the estimation of 2D processes.

II. Estimation and Identification Approaches to Inverse and Signal Reconstruction Problems. Among the problems of interest are the development of generalized tomographic methods for exact and approximate solution of 2D and 3D inverse scattering problems, and the development of algorithms operating at multiple spatial scales.

III. Computational Vision and Geometric Estimation and Reconstruction. Our approach here is via system- and estimation-theoretic formulations of problems involving reconstructing geometric objects from uncertain measurements of geometric features.

In the following three sections we briefly describe our results in each of these areas, and present our research directions. A list of publications supported in whole or in part by this grant is also included.

I. *Efficient Estimation and Signal Processing for Spatial Data*

Our work in this area has concentrated on two main topics: (i) boundary value problems for discrete-time noncausal descriptor systems; and (ii) recursive estimation and spectral estimation for 2D isotropic random fields. Our main results in these initial months have been for boundary value descriptor systems, and they are summarized here.

As described in the original proposal for this project, our work is motivated by the need to develop methods for dealing with noncausal models arising in spatial problems. As is often the case when examining questions for 2D processes, it was realized that the corresponding theory for 1D processes is incompletely developed. This has led us to focus on two-point boundary value descriptor systems (TPBVDSs) of the form:

$$Ex(k+1) = Ax(k) + Bu(k), \qquad 0 \leq k \leq N-1 \tag{1.1}$$

with the two-point boundary value condition

$$V_i x(0) + V_f x(N) = v \ . \tag{1.2}$$

Here E, A, and B are constant matrices, with E and A square, x and v are n-dimensional vectors, and u is an m-dimensional vector.

It is known that, without loss of generality, we can assume the system (1.1)-(1.2) is in *standard form*, i.e. it satisfies the following two properties: (i) there exist some scalars α and β such that

$$\alpha E + \beta A = I \ , \tag{1.3}$$

which implies that E and A commute; and (ii) the boundary matrices V_i and V_f are such that

$$V_i E^N + V_f A^N = I \ . \tag{1.4}$$

Our earlier work has established that there are two natural notions of recursion here, namely inward from the boundaries and outward toward the boundaries, with associated notions of reachability and observability. More recent results, described in [6], concern stability, its relation to the property of stochastic stationarity, and the study of these via generalized Lyapunov equations for TPBVDSs.

Since TPBVDSs are defined only over a finite interval, the concept of stability is not easy to formulate for these systems. Our concept of internal stability requires that, as the interval of definition N increases, the effects of the boundary conditions v on the states located close to the center of the interval should go to zero. To properly develop this concept, the notion of time invariance is introduced.

A TPBVDS is *time invariant* if the Green's function $G(k,l)$ appearing in the solution of the TPBVDS (1.1)-(1.2) depends only on the difference between arguments k and l, so that $G(k,l) = G(k-l)$. Unlike for causal systems, the fact that the matrices E and A are constant is not sufficient to guarantee that the TPBVDS (1.1)-(1.2) is time invariant. It was established in our earlier work that a TPBVDS is time invariant if and only if the matrices E and A commute with both V_i and V_f.

The following decomposition of a time invariant TPBVDS allows us to deduce a necessary and sufficient condition for internal stability:

(Decomposition of a time invariant TPBVDS into forwards, backwards and marginally stable components): Through the use of a state transformation, and by left multiplication of (1.1) and (1.2) by invertible matrices, an arbitrary time invariant TPBVDS can be decomposed

into three decoupled subsystems of the form

$$x_f(k+1) = A_f x_f(k) + B_f u(k) \quad , \quad V_{i1} x_f(0) + V_{f1} x_f(N) = v_1 \ , \qquad (1.5a)$$

$$x_b(k) = A_b x_b(k+1) - B_b u(k) \quad , \quad V_{i2} x_b(0) + V_{f2} x_b(N) = v_2 \ , \qquad (1.5b)$$

$$x_m(k+1) = U x_m(k) + B_m u(k) \quad , \quad V_{i3} x_m(0) + V_{f3} x_m(N) = v_3 \ , \qquad (1.5c)$$

where the matrices A_f and A_b have their roots inside the unit circle, and U has its roots on the unit circle. The subsystems (1.5a)-(1.5c) are time invariant and in standard form, and correspond respectively to the forwards, backwards and marginally stable components of the original TPBVDS (1.1)-(1.2).

