

# $\mathcal{H}^\infty$ and $\mathcal{H}^2$ Optimal Controllers for Periodic and Multi-Rate Systems\*

Petros G. Voulgaris

Munther A. Dahleh

Lena S. Valavani

Massachusetts Institute of Technology

Laboratory of Information and Decision Systems

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## Abstract

In this paper we present the solutions to the optimal  $\ell^2$  to  $\ell^2$  disturbance rejection problem ( $\mathcal{H}^\infty$ ) as well as to the LQG ( $\mathcal{H}^2$ ) problem in periodic systems using the lifting technique. Both problems involve a causality condition on the optimal LTI compensator when viewed in the lifted domain. The  $\mathcal{H}^\infty$  problem is solved using the Nehari's theorem whereas in the  $\mathcal{H}^2$  problem the solution is obtained using the Projection theorem. Exactly the same methods of solution can be applied in the case of multi-rate sampled data systems.

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# 1 Introduction

The study of periodically time varying systems is a topic of growing research. In [7] an equivalence between  $m$ -input,  $p$ -output, linear,  $N$ -periodic, causal, discrete systems and a class of discrete linear, time invariant, causal systems was established. Namely, this class consists of  $mN$ -input,  $pN$ -output, linear time invariant (LTI) systems with  $\lambda$ - transforms  $\hat{P}(\lambda)$  such that  $\hat{P}(0)$  is a block lower triangular matrix. This equivalence is strong in the sense that it preserves the algebraic structure (isomorphism) and the norm (isometry). This equivalence is termed “lifting” and the LTI system that lifting associates with the  $N$ -periodic system is called the “lifted” system. Hence, we can effectively use the theory of LTI systems to study periodic ones. In fact, the authors in [7] use this equivalence to prove that although the performance is not improved, periodic compensators for LTI plants offer significant advantages in terms of robustness to parametric uncertainty. Moreover, they argue that the optimal in  $\ell^2$  to  $\ell^2$  sense compensator for a  $N$ -periodic system is  $N$ -periodic. Indeed, as it is proved in [1] the above argument is true also in the worst case  $\ell^\infty$  to  $\ell^\infty$  sense. Hence, it can be easily inferred that the optimal controller for the  $N$ -periodic system can be obtained by solving the equivalent LTI problem. This problem however, includes a constraint on the optimal LTI compensator  $\hat{C}(\lambda)$ , namely  $\hat{C}(0)$  should be block lower triangular matrix so that  $C$  corresponds to a causal  $N$ -periodic controller.

The above problem was solved in [2, 3] in an optimal  $\ell^\infty$  to  $\ell^\infty$  sense. In this paper we present the solution to the same problem in an optimal  $\ell^2$  to  $\ell^2$  sense. The solution to this  $\mathcal{H}^\infty$  problem, consists of modifying the standard Nehari’s approach [6] in order to account for the additional causality constraint on the compensator. This modification yields to a finite dimensional convex optimization problem over a convex set that needs to be solved before applying the standard solution to the Nehari problem. The solution to the above convex finite dimensional problem can be obtained easily using standard programming techniques. Once this is done, we obtain the optimal LTI controller, and hence the optimal  $N$ -periodic controller for the original  $N$ -periodic system, by solving a standard Nehari’s problem.

Also in this paper, we encounter the problem of minimizing the  $\mathcal{H}^2$  norm of the system in the lifted domain by using a compensator  $\hat{C}(\lambda)$  with  $\hat{C}(0)$  a block lower triangular matrix. This corresponds to solving the LQG problem for the original periodic system. The solution to the latter problem is known and is obtained by solving a pair of periodic Riccati equations. The solution we present here for the  $\mathcal{H}^2$  problem in the lifted domain consists an alternative way of solving the LQG problem for periodic systems. This alternative solution involves only LTI systems and is obtained

by utilizing the Projection theorem in Hilbert spaces.

Finally, we demonstrate that exactly the same method of solution to the  $\mathcal{H}^\infty$  and  $\mathcal{H}^2$  problems in periodic systems applies when considering the same problems in multirate sampled data systems.

The paper is organized as follows. In section 2 we present some mathematical preliminaries together with some background on periodic systems. In section 3 the problems are defined and in section 4 we present their solutions. In section 5 we indicate how to solve the optimal performance problem in multirate systems and also how to treat the robustness issues in periodic and multirate systems. Finally, in section 6 we summarize and draw conclusions.

## 2 Notation and Preliminaries

In this paper the following notation is used:

$\rho(\cdot)$  : The spectral radius.

$\bar{\sigma}(\cdot)$  : The maximum singular value.

$\ell_m^{\infty, e}$  : The space of real  $m \times 1$  vector valued sequences.

$\ell_m^2$  : The space of real  $m \times 1$  vectors  $u$  each of whose components is an energy bounded real sequence  $(u_i(k))_{k=0}^\infty$ . The norm is defined as:

$$\|u\|_{\ell_m^2} \stackrel{\text{def}}{=} \left( \sum_{i=1}^m \sum_{k=0}^{\infty} (u_i(k))^2 \right)^{1/2}$$

$\hat{H}(\lambda)$  : The  $\lambda$ -transform of a right sided  $m \times n$  real sequence  $H = (H(k))_{k=0}^\infty$  defined as:

$$\hat{H}(\lambda) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} H(k) \lambda^k$$

$\mathcal{L}_{m \times n}^\infty$  : The space of all  $m \times n$  matrix valued functions  $F$  defined on the unit circle  $C$  with

$$\text{ess sup}_{\theta \in [0, 2\pi]} \bar{\sigma}(F(e^{j\theta})) \stackrel{\text{def}}{=} \|F\|_{\mathcal{L}^\infty} < \infty$$

