Problem Set 3

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1 Human Wealth, Financial Wealth and Consumption

The goal of the question is to derive the formulas on p13 of Topic 2. This is a partial equilibrium analysis that focuses on the problem of the consumer in a decentralized economy. We assume that the utility function of the consumer is $CRRA$:

$$u(C) = \frac{\sigma}{\sigma - 1}\left(C^{\frac{\sigma-1}{\sigma}} - 1\right)$$

We assume that there is **no uncertainty**. The dynamic budget constraint is:

$$S_{t+1} = R_t S_t + W_t - C_t$$

1.1 Integrating the budget constraint

Show that, for all $n > 1$

$$\frac{S_{t+n}}{R_{t+1} \times .. \times R_{t+n-1}} = \sum_{i=1}^{n-1} \frac{W_{t+i} - C_{t+i}}{R_{t+1}..R_{t+i}} + R_t S_t + W_t - C_t$$

1.2 The need for a No Ponzi Game condition

Suppose for simplicity that W_t is bounded above by some finite W :

$$\sup_t W_t = W < \infty$$

Pick **any** $C > 0$. Show that if there is no limit on what the consumer can borrow, the consumption profile $C_t = C$ for all t is sustainable. Show that when C is large enough, savings are negative and eventually grow in absolute value at a rate R_t in the sense that $\frac{S_{t+1}}{S_t} \simeq R_t$ for large t. This is called a Ponzi game. Finally show that the condition

$$\lim_{n\to\infty} \frac{S_{t+n}}{R_{t+1} \times .. \times R_{t+n-1}} \geq 0$$

eliminates such games (show it simply for $C_t = C$ constant). Interpret this condition.

1.3 The Intertemporal Budget Constraint

Why is it obvious that we can replace the No Ponzi game inequality by an equality? Using this equality, derive the intertemporal budget constraint:

$$\sum_{i=1}^{\infty} \frac{C_{t+i}}{R_{t+1}..R_{t+i}} + C_t = \sum_{i=1}^{\infty} \frac{W_{t+i}}{R_{t+1}..R_{t+i}} + W_t + R_t S_t$$

Interpret the different terms.

1.4 The Consumption Rule

Use the first order condition for consumption to show that:

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{C_{t+i}}{R_{t+1}..R_{t+i}} + C_t &= C_t \left(1 + D_t\right) \\ D_t &= \sum_{i=1}^{\infty} \beta^{i\sigma} \times \left(R_{t+1}..R_{t+i}\right)^{\sigma-1} \end{aligned}$$

Show that we recover the formula of page 13 in the log utility case. Interpret this equation. What happens to consumption today if there is an unexpected change in the sequence of future interest rates? Describe the different effects: the income/substitution effect and the wealth effect. (The wealth effect comes from $\sum_{i=1}^{\infty} \frac{W_{t+i}}{R_{t+1}..R_{t+i}} + W_t + R_t S_t$ and the income/substitution effect comes from D_t).

2 Consumption Asset Pricing Model

Financial asset pricing is about computing the value of a stream of risky cash flows. The $CAPM$ is one way to compute such a value.

Consider an investor who wants to maximize his expected utility by investing in a riskless bond and a risky stock. There are two periods, $t = 0, 1$. Uncertainty at time 1 is describe by the state: $\omega \in \Omega$. The return on the bond is R and the return on the stock is $\widetilde{R}(\omega)$. The investor is endowed with y_0 units of the numeraire at time 0. Let s be the savings and let x be the fraction of the savings invested in the stock. So the program of the investor is:

$$\max u\left(c_0\right) + \beta E\left[u\left(c_1\left(\omega\right)\right)\right]$$

subject to the budget constraints:

$$\begin{aligned} c_0 + s &= y_0 \\ s\left(\left(1-x\right)R + x\widetilde{R}\left(\omega\right)\right) &= c_1\left(\omega\right) \end{aligned}$$

Show that

$$\frac{1}{R} = \frac{E\left[\beta u'\left(c_1\left(\omega\right)\right)\right]}{u'\left(c_0\right)}$$

Interpret this equation.

Show that the time 0 price (p_0) of a stock that pays the dividends $y(\omega)$ at time 1 is (note that by definition $\widetilde{R}(\omega) = \frac{y(\omega)}{p_0}$):

$$p_0 = \frac{1}{u'(c_0)} E\left[\beta u'(c_1(\omega)) \times y(\omega)\right]$$

Assume for now that utility is quadratic:

$$u(c) = c - \gamma \frac{c^2}{2}$$

Consider two stocks A and B with the same expected dividends: $E[y_A(\omega)] = E[y_B(\omega)]$. Assume however that $y_A(\omega)$ is positively correlated with $c(\omega)$ while $y_B(\omega)$ is negatively correlated with $c(\omega)$.

