MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.04 **Spring Term 2003**

PROBLEM SET 9

1. **A reflectionless potential.** (20 points)

   Consider a particle of mass m moving in the potential

   $$V(x) = -\frac{\hbar^2 a^2}{m} \cosh^{-2}(ax),$$

   where a is a positive constant.

   (a) Show that this potential has the bound state

   $$\psi_0(x) = \frac{A}{\cosh(ax)},$$

   and find its energy. Normalize ψ_0 and sketch its graph. This turns out to be the *only* bound state for this potential.

   (b) Show that the function

   $$\psi_k(x) = A\left[\frac{ik - a\tanh(ax)}{ik + a}\right] e^{ikx}$$

   (where $k = \sqrt{2mE}/\hbar$) solves the Schrödinger equation for any positive energy E.
   Since $\tanh z \to -1$ as $z \to -\infty$, $\psi_k(x) \approx Ae^{ikx}$ for large negative x.
   This represents a wave coming in from the left with *no accompanying reflected wave* (that is, no term $\sim e^{-ikx}$). What is the asymptotic form of $\psi_k(x)$ for large positive x? What are the reflection and transmission coefficients R and T for this potential?

   (*Note:* $\cosh^{-2}(ax)$ is a famous example of a "reflectionless" potential – every incident particle, regardless of energy, passes right through.)

2. **Momentum space representation of a particle in a linear potential** (20 points)

   Consider the time independent Schrödinger equation for a particle which moves in a potential with a constant force,

   $$V(x) = ax.$$

   This represents, for example, the situation of a charged particle moving in a constant electric field. In the position space representation, the solution to this equation turns out to be rather complicated. The problem is easier to solve in the momentum space representation where the momentum and position operators are given by $\hat{p} = p$ and $\hat{x} = i\hbar\frac{\partial}{\partial p}$, respectively. (Note the commutation relation $[\hat{x}, \hat{p}] = i\hbar$ remains the same and is representation-independent.)

Find the energy eigenstates in the momentum space representation. Are the wave functions normalizable? Describe the spectrum of allowed energies.

3. **Uncertainty and harmonic oscillator** (20 points)

   Use the operator approach developed in lecture to prove that the n^{th} harmonic oscillator energy eigenstate obeys the following uncertainty relation:

$$\Delta x \Delta p = \frac{\hbar}{2}(2n+1).$$

4. **Properties of harmonic oscillator states** (20 points)

   (a) Show that the probability distribution of a particle in a harmonic oscillator potential returns to its original shape after the classical period $2\pi/\omega$. (You should prove this for *any* harmonic oscillator state, including non-stationary states.) What feature of the harmonic oscillator makes this true?

   (b) Use the raising operator to prove that the parity of harmonic oscillator states alternates between even and odd as n increases.

5. **WKB approximation.** (20 points)

   Use the WKB approximation to find the allowed energies E_n of an infinite square well with a "shelf" of height V_0, extending halfway across:

$$V(x) = \begin{cases} V_0 & 0 < x < L/2 \\ 0 & L/2 < x < L \\ \infty & \text{otherwise.} \end{cases}$$

   Express your answer in terms of V_0 and $E_n^0 \equiv (n\pi\hbar)^2/2mL^2$ (the n-th allowed energy for the "unperturbed" infinite square well, with no shelf). You may assume that $E_0^0 > V_0$, but do *not* assume that $E_n \gg V_0$.