MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.04 **Spring Term 2003**

PROBLEM SET 6

Reading: French & Taylor, Chapter 3. Liboff, Chapter 4 and 5.

1. **Practice with delta functions** (10 points). The Dirac delta function may be defined as

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0, \end{cases} \qquad \text{such that} \int_{-\infty}^{\infty} \delta(x) f(x) = f(0),$$

for any function $f(x)$. Evaluate the following integrals:

(a) $\int_{-3}^{1} (x^3 - 3x^2 + 2x - 1)\delta(x+2)dx$,

(b) $\int_{0}^{\infty} [\cos(3x) + 2]\delta(x - \pi)dx$,

(c) $\int_{-1}^{1} \exp(|x| + 3)\delta(x - 2)dx$.

2. **Ehrenfest's theorem** (15 points).

Using the time-dependent Schrödinger equation and starting with the integral expression derived in lecture for the expectation value of p,

$$\langle p \rangle = m\frac{d\langle x \rangle}{dt} = -i\hbar \int_{-\infty}^{\infty} \left(\Psi^* \frac{\partial \Psi}{\partial x} \right) dx,$$

show that

$$\frac{d\langle p \rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle.$$

This is known as *Ehrenfest's theorem.* It tells us that the quantum mechanical *expectation values* of momentum and potential energy obey Newton's second law.

3. **Properties of solutions to the time-independent Schrödinger equation.** (25 points).

Prove the following theorems regarding solutions to time-independent Schrödinger equation:

(a) In order for a "stationary state" solution to the Schrödinger equation,

$$\Psi(x, t) = \psi(x)e^{-iEt/\hbar},$$

to be normalizable, the eigenvalue E must be real. *Hint:* Write E as $E_0 + i\Gamma$ (with E_0 and Γ real), and show that for the normalization condition to be satisfied for all t, Γ must be zero.

(b) The spatial wave function $\psi(x)$ can always be taken to be *real* (unlike the total wave function $\Psi(x,t)$, which is necessarily complex). *Note:* This does not mean that every solution to the time-independent Schrödinger equation *is* real; instead, it says that if you have a solution that is *not* real, it can always be written as a linear combination of solutions (with the same energy) that *are* real.

(*Hint:* If $\psi(x)$ satisfies the time-independent Schrödinger equation for a given energy E, what can be said about its complex conjugate, $\psi^*(x)$? Find linear combinations of solutions that are necessarily real (by construction).)

(c) If $V(x)$ is an even function $[V(-x) = V(x)]$, then $\psi(x)$ can always be taken to be either even or odd.

(*Hint:* Find a linear combination of solutions that is explicitly even and one that is explicitly odd.)

4. **Minimum energy solutions to the time-indep. Schrödinger equation** (15 points).

(a) Show that E must exceed the minimum value of $V(x)$ for every normalizable solution to the time-independent Schrödinger equation. What is the classical analog to this statement? *Hint:* Rewrite the time-independent Schrödinger equation as

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi(x);$$

if $E < V_{min}$, then ψ and its second derivative always have the same sign. Argue that such a function cannot be normalized.

(b) Show explicitly that there is no acceptable solution to the time-independent Schrödinger equation for the infinite square well with $E = 0$ or $E < 0$. This is a special case of the general theorem in part (a), but this time do it by explicitly solving the Schrödinger equation and showing that you cannot meet the boundary conditions.

5. **Infinite square well revisited** (15 points).

Solve the time-independent Schrödinger equation with appropriate boundary conditions for an infinite square well of width a centered at $a/2$,

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{elsewhere.} \end{cases}$$

Check that the allowed energies are consistent with those derived in lecture for an infinite well of width a centered at the origin. Confirm that the wave functions $\psi_n(x)$ can be obtained from those found in lecture if one uses the substitution $x \to x + a/2$.

6. **Measurement and the infinite square well** (20 points).

Consider a particle in the infinite square well potential of the preceding problem. Suppose that at $t = 0$, the particle is described by the following wave function:

$$\Psi_A(x,0) = \sqrt{\frac{1}{6}}\psi_1(x) + \sqrt{\frac{1}{3}}\psi_2(x) + \sqrt{\frac{1}{2}}\psi_3(x)$$

where $\psi_1(x)$, $\psi_2(x)$, and $\psi_3(x)$ are the normalized eigenfunctions corresponding to the stationary states with energy eigenvalues E_1, $E_2(= 4E_1)$, and $E_3(= 9E_1)$, respectively. *Note:* Each part of this problem requires relatively little computation, but rather addresses the concepts covered so far.

(a) How does $\Psi_A(x, 0)$ evolve with time? That is, write down the expression for $\Psi_A(x, t)$.

(b) Calculate the expectation value of the energy, $\langle E\rangle$, for the particle described by $\Psi_A(x, t)$. Write your answer in terms of E_1. Does this quantity change with time?

(c) What is the probability of measuring the energy to equal $\langle E\rangle$ as a result of a single measurement at $t = 0$? At a later time $t = t_1$?

(d) What energy values will be observed as a result of a single measurement at $t = 0$ and with what probabilities? How do these probabilities change with time?

(e) The energy of the particle is found to be E_3 as a result of a single measurement at $t = t_1$. Write down the wave function $\Psi(x, t)$ which describes the state of the particle for $t > t_1$. What energy values will be observed and with what probabilities at a time $t_2 > t_1$?

(f) Construct another normalized wave function $\Psi_B(x, 0)$ which is linearly independent of $\Psi_A(x, 0)$, but yields the same value of $\langle E\rangle$ as well as the same set of measured energies with the same probabilities.

(g) Construct another normalized wave function $\Psi_C(x, 0)$ which is linearly independent of $\Psi_A(x, 0)$, yields the same value of $\langle E\rangle$, but gives a different set of measured energies (let's say that only two different energies are measured).

(h) How would you experimentally determine whether the particle is in a state described by Ψ_A, Ψ_B, or Ψ_C?