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**The Zames-Falb IQC for Critically Stable Systems**

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# The Zames-Falb IQC for Critically Stable Systems

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## Abstract

A feedback interconnection of neutrally stable linear time-invariant system and a nonlinearity with  $0 \leq x\varphi(x) \leq kx^2$  is called critical since the worst case linearization is at best neutrally stable. This makes the stability analysis of such systems particularly hard. It will be shown that an integrator and a sector bounded nonlinearity can be encapsulated in a bounded operator that satisfies several useful integral quadratic constraints. This gives powerful tools for stability analysis of critically stable systems.

Keywords: IQC, Stability, Unbounded Operators

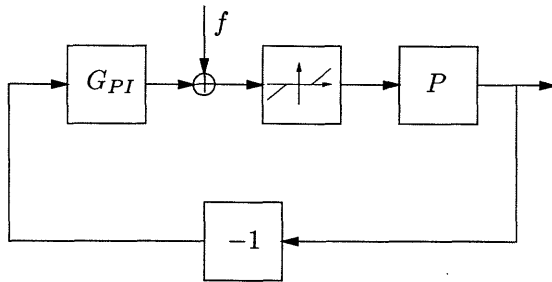


Figure 1: PI control with a deadzone actuator.

## 1 Introduction

Integral quadratic constraints (IQC) gives a unifying framework for problems in modern robust control. Modern robust control works with bounded operators and this allow us to consider  $L_2$ -gains and infinite dimensional state spaces. For example, delay operators can easily be considered, which is not the case when standard Lyapunov techniques are used to investigate asymptotic stability. However, the use of bounded operators implies loss of important cases when one or several operators in the system are unbounded. For example, hysteresis phenomena such as backlash defines operators that are unbounded on  $L_2$ . Another important case is the integrator in a PI controller, which is not  $L_2$ -bounded.

An important step was taken in [7, 5], where it was shown that it sometimes is possible to encapsulate an unbounded operator in an artificial feedback loop, which defines a bounded operator. Stability analysis can then be performed in the usual IQC framework, [6].

The purpose of this paper is to further expand the encapsulation technique to treat a general class of systems with integrators, nonlinearities, and possibly other perturbations.

Let us illustrate the idea with a simple example. The control system in Figure 1 consists of a stable plant that is regulated by a PI controller with transfer function

$$G_{PI}(s) = k_1 + \frac{k_2}{s}. \quad (1)$$

The actuator is assumed to be of deadzone type. The injected signal  $f$  can either be viewed as a disturbance or a signal that generates the initial conditions of the plant and the PI controller. Figure 2 shows the system in Figure 1 transformed to the standard form for robust control. The transfer function  $G = -G_{PI}P$  is unbounded due to the integrator pole at the origin.

In order to apply the usual IQC framework for stability analysis we need to somehow hide the unbounded part of  $G$ . A partial fraction expansion gives  $G(s) = k(G_0(s) - 1/s)$ , where  $k = -k_2P(0)$ . We can now transform the system

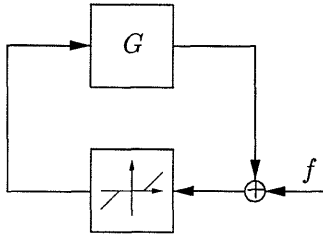


Figure 2: Transformation of the control system in Figure 1 into the standard form for robustness analysis. Note that  $G(s) = -G_{PI}(s)P(s)$  is unbounded.

as in Figure 3 where  $G_0$  is bounded, and

$$\varphi(x) = \begin{cases} k(x-1), & x > 1, \\ 0, & |x| \leq 1, \\ k(x+1), & x < -1. \end{cases}$$

The important point is that we have encapsulated the integrator in an operator defined by

$$w = \Delta_\varphi(v) \Leftrightarrow \begin{cases} \dot{z} = w, & z(0) = 0, \\ w = \varphi(v - z), \end{cases} \quad (2)$$

which is bounded if (and only if)  $k \geq 0$ , see the next section. Furthermore, we will derive several useful IQCs for the operator  $\Delta_\varphi$  that can be used for the stability analysis. Note that the method is not restricted to stability analysis of simple systems as in Figure 1. In fact, it is possible to consider systems consisting of encapsulated integrators together with various uncertainties and a nominal linear time invariant plant.

## 2 Preliminaries

We say that  $\varphi \in \text{sector}[0, k]$  if  $\varphi(0) = 0$  and if  $0 \leq \varphi(x)x \leq kx^2$ , for all  $x$ . The stronger assumption  $\varphi \in \text{slope}[0, k]$  means that  $\varphi(0) = 0$  and that the slope is restricted to the interval  $[0, k]$ , i.e.,

$$0 \leq \frac{\varphi(y_1) - \varphi(y_2)}{y_1 - y_2} \leq k, \quad \forall y_1 \neq y_2.$$

We will sometimes make the further assumption that  $\varphi$  is odd, i.e.,  $\varphi(-x) = -\varphi(x)$ .

We let  $\mathbf{L}_2^m[0, \infty)$  denote the vector space of square integrable  $\mathbf{R}^m$  valued

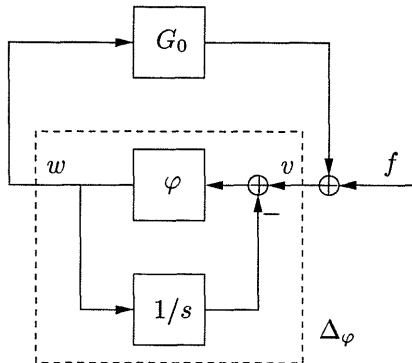


Figure 3: Encapsulation of the integrator.

functions. The inner product and norm on  $\mathbf{L}_2^m[0, \infty)$  are defined as

$$\langle f, g \rangle = \int_0^\infty f(t)^T g(t) dt,$$

$$\|f\| = \langle f, f \rangle^{1/2}.$$

The bi-infinite space  $\mathbf{L}_2^m(-\infty, \infty)$  is defined accordingly. The truncation operator  $P_T$  is defined by  $P_T f(t) = f(t)$  when  $t \leq T$  and  $P_T f(t) = 0$  when  $t > T$ . The extended space  $\mathbf{L}_{2e}^m[0, \infty)$  consists of all functions satisfying  $P_T f \in \mathbf{L}_2^m[0, \infty)$  for all  $T \geq 0$ .

