New Examples of Four Dimensional AS-Regular Algebras

by

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Abstract

This thesis deals with AS-regular algebras, first defined by Michael Artin and William Schelter in *Graded Algebras of Global Dimension 3.* All such algebras of dimension three have been classified, but the corresponding problem in higher dimensions remains open. We construct new examples of four dimensional AS-regular algebras, and provide some information about their module structure. Results are provided for proving the regularity of such algebras. In addition we classify the AS-regular algebras of dimension four satisfying certain conditions.

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Chapter 1

Introduction

In [4] the following class of "regular" algebras were introduced as putative noncommutative analogues of commutative polynomial rings.

Definition 1.0.1 *An Artin-Schelter regular algebra (AS-regular algebra) is a connected graded algebra A over a field k such that the following conditions hold:*

(i) A has finite global dimension d.

(ii) A is Gorenstein in the sense that there exists n such that $\mathrm{Ext}^n(k, A) = k$, and $\mathrm{Ext}^j(k, A) = 0$ for $j \neq n$.

(iii) A has finite GK-dimension, that is, the growth of the graded pieces of A is bounded by a polynomial.

By the dimension of an AS-regular algebra, we will mean the integer d appearing above.

Additionally, although work has been done in the more general case, we will restrict attention to the situation where A is generated in degree 1, and has defining relations in degree 2. This implies, in particular, that the integers d and n in the above definition are equal.

Among the simplest non-trivial such algebras are $k\langle x, y\rangle/(f)$ where $f = xy - \alpha yx$ (for $\alpha \neq 0$) or $f = (xy - yx - x^2)$, and these are actually all the possibilities of dimension 2. In [4], the components of the scheme parametrizing such algebras of dimension 3 were determined. This was done by parametrizing those resolutions which were potentially resolutions of k_A for some A, and then identifying the components where the corresponding algebra A was, generically regular.

However, even in the dimension three case, this left a number of questions unanswered. For example, there was not even any way to determine whether or not any particular specified algebra on one of the components was indeed AS-regular. Additionally, the arguments failed to provide much information about the actual properties of these algebras which were AS-regular.

This prompted more work, and eventually, in [2] and [3], it was realized that all such algebras possessed a normal element $\Omega \in A_3$ in degree three. The quotient by this element turned out, in the interesting case, to be the twisted coordinate ring of the degree 3 curve in $\mathbf{P}^2$ parametrizing point modules over the algebra. This was more easily understood, and useful information could be lifted from it to the original algebra. For example, it was shown in [3] that three dimensional regular algebras are in fact Auslander-regular, and that such an algebra is a finite module over its center exactly if σ is of finite order.

As of yet, no such general arguments have been successfully extended to AS-regular algebras of dimension 4. However, some partial results exist, and considerable effort has been expended proving the AS-regularity of various diverse families of algebras. Examples of this work can be found in [12], which generalizes the elliptic case of [2], [6], which considers extensions of algebras of [2] by a central element, [15], giving 4 dimensional AS-regular algebras mapping to the coordinate ring of a quadric surface in $\mathbf{P}^3$, [13], which presents some 4 dimensional AS-regular algebras which are not finite modules over their centers, and [8], which constructs new AS-regular algebras from algebras with a regular normalizing sequence.

In this thesis we will construct a four dimensional AS-regular algebra with finite point scheme, which possesses certain interesting properties beyond those of the algebras listed above. This algebra is described in by the following theorem.

Theorem 1.0.2 *There exists an AS-regular algebra A with the following properties:*

(i) A is not an Ore extension of a subalgebra.

(ii) A is has no non-trivial automorphisms.

(iii) A is not a finite module over its center, which is a polynomial ring generated by two elements of degree two.

(iv) A has a finite dimensional point scheme consisting of 13 *points.*

A was discovered by computer, and the same method can be used to exhibit many examples of algebras with similar properties.

Afterwards, we will partially extend one of the results of [4] to the global dimension four case. In particular, we will find the set of possible associated matrices, Q, which may appear for global dimension AS-regular algebras (up to a small number of cases which we cannot distinguish).

We will start by recalling some of the relevant background in the next chapter.