It is then shown in [6] that a time invariant TPBVDS is internally stable if and only if, in the decomposition (1.5) of the system, the boundary value matrices V_{i1} and V_{f2} are invertible and the system does not have any eigenmode on the unit circle (i.e. the marginally stable part in the decomposition (1.5) does not exist).

We have also examined the implications of the above results for stochastic systems of the form (1.1)-(1.2), where u(k) is a zero-mean white Gaussian noise with unit intensity, and where v is a zero-mean Gaussian random vector independent of u(k) for all k, and with covariance Q. A TPBVDS is said to be *stochastically stationary* if

$$M[x(k)x^T(l)] = R(k,l) = R(k-l) \ . \qquad (1.6)$$

If the TPBVDS (1.1)-(1.2) is stochastically stationary, then the *variance matrix* P(k) = R(k,k) of x(k) must obviously be constant, i.e., P(k) = P for all k.

An important issue for stationarity is again the fact that (1.1)-(1.2) is defined only over a finite interval. It is therefore of interest to see whether a given time invariant TPBVDS defined over a finite interval is extendible to a larger interval in some appropriate way. A time invariant TPBVDS given by (1.1)-(1.2) is said to be *extendible* if, given any interval $[K,L]$ containing $[0,N]$, i.e. such that $K \leq 0 \leq N \leq L$, there exists a TPBVDS over this larger interval with the same dynamics as in (1.1), but with new boundary matrices $V_i(K,L)$ and $V_f(K,L)$ such that the new extended system is time invariant and the Green's function $G(k-l)$ of the original system is the restriction of the Green's function $G_e(k-l)$ of the new extended system.

The class of time invariant, extendible TPBVDSs -- which we term *deterministically stationary* TPBVDSs -- turns out to be quite large. We have shown that, given an arbitrary time invariant TPBVDS defined over $[0,N]$, where it is assumed that $N > 2n$, there exists an "almost identical" extendible system. By "almost identical", we mean here that for any input sequence $u(l)$, the states $x(k)$ and $x'(k)$ of the two systems are identical for $k \in [n,N-n]$. Extendibility turns out to have a simple characterization in terms of E, A, V_i and V_f, see [6].

Assume now that the stochastic TPBVDS (1.1), (1.2) is deterministically stationary and in standard form. It has then been shown in [6] that the system is stochastically stationary if and only if the variance Q of the boundary vector v satisfies the generalized Lyapunov equation

$$EQE^T - AQA^T = V_i BB^T V_i^T - V_f BB^T V_f^T \ . \qquad (1.7)$$

Even if Q does not satisfy this condition, it turns out that, if the system (1.1)-(1.2) is internally stable, then the variance matrix of the states located close to the center of the interval $[0,N]$ converges to the solution P^* of the following generalized Lyapunov equation with $N = \infty$:

$$EPE^T - APA^T = (V_i E^N)BB^T(V_i E^N)^T - (V_f A^N)BB^T(V_f A^N)^T \qquad (1.8)$$

The above results contain significant extensions of our earlier ones for TPBVDSs. We expect that the concepts of internal stability and stochastic stationarity developed here will have the same far-ranging applications as the usual notions for standard systems.

II. *Estimation and Identification Approaches to Inverse and Signal Reconstruction Problems*

The theme of examining inverse problems from the viewpoint of estimation and identification is well represented by our work in [2], [7], which studies parameter estimation aspects of an *inverse conductivity* problem that was presented in our original proposal. We summarize the approach and conclusions of that work first, to indicate the substantial progress that has already been made on questions related to the theme of this section. This is followed by a summary of our recent progress on *approximate multidimensional inversion* methods, [9]-[12].

The inverse conductivity problem is stated as follows: estimate the conductivity within the unit square, a 2D scalar function, based on a set of experiments, each of which consists of applying a known potential distribution along the boundary and measuring the current normal to the boundary.

In seeking a computationally efficient algorithm, we explore the idea of solving the inverse problem at various spatial resolutions, starting at a very coarse resolution, then progressing to finer and finer resolutions. The main idea is that by starting at coarse resolutions involving fewer computations, then building up to finer and finer resolutions using information from previous resolutions, we arrive at a more computationally tractable algorithm, one in which the degrees of freedom are better controlled, and which converges faster than an algorithm aimed at solving the problem exclusively at the finest resolution. Moreover, it is plausible that a multi-resolution method would help in avoiding local minima, because by solving a sequence of problems at successively finer scales the estimate is likely to be guided towards the global minimum of the problem at the finest scale.