An operator  $\Delta : \mathbf{L}_{2e}^l[0, \infty) \rightarrow \mathbf{L}_{2e}^m[0, \infty)$  is causal if  $P_T \Delta = P_T \Delta P_T$  for all  $T \geq 0$ . A causal operator is bounded if there exists  $c > 0$  such that  $\|\Delta(v)\| \leq c\|v\|$ , for all  $v \in \mathbf{L}_2^l[0, \infty)$ . The smallest such constant is called the gain of  $\Delta$ .

### IQC Theory

In this paper we denote time-invariant quadratic forms on  $\mathbf{L}_2^{l+m}[0, \infty)$  by  $\sigma$ . A bounded and causal operator  $\Delta : \mathbf{L}_{2e}^l[0, \infty) \rightarrow \mathbf{L}_{2e}^m[0, \infty)$  is said to satisfy the IQC defined by  $\sigma$  ( $\Delta \in \text{IQC}(\sigma)$ ) if

$$\sigma(v, \Delta(v)) \geq 0, \quad \forall v \in \mathbf{L}_2^l[0, \infty).$$

We can usually let the quadratic form be defined in terms of a bounded and self-adjoint operator  $\Pi$ , i.e.,

$$\sigma_\Pi(v, \Delta(v)) = \left\langle \begin{bmatrix} v \\ \Delta(v) \end{bmatrix}, \Pi \begin{bmatrix} v \\ \Delta(v) \end{bmatrix} \right\rangle \geq 0,$$

for all  $v \in \mathbf{L}_2^m[0, \infty)$ . We will not always write out  $\Pi$  explicitly in this paper.

We consider systems consisting of several perturbations  $\Delta_1, \dots, \Delta_N$  interconnected through linear transfer functions. We assume that we have parametrizations,  $\Delta_{\tau i}$ , of the perturbations that satisfy

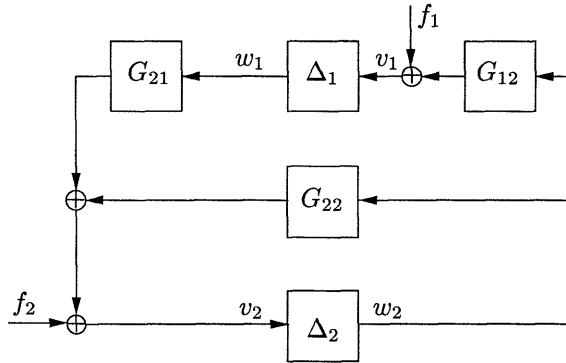


Figure 4: A system for IQC analysis.

- (i)  $\Delta_{1i} = \Delta_i$ ,
- (ii)  $\Delta_{\tau_i} : \mathbf{L}_{2e}^{l_i}[0, \infty) \rightarrow \mathbf{L}_{2e}^{m_i}[0, \infty)$  is bounded and causal for  $\tau \in [0, 1]$ ,
- (iii) there exists  $\gamma > 0$  such that

$$\|\Delta_{\tau_1\varphi}(v) - \Delta_{\tau_2\varphi}(v)\| \leq \gamma|\tau_1 - \tau_2| \cdot \|v\|,$$

for all  $v \in \mathbf{L}_2[0, \infty)$  and  $\tau_1, \tau_2 \in [0, 1]$ .

The parametrized system equation can be written<sup>1</sup>

$$\begin{aligned} w_i &= \Delta_{\tau_i}(v_i), \\ v_i &= \sum_{j=1}^N G_{ij}w_j + f_i, \end{aligned} \tag{3}$$

where  $G_{ij} \in \mathbf{RH}_{\infty}^{l_i \times m_j}$ , and  $f_i \in \mathbf{L}_{2e}^{l_i}[0, \infty)$ . Figure 4 gives a simple example.

**Definition 1.** Let  $v^T = (v_1^T, \dots, v_N^T)$  and define  $w$  and  $g$  similarly. The system in (3) is *well-posed* if it defines a causal map  $\mathbf{L}_2^l[0, \infty) \ni f \rightarrow (w, v) \in \mathbf{L}_{2e}^{l+m}[0, \infty)$ . Furthermore, the system is said to be  $\mathbf{L}_2$ -stable if there exists a positive constant  $c$  such that  $\|P_T w\| + \|P_T v\| \leq c\|P_T f\|$  for all  $T > 0$ , and  $f \in \mathbf{L}_{2e}^l[0, \infty)$ . We call  $c$  the gain of the system.

The main result of [6, 7] can be formulated as

**Theorem 1.** *Assume that the system in (3) is stable when  $\tau = 0$  and well posed for  $\tau \in [0, 1]$ . Under these conditions, if*

<sup>1</sup>Note that stability of the system in (3) also implies stability of

$$\begin{aligned} w_i &= \Delta_{\tau_i}(v_i) + g_i, \\ v_i &= \sum_{j=1}^N G_{ij}w_j + f_i, \end{aligned}$$

(see Definition 1 for a definition of stability). This follows since  $G_{ij}g_i \in \mathbf{L}_{2e}^{l_i}[0, \infty)$  whenever  $g_i \in \mathbf{L}_{2e}^{m_i}[0, \infty)$  and  $G_{ij}$  is bounded. We can thus include  $G_{ij}g_i$  in  $f_i$

(i) for  $\tau \in [0, 1]$ ,  $\Delta_{\tau i} \in IQC(\sigma_i)$ ,

(ii) there exists  $\varepsilon > 0$  such that

$$\sum_{i=1}^N \sigma_i \left( \sum_{j=1}^N G_{ij} w_j, w_i \right) \leq -\varepsilon \|w\|^2, \quad \forall w \in \mathbf{L}_2^m[0, \infty),$$

then the system (3) is  $\mathbf{L}_2$ -stable for all  $\tau \in [0, 1]$ .