Chapter 2

Definitions and Background

As stated in the introduction, our main subject of interest will be AS-regular algebras, as defined in 1.0.1. Additionally, in all that follows we will assume that A satisfies the further conditions that it is of global dimension 4, is generated by four elements of degree 1, and is defined by quadratic equations.

Now, to fix notation, for a graded algebra A, let V_A be the vector space A_1, TV_A the graded tensor algebra on A, and I_A the ideal defining A. When, as usual, confusion will not result, we will omit the subscripts. For an element $x^* \in V^*$ and $f \in TV_n$ we will let $x^* \vdash f$ denote the element of TV_{n-1} given by applying $x^* \otimes id \otimes \cdots \otimes id$ to f, and similarly for $f \dashv x^*$.

It is easy to see that for an AS-regular algebra as we are concerned with, the minimal free resolution resolution of ${}_Ak$

$$0 \to P_4 \overset{M_3}{\to} P_3 \overset{M_2}{\to} P_2 \overset{M_1}{\to} P_1 \overset{M_0}{\to} P_0 \to {}_Ak \to 0$$

is of the form

$$0 \to A(-4) \overset{M_3}{\to} A^4(-3) \overset{M_2}{\to} A^6(-2) \overset{M_1}{\to} A^4(-1) \overset{M_0}{\to} A \to {}_Ak \to 0 \qquad (2.0.2.1)$$

where the M_i are matrices with entries in V, acting by multiplication on the right on row vectors.

As is done in [4], with respect to the 3 dimensional regular algebras, certain useful invariants of such a resolution may be gleaned from the Gorenstein property. In order to do this we will consider products of successive matrices in such a resolution, considered as matrices with coefficients in TV.

In order to compare these products at the beginning and end of the resolution, we will use the following easy lemma.

Lemma 2.0.3 *Let A be a regular algebra with resolution of k as in 2.0.2.1, $U_i \subset V^{\otimes i}$ be the span of the entries in $M_{i-1} \cdots M_0$ (considered as elements of TV_i), and $U_i' \subset V^{\otimes i}$ the span of the entries in $M_3 \cdots M_{3-i+1}$. Then $U_i = U_i'$ for all i.*

Proof. Since 2.0.2.1 is a complex, for every i the elements of $M_{i+1}M_i$ go to zero in A_2 and so are in I. Hence the entries of U_i lie in each of the subspaces $I \otimes V^{\otimes i-2}$, $V \otimes I \otimes V^{\otimes i-3}$, $\dots V^{\otimes i-2} \otimes I$, and hence in their intersection, which we denote by S_i. We will now show the opposite inclusion by induction. Clearly $V = U_0 = S_0$. Now say that $U_{n-1} = S_{n-1}$, and let f be an element of S_n. Let $\langle x_1, \dots, x_4 \rangle$ be a basis for V, then we may write $f = \Sigma x_i f_i$ where each f_i is an element of S_{n-1} and hence of U_{n-1} by hypothesis, let p_i be the degree $n-1$ element of P_{n-1} corresponding to f_i. Define a map of left A-modules $\phi : A(-n) \to P_{n-1}$ by sending $1 \in A_0$ to $\Sigma_i x_i p_i$. Then, if we consider ϕ as a 1 by rank(P_{n-1}) matrix, then $\phi M_{n-2} \cdots M_0$ is just the matrix $[f]_{1,1}$, and the image of ϕ is in the kernel of M_{n-2} since f is in S_n. Thus ϕ factors through the map M_{n-1} (since $A(-n)$ is projective and P_* is a resolution), and we see that $f \in U_n$.

Applying $\mathrm{Hom}_A(-, A)$ to the free resolution of ${}_Ak$, (2.0.2.1), gives the the complex of right modules

$$0 \leftarrow k_A \leftarrow A(4) \xleftarrow{M_3} A^4(3) \xleftarrow{M_2} A^6(2) \xleftarrow{M_1} A^4(1) \xleftarrow{M_0} A \leftarrow 0$$

where the maps now represent left multiplication by the same matrices (the A^i being thought of now as column vectors). By the Gorenstein hypothesis on A, this complex gives a resolution of k_A.

