MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.04 **Spring Term 2003**

PROBLEM SET 11

Reading: Liboff, 9.1 – 9.3. French & Taylor, 10.1 – 10.7 and 11.1 – 11.2.

1. **Raising and lowering operators for the 2-dim harmonic oscillator.** (20 points)

   Consider a two dimensional harmonic oscillator with a potential

   $$V(x,y) = \frac{1}{2}m\omega^2(x^2+y^2).$$

   We can use two sets of raising and lowering operators $a, a^\dagger$, $b, b^\dagger$ to write

   $$x = \frac{1}{\sqrt{2}\alpha}(a+a^\dagger), p_x = \frac{\alpha\hbar}{\sqrt{2}i}(a-a^\dagger), y = \frac{1}{\sqrt{2}\alpha}(b+b^\dagger), p_y = \frac{\alpha\hbar}{\sqrt{2}i}(b-b^\dagger).$$

   (a) Evaluate the commutators

   $$[a,a^\dagger], \quad [a,b], \quad [a,b^\dagger], \quad [a^\dagger,b], \quad [a^\dagger,b^\dagger], \quad [b,b^\dagger].$$

   (b) Express the Hamiltonian operator as well as the angular momentum operator, $L_z = xp_y - yp_x$, in terms of the raising and lowering operators.

   (c) Use these expressions to evaluate the commutator $[H, L_z]$.

2. **Components of angular momentum.** (20 points)

   Since the commutator $[L_x, L_z] \neq 0$, this means there is not a complete set of simultaneous eigenstates of L^2, L_x, and L_z. Show that there are no simultaneous eigenstates of L^2, L_x, and L_z except for $l = 0$. Prove this in the following two ways:

   (a) Use the generalized uncertainty principle to show that such a state must have $m_x = m_z = 0$. In this case, what is the condition on L_y^2 and what does this imply?

   (b) Use the decomposition of L_x into raising and lowering operators of L_z.

3. **Angular momentum eigenfunctions.** (20 points)

   Consider the angular momentum operators

   $$\vec{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right], \qquad L_z = -i\hbar\frac{\partial}{\partial\varphi},$$

   in polar coordinates, as well as the spherical harmonics $Y_{l,m}(\theta,\varphi)$ for the following values of l and m

   $$Y_{0,0} = \sqrt{\frac{1}{4\pi}}, \qquad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \qquad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}.$$

(a) Show that the functions are properly normalized and orthogonal.

(b) Show that the above functions $Y_{l,m}(\theta, \varphi)$ are eigenfunctions of both $\vec{L}^2$ and L_z and compute the corresponding eigenvalues.

4. **Rigid rotator state.** (15 points)

   Liboff, Problem 9.26.

5. **Measurement of L_x on an eigenstate of L_z.** (10 points)

   Liboff, Problem 9.27.

   You need not show much work in this problem. Read the argument in the text of Liboff (starting from the bottom of page 378 through page 380). Write down an expression similar to equation 9.101, but for the the state $Y_{1,-1}$. Then answer the question.

6. **Electron bound to a radioactive nucleus.** (15 points)

   Consider an electron in the ground state of tritium, H^3, whose nucleus consists of one proton and two neutrons. A neutron can decay radioactively into a proton, an electron, and an anti-neutrino. When the electron and the anti-neutrino are emitted, they leave a He^3 nucleus behind which consists of two protons and one neutron. Assume the radioactive decay happens instantaneously. Calculate the probability that the electron of the original tritium atom remains in the ground state of He^3. (Ignore all effects of the emitted electron and anti-neutrino. Also, assume that the proton and neutron have the same mass.)