*Remark 1.* The second condition can be formulated as a frequency domain condition. Assume  $\sigma_i = \sigma_{\Pi_i}$ , where  $\Pi_i \in \mathbf{RL}_{\infty}^{(l_i+m_i) \times (l_i+m_i)}$  and let

$$G_i = [G_{i1}, \dots, G_{iN}],$$

$$E_i = \begin{bmatrix} 0_{m_i \times \sum_{k=1}^{i-1} m_k} & I_{m_i} & 0_{m_i \times \sum_{k=i+1}^N m_k} \end{bmatrix}.$$

Then (ii) in Theorem 1 can be formulated as

$$\sum_{i=1}^N \begin{bmatrix} G_i(j\omega) \\ E_i \end{bmatrix}^* \Pi_i(j\omega) \begin{bmatrix} G_i(j\omega) \\ E_i \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty].$$

### 3 IQCs for the Encapsulation

We will in this section derive several useful IQCs for the encapsulation,  $\Delta_{\varphi}$ , defined in (2). However, we first prove the boundedness of  $\Delta_{\varphi}$ .

**Lemma 1.** *Let  $\varphi \in \text{sector}[0, k]$ , where  $k > 0$ . Then  $\Delta_{\varphi}$  is bounded on  $\mathbf{L}_2[0, \infty)$  with gain not greater than  $k$ .*

*Remark 2.* The proof of the lemma also shows that  $|z(T)| \leq \sqrt{2k} \|P_T v\|$  for all  $T \geq 0$ .

*Remark 3.* It follows from (4) in the proof that  $\Delta_{\varphi}$  satisfies the IQC defined by

$$\sigma_{\text{circ}}(v, w) = \langle v, w \rangle - \frac{1}{k} \|w\|^2$$

when  $k > 0$ . This IQC corresponds to a circle criterion.

*Proof.* Multiplying the differential equation in (2) by  $\dot{z}$  gives the inequality

$$\dot{z}^2 = \dot{z}\varphi(v - z) \leq k\dot{z}(v - z)$$

Integration gives

$$\begin{aligned} \int_0^T \dot{z}^2 dt &\leq k \int_0^T \dot{z}v dt + \frac{k}{2} (z(0)^2 - z(T)^2) \\ &\leq k \int_0^T \dot{z}v dt \\ &\leq k \sqrt{\int_0^T \dot{z}^2 dt} \sqrt{\int_0^T v^2 dt}, \end{aligned} \tag{4}$$

since  $z(0) = 0$ . We get  $\|w\| = \|\dot{z}\| \leq k\|v\|$ , as  $T \rightarrow \infty$ . This proves the lemma.  $\square$

If we can assume that the input  $v$  to the encapsulated operator is differentiable, then it is possible to derive Popov IQCs for  $\Delta_\varphi$ . The Popov IQCs corresponds to unbounded quadratic forms on  $\mathbf{L}_2$  (since differentiability of the input  $v$  is required). However, the standard IQC theory can easily be extended to treat this case, see [6, 3].

**Lemma 2.** *Assume that  $\varphi \in \text{sector}[0, k]$  is continuous. Then  $\Delta_\varphi$  satisfies the IQC defined by*

$$\sigma_{\text{Pop}}(v, w) = \lambda \langle w, \dot{v} - w \rangle,$$

where  $\lambda \geq 0$ . If  $\varphi \in \text{slope}[0, k]$  then we can take  $\lambda \in \mathbf{R}$ .

*Proof.* See the appendix.  $\square$

*Remark 4.* We obtain a useful IQC by combining the sector IQC in Remark 3 with the Popov IQC above. We get  $\sigma_\Pi = \sigma_{\text{circ}} + \sigma_{\text{Pop}}$ , where

$$\Pi(j\omega) = \begin{bmatrix} 0 & 1 - j\omega\lambda \\ 1 + j\omega\lambda & -\frac{2}{k} - 2\lambda \end{bmatrix}.$$

The next theorem, which is the main result of this paper, provides a set of IQCs for  $\Delta_\varphi$  that corresponds to Zames and Falb's IQC for slope restricted nonlinearities, [9].

**Theorem 2.** *Let  $\varphi \in \text{slope}[0, k]$ . Then  $\Delta_\varphi$  satisfies the IQC defined by*

$$\sigma_{\text{ZF}}(v, w) = \left\langle w, (I - H)(v - \frac{1}{k}w) \right\rangle + \langle w, Fw \rangle,$$

where

$$F(s) = \frac{H(s) - H(0)}{s},$$

and where  $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt$  for some  $h : \mathbf{R} \rightarrow \mathbf{R}$  with

(i)  $h(t) \geq 0$  for all  $t \in \mathbf{R}$ ,

(ii)  $\|h\|_1 = \int_{-\infty}^{\infty} |h(t)| dt \leq 1$ .

Furthermore, if  $\varphi$  is odd then we only need to impose the second constraint (ii).

*Proof.* See the appendix.  $\square$

*Remark 5.* Note that  $\sigma_{\text{ZF}} = \sigma_\Pi$ , where

$$\Pi(j\omega) = \begin{bmatrix} 0 & 1 - H(j\omega)^* \\ 1 - H(j\omega) & -\frac{2}{k} \text{Re}(1 - H(j\omega) - kF(j\omega)) \end{bmatrix}$$

This is a bounded operator since the singularity of  $F(s)$  is removable.



## 4 Parametrization of $\Delta_\varphi$

An important issue in the application of Theorem 1 is to find a suitable parametrization of  $\Delta_\varphi$ . The parametrization  $\Delta_{\tau\varphi} = \tau\Delta_\varphi$  can be used when  $\varphi \in \text{sector}[0, k]$ , i.e., when the IQC from Remark 4 is used with  $\lambda \geq 0$ . This does not work if we want to use the full power of the IQCs in Lemma 2 and Theorem 2 for the case when  $\varphi \in \text{slope}[0, k]$ . The reason is that the lower right corner of the corresponding  $\Pi$  may not be negative semidefinite and we have to impose the additional assumptions  $\lambda \geq 0$  and

$$1 - \text{Re}(H(j\omega) + kF(j\omega)) \geq 0, \quad \forall \omega$$

in order to satisfy condition (i) in Theorem 1. Indeed, this additional constraint on the IQCs would make the analysis more conservative.

The next theorem gives an alternative parametrization such that we avoid these additional constraints on the IQCs.

