Fluctuations. Shocks, Uncertainty, and the Consumption/Saving Choice

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Want to start with a model with two ingredients:

- Shocks, so uncertainty. (Much of what happens is unexpected).

  Natural shocks if we want to get good times, bad times: Productivity shocks.

- Basic intertemporal choice: Consumption/saving

So take familiar **Ramsey** model, add technological shocks, and by implication uncertainty.

Model clearly cannot go very far. No movements in employment, and many other problems.

But a good starting point. Shocks/Propagation mechanisms. The nature of consumption smoothing. Comovement in consumption, and investment.

And a simple structure to discuss a number of basic conceptual and methodological issues. How to solve? Equivalence between centralized or decentralized economy.

1 The optimization problem

$$\max E[\sum_{0}^{\infty} \beta^i U(C_{t+i}) | \Omega_t]$$

subject to:

$$C_{t+i} + S_{t+i} = Z_{t+i} F(K_{t+i}, 1)$$

$$K_{t+i+1} = (1 - \delta) K_{t+i} + S_{t+i}$$

How to think of this system? We shall give later a decentralized economy interpretation. But easier to solve this way

Separability. Exponential discounting. Why? What about non exponential/hyperbolic for example.

Expectation based on information at time t.

No growth. If we wanted growth, would want a balanced path, so Harrod neutral progress. $Z_tF(K_t, A_tN)$. Can think of all variables divided by A_t. (See hand out by Thomas)

2 Deriving the first order conditions

The easiest way to derive them is the old fashioned way. Lagrange multipliers. Put the two constraints together. Associate $\beta^t\lambda_t$ with the constraint at time t (Why do that? For convenience: To get the marginal value of capital as of time $t+j$, not as of time t):

$E[U(C_t) + \beta U(C_{t+1}) - \lambda_t(K_{t+1} - (1-\delta)K_t - Z_tF(K_t, 1) + C_t) - \beta\lambda_{t+1}(K_{t+2} - (1-\delta)K_{t+1} - Z_{t+1}F(K_{t+1}, 1) + C_{t+1}) + ... \mid \Omega_t]$

So the First Order Conditions are given by:

$$C_t \;:\; E[\, U'(C_t) = \lambda_t \mid \Omega_t]$$

$$K_{t+1} \;:\; E[\, \lambda_t = \beta\lambda_{t+1}(1 - \delta + Z_{t+1}F_K(K_{t+1}, 1) \mid \Omega_t]$$

Define $R_{t+1} \equiv 1 - \delta + Z_{t+1}F_K(K_{t+1}, 1)$. And use the fact that C_t, λ_t are known at time t, to get:

$$U'(C_t) = \lambda_t$$

$$\lambda_t = E[\beta R_{t+1}\lambda_{t+1} \mid \Omega_t]$$

Interpretation

- The marginal utility of consumption must equal to the marginal value of capital. (wealth)
- The marginal value of capital must equal to the expected value of the

marginal value of capital tomorrow times the expected gross return on capital, times the subjective discount factor.

Or, merging the two:

$$U'(C_t) = E\ [\ \beta R_{t+1} U'(C_{t+1}) \mid \Omega_t]$$

This is the **Keynes-Ramsey condition**: Smoothing and tilting.

To see it more clearly, use the constant elasticity function (which in the context of uncertainty, also corresponds to the CRRA function):

$$U(C) = \frac{\sigma}{\sigma - 1} C^{(\sigma-1)/\sigma)}$$

Then:

$$C_t^{-1/\sigma} = E[\ \beta R_{t+1} C_{t+1}^{-1/\sigma} \mid \Omega_t]$$

Or, as C_t is known at time t:

$$E[(\frac{C_{t+1}}{C_t})^{-1/\sigma} \beta R_{t+1} | \Omega_t] = 1$$

Can play a lot with this formula, and this is what is done in finance. But, for intuition's sake, ignore uncertainty altogether:

$$\frac{C_{t+1}}{C_t} = (\beta R_{t+1})^{\sigma}$$

As $\sigma \rightarrow 0$, then $C_{t+1}/C_t \rightarrow 1$.

As $\sigma \rightarrow \infty$, then $C_{t+1}/C_t \rightarrow +/-\infty$.

Interpretation.

3 The effects of shocks. Using the FOCs and intuition

Look at the non stochastic steady state, $Z \equiv 1$:

$$C_t = C_{t+1} \Rightarrow R = (1 - \delta + ZF_K(K^*, 1)) = 1/\beta \Rightarrow K^*$$

This is the modified golden rule

$$ZF(K^*, 1) - \delta K^* = C$$

- Consider an additive permanent shift in $F(.,.)$. Not very realistic, but useful. No change in steady state. No change in F_K for given K, so no tilting. Consumption increases one for one with the shock.
- Consider a permanent multiplicative shock. Z up permanently. Then, in the new steady state: K is higher. Positive investment. So C must increase by less than $ZF(K, 1)$. Can C go down? Yes, if σ is high enough. Why? Smoothing: up. Tilting: down.
- Why if transitory? C goes up by less. So more investment, but for less time.