$\mathcal{L}_{m \times n}^2$  : The space of matrix valued functions  $F$  defined on the unit circle  $C$  with

$$(2\pi)^{-1} \int_0^{2\pi} \text{trace}(F^T(e^{-j\theta})F(e^{j\theta}))d\theta \stackrel{\text{def}}{=} \|F\|_{\mathcal{L}^2} < \infty$$

$\mathcal{H}_{m \times n}^\infty$  : The space of all  $m \times n$  matrix valued functions  $F$  analytic in the open unit disk  $D$  with

$$\sup_{r \in (0, 1)} \max_{\theta \in [0, 2\pi]} \bar{\sigma}(F(re^{j\theta})) \stackrel{\text{def}}{=} \|F\|_{\mathcal{H}^\infty} < \infty$$

$\mathcal{H}_{m \times n}^2$  : The space of all  $m \times n$  matrix valued functions  $F$  analytic in the open unit disk  $D$  with

$$\sup_{r \in (0,1)} (2\pi)^{-1} \int_0^{2\pi} \text{trace}(F^T(re^{-j\theta})F(re^{j\theta}))d\theta \stackrel{\text{def}}{=} \|F\|_{\mathcal{H}^2} < \infty$$

$\mathcal{B}_{TV}^{m \times n}$  : The space of all linear bounded and causal maps from  $\ell_n^2$  to  $\ell_m^2$ . We refer to these operators as  $\ell^2$ -stable.

$\mathcal{B}_{TI}^{m \times n}$  : The subspace of  $\mathcal{B}_{TV}^{m \times n}$  consisting of the maps that commute with the shift operator (i.e. the time invariant maps).

$\Pi_m^k$  : The  $k^{\text{th}}$ -truncation operator on  $\ell_m^{\infty, e}$  defined as:

$$\Pi_m^k : \{u(0), u(1), \dots\} \longrightarrow \{u(0), \dots, u(k), 0, 0, \dots\}$$

$\Lambda_m$  : The right shift operator on  $\ell_m^{\infty, e}$  i.e.

$$\Lambda_m : \{a(0), a(1), \dots\} \longrightarrow \{0, a(0), a(1), \dots\}$$

**Note:** We will often drop the  $m$  and  $n$  in the above notation when the dimension is not important or when it is clear from the context. Also, subscripts on the norms are dropped when there is no ambiguity.

We now present some definitions and facts that are used in this paper.

**Definition 2.1** Let  $f : \ell_m^{\infty, e} \longrightarrow \ell_n^{\infty, e}$  be an operator.  $f$  is called causal if

$$\Pi_n^k f u = \Pi_n^k f \Pi_m^k u, \quad \forall k = 0, 1, 2, \dots,$$

$f$  is called strictly causal if

$$\Pi_n^k f u = \Pi_n^k f \Pi_{k-1}^m u, \quad \forall k = 0, 1, 2, \dots,$$

Let  $f$  represent the input-output map from  $\ell_m^{\infty, e}$  to  $\ell_p^{\infty, e}$  of a linear causal time varying system.

**Definition 2.2** The map  $f$  is  $N$ -periodic if and only if it commutes with the  $N$ th power of the right shift; i.e.

$$f(\Lambda_m)^N = (\Lambda_p)^N f \quad .$$

If  $f$  is a linear  $N$ -periodic finite dimensional operator associated with the the state space description

$$x(t+1) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t),$$

$t = 0, 1, 2, \dots$  with  $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$  being  $N$ -periodic matrices then

**Definition 2.3** The pair  $(A(\cdot), B(\cdot))$  is called stabilizable if there exists a bounded matrix function  $K(\cdot)$  such that the system  $x(t+1) = (A(t) - B(t)K(t))x(t)$  is exponentially stable [4]. Similarly, the pair  $(A(\cdot), C(\cdot))$  is detectable if there is a bounded matrix function  $L(\cdot)$  such that the system  $x(t+1) = (A(t) - L(t)C(t))x(t)$  is exponentially stable.

The following facts can be found in [7]. Let  $f$  be  $N$ -periodic and  $W_m$  represent the isomorphism

$$W_m : \ell_m^{\infty, e} \rightarrow \ell_{mN}^{\infty, e}$$

$$a = \{a(0), a(1), \dots\} \rightarrow W_m(a) = \left\{ \begin{pmatrix} a(0) \\ a(1) \\ \vdots \\ a(N-1) \end{pmatrix}, \begin{pmatrix} a(N) \\ a(N+1) \\ \vdots \\ a(2N-1) \end{pmatrix}, \dots \right\}$$

Define the map  $L$  as

$$L(f) = W_p f W_m^{-1}$$

where  $W_p$  is defined similarly to  $W_m$ . Then  $L(f)$  represents a system with inputs in  $\ell_{mN}^{\infty, e}$  and outputs in  $\ell_{pN}^{\infty, e}$ . Moreover, as shown in [7]  $L(f)$  is LTI and the following hold:

**Fact 2.1** Given a  $m$ -input  $p$ -output linear causal  $N$ -periodic system  $f$ , one can associate via the map  $L$  a unique causal  $(pN \times mN)$  LTI system  $L(f)$  with a transfer matrix  $\hat{F}(\lambda)$ . Conversely, any  $(pN \times mN)$  transfer matrix  $\hat{F}(\lambda)$  with  $\hat{F}(0)$  block lower triangular can be associated by  $L^{-1}$  with a unique  $m$ -input,  $p$ -output, linear, causal,  $N$ -periodic system.