**Lemma 3.** *Assume that  $\varphi \in \text{slope}[0, k]$ , where  $k > 0$ . The parametrized operator defined by*

$$w = \Delta_{\tau\varphi}(v) \Leftrightarrow \begin{cases} \dot{z} = w, & z(0) = 0 \\ w = \tau\varphi(v - z) \end{cases} \quad (5)$$

*satisfies the following properties:*

- (i)  $\Delta_{0\varphi} = 0$  and  $\Delta_{1\varphi} = \Delta_\varphi$ ,
- (ii)  $\Delta_{\tau\varphi} : \mathbf{L}_{2e}[0, \infty) \rightarrow \mathbf{L}_{2e}[0, \infty)$  is bounded and causal for  $\tau \in [0, 1]$ ,
- (iii)  $\Delta_{\tau\varphi} \in \text{IQC}(\sigma_{ZF})$ , for  $\tau \in [0, 1]$ ,
- (iv)  $\Delta_{\tau\varphi} \in \text{IQC}(\sigma_{Pop})$ , for  $\tau \in [0, 1]$ ,  $\lambda \in \mathbf{R}$ ,
- (v) there exists  $\gamma > 0$  such that

$$\|\Delta_{\tau_1\varphi}(v) - \Delta_{\tau_2\varphi}(v)\| \leq \gamma|\tau_1 - \tau_2| \cdot \|v\| \quad (6)$$

for all  $v \in \mathbf{L}_2[0, \infty)$  and  $\tau_1, \tau_2 \in [0, 1]$ .

*Proof.* See the appendix. □

*Remark 6.* An interesting detail of the proof is that it is obtained by using a non-quadratic Lyapunov function. It appears that quadratic Lyapunov functions fail to prove the statement.

## 5 Applications

Let us consider the introductory example again, see Figure 1.

**Proposition 1.** *Let  $G(s) = -G_{PI}(s)P(s)$ , where  $P$  is assumed to be proper and stable and the PI controller is defined in (1). We assume that  $f \in \mathbf{L}_2[0, \infty)$ . If*

- (i)  $\lim_{s \rightarrow 0} sG(s) < 0$   
(ii) there exists  $\varepsilon > 0$  such that

$$\operatorname{Re}(1 - H(j\omega))(G(j\omega) + 1) \leq -\varepsilon, \quad \forall \omega \neq 0$$

for some  $h : \mathbf{R} \rightarrow \mathbf{R}$  with  $\|h\|_1 \leq 1$ ,

then the system in Figure 1 is stable in the sense that the state vector,  $x$ , corresponding to the plant  $P$ , and the integrator state<sup>2</sup>,  $x_I$ , satisfy

- a.  $x, \dot{x} \in \mathbf{L}_2^n[0, \infty)$ ,  
b.  $\dot{x}_I \in \mathbf{L}_2[0, \infty)$  and

$$\int_0^\infty D^2(x_I, [-1, 1]) dt < \infty,$$

where  $D(\cdot, [-1, 1])$  is the minimum distance function.

*Remark 7.* The stability conclusion implies that  $x(t) \rightarrow 0$  and  $x_I(t) \rightarrow [-1, 1]$  as  $t \rightarrow \infty$ .

*Remark 8.* Note that the injected signal  $f$  can be used to generate any initial condition for a state space representation of  $G = -G_{PI}P$ . Indeed, consider any controllable state space realization  $G(s) = C(sI - A)^{-1}B$ , where  $A$  has a simple eigenvalue at the origin and all other eigenvalues are strictly in the left half plane. The corresponding state space representation of the system in Figure 1 becomes

$$\begin{aligned} \dot{z} &= Az + Bu, \quad z(0) = 0, \\ u &= \varphi(Cz + f), \end{aligned}$$

where  $\varphi$  represents the deadzone nonlinearity. Then

$$f(t) = \begin{cases} -Cz(t) + \operatorname{sign}(u(t)) + u(t), & t \in [0, t_0] \\ 0, & t > t_0 \end{cases}$$

where

$$\begin{aligned} u(t) &= B^T e^{A^T(t_0-t)} W(t_0)^{-1} z_0, \\ W(t_0) &= \int_0^{t_0} e^{A(t_0-t)} B B^T e^{A^T(t_0-t)} dt, \end{aligned}$$

can be used to generate the initial condition  $z(t_0) = z_0$ , for any  $t_0 > 0$ . This follows from standard controllability arguments, see for example [1].

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<sup>2</sup>We assume that the integrator is implemented as  $I(t) = \int_0^t k_2 eds$ , where  $I$  denotes the integral part of the controller output.

*Proof of Proposition 1.* We transform the system as in Figure 3. We have  $G_0(s) = \frac{1}{k}G(s) + \frac{1}{s}$  and the encapsulation  $\Delta_\varphi$  satisfies the IQC in Lemma 2. Let us apply Theorem 1 with the parametrization  $\Delta_{\tau\varphi}$  in (5). It follows from Lemma 3 that we only need to verify condition (ii) in Theorem 1. We have

$$\begin{aligned} \frac{k}{2} \begin{bmatrix} G_0(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G_0(j\omega) \\ I \end{bmatrix} &= \text{Re}(kG_0(j\omega) - 1)(1 - H(j\omega)) \\ &\quad - \text{Re} \frac{k}{j\omega} (1 - H(j\omega)) \\ &= \text{Re}(G(j\omega) - 1)(1 - H(j\omega)) \leq -\varepsilon, \end{aligned}$$

for all  $\omega \neq 0$ . We can thus conclude that the system is  $\mathbf{L}_2$  stable. The states corresponding to the plant  $P$  are included in  $G_0$  and stability conclusion *a* follows. For stability conclusion *b* we notice that the state,  $z$ , in  $\Delta_\varphi$  corresponds to the integrator state  $x_I$ . Stability of the system in Figure 3 implies that  $\varphi(v-z)$  is in  $\mathbf{L}_2[0, \infty)$ . Hence,  $\int_0^\infty D^2(v-z, [-1, 1])dt < \infty$ . The conclusion follows since

$$\int_0^\infty D^2(z, [-1, 1])dt \leq 2 \int_0^\infty D^2(v-z, [-1, 1])dt + 2\|v\|^2 < \infty.$$

□

The encapsulation technique is also useful in the analysis of more complex systems. We illustrate with a slight extension of the introductory example.