Finally, note that the definition of S depends only on the ideal I. Thus, by the same argument, in the resolution of k_A, the span of the entries of the successive products $\Pi_{k=0}^{j} M_{3-k}$

g acts on:	P_3	P_2	P_1
Q_1	$g^T Q_1$	Q_1	$Q_1 g$
Q_2	Q_2	$g^T Q_2 g$	Q_2
Q_3	$Q_3 g$	Q_3	$g^T Q_3$

Table 2.1: Effect of change of basis on Q_i

are, respectively, equal to S_j thus proving the lemma. ■

In light of this lemma, we have that $U_2 = U_2' = I_A$, and $U_3 = U_3'$, which we shall denote simply by U. We shall denote by w_A the lone entry of $M_3 \dots M_0$.

It also follows from the lemma that for any given resolution P_*, as in 2.0.2.1, there exist unique invertible matrices $Q_1(P_*)$, $Q_2(P_*)$, $Q_3(P_*)$ (or simply Q_i when it is clear what resolution is referred to) such that

$$M_3^T = Q_1 M_0 \tag{2.0.3.1}$$

$$(M_3 M_2)^T = Q_2 M_1 M_0 \tag{2.0.3.2}$$

$$(M_3 M_2 M_1)^T = Q_3 M_2 M_1 M_0 \tag{2.0.3.3}$$

Now we note that changing the basis of P_i by the linear transformation $g \in GL$

$$\begin{tikzcd}
\cdots \arrow[r] & P_{i+1} \arrow[r, "M_i"] \arrow[dr, "g \circ M_i"'] & P_i \arrow[r, "M_{i-1}"] \arrow[d, "g"] & P_{i-1} \arrow[r] & \cdots \\
& & P_i \arrow[ur, "M_{i-1} \circ g^{-1}"'] & &
\end{tikzcd}$$

gives a new resolution with the maps M_{i-1} and M_i replaced, respectively, by $g^{-1} M_{i-1}$ and $M_i g$. Clearly this also has the affect of changing the linear transformation Q_i associated with the resolution. This effect is summarized in table 2.

Using these actions we can put P_* in a special form in the following manner. First we may change the basis of P_1 so that $Q_1 = Id$. Now, if we subsequently change the basis of P_3 by g, then in order to maintain the identity $Q_1 = Id$ we much change the basis of P_1 by $(g^T)^{-1}$, and these two changes together have the effect of conjugating Q_3 by g. This prompts the following definition.

Definition 2.0.4 *We say that a resolution P_* corresponding to a regular algebra A is normalized if $Q_1 = Id$, and Q_3 is in Jordan normal form. Furthermore, in this situation we will let $(x_i)_i$ be the basis of A_1 given by the entries of M_0, and for convenience write M, N, $\mathbf{x}$, P and Q for M_2, M_1, M_0, Q_2 and Q_3, respectively.*

Finally, we note a relationship between the matrices P and Q. If we let $\phi : V \to V$ be the map taking the basis given by $\mathbf{x}^T$ to the basis given by $\mathbf{x}^T Q^{-1}$, $\psi : I \to I$ the map taking the basis $(N\mathbf{x})^T$ to the basis $(N\mathbf{x})^T P^T P^{-1}$, and $\rho : V^{\otimes 4} \to V^{\otimes 4}$ the map taking $x_{i_1}x_{i_2}x_{i_3}x_{i_4}$ to $x_{i_4}x_{i_1}x_{i_2}x_{i_3}$. Also let $\phi_{i,n} : V^{\otimes n} \to V^{\otimes n}$ be the map acting by ϕ in the ith factor and the identity elsewhere, and similarly for $\psi_{i,n} : V^{\otimes i-1} \otimes I \otimes V^{\otimes n-i-1} \to V^{\otimes i-1} \otimes I \otimes V^{\otimes n-i-1}$. Then we see that since

$$w = \mathbf{x}^T MN\mathbf{x} = \mathbf{x}^T Q^{-1} QMN\mathbf{x} = \mathbf{x}^T Q^{-1}(\mathbf{x}^T MN)^T$$

that we have $\phi_{1,4} \circ \rho(w) = w$. And since

$$\begin{aligned} \mathbf{x}^T MN\mathbf{x} &= (\mathbf{x}^T M)P^{-1}(PN\mathbf{x}) && (2.0.4.1)\\ &= (PN\mathbf{x})^T P^{-1}(\mathbf{x}^T M)^T && (2.0.4.2)\\ &= (N\mathbf{x})^T P^T P^{-1}(\mathbf{x}^T M)^T && (2.0.4.3) \end{aligned}$$

