MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.04 **Spring Term 2003**

PROBLEM SET 7

Reading: French & Taylor, Chapter 4 and 8.

1. **"Sloshing" superposition state in the infinite potential well.** (20 points)

   Consider a particle of mass m that is in a superposition state of the first two eigenstates of an infinite potential well of width a,

   $$\Psi(x,t) = \sqrt{\frac{1}{a}} \sin\left(\frac{\pi x}{a}\right) e^{-i\omega_1 t} + \sqrt{\frac{1}{a}} \sin\left(\frac{2\pi x}{a}\right) e^{-i\omega_2 t}, \qquad (1)$$

   for $0 < x < a$.

   (a) Verify that the normalization factor $\sqrt{1/a}$ holds for all times $t > 0$.

   (b) Calculate and sketch the probability distribution $|\Psi|^2$ for $t = \pi\hbar/[2(E_2 - E_1)]$, that is, after one quarter-cycle repetition for this superposition.

   (c) Find the (time-dependent) probability for the particle to be located in the left half of the well.

   (d) Find the expectation value $\langle x(t) \rangle$ of the particle's position.

   (e) Show that the probability density $|\Psi|^2$ at $x = a/2$ is independent of time.

   (f) As seen above, the probability density at the center position $x = a/2$ is time-independent. However, the probability distribution for the particle "sloshes" back and forth between the left and right halves of the well. Briefly discuss how this occurs.

2. **Infinite well superposition state.** (20 points)

   A particle of mass m is initially in the ground eigenstate of a one-dimensional infinite potential well extending from $x = 0$ to $x = a/2$. Its spatial wave function, correctly normalized, is given by

   $$\psi(x) = \begin{cases} \frac{2}{\sqrt{a}} \sin\left(\frac{2\pi x}{a}\right) & 0 < x < a/2 \\ 0 & \text{elsewhere} \end{cases} . \qquad (2)$$

   Suddenly, at time $t = 0$, the right-hand wall of the well is moved to $x = a$. You may assume that the wave function remains the same immediately after the change (Ψ is continuous in time). However, note that the eigenstates of the new well are not the same as the eigenstates of the old well.

(a) What is the probability that the particle is in the second ($n = 2$) state of the new well, immediately after the change? (Note that the wavelength within the well, and hence the energy, for this state is the same as for the initial state in the old well.)

(b) What is the probability that the particle would be found in the ground state of the new well, immediately after the change?

3. **Free particle as a Gaussian wave packet.** (20 points)

A free particle of mass m has the initial wave function

$$\Psi(x,0) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} e^{-ax^2},$$

where a is a real, positive constant. Note that unlike the plane wave eigenfunctions of the free particle Hamiltonian, this Gaussian wave packet *is* normalizable.

(a) Compute the Fourier transform of $\Psi(x,0)$ and show that it equals:

$$\tilde{\psi}(k) = \left(\frac{1}{2a\pi}\right)^{\frac{1}{4}} e^{-\frac{k^2}{4a}}.$$

What is the most probable momentum of this particle? (*Hint:* You should complete the square in the exponent to perform the integration.)

(b) Use your answer from part (a) to show that the time-evolution is given by:

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{e^{-ax^2/[1+(2i\hbar at/m)]}}{\sqrt{1+(2i\hbar at/m)}}.$$

Hint: Again, use the trick of completing the square in the exponent.

(c) Calculate the probability distribution $|\Psi(x,t)|^2$. As seen in lecture in the context of wave packets, the probability distribution is a Gaussian that spreads-out over time.

4. **Qualitative bound state solutions** (20 points).

(1) French & Taylor, Problem 3-15

(a) Draw the wave functions associated with E_1, E_2, and E_3.

(b)-(e) Draw the ground state wave function only.

(f) Draw two wave functions for: small barrier height and infinite height.

(g) Draw one wave function for a narrow central barrier.

State in words what happens when the barrier grows to the full width.

(2) French & Taylor, Problem 3-16

5. **Odd-Parity solutions for the finite square well** (20 points).

Consider the finite square well potential,

$$V(x) = \begin{cases} -V_0 & |x| < a \\ 0 & |x| > a. \end{cases}$$

We examined the even-parity bound state solutions in lecture. Analyze the odd-parity bound state solutions. Derive the transcendental equation for the allowed energies, and solve it graphically. Examine the two limiting cases of (i) a wide, deep well (z_0 large) and a shallow, narrow well (z_0 small), where $z_0 = (a/\hbar)\sqrt{2mV_0}$. Is there always at least one odd bound state?