**Example** Consider the system with a PI controller and a deadzone actuator in Figure 1 for the case when there is an uncertain time delay in the system. We assume that

$$\begin{aligned} P(s) &= P_0(s)e^{-sT}, \quad T \in [0, T_0], \\ G_{PI}(s) &= K_1 + K_2/s, \end{aligned}$$

where  $P_0(s)$  is stable and proper. We are interested in finding a bound on the maximal time delay  $T_0$  such that stability for the closed loop system is ensured.

Straightforward manipulations give

$$-G_{PI}P = -G_{PI}P_0(1 + e^{-sT} - 1) = K(G_{22} - 1/s) + G_{21}\Delta_T G_{12} \quad (7)$$

where

$$\begin{aligned} K &= -K_2 P_0(0), & G_{22} &= -\frac{1}{K} G_{PI} P_0 + \frac{1}{s} \\ G_{21} &= -\frac{K_1 s + K_2}{K(s+a)}, & G_{12} &= (s+a) P_0 \\ \Delta_T(s) &= \frac{e^{-sT} - 1}{s}, & a &> 0, \end{aligned}$$

where we note that  $\Delta_T \in \mathbf{H}_\infty$ , i.e., it is a bounded operator. We encapsulate the integrator in (7) with the deadzone nonlinearity exactly as in the introductory example. The resulting system can be represented as in Figure 4 with  $\Delta_1 = \Delta_T$ ,  $\Delta_2 = \Delta_\varphi$ ,  $f_1 = 0$ , and  $f_2 = f$ . We need to find suitable IQCs for  $\Delta_1$  and  $\Delta_2$  in order to apply Theorem 1. By Lemma 2 we have  $\Delta_2 \in \text{IQC}(\Pi_2)$ , where  $\Pi_2$  is the matrix in Remark 5. An IQC for  $\Delta_T$  can be obtained by using an idea in [6]. Let

$$\Psi_{T_0}(\omega) = \max_{T \in [0, T_0]} \left| \frac{e^{-j\omega T} - 1}{j\omega} \right|^2 = \begin{cases} 4 \sin^2(\omega T_0/2)/\omega^2, & |\omega| \leq \pi/T_0 \\ 4/\omega^2, & |\omega| > \pi/T_0 \end{cases}$$

Then we have  $\Delta_1 = \Delta_T \in \text{IQC}(\Pi_1)$ , where

$$\Pi_1(j\omega) = x(j\omega) \begin{bmatrix} \Psi_{T_0}(\omega) & 0 \\ 0 & -1 \end{bmatrix},$$

and where  $x(j\omega) = \overline{x(j\omega)} \geq 0$ . For numerical computations we often want  $\Pi_1$  and  $\Pi_2$  to be rational. We can use

$$\widehat{\Psi}_{T_0}(\omega) = T_0^2 \frac{1 + 0.08(\omega T_0)^2}{1 + 0.13(\omega T_0)^2 + 0.02(\omega T_0)^4} \quad (8)$$

as an rational upper bound of  $\Psi_{T_0}$ . It follows from Remark 1 after Theorem 1 that the system is stable if

$$\begin{bmatrix} 0 & G_{12} \\ I & 0 \end{bmatrix}^* \Pi_1 \begin{bmatrix} 0 & G_{12} \\ I & 0 \end{bmatrix} + \begin{bmatrix} G_{21} & G_{22} \\ 0 & I \end{bmatrix}^* \Pi_2 \begin{bmatrix} G_{21} & G_{22} \\ 0 & I \end{bmatrix} < 0, \quad (9)$$

for all  $\omega \in [0, \infty]$ .

Let us consider the case when  $K_1 = 3$ ,  $K_2 = 0.3$ , and

$$P_0(s) = \frac{1}{s^2 + s + 1}.$$

With  $a = 1$  we get

$$G_{12}(s) = \frac{s+1}{s^2+s+1}, \quad G_{21}(s) = \frac{10s+1}{s+1}, \quad \text{and} \quad G_{22}(s) = \frac{s-9}{s^2+s+1}.$$

The stability criterion in (9) is satisfied when  $T_0 = 0.22$ ,  $X = 20$ , and  $H(s) = 1/(s+1)$ , i.e.,

$$\Pi_1(j\omega) = 20 \begin{bmatrix} \widehat{\Psi}_{0.22}(\omega) & 0 \\ 0 & -1 \end{bmatrix},$$

$$\Pi_2(j\omega) = \begin{bmatrix} 0 & \frac{j\omega}{j\omega - 1} \\ \frac{j\omega}{j\omega + 1} & -2 \frac{10\omega^2/3 + 1}{\omega^2 + 1} \end{bmatrix}.$$

Figure 5 shows the Nyquist curves for the open loop system  $G_{PI}P_0e^{-sT_0} = (3s+0.3)/(s^3+s^2+s)e^{-sT_0}$ , when  $T_0$  is 0, 0.10, and 0.22. It can be shown that the maximum allowable time delay in the linear case is  $T_0 = 0.35$ . We see that even very simple multipliers give a reasonable bound on  $T_0$ .

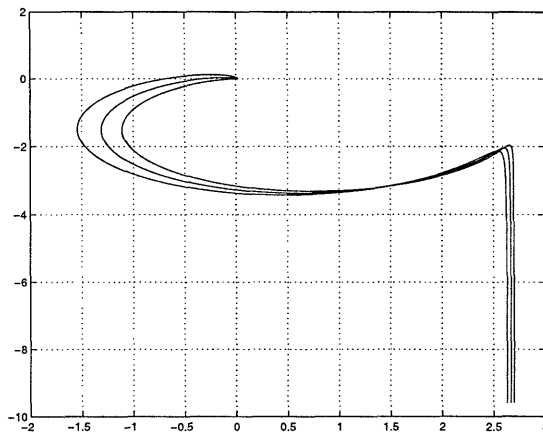


Figure 5: Nyquist plots of the open loop system  $G_{PI}P_0e^{-sT_0} = (3s + 0.3)/(s^3 + s^2 + s)e^{-sT_0}$ , when  $T_0$  is 0, 0.10, and 0.22.

## 6 Conclusions

We have obtained useful tools for stability analysis of critically stable systems. Our results are more general and less conservative than previous approaches based on state space techniques, see e.g., [8, 4].