**Note:** In the above fact,  $\hat{F}(0)$  block lower triangular means that

$$\hat{F}(0) = \begin{pmatrix} F_{00} & 0 & 0 & \dots & 0 \\ F_{10} & F_{11} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ F_{N0} & F_{N1} & F_{N2} & \dots & F_{NN} \end{pmatrix}$$

where each submatrix  $F_{ij}$  has dimension  $p \times m$ . This is exactly what is meant when the term block lower triangular is used hereafter. Also, we will use the term "lifting" to indicate the action of  $L$  to a

system  $f$  (i.e.  $F$  is the lifted system or  $F$  is the lift of  $f$ ). Finally, to avoid proliferation of notation, we hereafter use  $W$  generically instead of  $W_n$  where the subscript  $n$  specifies the dimension of the elements of the sequence that this isomorphism acts on.

**Fact 2.2**  $L$  preserves the algebraic properties and the norm. In particular,

$$\sup_{u \in \mathcal{B}\mathcal{L}_m^2} \|fu\|_{\mathcal{L}_p^2} = \sup_{w \in \mathcal{B}\mathcal{L}_{mN}^2} \|Fw\|_{\mathcal{L}_{pN}^2}$$

Hence,  $f$  is input-output stable if and only if  $F = L(f)$  is stable.

Suppose in addition, that  $f$  is finite dimensional with a stabilizable and detectable state space description. In [1, 9] it is shown that we can obtain a doubly coprime factorization (dcf) of  $f$  by obtaining a dcf of the lifted system  $F$ . The key observation in [1] is that the factors of  $F$  obtained using the standard formulas in [6] possess the property of being block lower triangular at  $\lambda = 0$ . Hence, since  $L$  is an isomorphism, we can obtain a dcf of  $f$ , the factors being the images of the inverse map of the lifted LTI factors of  $F$  and therefore  $N$ -periodic. Also assuming well-posedness [6, 4] we can characterize all stabilizing controllers  $c$  in terms of these factors. In summary we have:

**Fact 2.3** Let  $F = N_l D_l^{-1} = D_r^{-1} N_r$  and

$$\begin{pmatrix} X_r & -Y_r \\ -N_r & D_r \end{pmatrix} \begin{pmatrix} D_l & Y_l \\ N_l & X_l \end{pmatrix} = I$$

represent a dcf of  $F$  where the factors are given as in [6]. Then, the following represent a dcf of  $f$ :

$$f = n_l d_l^{-1} = d_r^{-1} n_r$$

$$\begin{pmatrix} x_r & -y_r \\ -n_r & d_r \end{pmatrix} \begin{pmatrix} d_l & y_l \\ n_l & x_l \end{pmatrix} = I$$

where  $n_l = L^{-1}(N_l)$ ,  $d_l = L^{-1}(D_l)$ ,  $x_l = L^{-1}(X_l)$ ,  $y_l = L^{-1}(Y_l)$ ,  $n_r = L^{-1}(N_r)$ ,  $d_r = L^{-1}(D_r)$ ,  $x_r = L^{-1}(X_r)$ ,  $y_r = L^{-1}(Y_r)$  are in  $\mathcal{B}_{TV}$  and  $N$ -periodic. Moreover, all stabilizing time varying controllers  $c$  of  $f$  are given by

$$c = (y_l - d_l q)(x_l - n_l q)^{-1} = (x_r - q n_r)^{-1}(y_r - q d_r)$$

where  $q \in \mathcal{B}_{TV}$ .

Note that in the above fact it is easy to check that  $q$  is  $N$ -periodic if and only if  $c$  is  $N$  periodic. Finally, along the lines of [?], it is shown in [1] that the optimal performance in periodic plants is achieved with periodic controllers, namely:

**Fact 2.4** Let  $H_p, U_p, V_p$  be  $N$ -periodic and stable causal linear operators. Then

$$\inf_{Q \in \mathcal{B}_{TV}} \|H_p - U_p Q V_p\| = \inf_{Q_p} \|H_p - U_p Q_p V_p\|$$

where  $Q_p$  is  $N$ -periodic and in  $\mathcal{B}_{TV}$ .

### 3 Problem Definition

The standard block diagram for the disturbance rejection problem is depicted in Figure 1. In this figure,  $P_p$  denotes some fixed linear causal  $N$ -periodic plant,  $C_v$  denotes a time varying compensator (not necessarily periodic), and the signals  $w, z, y$ , and  $u$  are defined as follows:  $w$ , exogenous disturbance;  $z$ , signals to be regulated;  $y$ , measured plant output; and  $u$ , control inputs to the plant.  $P_p$  can be thought as a four block matrix each block being a linear causal  $N$ -periodic system. The following common assumptions are made:

**Assumption 3.1** The system of figure 1 is well-posed [6, 4].

**Assumption 3.2**  $P_p$  is finite dimensional and stabilizable.

Assumption 3.2 means that  $P_p$  has the state space description

$$P_p \sim (A(\cdot), (B_1(\cdot) B_2(\cdot)), \begin{pmatrix} C_1(\cdot) \\ C_2(\cdot) \end{pmatrix}, \begin{pmatrix} D_{11}(\cdot) & D_{12}(\cdot) \\ D_{12}(\cdot) & D_{22}(\cdot) \end{pmatrix})$$

with all the matrices above being  $N$ -periodic and the pairs  $(A(\cdot), B_2(\cdot))$  and  $(A(\cdot), C_2(\cdot))$  being stabilizable and detectable respectively. Also, a sufficient condition for assumption 3.1 to hold is that  $D_{22}(\cdot) = 0$ .