we have $\psi_{1,4} \circ \rho^2(w) = w$. Thus $\psi_{1,4}\rho^2(w) = (\phi_{1,4} \circ \rho)^2(w)$ and, since $\rho \circ \phi_{1,4} = \phi_{2,4} \circ \rho$, this equals $\phi_{1,4} \circ \phi_{2,4} \circ \rho^2(w)$. Hence $\phi_{1,2} \circ \phi_{2,2} = \psi_{1,2}$. This shows, if it was not clear already, that the action of Q on A_1 extends to an automorphism of A, since it preserves I. Moreover it gives a condition on P in terms of Q, for example if Q is diagonal, then the action $\phi_{1,2} \circ \phi_{2,2}$ on I will be diagonal as well, hence also $P^T P^{-1}$, and in particular if Q is scalar then P will necessarily be symmetric.

Chapter 3

A New AS-Regular Algebra

3.1 Known Examples of Algebras with Finite Point Scheme

One difference between the case of AS-regular algebras of dimension three and four is that while in the former case the dimension of the associated scheme of (truncated) point modules is always at least one dimensional, as can be seen from a simple count of the relations cutting it out of $\mathbf{P}^2 \times \mathbf{P}^2$, in the latter case this calculation shows that the point scheme need only be zero dimensional. While several of the interesting families of four dimensional regular algebras still do have as least one dimensional point schemes, such as those which correspond to families which already exist in the three dimensional case (see for example [6] and [12]), this is presumably not the general situation. And indeed, a significant amount of effort has been expended constructing examples of AS-regular algebras for which the dimension of the point scheme achieves it's lower bound.

For example, in [14], and continuing in [10], families of Clifford algebras are constructed which are shown to have a zero dimensional point scheme in general. Moreover, certain deformations of such algebras are exhibited with both a zero dimensional point scheme (in fact, consisting of only one point) and a 1-dimensional line schemes. All of these algebras are shown to be AS-regular of dimension 4. Further, in [13], similiar families of regular algebras are considered which can be shown to be infinite modules over their centers (giving a counterexample to the naive generalization of a theorem that says when this should be the case for dimension 3 AS-regular algebras). Finally, in [8] a method is given for "dualizing" an AS-regular algebras which possesses a regular normal sequence in degree 1. In the examples considered, the construction turns out to yield an algebra with a finite point scheme.

A common feature of these constructions, however, is that the resulting algebras are Ore extensions of lower dimensional regular algebras ([8] does not strictly conform to this mold, though in that case the input algebra is similarly constrained, in that the source algebra must be even more special than an Ore extension). Indeed, the fact that they possess this property is not merely incidental, but an essential ingredient in proving their regularity, the proof of which proceeds by first deriving Auslander regularity from general properties of Ore extensions, and then using the results of [7] which derive Artin-Schelter from Auslander regularity. In particular, as the proof just outlined shows, such algebras are automatically Auslander regular.

In light of the fact that the equivalence of Auslander and AS-regularity is one of the important results concerning three dimensional algebras, and that the corresponding question in dimension four remains open, it seems a natural question to ask whether we can produce slightly more general AS-regular algebras which are at least not a priori Auslander regular.

3.2 A Skew Polynomial Ring

In this section, we will consider the family of algebras $A = TV/I$ where I is the four parameter ideal generated by the following six quadratic relations:

$$\begin{aligned}
f_6 &= x_4x_3 - x_3x_4 - ax_1x_2 \\
f_5 &= x_4x_2 - bx_3x_3 + x_2x_4 \\
f_4 &= x_4x_1 - cx_3x_3 + x_1x_4 \\
f_3 &= x_3x_2 - x_2x_3 + \frac{bd}{c}x_2x_4 - \frac{b^2d}{c^2}x_1x_4 \\
f_2 &= x_3x_1 - x_1x_3 + \frac{bd}{c}x_1x_4 - dx_2x_4 \\
f_1 &= x_2x_1 + x_1x_2
\end{aligned}$$

Though we will never make use of the fact, we remark that the above family was discovered by computer, and a large number of similar algebras could easily be exhibited. In particular a computer was used to solve the equations defining the space of skew polynomial rings, restricted to a sufficently small subspace to be tractable. The solutions were then sifted to find those representing regular algebras with properties of interest. We will describe the approach more specifically once we have recalled the definition of a skew polynomial ring.