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## Appendix

*Proof of Lemma 2:* Let  $y = v - z$ . We have  $\dot{y} \in \mathbf{L}_2[0, \infty)$  since  $\dot{v} \in \mathbf{L}_2[0, \infty)$  by assumption. We will for simplicity assume that  $v(0) = 0$ . This can be achieved by transforming the system as in Corollary 4 in [6]. The case when  $v(0) \neq 0$  can be treated along the lines of [3].

It follows from the boundedness of  $\Delta_\varphi$  that  $\varphi(y) \in \mathbf{L}_2[0, \infty)$ , so by the assumed continuity of  $\varphi$ , we have  $\lim_{t \rightarrow \infty} \varphi(y(t)) = 0$ . Let

$$\Psi(y) = \int_0^y \varphi(s) ds$$

It is clear that  $\Psi(y) \geq 0$ . Hence, if  $\lambda \geq 0$  we have

$$\lambda \int_0^\infty \varphi(y) \dot{y} dt = \lambda \Psi(y(\infty)) \geq 0$$

since  $y(0) = 0$ . Hence, with  $y = v - z$ ,  $w = \dot{z} = \varphi(y)$ , we get

$$\sigma_{\text{Pop}}(v, w) = \lambda \langle w, \dot{v} - w \rangle \geq 0,$$

which proves the claim. If  $\varphi \in \text{slope}[0, k]$  then  $\lim_{t \rightarrow \infty} \Psi(y(t)) = 0$  and the IQC also holds when  $\lambda \leq 0$ .

*Proof of Theorem 2:* The next lemma is a key ingredient in the proof. We will assume that  $\mathbf{L}_2[0, \infty) \subset \mathbf{L}_2(-\infty, \infty)$  is defined such that  $f \in \mathbf{L}_2[0, \infty)$  has  $f(t) = 0$  when  $t < 0$ . Similarly, any  $f \in \mathbf{L}_{2e}[0, \infty)$  is extended to be defined on  $\mathbf{R}$  with  $f(t) = 0$  when  $t < 0$ .

**Lemma 4.** *Assume that  $x, y \in \mathbf{L}_{2e}[0, \infty)$  are a monotonic pair in the sense that for any  $t_1, t_2 \in \mathbf{R}$ , we have the implication  $x(t_1) \leq x(t_2) \Rightarrow y(t_1) \leq y(t_2)$ , then*

$$\int_{-\infty}^T x(t)y(t)dt \geq \int_{-\infty}^T x(t-\tau)y(t)dt,$$

for all  $T \geq 0$  and for all  $\tau \in \mathbf{R}$ .

*Proof.* This follows from Hardy, Littlewood, and Polya's rearrangement inequality, [2].  $\square$

Consider the equations that defines  $\Delta_\varphi$ :

$$\begin{cases} \dot{z} = w, & z(0) = 0, \\ w = \varphi(v - z), \end{cases}$$

If  $v \in \mathbf{L}_2[0, \infty)$ , then it follows from Lemma 1 that also  $w \in \mathbf{L}_2[0, \infty)$ . Let  $y = v - z$ . It follows from the slope condition of  $\varphi$  that  $w$  and  $y - \frac{1}{k}w$  satisfy the monotonicity condition in Lemma 4 and thus

$$\int_{-\infty}^T w(t)(y(t) - \frac{1}{k}w(t))dt \geq \int_{-\infty}^T w(t-\tau)(y(t) - \frac{1}{k}w(t))dt,$$

for all  $T \geq 0$  and all  $\tau \in \mathbf{R}$ .

We can get an additional inequality if  $\varphi$  is odd. For fixed,  $\tau$ , let  $\theta$  be defined such that  $\theta(t)\theta(t-\tau) = -1$  and  $\theta(t)^2 = 1, \forall t$ . Then if  $\tilde{y} = \theta(t)y(t)$ , we have  $\text{sign}(\tilde{y}(t)\tilde{y}(t-\tau)) = -\text{sign}(y(t)y(t-\tau)), \forall t$ . Using that  $\tilde{w}(t) = \varphi(\tilde{y}(t)) =$

$\theta(t)\varphi(y(t))$  gives

$$\begin{aligned} \int_{-\infty}^T w(t)(y(t) - \frac{1}{k}w(t))dt &= \int_{-\infty}^T \tilde{w}(t)(\tilde{y}(t) - \frac{1}{k}\tilde{w}(t)) \\ &\geq \int_{-\infty}^T \tilde{w}(t - \tau)(\tilde{y}(t) - \frac{1}{k}\tilde{w}(t))dt \\ &= - \int_{-\infty}^T w(t - \tau)(y(t) - \frac{1}{k}w(t))dt. \end{aligned}$$

Using these inequalities with  $y = v - z$  gives

$$\begin{aligned} \int_{-\infty}^T w(t)(v(t) - \frac{1}{k}w(t))dt &\geq \pm \int_{-\infty}^T w(t - \tau)(v(t) - \frac{1}{k}w(t))dt \\ &\quad + \int_{-\infty}^T z(t)(w(t) \mp w(t - \tau))dt, \quad (10) \end{aligned}$$

where the inequality with the ‘‘upper signs’’ is valid for all  $\varphi \in \text{slope}[0, k]$  and the other inequality holds if  $\varphi$  in addition is odd.

The last term in (10) causes some worries since it contains  $z(t)$ , which may not be in  $\mathbf{L}_2[0, \infty)$ . However, we will next see that a partial integration of the last term gives benign terms.

Let  $u(t) = \int_{-\infty}^t [w(s) - w(s - \tau)]ds = Fw$ , where

$$F(s) = \frac{1 - e^{-s\tau}}{s}$$

is a bounded operator on  $\mathbf{L}_2(-\infty, \infty)$ . This means that  $u \in \mathbf{L}_2(-\infty, \infty)$  and since also  $\dot{u} \in \mathbf{L}_2(-\infty, \infty)$  we have  $u(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Partial integration gives

$$\int_{-\infty}^T z \dot{u} dt = z(T)u(T) - z(-\infty)u(-\infty) - \int_{-\infty}^T z u dt.$$

If we use that

1.  $u(T) \rightarrow 0$ , as  $T \rightarrow \infty$ , and  $z(T)$  is bounded,
2.  $z(t) = 0$  for  $t \leq 0$  and  $u(t) = 0$  for  $t \leq \tau$ ,

then the above inequality becomes

$$\int_{-\infty}^{\infty} z \dot{u} dt = - \int_{-\infty}^{\infty} z u dt,$$

as  $T \rightarrow \infty$ .