Now let  $T_{zw}$  represent the resulting map from  $w$  to  $z$  for a given compensator  $C_v$ .

The first problem we want to solve is when  $w$  is assumed to be any  $\ell^2$  disturbance and we are interested in minimizing the worst case energy of  $z$ . Namely, our objective can be stated as (OBJ $^\infty$ ): Find  $C_v$  such that the resulting closed loop system is stable and also the induced norm  $\|T_{zw}\|$  over  $\ell^2$  is minimized.

The second problem we want to solve is when we assume that  $w$  is a stationary zero mean Gaussian white noise with  $E[ww^T] = I$  and we seek to minimize the average noise power in  $z$ . So our objective is stated as (OBJ $^2$ ):

Find  $C_v$  such that the resulting closed loop system is stable and also

$$\lim_{M \rightarrow \infty} (1/2M) \sum_{k=-M}^{M-1} \text{trace}(E[z(k)z^T(k)])$$

Figure 1: Block Diagram for Disturbance Rejection.

is minimized.

In the sequel we show that these problems can be turned into equivalent LTI problems. Towards this end, we lift each of the four blocks of  $P_p$  and let  $P$  denote the resulting LTI plant. Consider now, the disturbance rejection problem for the system  $P$  and let  $\tilde{T}_{zw}$  denote the map from  $\tilde{w} = Ww$  to  $\tilde{z} = Wz$ . Note that assumption 3.1 guarantees the well-posedness of the LTI problem. Let  $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ , and let the following represent a doubly coprime factorization of  $P_{22}$ :  $P_{22} = N_l D_l^{-1} = D_r^{-1} N_r$  and

$$\begin{pmatrix} X_r & -Y_r \\ -N_r & D_r \end{pmatrix} \begin{pmatrix} D_l & Y_l \\ N_l & X_l \end{pmatrix} = I$$

where the factors are obtained by the formulas in [6] (which preserve the block lower triangular structure.) Then it is well known [12, 5, 11, 4] that all the feasible maps are given as  $\tilde{T}_{zw} = \tilde{H} - UQV$  where  $H, U, V \in \mathcal{B}_{TI}$  and  $Q \in \mathcal{B}_{TV}$ . Moreover,  $H, U, V$  are determined by  $P$ . Now, the following lemma shows the aforementioned equivalence.

**Lemma 3.1** *The  $(OBJ^\alpha)$  with  $\alpha = \infty$  or 2 is equivalent to the problem  $(OPT^\alpha)$ :*

$$\inf_{Q \in \mathcal{H}^\alpha} \|H - UQV\|_{\mathcal{H}^\alpha}$$

*subject to  $Q(0)$  is block lower triangular. Moreover, if  $Q_o$  is the optimal solution to the above problem then the optimal compensator for the periodic plant is the inverse lift of*

$$C = (Y_l - D_l Q_o)(X_l - N_l Q_o)^{-1} = (X_r - Q_o N_r)^{-1}(Y_r - Q_o D_r).$$

**Proof** For the case  $\alpha = \infty$  we use fact 2.3 and fact 2.4 to obtain

$$\inf_{C_p} \|T_{zw}\| = \inf_{Q_p} \|H_p - U_p Q_p V_p\|$$



where  $Q_p$  is  $N$ -periodic, stable, and  $H_p = L^{-1}(H), U_p = L^{-1}(U), V_p = L^{-1}(V)$ . Now, by facts 2.1 and 2.2 it follows that

$$\inf_{Q_p} \|H_p - U_p Q_p V_p\| = \inf_{Q=L(Q_p)} \|H - UQV\|.$$

For the case  $\alpha = 2$  it is known from the classical general LQG solution (for example [?]) that the optimal solution consists of solving a pair of periodic Riccati equations. Specifically, using the asymptotic solutions of the Riccati equations a state estimator and a feedback gain is obtained; then putting together these two elements yields the optimal controller. Now, from the results in [?, ?] and by assuming that the pairs  $(A(\cdot), B_1(\cdot)), (A(\cdot), C_1(\cdot))$  are stabilizable and detectable respectively we conclude that the asymptotic solution of the Riccati equations are periodic and hence the optimal controller is periodic. In fact, the stabilizability and detectability assumptions can be further weakened ([?, ?]). Therefore, it is enough to search over periodic controllers. But then by looking at the lifted problem this is a standard LQG problem in LTI systems. Moreover since  $E[\tilde{w}\tilde{w}^T] = I$  and

$$\lim_{M \rightarrow \infty} (1/2M) \sum_{k=-M}^{M-1} \text{trace}(E[z(k)z^T(k)]) = (1/N) \lim_{M \rightarrow \infty} (1/2M) \sum_{k=-M}^{M-1} \text{trace}(E[\tilde{z}(k)\tilde{z}^T(k)])$$

then by solving the LTI problem subject to the causality conditions on the LTI controller will result to the optimal controller for the periodic system. But the LTI problem is

$$\inf_{Q \in \mathcal{H}^\infty} \|H - UQV\|_{\mathcal{H}^2}$$

subject to  $Q(0)$  being block lower triangular and therefore the proof is complete. ■

It is problems (OPT $^\alpha$ ) with  $\alpha = \infty$  and 2 that we approach in the following section.

