MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.04 **Spring Term 2003**

PROBLEM SET 3

Reading: Review French & Taylor, Chapters 1 & 2.
Liboff, Chapter 2.

1. **Line shift due to recoil** (20 points).

   The electron in a hydrogen atom jumps from the first excited state ($n = 2$) to the ground state ($n = 1$) and emits a photon. Before the transition, the hydrogen atom is at rest with zero momentum. The emitted photon will cause the hydrogen atom to recoil. Since the recoiled atom carries away a small amount of energy, the energy of an emitted photon is actually slightly less than the energy difference of the second and first orbits, $E_2 - E_1$.

   (a) As a first order approximation, assume the energy of the emitted photon is given by $E_2 - E_1$. Calculate the momentum of the emitted photon and the momentum of the recoiled hydrogen atom.

   (b) Find the kinetic energy of the recoiled atom.

   (c) Use the result in (b) to calculate the *actual* energy $h\nu$ of the emitted photon. Find the numerical value of the dimensionless ratio

   $$\frac{(E_2 - E_1) - h\nu}{E_2 - E_1}.$$

2. **A one-electron uranium atom** (20 points).

   A neutral uranium atom has 92 electrons surrounding a nucleus containing 92 protons.

   (a) Given that the Bohr radius for the lowest energy in hydrogen is $a_0 = 0.5$ Å, derive an approximate numerical value on the basis of Newtonian mechanics for the radius of the smallest Bohr orbit about the uranium nucleus.

   (b) In a violent nuclear event, a uranium nucleus is stripped of all 92 electrons. The resulting bare nucleus then captures a single free electron from its surroundings. Given that the ionization energy for hydrogen is 13.6 eV, derive an approximate numerical value for the maximum energy of the photon that can be given off as the uranium nucleus captures this first electron.

   (c) Calculate the value of v/c for the first Bohr orbit in U^{92} according to Newtonian mechanics. This will show you that relativistic dynamics should really be used in this problem.

3. **Example of the Wilson-Sommerfeld quantization rule** (10 points).

   Use the Wilson-Sommerfeld quantization rule to calculate the lowest three quantized heights (in cm) and energies (in eV) for an electron "bouncing" on the surface of the Earth. Assume that the electron only feels the gravitational force, and collides elastically with the Earth's surface when it bounces.

4. **Energy of rotating molecules** (10 points).

   As a model for a diatomic molecule (e.g., H_2 or N_2 or O_2), consider two point particles, each of mass m, connected by a rigid massless rod of length r_0. Suppose that this molecule rotates about an axis perpendicular to the rod through its midpoint. Show that, if the Bohr quantization condition on angular momentum is applied, the quantized values of the energy of rotation are given by

   $$E_n = \frac{n^2 h^2}{4\pi^2 m r_0^2},$$

   with $n = 1, 2, 3, \ldots$

5. **Hydrogen: a structure of minimum energy** (20 points).

   A simple but sophisticated argument holds that the hydrogen atom has its observed size because this size minimizes the total energy of the system. The argument rests on the assumption that the lower-energy state corresponds to a physical size comparable to the De Broglie wavelength of the electron. *Larger* size means larger De Broglie wavelength, hence smaller momentum and kinetic energy. In contrast, *smaller* size means lower potential energy, since the potential well is deepest near the proton. The observed size is a compromise between kinetic and potential energy that minimizes the total energy of the system. Develop this argument explicitly, as follows:

   (a) Write down the classical expression for the total energy of the hydrogen atom with an electron of momentum p in a circular orbit of radius r. Keep kinetic and potential energies separate.

   (b) *Failure of classical energy minimization.* Use the force law to obtain the total energy as a function of radius. What radius corresponds to the lowest possible energy?

   (c) For the lowest-energy state, demand that the orbit circumference be one De Broglie wavelength. Obtain an expression for the total energy as a function of radius. Note how a larger radius decreases the kinetic energy and increases the potential energy, whereas a smaller radius increases the kinetic energy and decreases the potential energy.

   (d) Take the derivative of the energy versus radius function and find the radius that minimizes the total energy. Show that the resulting radius is the Bohr radius a_0, and that the resulting energy is that calculated by Bohr for the lowest-energy state.

Comment: This *exact* result depends on setting the De Broglie wavelength equal to the *circumference* of the orbit. In fact the argument we are following specifies only an *approximate* correspondence between system size and De Broglie wavelength. We could have chosen "system size" to mean radius or diameter and then derived a size differing from that calculated above by factors of 2 or π, which is as near to the observed values as one has any right to expect!

6. **Fourier transform of a triangular wave packet** (20 points).

   Calculate the Fourier transform

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

   of the triangular wave packet

$$f(x) = \begin{cases} 1 + \frac{x}{b} & -b \le x \le 0 \\ 1 - \frac{x}{b} & 0 < x \le b \\ 0 & \text{elsewhere.} \end{cases}$$

   Draw qualitative graphs of $f(x)$ and $\tilde{f}(k)$. Next to each graph, write down its approximate "width." (*Hint*: For $\tilde{f}(k)$, one might define a suitable width as the spacing between its first two zeros.) We will revisit this problem in next week's assignment.