To state this more concretely, consider the unit square, in which we wish to estimate the 2D conductivity function. Within this domain the equations governing the physics of this problem are Gauss's Law,

$$\nabla \cdot J = 0 \qquad (2.1)$$

where J is the vector current density function in 2D, and Ohm's Law,

$$J = \sigma(x,y)E \qquad (2.2)$$

where $\sigma(x,y)$ is the unknown conductivity function and E is the electric field, which can be related to the potential function as $E = \nabla\phi(x,y)$. The fundamental problem is to estimate $\sigma(x,y)$ within the unit square by applying potentials along the boundary and taking measurements of the normal current along the boundary.

The mathematical physics for the problem can be concisely summarized by a partial differential equation (PDE) which must be satisfied within the unit square. The excitations for each experiment provide boundary conditions on the PDE. If we substitute (2.2) into (2.1) we obtain the following PDE:

$$\nabla \cdot \sigma(x,y)\nabla\phi_i(x,y) = 0 \qquad (2.3)$$

for $0 \leq x \leq 1$, $0 \leq y \leq 1$ with boundary conditions,

$$\phi_i(s) = B_i(s) \qquad (2.4)$$

where $s \epsilon \Gamma$, Γ being the boundary of the unit square, and B_i is the applied potential function along Γ. The subscript i indexes the boundary conditions and potential for a particular experiment. The available measurements can be concisely described by the following observation equation,

$$R_i(s) = \sigma(s)\frac{\partial \phi_i(x,y)}{\partial n}\Big|_s + n_i \tag{2.5}$$

where R_i is the observation function, i indexes the particular experiment, and n_i is the additive noise function associated with the observation.

Our approach to estimating $\sigma(x,y)$ begins with a 2D piecewise constant model for $\sigma(x,y)$. The domain, the unit square, becomes a grid of pixels in which the value of $\sigma(x,y)$ within each pixel is a constant. The basic structure of the estimation problem is then the following.

Within each square we must estimate the value of $\sigma(x,y)$ in that square and, as a nuisance parameter, the potential along the edges non-adjacent to the overall boundary, Γ, of the large unit square. The potential along Γ is applied and therefore known. Our observations consist of the normal currents along the edges. In the cases where the edges are adjacent to the boundary the observations consist of actual measurements. In the remaining interior edges we have what we can think of a pseudo-measurements consisting of quantities associated with adjacent squares. This overall structure is suggestive of a highly distributed algorithm for performing the optimization necessary in computing the maximum likelihood estimate of $\sigma(x,y)$.

Our multi-resolution algorithm consists of a sequence of iterative relaxation algorithms generating estimates at successively finer spatial scales. The estimators at each scale have the same form, and consist of an alternating sequence of *linear* estimators, each of which is highly parallelizable. Proofs of convergence of these algorithms are given in [2].

The effects of fine-scale variations on coarse-scale estimation are studied by estimating a constant background when in fact the true background is spatially varying. Based on a linearization of the PDE, we derive approximations to the bias and mean-square error for this case, and determine performance characteristics based on Monte-Carlo simulations.

We also derive the Cramer-Rao lower bound on the mean-square error of the estimate in the general case, and numerically compute this bound for the four-pixel case, using various excitation schemes and various conductivity backgrounds. For this case and the case in which the conductivity is piecewise constant in sixteen distinct regions, the performance of the algorithm is demonstrated on synthetic data. The speed of convergence and accuracy of our algorithm when information from the previous scale is used is compared to the performance when this information is ignored. We also examine the ill-posedness of the problem by investigating the presence of local minima of the cost function.

The conclusion derived from our work so far is that the multi-resolution approach has several attractive and promising features, and suggests interesting problems for continued research.

In the area of multidimensional inverse problems, we have pursued our earlier efforts on the development of efficient linearized inversion methods. For inversion methods based on wavefield backpropagation, we had developed earlier an inversion technique for solving the distorted-wave Born inversion problem with an arbitrary reference profile for a constant density acoustic medium, where the objective is to reconstruct the medium velocity as a function of space. Our efforts during the past year have focused on extending this inversion method to multiparameter inversion problems. In [9], the case of a variable density acoustic medium was examined, where both the density and velocity need to be reconstructed. In this case, the scalar image obtained by wavefield backpropagation for a single experiment is not sufficient to reconstruct both the velocity and density profiles, and at least two experiments must be performed, where the source location is changed from one experiment to the next. A least-squares technique was then used to reconstruct separately the Fourier transforms of the velocity and density perturbations. A similar approach was used in [10] to study the Born inversion problem for EM waves in a lossless 3D medium