## 4 Problem Solution

First, by performing an inner outer factorization [6] for  $U, V$  we obtain

$$U = U_i U_o, \quad V = V_o V_i$$

where the subscript  $i$  stands for “inner” and  $o$  for “outer”; i.e.  $\hat{U}_i^T(\lambda^{-1})\hat{U}_i(\lambda) = I$  and  $\hat{V}_i(\lambda^{-1})\hat{V}_i^T(\lambda) = I$ . We will also make the simplifying technical assumption that  $\hat{U}(\lambda), \hat{V}(\lambda)$  do not lose rank on the unit circle and hence  $U_o, V_o$  have stable right and left inverses respectively. Then we proceed by reflecting the causality constraints of  $Q(0)$  on  $U_o Q V_o$ . Towards this end let  $Z = U_o Q V_o$ ; the following proposition from [2, 3] shows how  $Z$  is affected due to the constraints on  $Q$ .

**Proposition 4.1** Let  $Z \in \mathcal{H}^\infty$  then

$$\exists Q \in \mathcal{H}^\infty \quad \text{with } Q(0) \text{ block lower triangular and } Z = U_o Q V_o$$

if and only if

$$Z(0) \in S_A = \{U_o(0)AV_o(0) : A \text{ block lower triangular matrix}\}.$$

■

Proposition 4.1 shows that only  $Z(0)$  is constrained to lie in a certain finite dimensional subspace (i.e.  $S_A$ ) otherwise  $Z$  can be arbitrary in  $\mathcal{H}^\infty$ . One can easily find a basis for this subspace by considering  $S_A^\perp$  and finding a basis for this subspace. This is done in [2, 3] and we briefly repeat the procedure here: For each element  $j$  of  $Q(0)$  with indices  $(l_j, m_j)$  that has to equal 0 (i.e. the elements that are not in the block lower triangular portion of  $Q(0)$ ) we associate a matrix  $R_j$  with the same dimension as  $Q(0)$  that has all its entries but one equal to 0. The non-zero entry is taken to equal 1 and its indices are precisely the ones that correspond to  $j$  i.e.  $(l_j, m_j)$ . Then if  $r$  is the number of the elements in  $Q(0)$  that are necessarily equal to 0 then the annihilator subspace  $S_A^\perp$  is

$$S_A^\perp = \{B : U_o^T(0)BV_o^T(0) \in \text{span}(R_1, R_2, \dots, R_r)\}.$$

A basis  $\{B_1, B_2, \dots, B_{j_B}\}$  for this subspace can be found in a routine way (see [3] for details.) When  $U_o, V_o$  are square the computation of the basis is immediate. Namely,

$$B_j = U_o^{-T}(0)R_jV_o^{-T}(0) \quad j = 1, \dots, r.$$

Now, if  $\langle \bullet, \bullet \rangle$  denotes the inner product in the finite dimensional Euclidean space of matrices then

$$Z(0) \in S_A \text{ if and only if } \langle Z(0), B_j \rangle = 0 \quad \forall j = 1, \dots, j_B.$$

In view of the above proposition the optimization problem becomes :

$$\inf_{Z \in \mathcal{H}^\infty, Z(0) \in S_A} \|H - U_i Z V_i\|_{\mathcal{H}^\alpha}.$$

We are now ready to approach the problems for each case  $\alpha = \infty$  and  $\alpha = 2$  separately.

## 4.1 The $\mathcal{H}^\infty$ problem

We solve this problem by modifying the standard Nehari's approach [6] in order to account for the additional causality constraint on the compensator. This modification yields to a finite dimensional convex optimization problem over a convex closed set that needs to be solved before applying the standard solution to the Nehari problem. For simplicity, we assume that  $U_i, V_i$  are square. We will come back to the general 4-block problem later.

The solution to the 1-block case is as follows: Let  $R = U_i^* H V_i^*$  where  $\hat{U}_i^*(\lambda) = \hat{U}_i^T(\lambda^{-1}), \hat{V}_i^*(\lambda) = \hat{V}_i^T(\lambda^{-1})$  and define for each  $J \in S_A$  the system  $R_J$  as:

$$\hat{R}_J(\lambda) = \lambda^{-1}(\hat{R}(\lambda) - J).$$

The following theorem shows how to obtain the optimal solution to

$$\mu_{\mathcal{H}^\infty} = \inf_{Z \in \mathcal{H}^\infty, Z(0) \in S_A} \|H - U_i Z V_i\|_{\mathcal{H}^\infty}.$$

**Theorem 4.1** For each  $J \in S_A$ , let  $\Gamma_{R_J}$  represent the Hankel operator [6] with symbol  $R_J$ ; then the following hold:

1.  $\mu_{\mathcal{H}^\infty} = \inf_{Z \in \mathcal{H}^\infty, Z(0) \in S_A} \|R - Z\|_{\mathcal{H}^\infty} = \inf_{J \in S_A} \|\Gamma_{R_J}\|$ ,
2.  $\inf_{J \in S_A} \|\Gamma_{R_J}\|$  is a finite dimensional optimization of a convex and continuous functional on a convex closed set,
3. if  $J_o$  is the solution to the proceeding convex programming problem and  $X_o$  is the solution to the standard Nehari problem

$$\inf_{X \in \mathcal{H}^\infty} \|R_{J_o} - X\|$$

then the optimal solution  $Z_o$  is given by

$$\hat{Z}_o(\lambda) = J_o + \lambda \hat{X}_o(\lambda).$$

**Proof** For the first part note that since  $\hat{U}_i(\lambda), \hat{V}_i(\lambda), \lambda I$  are inner then

$$\|H - U_i Z V_i\| = \|U_i^* H V_i^* - Z\| = \|R - Z\|$$

Writting  $\hat{Z}(\lambda) = Z(0) + \lambda \hat{\bar{Z}}(\lambda)$  with  $\bar{Z}$  arbitrary in  $\mathcal{H}^\infty$  we have

$$\|R - Z\| = \left\| \hat{R}(\lambda) - Z(0) - \lambda \hat{\bar{Z}}(\lambda) \right\| = \left\| \lambda^{-1}(\hat{R}(\lambda) - Z(0)) - \hat{\bar{Z}}(\lambda) \right\|.$$

Now, if  $J \in S_A$  then from Nehari's theorem we have:

$$\inf_{\bar{Z} \in \mathcal{H}^\infty} \|R_J - \bar{Z}\| = \|\Gamma_{R_J}\|$$

hence the first part of the proof follows.