Show that $p_{0,B} > p_{0,A}$ You may want to use the formula $E[AB] = cov(A,B) + E[A]E[B]$. What is the intuition for this result? How would you generalize it to the case of a more general utility function?

3 Q theory of investment

Consider the stochastic infinite horizon model of a firm facing adjustment costs to investment. The firm generates the random cash flows:

$$\Pi_t = \Pi(K_t, I_t, Z_t)$$

Where K is the firm capital stock, I its investment and Z the shocks. Capital accumulates according to

$$K_{t+1} = (1-\delta)K_t + I_t$$

For simplicity, we assume that the cash flows are discounted using a constant risk free rate:

$$V_t = \Pi_t + E_t\left[\sum_{i=1}^{\infty} R^{-i} \times \Pi_{t+i}\right]$$

The firm maximizes V_t.

Assume that the program of the firm is concave and show that the first order condition is:

$$\frac{\partial \Pi_t(K_t, I_t)}{\partial I_t} + E_t\left[\sum_{i=1}^{\infty} R^{-i}(1-\delta)^{i-1} \frac{\partial \Pi_{t+i}(K_{t+i}, I_{t+i})}{\partial K_{t+i}}\right] = 0$$

This is the q theory of investment. The first term is the marginal cost of one extra unit of investment today. The second term is the marginal revenue -i.e., the present discounted value of the future marginal product of capital. We call it q_t:

$$q_t = E_t\left[\sum_{i=1}^{\infty} R^{-i}(1-\delta)^{i-1} \frac{\partial \Pi_{t+i}(K_{t+i}, I_{t+i})}{\partial K_{t+i}}\right]$$

3.1 A simple example of adjustment costs

Consider the case where

$$\Pi(K_t, I_t, Z_t) = Z_t K_t - I_t \left(p_{I,t} + A\left(\frac{I_t}{K_t}\right)\right)$$

$A(.)$ is the adjustment cost function and $p_{I,t}$ is the price of the investment goods. Interpret each term of this functional form. When is it likely to be a good description of reality? Suppose that $A(.)$ is increasing and convex. Derive and interpret the first order conditions. Show that they imply a simple mapping from q_t to the investment decision I_t. What are the factors affecting q_t? How would investment respond to a change in q_t with and without adjustment costs?

3.2 Marginal and Average Q

The problem is that q is not observable (explain why). However, we can observe Tobin's Q, or average Q. It is defined as the market value of the company (debt + equity) divided by the book value of its capital stock. In fact, because of the timing convention of the model, we need to define it as the market value net of the current period cash flows

$$Q_t = \frac{V_t - \Pi_t}{K_{t+1}}$$

The goal of this question is to show that under constant returns to scale, average Q is equal to marginal q. We will use the Lagrange multiplier method.

Define the Lagrangian

$$E_t\left[\sum_{i=o}^{\infty} R^{-i} \times \left(\Pi_{t+i} + q_{t+i}\left((1-\delta)K_{t+i} + I_{t+i} - K_{t+i+1}\right)\right)\right]$$

Show that the first order conditions are

$$\begin{aligned} \frac{\partial \Pi_t(K_t, I_t)}{\partial I_t} + q_t &= 0 \\ q_t &= E_t\left[\frac{1}{R} \times \left(\frac{\partial \Pi_{t+1}}{\partial K_{t+1}} + (1-\delta) q_{t+1}\right)\right] \end{aligned}$$

Assume that $\Pi_t(K_t, I_t)$ has constant returns to scale. Show that this implies that

$$\frac{\partial \Pi_{t+1}}{\partial K_{t+1}} \times K_{t+1} = \Pi_{t+1} - \frac{\partial \Pi_{t+1}}{\partial I_{t+1}} \times I_{t+1}$$

Use this formula together with the $FOCs$ and the capital accumulation to show that:

$$q_t K_{t+1} = E_t\left[\frac{1}{R}\Pi_{t+1} + \frac{1}{R} q_{t+1} K_{t+2}\right]$$

Solve forward to show that

$$q_t = Q_t$$

3.3 Testing the Theory

How would you test this theory? Why kind of data would you need? Suppose that you run the following regression:

$$\frac{I_t}{K_t} = a + bQ_t + \varepsilon_t$$

How is b related to adjustment costs? Empirically, Q is not very good at explaining investment, neither at the firm level, nor at the aggregate level. In fact, current (or past) profits have much more explanatory power than Q. How would you interpret these findings? (think of monopoly power and increasing returns to scale, credit constraints, and stock market bubbles).

3.4 Extra credit

Derive the same result ($q = Q$) using dynamic programming (if you do it right, it takes 2 lines).