In parallel with this first approach, we have also continued our work on tomographic inversion methods. During the past year, we have extended this basic approach to the joint reconstruction of the velocity and density in an acoustic medium [11], and to the elastic case [12]. For these problems, several experiments with probing plane waves at various angles are necessary, and a detailed analysis of the robustness properties of our inversion procedure was performed. This analysis shows that joint inversion problems in geophysics are rather ill-conditioned, and redundant data is important for improving performance. The theoretical predictions have been confirmed with extensive numerical simulations using synthetic data.

We conclude this section by mentioning our survey in [5] of some key papers in subspace methods for high-resolution spectral estimation. Of particular interest is the increasing use of "total least squares" methods in situations where ordinary least squares would traditionally have been used. We intend to further examine the use of subspace methods and total least squares for inversion problems (continuing and extending the work of Weiss, Willsky and Levy, which was carried out under partial support from our earlier NSF Grant ECS-8312921; see Asilomar, November 1987).

III. *Computational Vision and Geometric Reconstruction*

Our third area of research involves the formulation and treatment of problems in image analysis and computational vision from the viewpoint of estimation, systems theory and optimization. The major effort in the initial months of the project has been on reconstructing geometric objects from partial information, and this is summarized below. We expect in the coming months to build on this work and treat similar problems involving the movement and shape evolution of geometric objects.

The report [1] continues the earlier NSF-supported work, under Grant ECS-8312921, of Rossi and Willsky (which recently won the ASSP Paper Award from the Acoustics, Speech and Signal Processing Society of the IEEE). The main interest of [1] is in evaluating how accurately coarse features of object geometry -- particularly size, elongation and orientation -- can be estimated from noisy tomographic data. In many applications, knowledge of these features suffices if it can be obtained much more efficiently, safely or robustly than high-resolution images. A maximum-likelihood parameter estimation formulation is used, and estimation performance is analyzed by evaluation of the Cramer-Rao lower bound on the error variances of the estimates. It is demonstrated, for measurements available at all projection angles and at a given noise level, that: (i) the size and orientation of an object are more accurately determined than its elongation; and (ii) reliable estimation of orientation requires a minimum elongation, which depends inversely on the signal-to-noise ratio.

In [3], [4], [13], [14], we address the problem of image reconstruction from noisy and limited-angle or sparse-angle tomographic projections. The basic approach is first to estimate the *full* 2D Radon transform, or sinogram, of the object. This estimation uses prior knowledge of object mass, center of mass, and convex support, as well as information about fundamental constraints, smoothness, and periodicity properties of the Radon transform. The sinogram prior probability is given by a Markov random field (MRF), which reflects this prior information. The object is then reconstructed using convolution-backprojection applied to the estimate of the full Radon transform.

The algorithm presented in [4] for maximum *a posteriori* (MAP) estimation of the sinogram is implemented using a primal-dual constrained optimization procedure. The partial differential equation that governs the primal phase is solved using an efficient local relaxation algorithm due to Kuo, Levy and Musicus (*SIAM J. Sci. Stat. Comp.*, 8, 550, 1987). The dual phase involves only a simple Lagrange multiplier update.

The geometric information reflected in the MRF formulation is estimated hierarchically via new set reconstruction algorithms developed in [4]. These algorithms are based on probabilistic estimation formulations that incorporate prior information about the size, position and shape of the object. In particular, knowledge of the eccentricity, orientation and boundary curvature may be used. New and interesting insights on the problem of convex set reconstruction from support line measurements emerge from this work.

Problems of reconstruction from "shadows", i.e. from indicator sets of orthogonal projections, were also posed in our original proposal, and progress has been made on this, as described in [15], [16]. In [15] we present extensions of results of Van Hove and Verly (*IEEE ICASSP*, 1985) that relate curvatures of shadows to curvatures of the projected object. Our derivation uses the well-known fact that

$$(I \quad 0)\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix} = (A-BD^{-1}C)^{-1} \qquad (3.1)$$

to streamline considerably the earlier derivation. Because of this, we have been able to generalize to arbitrary dimensions some of the major results shown earlier only for projections from 3D to 2D.