To summarize. Positive shocks to technology. Investment up. Consumption: probably, but not necessarily. So far, can fit basic facts.

What about other shocks? Suppose change in discount factor. β goes down: like the future less. (Deeper issues of uncertainty in discount rate). What would it do?

Will tilt consumption path towards consumption today. So consumption today will go up. But, as production has not changed, investment will go down. Clearly robust. Not good news for taste shocks in this class of models.

4 The effects of shocks. Actually solving the model

Solving the model is tough. Various approaches.

- Ignore uncertainty, go to continuous time, and use a phase diagram.

- Log linearize, and get an explicit solution (numerically, or analytically).
- Set it up as a stochastic dynamic programming problem, and solve numerically.
- Find special cases which solve explicitly.

4.1 Continuous time, ignoring uncertainty

Set up the model in continuous time. BF, Chapter 2. Hard to handle uncertainty, so pretend that people act as if they were certain.

Can then use a phase diagram to characterize the dynamic effects of shocks. Often very useful.

Assume that

$$\max \int_0^\infty e^{-\theta t} U(C_t)$$

subject to:

$$\dot{K_t} = ZF(K_t, 1) - \delta K_t - C_t$$

Then Keynes-Ramsey FOC:

$$\dot{C_t} / Ct = \sigma(ZF_K(K_t, 1) - \delta - \theta)$$

Can represent this differential system in a phase diagram. If do this, then show the solution is a saddle path. Given K, determines the value of C

In this system, show the effect of a permanent (unexpected) increase in Z. Show whether C goes up or down is ambiguous and depends on σ.

4.2 Log linearization

Look at the system composed of the FOC and the accumulation equation. Can think of it as a non linear difference system:

Consumption C_t depends on K_{t+1} and C_{t+1}.

Capital K_{t+1} depends on K_t and C_t

The non linear aspect makes it difficult to solve. If linearize, or log linearize , then becomes much easier. (Can intervert the expectation and the sum) (Why log linearize rather than linearize? For the same reason as elasticities are often more useful than derivatives). (See Campbell for a detailed derivation)

So, log linearizing around the steady state gives:

$$c_t = E[c_{t+1}|\Omega_t] - \sigma E[r_{t+1}|\Omega_t]$$

$$(R/F_K)E[r_{t+1}|\Omega_t] = (F_{KK}K/F_K)k_{t+1} + E[z_{t+1}|\Omega_t]$$

$$k_{t+1} = Rk_t - (C/K)c_t + (F/K)z_t$$

where small letters denote proportional deviations from steady state.

(How to derive these relations? Totally differentiate each equation. Then, multiply and divide derivatives to get to elasticities. For example, take the Euler equation:

$$E[U''\,dC_t = U''\,dC_{t+1} + U'\beta\,dR_{t+1} \mid \Omega_t] = 1$$

Divide both sides by U', and use $\beta R = 1$, to get

$$E[(U''C/U')dC_t/C = (U''C/U')dC_{t+1}/C + dR_{t+1}/R \mid \Omega_t] = 1$$

and use $1/\sigma = -U''C/U'$ to get the expression above.)

Then, replacing Er_{t+1} by the second expression, and replacing k_{t+1} by its value from the third expression, we get a linear system in c_t, c_{t+1}, k_{t+1}, k_t and z_t.

This difference system can be solved in a number of ways. Undetermined coefficients. If you had to guess:

$$c_t \text{ linear in } k_t, z_t, E[z_{t+1} \mid \Omega_t], E[z_{t+2} \mid \Omega_t]...$$

Or solve explicitly, using matrix algebra: BK, or some of the methods in BF. (See handout, and the Matlab program written by Thomas Philippon (RBC.m)).

Assume further that we are willing to assume a process for z_t. For example:

$$z_t = \rho z_{t-1} + \epsilon_t$$

Then, all expectations of the future depend only on z_t. So, consumption is given by:

$$c_t \text{ linear in } k_t, z_t$$

Consumption rule: Consumption depends on k_t and z_t. This takes us to stochastic dynamic programming.

4.3 Stochastic dynamic programming.

If we are willing to assume a specific process for z_t, then we can use SDP.

Suppose for example that Z_t follows a Markov process. So that all we need to know to predict future values of Z is Z_t. Then the value of the program depends only on K_t and Z_t. (Why?)

So write it as $V(K_t, Z_t)$:

$$V(K_t, Z_t) = \max_{C_t, C_{t+1}, \ldots} E[\sum_0^\infty \beta^i U(C_{t+i}) \mid \Omega_t]$$

subject to:

$$K_{t+i+1} = (1-\delta)K_{t+i} + Z_{t+i}F(K_{t+i}, 1) - C_{t+i}$$

The insight of SDP is that we can rewrite this infinite horizon problem as a two-period problem:

$$V(K_t, Z_t) = \max_{C_t, K_{t+1}} [U(C_t) + \beta E[V(K_{t+1}, Z_{t+1}|\Omega_t]$$

subject to:

$$K_{t+1} = (1-\delta)K_t + Z_tF(K_t, 1) - C_t$$

If we knew the form of the value function, then would be straightforward. We would get the rule:

$$C_t = C(K_t, Z_t)$$

We obviously do not know the value function. But it turns out to be easy to derive it numerically:

- Start with any function $V(.,.)$, call it $V_0(.,.)$.
- Use it as the function on the right hand side. Solve for $C_0(.,)$.
- Solve for the implied $V_1(.,.)$ on the left hand side
- Use $V_1(.,.)$ on the right hand side, derive $C_1(.,.)$, and iterate.