Similarly,

$$\int_{-\infty}^{\infty} z(t)[w(t) + w(t - \tau)]dt = 2 \int_{-\infty}^{\infty} z \dot{z} dt - \int_{-\infty}^{\infty} z u dt \geq \int_{-\infty}^{\infty} w u dt.$$

Using this in (10) gives

$$\int_{-\infty}^{\infty} (w(t) \mp w(t - \tau))(v(t) - \frac{1}{k}w(t))dt \pm \int_{-\infty}^{\infty} w(t)u(t)dt \geq 0, \quad (11)$$

for all  $\tau \in \mathbf{R}$ .

Let us first consider the general case when  $\varphi$  is not necessarily odd. Multiplying the inequality in (10) with “upper signs” with  $h(-\tau)$  and integrating with respect to  $\tau$  gives

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(-\tau)w(t)(v(t) - \frac{1}{k}w(t))dtd\tau - \\ &\quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(-\tau)(w(t - \tau)(v(t) - \frac{1}{k}w(t))dtd\tau + \\ &\quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(-\tau)w(t)u(t)dtd\tau \\ &= \left\langle (\|h\|_1 - H^*)w, v - \frac{1}{k}w \right\rangle + \langle F^*w, w \rangle \\ &\leq \left\langle w, (I - H)(v - \frac{1}{k}w) \right\rangle + \langle w, Fw \rangle, \end{aligned}$$

where the last inequality follows from the observation in Remark 3 and since  $\|h\|_1 \leq 1$ .

Finally, for the case when  $\varphi$  is odd we multiply the inequalities in (11) with  $|h(-\tau)|$ . The terms with arbitrary sign for given  $\tau$  can be multiplied by  $h(-\tau) = |h(-\tau)|\text{sign}(h(-\tau))$ . Integration with respect to  $\tau$  gives (where  $y = v - \frac{1}{k}w$ )

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(-\tau)|w(t)y(t)dtd\tau - \\ &\quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(-\tau)|\text{sign}(h(-\tau))(w(t - \tau)y(t))dtd\tau + \\ &\quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(-\tau)|\text{sign}(h(-\tau))w(t)u(t)dtd\tau \\ &\leq \left\langle (\|h\|_1 I - H^*)w, v - \frac{1}{k}w \right\rangle + \langle F^*w, w \rangle \\ &\leq \left\langle w, (I - H)(v - \frac{1}{k}w) \right\rangle + \langle w, Fw \rangle. \end{aligned}$$

*Proof of Theorem 3:* It is clear that  $\Delta_{0\varphi} = 0$  and  $\Delta_{1\varphi} = \Delta_\varphi$ . Causality of  $\Delta_{\tau\varphi}$  is obvious and boundedness follows since  $\tau\varphi \in \text{sector}[0, \tau k]$ , which by Lemma 1 implies that  $\|\Delta_{\tau\varphi}\| \leq \tau k$ . Condition (iii) and (iv) follows from the same argument.

The proof of condition (v) relies on the following lemma.



**Lemma 5.** *Consider*

$$\dot{y}(t) = -k(t)y(t) + \dot{f}(t), \quad y(0) = f(0) = 0,$$

where  $k(t) \in [0, 1]$ . Then there exists  $c > 0$  such that

$$\int_0^T |ky|^2 dt \leq c \int_0^T |f\dot{f}| dt. \quad (12)$$

*Proof.* We will construct a continuous function  $V(t)$  satisfying  $V(t) \geq 0$ ,  $\forall t \geq 0$ ,  $V(0) = 0$ , and

$$\frac{dV}{dt} \leq c_1 |f\dot{f}| - c_2 |ky|^2, \quad \forall t \geq 0, \quad (13)$$

for some positive constants  $c_1$  and  $c_2$ . Integration of (13) gives

$$c_2 \int_0^T |ky|^2 dt \leq c_1 \int_0^T |f\dot{f}| dt + V(0) - V(t) \leq c_1 \int_0^T |f\dot{f}| dt,$$

since  $V(0) = 0$  and  $V(t) \geq 0$ , which proves the lemma.

It remains to construct  $V(t)$  with the stated properties. Let us partition the  $(y, f)$  plane into the regions

$$\begin{aligned} R_1 &= \{(y, f) : y \geq 0, f \leq y/4\} \cup \{(y, f) : y < 0, f \geq y/4\}, \\ R_2 &= \{(y, f) : y \geq 0, f > y/4\} \cup \{(y, f) : y < 0, f < y/4\}, \end{aligned}$$

see Figure 6. Then define

$$V(t) = \begin{cases} (y(t) - f(t))^2/2, & (y(t), f(t)) \in R_1 \\ (0.5y(t) + f(t))^2/2, & (y(t), f(t)) \in R_2 \end{cases} \quad (14)$$

It is clear that  $V(t)$  is continuous,  $V(t) \geq 0$ , and finally that  $V(0) = 0$ , since it assumed that  $y(0) = f(0) = 0$ . It remains to prove (13). In region  $R_1$  we need

$$\dot{V} = -(y - f)ky \leq c_1 |f\dot{f}| - c_2 |ky|^2$$

for all  $(y, f) \in R_1$  and  $0 \leq k \leq 1$ . We assume  $c_2 > 0$ , so it follows by convexity that we only need to verify the inequality for  $k = 0$  and  $k = 1$ . The case  $k = 0$  is trivial and for  $k = 1$  we get the constraint  $(c_2 - 1)y^2 + fy \leq c_1 |f\dot{f}|$ . This constraint holds if  $c_2 \leq 3/4$  since  $fy \leq y^2/4$  in  $R_1$ .

In region  $R_2$  we need

$$\dot{V} = (0.5y + f)(-0.5ky + 1.5\dot{f}) \leq c_1 |f\dot{f}| - c_2 |ky|^2 \quad (15)$$

for all  $(y, f) \in R_2$  and  $0 \leq k \leq 1$ . Convexity in  $k$  implies, that we only need to verify the cases  $k = 0$  and  $k = 1$ .