The second part is showed by computing  $\|\Gamma_{R_J}\|$  using state space formulae. In particular we are going to compute the controllability and observability grammians [6] associated with  $\Gamma_{R_J}$ . Towards this end let  $R$  correspond via the Fourier transform to the double-sided (since  $R$  is not necessarily stable) sequence  $(R(i))_{i=-\infty}^{\infty}$  then  $R_J$  will correspond to  $(R_J(i))_{i=-\infty}^{\infty}$  with  $R_J(i) = R(i+1) \quad \forall i \neq -1$  and  $R_J(-1) = R(0) - J$ . Let now  $\bar{G}$  represent the stable (causal) system associated with the pulse response  $\{0, R(-1), R(-2), \dots\}$  and let  $(\bar{A}, \bar{B}, \bar{C}, 0)$  be a minimal state space description of it. Let also  $G$  represent the stable system associated with the pulse response  $\{0, R_J(-1), R_J(-2), \dots\}$  i.e.  $G$  is the anticausal part of  $R_J$  but viewed as a causal (one-sided) system. Then it easy to check that  $G$  has the state space description  $(A, B, C, 0)$  with

$$A = \begin{pmatrix} \bar{A} & \bar{B} \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad C = (\bar{C} \quad \bar{J})$$

where  $\bar{J} = R(0) - J$ . Finally, let  $W_c, W_o$  be the controllability and observability grammians for  $G$  i.e.

$$W_c = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k$$

$$W_o = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k.$$

Then  $W_c$  and  $W_o$  are the solutions to the Lyapunov equations :

$$W_c - A W_c A^T = B B^T, \quad W_o - A^T W_o A = C^T C.$$

Similarly, let  $\bar{W}_c, \bar{W}_o$  be the controllability and observability grammians for  $\bar{G}$ .

Following [6] we have that  $\|\Gamma_{R_J}\| = \rho^{1/2}(W_c^{1/2} W_o W_c^{1/2})$ . Using the state space description we compute

$$W_c = \begin{pmatrix} \bar{W}_c & 0 \\ 0 & I \end{pmatrix}$$

$$W_o = \begin{pmatrix} \bar{C}^T \bar{C} & \bar{C}^T \bar{J} \\ \bar{J}^T \bar{C} & \bar{J}^T \bar{J} \end{pmatrix} + K$$

where

$$K = \sum_{k=1}^{\infty} (A^T)^k C^T C A^k = \begin{pmatrix} \bar{W}_o - \bar{C}^T \bar{C} & \bar{A}^T \bar{W}_o \bar{B} \\ \bar{B}^T \bar{W}_o \bar{A} & \bar{B}^T \bar{W}_o \bar{B} \end{pmatrix}.$$

Note that  $K$  does not depend on  $J$ . Also, since  $K \geq 0$  then  $K = K^{1/2}K^{1/2}$  with  $K^{1/2} \geq 0$ . Now, proceeding with the computations and rearranging certain terms we obtain:

$$W_c^{1/2}W_oW_c^{1/2} = \begin{pmatrix} I & 0 \\ 0 & \bar{J}^T \end{pmatrix} M^T M \begin{pmatrix} I & 0 \\ 0 & \bar{J} \end{pmatrix} + L^T L$$

where

$$M = \begin{pmatrix} \overline{CW}_c^{1/2} & I \\ 0 & 0 \end{pmatrix}, \quad L = K^{1/2}W_c^{1/2}.$$

Hence

$$\|\Gamma_{R,J}\| = \rho^{1/2}(W_c^{1/2}W_oW_c^{1/2}) = \bar{\sigma}\left[\begin{pmatrix} M\bar{H} \\ L \end{pmatrix}\right]$$

with  $\bar{H} = \begin{pmatrix} I & 0 \\ 0 & \bar{J} \end{pmatrix}$ . Therefore

$$\inf_{J \in S_A} \|\Gamma_{R,J}\| = \inf_{\bar{H} \in \bar{S}} \bar{\sigma}\left[\begin{pmatrix} M\bar{H} \\ L \end{pmatrix}\right].$$

where

$$\bar{S} = \left\{ \begin{pmatrix} I & 0 \\ 0 & \bar{J} \end{pmatrix} : \bar{J} = R(0) + J, \quad J \in S_A \right\}.$$