Under fairly general conditions, this will converge to the value function and the optimal consumption rule. Various numerical issues. Need a grid for K, Z. But conceptually straightforward. (On this, read Ljungqvist and Sargent, Chapters 2 and 3)

If do this (see Matlab exercise DP.m written by Thomas), then can derive the consumption surface as a function of K, Z. Can see how C moves with Z for different values of σ and verify our intuition from the phase diagram.

4.4 Special cases

Finally, one can find special cases which have an explicit solution. For this model, a well known and well examined special case is (see BF, Chapter 7, or LS, Chapter 2):

$$U(C_t) = \log C_t$$

$$Z_t F(K_t, 1) = Z_t K_t^{\alpha} \text{ (Cobb Douglas)}$$

$$\delta = 1 \text{ (Full depreciation)}$$

The last assumption is clearly the least palatable. Under these assumptions: (we do not need to specify the process for Z_t)

$$C_t = (1 - \alpha\beta) Z_t K_t^{\alpha}$$

A positive shock affects investment and consumption in the same way. Both increase in proportion to the shock.

An even more drastic short cut is simply to give up the infinite horizon structure, and think of a two period optimization problem. Often, this is a very useful step, and gives much of the intuition.

5 The decentralized economy

So far, we have looked at a central planning problem. But given the assumptions, there is a competitive equilibrium which replicates it.

Useful to look at the decentralized economy, with many identical consumers/workers. There are many identical firms

There are then many ways of describing the economy. Firms may buy and hold the capital, or rent it from consumers. They can finance themselves by debt, or equity, and so on. Here assume all capital held by consumers, who rent it to firms.

The goods, labor, capital services markets are competitive. Firms rent labor and capital services in the labor and capital market.

Consumers

- Each one has the same preferences as above.
- Each supplies one unit of labor inelastically in a competitive labor market, at wage W_t
- Each one can save by accumulating capital. Capital is rented out to firms every period in a competitive market for rental services, at net rental rate (rental rate net of depreciation) r_t,
- Each one owns an equal share of all firms in the economy, But, as the firms operate under constant returns, profits are zero, so we can ignore that.
- The budget constraint of consumers is therefore given by:

$$K_{t+1} = (1 + r_t)K_t + W_t - C_t$$

- So the first order condition is:

$$U'(C_t) = E[\ (1 + r_{t+1})\beta U'(C_{t+1} \mid \Omega_t]$$

Firms

- Firms have the same technology as above, namely $Z_t F(K_t, N_t)$.
- They rent labor and capital. Their profit is therefore given by

$$\pi_t = Y_t - W_t N_t - (r_t + \delta)K_t$$

  The last term in parentheses is the gross rental rate.

- They maximize the present value of profits, discounted at the interest rate. So:

$$\max E[\pi_t + \sum_{i=1}^{\infty} \Pi_{j=1}^{j=i}(1 + r_{t+j})^{-1}\pi_{t+j} \mid \Omega_t]$$

- Profit maximization implies:

$$W_t = F_N(K_t, N_t)$$

$$r_t + \delta = F_K(K_t, N_t)$$

Finally, labor market equilibrium implies

$$N_t = 1$$

Now, it is straightforward to show that the equilibrium is the same as in the central planning problem:

Using the relation between rental rate, and marginal product of capital, and replacing in the first order conditions:

$$U'(C_t) = E[(1 - \delta + F_K(K_{t+1}, 1))\beta U'(C_{t+1} \mid \Omega_t]$$

Using the expressions for the wage and the rental rate in the budget constraint of consumers gives:

$$K_{t+1} = (1 - \delta + F_K(K_t, 1))K_t + F_N(K_t, 1) - C_t$$

Or

$$K_{t+1} = (1 - \delta)K_t + F(K_t, 1) - C_t$$

What is learned? Gives a different interpretation. Think about the consumers after a positive shock to Z_t. They anticipate higher wages, but also higher interest rates. What do they do?

We know we cannot typically solve for consumption. (Could not before, cannot now). But again, can cheat or consider special cases.

For example, ignore uncertainty, assume log. and show (along the lines of BF, p50) that:

$$C_t = (1 - \beta)[(1 + r_t)K_t + H_t]$$

where

$$H_t \equiv [W_t + \sum_{i=1}^{\infty} \Pi_{j=1}^{j=i}(1 + r_{t+j})^{-1}W_{t+j}]$$

So consumers look at human wealth, the present value of wages, plus non human wealth, capital. They then consume a constant fraction of that total wealth. Whatever they do not consume, they save.

Now think again about the effects of a technological shock. What are the effects at work?