In particular, the results in [16] deal with finding an ellipsoid from shadows. We consider determining the *symmetric* n × n matrix H from p quadratic-form observations

$$H(i) = S(i)^T HS(i) \quad , \; S(i)^T S(i) = I \quad , \quad 1 \leq i \leq p \qquad (3.2)$$

where S(i) is $n \times m_i$. For a given set of "probes" S(i), a necessary and sufficient condition for H to be uniquely determined, along with a procedure for computing it, is given in [16]. These results are obtained in a natural way by embedding the problem in the space of symmetric n × n matrices. This gives rise to several insights that would be lost if (3.2) was simply treated as a system of linear equations in n(n+1)/2 unknowns.

The generic implications of our results include the following:

(i) Shadows of an ellipsoid (of arbitrary dimension) on three hyperplanes generically suffice to recover the ellipsoid.

(ii) Shadows of an ellipsoid (of arbitrary dimension) on two hyperplanes and on a line generically suffice to recover the ellipsoid.

(iii) Shadows of an n × n ellipsoid on n(n+1)/2 lines generically suffice to recover the ellipsoid.

Our results also permit the treatment of arbitrary non-generic cases, however.

PUBLICATIONS

The publications listed below represent papers and reports supported in part by NSF Grant ECS-8700903.

1. D.J. Rossi and A.S. Willsky, "Object Shape Estimation from Tomographic Measurements -- A Performance Analysis," Report LIDS-P-1710, Lab. for Info. and Dec. Systems, MIT, October 1987 (to be submitted for publication).

2. K.C. Chou, "A Multi-Resolution Approach to an Inverse Conductivity Problem," S.M. Thesis, MIT EECS Dept.; Report LIDS-TH-1720, Lab. for Info. and Dec. Systems, MIT, December 1987.

3. J.L. Prince and A.S. Willsky, "A Projection Space Map Method for Limited Angle Reconstruction," Report LIDS-P-1731, Lab. for Info. and Dec. Systems, MIT, February 1988; IEEE Int. Conf. Acoustics, Speech and Sig. Proc., New York, April 1988.

4. J.L. Prince, "Geometric Model-Based Estimation from Projections," Ph.D. Thesis, MIT EECS Dept.; Report LIDS-TH-1735, Lab. for Info. and Dec. Systems, MIT, February 1988.

5. T.M. Chin, "Improved Frequency Estimation by Identification of Signal and Noise Subspaces," Area Exam Report, MIT EECS Dept.; Report LIDS-P-1761, Lab. for Info. and Dec. Systems, MIT, April 1988.

6. R. Nikoukhah, B.C. Levy and A.S. Willsky, "Stability, Stochastic Stationarity and Generalized Lyapunov Equations for Two-Point Boundary-Value Descriptor Systems," Report LIDS-P-1758, Lab. for Info. and Dec. Systems, MIT, April 1988 (to be submitted for publication).

7. K.C. Chou and A.S. Willsky, "A Multi-Resolution Probabilistic Approach to 2D Inverse Conductivity Problems," Report LIDS-P-1763, Lab. for Info. and Dec. Systems, MIT, April 1988 (to be submitted for publication).

Papers In Preparation

8. R. Nikoukhah, A.S. Willsky and B.C. Levy, "Reachability, Observability and Minimality for Stationary Two-Point Boundary Value Descriptor Systems," Technical Report, Lab. for Info. and Dec. Systems, MIT,

9. B.C. Levy, "Joint Velocity and Density Born Inversion by Reverse-Time Wavefield Extrapolation," Technical Report, EECS Department, University of California, Davis.

10. B.C. Levy, "Variable Background Born Inversion for a 3-D Lossless Electromagnetic Medium by Reverse-Time Wavefield Extrapolation," Technical Report, EECS Department, University of California, Davis.

11. A. Ozbek and B.C.Levy, "Born Inversion of Velocity and Density Profiles for Multidimensional Acoustic Waves," Technical Report, Lab. for Info. and Dec. Systems, MIT.

12. A. Ozbek, "Tomographic Methods for Multidimensional Velocity Inversion," Ph.D. Dissertation, MIT EECS Dept.; expected date of completion: August 1988.

13. J.L. Prince and A.S. Willsky, "A Geometric Projection-Space Reconstruction Algorithm: Parts I and II."

14. J.L. Prince and A.S. Willsky, "Convex Shape Estimation Using Noisy Support Lines and Priors."

15. W.C. Karl and G.C. Verghese, "Generalized Projection-Slice Results."

16. W.C. Karl and G.C. Verghese, "Determining an Ellipsoid from Shadows."