Clearly, since  $S_A$  is a space then  $\bar{S}$  is a convex set. Moreover if  $\bar{H}_1, \bar{H}_2 \in \bar{S}$ , and  $t \in [0, 1]$  we have

$$\bar{\sigma}\left[\begin{pmatrix} M(t\bar{H}_1 + (1-t)\bar{H}_2) \\ L \end{pmatrix}\right] = \bar{\sigma}\left[\begin{pmatrix} tM\bar{H}_1 \\ tL \end{pmatrix} + \begin{pmatrix} (1-t)M\bar{H}_2 \\ (1-t)L \end{pmatrix}\right]$$

or

$$\bar{\sigma}\left[\begin{pmatrix} M(t\bar{H}_1 + (1-t)\bar{H}_2) \\ L \end{pmatrix}\right] \leq t\bar{\sigma}\left[\begin{pmatrix} M\bar{H}_1 \\ L \end{pmatrix}\right] + (1-t)\bar{\sigma}\left[\begin{pmatrix} M\bar{H}_2 \\ L \end{pmatrix}\right]$$

which shows that  $\bar{\sigma}\left[\begin{pmatrix} M\bar{H} \\ L \end{pmatrix}\right]$  is convex in  $\bar{H}$  and consequently in  $J$ . Also, continuity of  $\bar{\sigma}\left[\begin{pmatrix} M\bar{H} \\ L \end{pmatrix}\right]$  with respect to  $J$  is apparent and therefore our second claim is proved.

The third part of the theorem is immediate given that a bounded minimizer  $J_o$  of the convex minimization in the second part can be found in  $S_A$  (note  $S_A$  is unbounded). In fact, this is always the case and the proof of it follows from the fact that the optimal solution  $Z$  has to be bounded: Clearly the selection  $Z = 0$  is a legitimate one since  $Z(0) \in S_A$ . Hence  $\mu_{\mathcal{H}\infty} \leq \|H\|$ ; if now  $\|Z\| > 2\|H\|$  then

$$\|H - U_i Z V_i\| \geq \|Z\| - \|H\| > \|H\| \geq \mu_{\mathcal{H}\infty}.$$

Therefore for  $Z$  to be a minimizer it is necessary that  $\|Z\| \leq 2\|H\|$  and hence  $\bar{\sigma}[Z(0)] \leq 2\|H\|$  which implies that the search for the optimal  $J$  can be constrained in a closed and bounded subset  $\bar{S}_A$  of  $S_A$ . Namely,

$$\bar{S}_A = \{J \in S_A : \bar{\sigma}[J] \leq 2\|H\|\}.$$

But then the continuity of the cost implies that an optimal  $J_o$  can be found in  $\bar{S}_A$  which is bounded. ■

The previous theorem indicates what is the additional convex minimization problem that has to be solved in order to account for the causality constraint on  $Q(0)$ . The following corollary is a direct consequence from the proof of the Theorem 4.1.

**Corollary 4.1** *With the notation as in the proof of Theorem 4.1 the convex minimization problem of item 2 in Theorem 4.1 is*

$$\min_{J \in \bar{S}_A} \rho^{1/2} (W_c^{1/2} W_o W_c^{1/2})$$

with

$$W_c = \begin{pmatrix} \bar{W}_c & 0 \\ 0 & I \end{pmatrix}, \quad W_o = \begin{pmatrix} 0 & \bar{C}^T \bar{J} \\ \bar{J}^T \bar{C} & \bar{J}^T \bar{J} \end{pmatrix} + \begin{pmatrix} \bar{W}_o & \bar{A}^T \bar{W}_o \bar{B} \\ \bar{B}^T \bar{W}_o \bar{A} & \bar{B}^T \bar{W}_o \bar{B} \end{pmatrix}$$

and  $\bar{J} = R(0) + J$ . ■

The above convex programming problem can be solved with descent algorithms. In [?] the authors treating a problem of  $\mathcal{H}^\infty$  optimization with time domain constraints arrive at a similar finite dimensional convex programming problem. As they indicate the cost might not be differentiable at all points and therefore methods of non-differentiable optimization are called for. Although these generalized descent methods might be slow they have guaranteed convergence properties. In [?] and the references therein alternatives are given to improve the convergence rate.

The full 4-block problem i.e. when  $U_i$  and/or  $V_i$  are not square is treated analogously as in the standard Nehari approach [6] with the so-called  $\gamma$ -iterations. In particular, using exactly the same arguments as in [6] the same iterative procedure can be established where at each iteration step a 1-block (square) problem with the additional causality constraints on the free parameter  $Q$  needs to be solved. Hence, the aforementioned procedure of solving the  $\mathcal{H}^\infty$  constrained problem is complete.

## 4.2 The $\mathcal{H}^2$ problem

The solution to this problem is obtained by utilizing the Projection theorem as follows: Let again  $R = U_i^* H V_i^*$  and let  $Y = \{Y(0), Y(1), Y(2), \dots\}$  represent the projection of  $R$  onto  $\mathcal{H}^2$  i.e.  $Y = \Pi_{\mathcal{H}^2}(R)$ . Also, consider the finite dimensional Euclidean space  $E$  of real matrices with dimensions equal to those of  $Y(0)$  and let  $\Pi_{S_A}$  represent the projection operator onto the subspace  $S_A$  of  $E$ . Then

**Theorem 4.2** *The optimal solution  $Z_o$  for the problem*

$$\mu_{\mathcal{H}^2} = \inf_{Z \in \mathcal{H}^\infty, Z(0) \in S_A} \|H - U_i Z V_i\|_{\mathcal{H}^2}$$

is given by

$$Z_o = \{\Pi_{S_A}(Y(0)), Y(1), Y(2), \dots\}.$$

**Proof** The proof follows from a direct application of the Projection theorem in the Hilbert space  $\mathcal{L}^2$ . Let  $\mathcal{H}_S = \{Z : Z \in \mathcal{H}^2, Z(0) \in S_A\}$  then  $\mathcal{H}_S$  is a closed subspace of  $\mathcal{L}^2$ . Also, let  $\langle \bullet, \bullet \rangle$  denote the inner product in  $\mathcal{L}^2$ . Viewing  $U_i$  and  $V_i$  as operators on  $\mathcal{L}^2$  we have that  $Z_o$  is the optimal solution if and only if

