8.04 Quantum Physics I Spring 2003

Exam # 2 (4/24/2003)

Selected Formulae

Time-independent Schrödinger Equation:

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) = E\psi(x). \tag{1}$$

Momentum operator (in position representation):

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}. \tag{2}$$

Infinite Potential Well (One Dimension):

$$V(x) = \begin{cases} 0 & \text{where } -a < x < a, \\ \infty & \text{otherwise.} \end{cases} \tag{3}$$

$$E_n = \frac{\hbar^2\pi^2 n^2}{8ma^2}, \quad n \in \{1, 2, 3, \cdots\}, \tag{4}$$

$$\psi_n(x) = \frac{1}{\sqrt{a}}\begin{cases} \cos\left(\frac{n\pi x}{2a}\right) & \text{where } n \text{ is odd,} \\ \sin\left(\frac{n\pi x}{2a}\right) & \text{where } n \text{ is even.} \end{cases} \tag{5}$$

Useful integral expression:

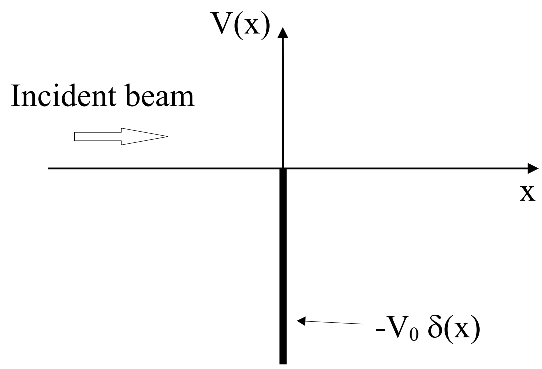
$$\int_{-\infty}^{\infty} e^{-\alpha y^2}\, dy = \sqrt{\frac{\pi}{\alpha}} \tag{6}$$

1. **Problem 1: (20 points)**

   A beam of particles, each of mass m and energy $E > 0$, is incident from the left on a delta function well located at the origin:

   $$V(x) = -V_0\delta(x)$$

   with $V_0 > 0$.



   Calculate the fraction of particles in the incident beam that are reflected by this potential. That is, find the reflection coefficient R. Write your answer in terms of E, m, V_0 and fundamental constants. (Be sure to check that your answer makes sense in the $V_0 \to 0$ limit.)

2. **Problem 2: (20 points)**

A particle of mass m is in a one-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$. The Hamiltonian for this system may be written

$$\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + \tfrac{1}{2})$$

$$\text{where } \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\,\hat{p}\right) \text{ is the lowering operator}$$

$$\text{and } \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\,\hat{p}\right) \text{ is the raising operator.}$$

The ground state wave function is given by $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-m\omega x^2/2\hbar}$.

(a) (10 points) Starting with $\psi_0(x)$, use the operator method to explicitly construct the wave function for the first excited state $\psi_1(x)$. Don't worry about normalizing this wave function; that is, collect all constant factors into an overall constant C_1.

(b) (10 points) Suppose a particle is initially in the ground state of the harmonic oscillator potential. The potential is suddenly modified such that the oscillation frequency is doubled, $\omega' = 2\omega$. (Assume that the change to the potential is so sudden that the particle remains in the ground state of the original potential immediately after the change.) Calculate the probability that a measurement of the particle's energy immediately after the change will find the particle in the ground state of the new oscillator potential.

3. **Problem 3: (30 points)**

A particle of mass m is in a one-dimensional infinite potential well given by

$$V(x) = \begin{cases} 0 & \text{where } -a < x < a, \\ \infty & \text{otherwise.} \end{cases}$$

Suppose that at time $t = 0$, the particle is described by the following wave function:

$$\Psi(x,0) = \left[\frac{1}{2}\psi_1(x) + \frac{\sqrt{2}}{2}\psi_2(x) + \frac{1}{2}\psi_3(x)\right]$$