Note first of all that if we let $g_6 = -f_6 - af_1$, $g_5 = f_5$, $g_4 = f_4$, $g_3 = -f_3 + \frac{bd}{c} f_5 - \frac{b^2 d}{c^2} f_4$, $g_2 = -f_2 + \frac{bd}{c} f_4 - df_5$, and $g_1 = f_1$ then we have another set of generators for the same ideal:

$$\begin{aligned} g_6 &= x_3x_4 - x_4x_3 + ax_2x_1 \\ g_5 &= x_2x_4 + x_4x_2 - bx_3x_3 \\ g_4 &= x_1x_4 + x_4x_1 - cx_3x_3 \\ g_3 &= x_2x_3 - x_3x_2 + \tfrac{bd}{c} x_4x_2 - \tfrac{b^2d}{c^2} x_4x_1 \\ g_2 &= x_1x_3 - x_3x_1 + \tfrac{bd}{c} x_4x_1 - dx_4x_2 \\ g_1 &= x_1x_2 + x_2x_1 \end{aligned} \tag{3.2.0.1}$$

So in particular the map $A \to A^{op}$ given by $x_i \mapsto x_i$ for $i = 1, \ldots, 4$ gives an isomorphism of rings.

In proving that some properties hold generically, we sometimes will need to consider only one algebra. In these cases we consider the specialization $(a = b = c = d = 1)$ A' given by the relations:

$$\begin{gathered} x_4x_3 - x_3x_4 - x_1x_2 \\ x_4x_2 + x_2x_4 - x_3x_3 \\ x_4x_1 + x_1x_4 - x_3x_3 \\ x_3x_2 - x_2x_3 + x_2x_4 - x_1x_4 \\ x_3x_1 - x_1x_3 - x_2x_4 + x_1x_4 \\ x_2x_1 + x_1x_2 \end{gathered} \tag{3.2.0.2}$$

where all of the parameters have been set equal to 1.

We will show that the algebra A is a four dimensional regular algebra, which is neither an iterated Ore extension nor a twist of one. We will show in addition that the algebra is suitably "generic", in the sense that certain associated schemes of modules are of appropriately low dimension.

In order to get some handle on the behaviour of this algebra, we will start by showing that it is a skew polynomial ring, as defined in [4]. First let us recall some relevant definitions from the literature.

Let $>_{\text{lex}}$ be the lexicographical order on monomials in x_i. That is $x_{i_0} \ldots x_{i_d} >_{\text{lex}} x_{j_0} \ldots x_{j_d}$ iff there exits n such that $i_n > j_n$, and $i_k = j_k$ for $k < n$. Also, let $>_{\text{lex}}^{\text{op}}$ be the order defined by $>_{\text{lex}}$ on opposite monomials, that is, $a >_{\text{lex}}^{\text{op}} b$ iff $a^{op} >_{\text{lex}} b^{op}$.

Given such an order, we may think of an element of $V^{\otimes n}$ as a rule for replacing the $>_{\text{lex}}$-greatest monomial appearing in it by a sum of smaller monomials. In applying a set of such reduction rules to a given polynomial, there can obviously be choices involved in which replacement rules to apply, and it is not the case that the result is necessarily unambiguous. This motivates the following:

Definition 3.2.1 *Let $\{h_i\}_{i=0}^d$ be a set of generators of a homogeneous ideal in TV. Then we will say that the h_i are a complete set of replacements iff the result of fully reducing any element of TV is unambiguous.*

We note that this is a merely a noncommutative formulation of the notion of a Grobner basis found in commutative algebra, and indeed, most of the basic arguments carry over easily. The principal difference in this case is that the noncommutative version of Buchberger's algorithm is not guaranteed to terminate (as a noncommutative polynomial ring in more than one variable is not noetherian), so a complete set of replacements can *a priori* be infinitely large.