$$\langle H - U_i Z_o V_i, U_i Z V_i \rangle = 0 \quad \forall Z \in \mathcal{H}_S$$

or equivalently

$$\langle U_i^* H V_i^* - Z_o, Z \rangle = 0 \quad \forall Z \in \mathcal{H}_S$$

or equivalently

$$\Pi_{\mathcal{H}_S}(U_i^* H V_i^* - Z_o) = 0.$$

But

$$\Pi_{\mathcal{H}_S}(U_i^* H V_i^*) = \{\Pi_{S_A}(Y(0)), Y(1), Y(2), \dots\} \in \mathcal{H}^\infty$$

which completes the proof. ■

The above theorem states that only the first component of the classical solution  $Y$  is affected.

## 5 Applications

### 5.1 Optimal Performance in Multi-Rate Systems

As it is indicated in [3] the problem of optimal disturbance rejection in multirate sampled data systems is of the same type as in periodic. Again, a lifting method [8, 9, 10] is applied to transform the problem to an equivalent LTI one. The lifting technique induces a causality constraint on the equivalent LTI optimal compensator  $\hat{C}(\lambda)$  that is reflected on the feedforward term  $\hat{C}(0)$ . This constraint is of the same type with the periodic case. Namely, certain elements of  $\hat{C}(0)$  have to equal to 0. This restriction transforms to exactly the same restriction [9, 10] on the feedforward term  $Q(0)$  of the Youla parameter  $Q$ . Using the same arguments as in the periodic case these constraints can be reflected to constraint on  $Z(0)$ ; the latter are of the form  $Z(0) \in S_{A_1}$  where  $S_{A_1}$  is a completely determined finite dimensional space. Hence, exactly the same results hold as in the periodic case for the optimal  $\mathcal{H}^\infty$  and  $\mathcal{H}^2$  performance.

## 5.2 Robust Stabilization of Periodic and Multi-Rate Systems

In this subsection we consider the problem of finding necessary and sufficient conditions for stability in the presence of unstructured perturbations in the feedback loop of periodic systems. In particular we encounter the following problem: Let  $G_p$  be a periodic system in  $\mathcal{B}_{TV}$  and  $\Delta_p$  be an unknown stable periodic perturbation in the class

$$\mathcal{D} = \{\Delta_p \in \mathcal{B}_{TV} : \|\Delta_p\| < 1\}.$$

The question now is under what conditions the pair  $(G_p, \Delta_p)$  under feedback results in a  $\ell^2$  stable system. A sufficient condition is the small gain condition [4] i.e.  $\|G\| \leq 1$ . In fact as we will indicate in what follows, this is necessary as well: Let  $G, \Delta$  represent the lifts of  $G_p, \Delta_p$  respectively and let  $\Lambda$  represent the delay operator (i.e. the shift operator.) If  $\|G\| > 1$  then  $\|\Lambda G\| = \|G\| > 1$  and hence there exists [?] a  $\Delta_1$  with  $\|\Delta_1\| \leq 1$  such that the pair  $(\Lambda G, \Delta_1)$  in feedback is unstable. But then the pair  $(G, \Delta^\circ)$  is unstable with  $\Delta^\circ = \Lambda \Delta_1$ ; moreover since  $\|\Delta^\circ\| = \|\Delta_1\| \leq 1$  and also  $\Delta^\circ$  is strictly proper then by fact 2.2 we conclude that if  $\|G_p\| > 1$  there is a destabilizing  $\Delta_p^\circ \in \mathcal{D}$  with  $\Delta_p^\circ$  being the inverse lift of  $\Delta^\circ$ .

In the discussion above  $G_p$  will depend in general on the particular controller  $C_v$  we pick for the nominal unperturbed system. Hence if we wish to maximize the tolerance to perturbations in the class  $\mathcal{D}$  the problem we have to solve is

$$\inf_{C_v \text{ stabilizing}} \|G_p(C_v)\|$$

which is a performance problem in periodic systems that can be solved with the method indicated in section 4. Finally, the situation is exactly the same when we encounter the robustness problem for multirate systems: The small gain condition is necessary as well as sufficient for stability in the presence of unstructured perturbations.

## 6 Conclusions

In, this paper we presented the solutions to the optimal  $\ell^2$  to  $\ell^2$  disturbance rejection problem ( $\mathcal{H}^\infty$ ) as well as the solution to the LQG ( $\mathcal{H}^2$ ) problem in periodic systems using the lifting technique. Both problems involved a causality condition on the optimal LTI compensator when viewed in the lifted domain. The  $\mathcal{H}^\infty$  problem was solved using the Nehari's theorem whereas in the  $\mathcal{H}^2$  problem the solution was obtained using the Projection theorem. In particular, the  $\mathcal{H}^\infty$  problem was solved by modifying the standard Nehari's approach in order to account for the additional



causality constraint on the compensator. This modification yielded to a finite dimensional convex optimization problem over a convex set that needs to be solved before applying the standard solution to the Nehari problem. The solution to the above convex finite dimensional problem can be obtained easily using standard programming techniques. In the  $\mathcal{H}^2$  case the solution was obtained from the optimal (standard) unconstrained problem by projecting only the feedforward term of the standard solution to the allowable subspace. Also we demonstrated that exactly the same methods can be applied to solve the performance problems in the case of multi-rate sampled data systems. Finally, we indicated that the small gain condition is necessary as well as sufficient for  $\ell^2$ -stability in periodic and multirate systems.

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