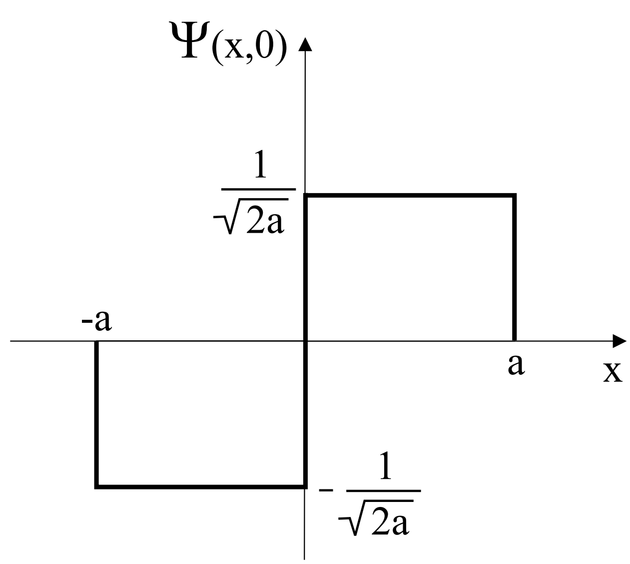
where $\psi_n(x)$ denotes a normalized energy eigenfunction with eigenvalue $E_n = \frac{\hbar^2\pi^2 n^2}{8ma^2}$.

(a) (4 points) Write down the time-dependent wave function $\Psi(x,t)$ for $t > 0$. Leave your answer in terms of ψ_1, ψ_2, and ψ_3.

(b) (7 points) Calculate the expectation value of the energy, $\langle E \rangle$, for the particle described by $\Psi(x,t)$. Does this quantity change with time?

(c) (7 points) What energy values will be observed as a result of a single measurement at time $t = 0$ and with what probabilities? How do these probabilities change with time?

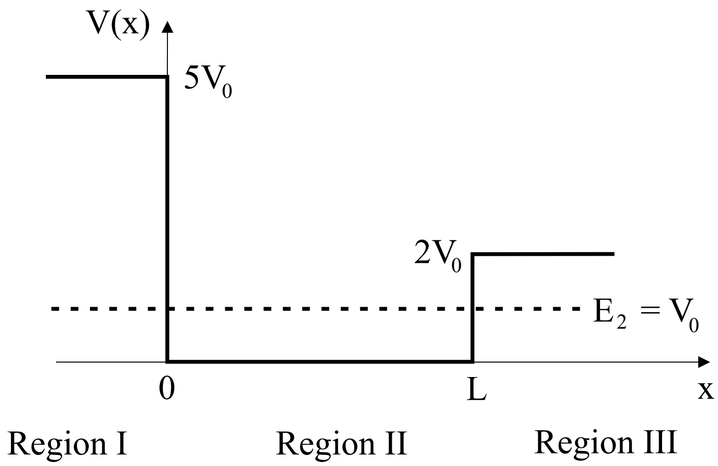
(d) (12 points) Instead, suppose that at time $t = 0$, the system is prepared so that the particle has the following wave function:



If a measurement of energy is made at time $t = 0$, what energy values will be measured and with what probabilities? (Hint: Is the above function even or odd?)

4. **Problem 4: (30 points)**

A rectangular potential well is bounded by a wall of height $5V_0$ on one side and a wall of height $2V_0$ on the other, as shown in the figure. The well has a width L, choosen such that the *second* energy state for a particle of mass m has energy $E_2 = V_0$. (The state of *lowest* energy is defined to be the first state.)



(a) (8 points) Make a qualitative plot of the wave function $\psi_2(x)$. (Draw this in your examination booklet, not on the above figure.) What are the relative rates of decrease of $|\psi_2|$ with x in regions I and III? Does the node of this wave function occur to the right or the left of the center of the well?

(b) (7 points) By comparing your qualitative plot of $\psi_2(x)$ with the wave function for the second energy level in the infinite square well, we can infer that the mass m of the particle is less than a certain quantity involving L, V_0, and $\hbar$. What is this upper limit for m?

(c) (7 points) In order to satisfy the boundary condition that $\frac{d\psi}{dx}$ be continuous, the wave function must be "headed" toward the axis at $x = 0$ and at $x = L$. For ψ_2, this means that $\lambda_2/2$ must be less than L. (Convince yourself of this.) (Here, λ_2 is the wavelength of the wave function ψ_2 in region II.) What *lower* limit does this imply for m?

(d) (8 points) By *carefully* extending the argument used in (c), can you rule out the existence of a fourth energy level for a bound state in this potential? Write down your argument for the existence or non-existence of a fourth bound energy level.