Using this notion of a complete set of replacements, we make the following definition:

Definition 3.2.2 *An algebra $B = TV/I$ as above will be called a skew polynomial ring if the ideal I is generated by a complete set of replacements of the form $\{h_{ij}\}_{n \geq i > j \geq 0}$, where the leading monomial of h_{ij} is $x_i x_j$. We shall also sometimes refer to such a ring as left skew polynomial when we are interested in comparing this with the opposite property, which we shall refer to as right skew polynomial.*

And we note that it is not difficult to see that the following definition would be equivalent.

Definition 3.2.3 *Let V be the vector space with basis $\{x_i\}_{i=1}^n$. We will say that an algebra $B = TV/I$, defined by a homogeneous ideal I, is a skew polynomial ring if the monomials of the form $x_{j_1}x_{j_2}\dots x_{j_k}$, with $j_1 \leq j_2 \leq \cdots \leq j_k$, give a basis for the algebra as a vector space, and none of these monomials is in the span of $>_{\text{lex}}$-smaller monomials.*

To avoid confusion, we should note here that the term "skew polynomial ring" is well travelled in the literature, and has been used to refer to, among other things, iterated Ore extensions. The definition we have given above is, however, strictly more general. This is easy to see, and we will in any case be providing an example.

Moreover, while being somewhat more general, algebras satisfying the skew polynomial condition continue to possess properties good for our purposes. For example, it is easy to see that a skew polynomial ring on n generators has the same Hilbert series as a commutative polynomial ring on the same generators (not least because a commutative polynomial ring is a skew polynomial ring). Also, when dealing with skew polynomial rings, the results of [1] will provide us with a free resolution from which we may extract homological information.

Definition 3.2.4 *Let M be a set of monomials in variables y_j, such that no monomial in M divides any other (that is, there do not exist m and m' in M such that $m = am'b$). Then by a k-chain of M we will mean a monomial $y_{j_1}\dots y_{j_n}$ for which there exist integers a_j and b_j for $j = 1,\dots,k$ satisfying the following conditions:*

- *$a_1 < a_2 < b_1 \leq a_3 < b_2 \leq a_4 < \cdots < b_{k-2} \leq a_k < b_{k-1} < b_k$, or, in other words, a_j and b_j are strictly increasing sequences with $a_j < b_{j-1} \leq a_{j+1}$.*
- *$y_{j_{a_i}}y_{j_{a_i+1}}\dots y_{j_{b_i}}$ is an element of M for every $1 \leq i \leq k$.*
- *No proper factor of $y_{j_1}\dots y_{j_n}$ is a k-chain.*

To give a simple example, to illustrate the concept, if M consists of the monomials y_1y_2,y_2y_3 and y_2y_4, then the 2-chains would be $y_1y_2y_3$ and $y_1y_2y_4$. Numerous other examples are provided in the above mentioned paper, [1], wherein it is also shown that given a k-chain, the integers a_j, and b_j are uniquely determined.

In the cases we will consider, the set of monomials appearing in the above definition will consist of of the leading terms of a complete set of replacements for an algebra In general, one might need to discard redundant replacements in order that the set of leading monomials satisfy the condition of the definition, but this will not be an issue in the case of skew polynomial rings.

The main theorem of [1] defines a free resolution of the simple module k_B in terms of the chains associated to the algebra.

Theorem 3.2.5 *Let $B = TV/I$ be a k-algebra, with a minimal complete set of replacements corresponding to the relations h_i, let m_i be the $>_{\text{lex}}$-greatest monomial of h_i, $M = \{m_i\}$, and let $W^{(k)}$ be the vector space with basis the k-chains of M (and for convenience set $W^{(0)} = V$). Then k_B has a free resolution of the form*

$$0 \longleftarrow k_B \longleftarrow B \xleftarrow{\delta_0} W^{(0)} \otimes B \xleftarrow{\delta_1} W^{(1)} \otimes B \xleftarrow{\delta_2} W^{(2)} \otimes B \xleftarrow{\delta_3} \dots \tag{3.2.5.1}$$

where each free summand of each term in the resolution is in the degree of its corresponding chain.

Moreover, we may compare elements not necessarily in the same free module of the resolutions by mapping them all into TV, via the maps sending the element b of the summand corresponding to the chain c to $c \otimes b \in TV$. Comparing elements in this way, the maps δ_j have the additional property that for $b \in W^{(k)} \otimes B$, we have $b \geq_{\text{lex}} \delta_k(b)$.