Consider the case  $k = 0$ . The left hand side of (15) can be bounded above by  $1.5(0.5|y| + |f|)|\dot{f}| \leq 4.5|f\dot{f}|$ , since  $|y| \leq 4|f|$  in  $R_2$ . Hence, if  $c_1 \geq 4.5$ , then (15) holds for the case  $k = 0$ .

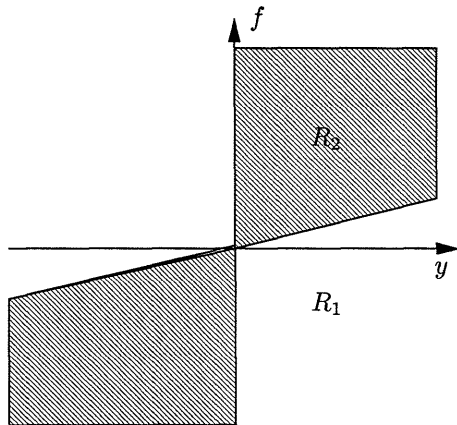


Figure 6: Regions for defining  $V(t)$ .

For the case  $k = 1$  we have

$$-\frac{1}{2}(0.5y + f)y + \underbrace{\frac{3}{2}(0.5y + f)f - c_1|ff|}_{\leq 0 \text{ if } c_1 \geq 4.5} \leq -c_2|y|^2.$$

This inequality holds if  $c_2 \geq 3/8$  and  $c_1 \geq 4.5$ , since  $fy \geq y^2/4$  in  $R_2$ .

We have thus proved that  $V$  in (14) satisfies the inequality in (13) if  $c_1 \geq 4.5$  and  $3/8 \leq c_2 \leq 3/4$ .  $\square$

We will now prove (v). Let

$$\begin{aligned} \dot{z}_1 &= \tau_1 \varphi(v - z_1), & z_1(0) &= 0, \\ \dot{z}_2 &= \tau_2 \varphi(v - z_2), & z_2(0) &= 0, \end{aligned}$$

and consider the difference  $\dot{\delta} = \dot{z}_1 - \dot{z}_2$ . We need to prove that  $\|\dot{\delta}\| \leq \gamma|\tau_1 - \tau_2| \cdot \|v\|$ , for some  $\gamma > 0$ . We have

$$\begin{aligned} \dot{\delta} &= \tau_1(\varphi(v - z_1) - \varphi(v - z_2)) + (\tau_1 - \tau_2)\varphi(v - z_2) \\ &= -\tilde{k}(t)\delta(t) + \frac{\tau_1 - \tau_2}{\tau_2}\dot{z}_2, \end{aligned} \tag{16}$$

where  $\tilde{k}(t) \in [0, \tau_1 k]$ . The first term in the last equality follows from the slope condition,  $\varphi \in \text{slope}[0, k]$ . The case when  $\tau_1 = 0$  is trivial since then  $\|\dot{\delta}\| \leq \tau_2 k \|\cdot\| \|v\|$ . It is thus no restriction to assume that  $0 < \tau_1 < \tau_2 < 1$ .

If we change time scale so that  $t \rightarrow s = \tau_1 kt$ , and define

$$y(s) = \delta(t), \quad f(s) = \frac{\tau_1 - \tau_2}{\tau_2} z_2(t), \quad \hat{k}(s) = \frac{1}{k\tau_1} \tilde{k}(t),$$

then (16) becomes  $\frac{dy}{ds} = \hat{k}(s)y(s) + \frac{df}{ds}$ , where  $k(s) \in [0, 1]$ , and  $y(0) = f(0) = 0$ . An application of Lemma 5 shows that there exists  $c > 0$  such that (12) holds in the new time scale. Transformation back to the original time scale gives

$$\int_0^T |\tilde{k}\delta|^2 dt \leq c \left( \frac{\tau_2 - \tau_1}{\tau_2} \right)^2 k\tau_1 \int_0^T |z_2 \dot{z}_2| dt,$$

for all  $T > 0$ . The only essential remaining step of the proof is to show that the integral on the right hand side can be bounded by  $\|v\|^2$ . To do this we first observe that

$$\dot{z}_2^2 = \tau_2 \varphi(v - z_2) \dot{z}_2 \leq \tau_2 k(v - z_2) \dot{z}_2,$$

and thus,

$$z_2 \dot{z}_2 \leq v \dot{z}_2 - \frac{1}{\tau_2 k} \dot{z}_2^2 \leq \frac{\tau_2 k}{4} v^2. \quad (17)$$

We define the positive and negative parts of  $z_2 \dot{z}_2$  as

$$(z_2 \dot{z}_2)_\pm(t) = \begin{cases} \pm z_2 \dot{z}_2(t), & \pm z_2 \dot{z}_2(t) \geq 0, \\ 0, & \pm z_2 \dot{z}_2(t) < 0 \end{cases}$$

From (17) we get  $(z_2 \dot{z}_2)_+ \leq \frac{\tau_2 k}{4} v^2$ , which implies that  $(z_2 \dot{z}_2)_+$  is integrable with  $\int_0^\infty (z_2 \dot{z}_2)_+ dt \leq \frac{\tau_2 k}{4} \|v\|^2$ . The relation

$$\int_0^T (z_2 \dot{z}_2)_+ dt - \int_0^T (z_2 \dot{z}_2)_- dt = \int_0^T z_2 \dot{z}_2 dt = \frac{1}{2} (z_2(T)^2 - z_2(0)^2) \geq 0,$$

shows that also  $\int_0^\infty (z_2 \dot{z}_2)_- dt \leq \frac{\tau_2 k}{4} \|v\|^2$ , and thus  $\int_0^\infty |z_2 \dot{z}_2| dt \leq \frac{\tau_2 k}{2} \|v\|^2$ . Hence,

$$\begin{aligned} \|\delta\| &= \left\| -\tilde{k}\delta + \frac{\tau_1 - \tau_2}{\tau_2} \dot{z}_2 \right\| \\ &\leq \left( \frac{c^{1/2}}{2} \sqrt{\frac{\tau_1}{\tau_2}} + 1 \right) k |\tau_1 - \tau_2| \cdot \|v\| \leq \gamma |\tau_1 - \tau_2| \cdot \|v\|, \end{aligned}$$

where  $\gamma = (c^{1/2}/2 + 1)k$ . This concludes the proof.

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