Proof. The resolution guaranteed by the theorem is produced by simultaneously constructing the maps of the resolution, and splittings (as vector spaces) of their kernels, using the the Artinian property of the monomial order. We will briefly sketch this process in the highly simplified case of a skew polynomial ring.

Let use denote by $[f]$ the generator of the free summand corresponding to the j-chain $f = x_{d_1} \cdots x_{d_{j+1}}$.

Now we will say how to construct the δ_j and i_j.

Suppose, first, that the map δ_j has been constructed, and say that g is the minimal element of the kernel of δ_{j-1} on which i_j has not yet been defined. Let $m = \left[x_{d_1} \ldots x_{d_j}\right] x_{d_{j+1}} \ldots x_{d_s}$, with $x_{d_{j+1}} \ldots x_{d_s}$ reduced, be the maximum monomial appearing in g. Then since g is in the kernel, δ_{j-1} must reduce the size of the maximum monomial, but this is possible only if $x_{d_j} x_{d_{j+1}}$ is reducible, i.e., if $x_{d_1} \ldots x_{d_{j+1}}$ is a j-chain. But in this case $g' = g - \delta_j(\left[x_{d_1} \ldots x_{d_{j+1}}\right] x_{d_{j+2}} \ldots x_{d_s}$ has all monomials smaller than m, and so we may define $\imath_j$ by $i_j(g) = \left[x_{d_1} \ldots x_{d_{j+1}}\right] x_{d_{j+2}} \ldots x_{d_s} + i_j(g')$.

On the other hand, say the maps δ_j and splittings i_j have been constructed for for $j < k$. Then we may define δ_j by, for $f = x_{d_1} \ldots x_{d_{j+1}}$ and $f_L = x_{d_1} \ldots x_{d_j}$:

$$\delta_j([f]) = [f_L]\, x_{d_{j+1}} - i_{j-1}\delta_{j-1}([f_L]\, x_{d_{j+1}})$$

With δ_* and i_* defined this way we note that δ_j certainly maps into the kernel, since i_{j-1} is a splitting, and the complex we get must be exact, as we have a splitting of each kernel.

For a more detailed and general account see [1]. ■

We remark that in the preceding argument no particular properties of the monomial order were used, such as would distinguish $>_{\text{lex}}$ from $>_{\text{lex}}^{\text{op}}$. Thus we see that a similar process can also be used to construct a free resolution of ${}_A k$ by left modules.

Since we know that all of the k-chains for a skew polynomial ring are of degree $k+1$ (they are simply the products of $k+1$ decreasing variables), we can see that for such an algebra, all the maps in the above resolution are of degree 1, so the resolution is minimal. We also see that the terms in the resolution have the correct ranks for a skew polynomial ring.

We now turn to applying these results to A.

Proposition 3.2.6 *The algebra A is a skew polynomial ring with respect to the order $x_4 > x_3 > x_2 > x_1$ on the variables, and the given relations.*

Proof. In order to verify that the given relations constitute a complete set of replacements, it is enough, as in the commutative case, to verify that all degree three monomials have unambiguous reductions (see [5]). This is easily done by computer, but to illustrate, we check one of the easy cases here.

The monomial $x_3x_2x_1$ can be reduced by the given relations in two ways. Using relation f_4 we can substitute for the initial factor of x_3x_2 to yield $-(\frac{bdx_2x_4x_1}{c} - x_2x_3x_1 - \frac{b^2dx_1x_4x_1}{c^2})$, or we may use the relation f_6 to substitute for the terminal factor of x_2x_1 to give us $-x_3x_1x_2$. We must reduce these two expressions further, in order to see that they may both be reduced to the same thing.

For the first expression we have:

$$\begin{aligned}
&-\frac{bdx_2x_4x_1}{c} + x_2x_3x_1 + \frac{b^2dx_1x_4x_1}{c^2})\\
=&(\frac{b}{c}x_1 - x_2)\frac{bdx_4x_1}{c} + x_2x_3x_1\\
=&(\frac{b}{c}x_1 - x_2)(bdx_3x_3 - \frac{bdx_1x_4}{c}) + x_2x_3x_1 && (f_3)\\
=&\frac{b^2d}{c}x_1x_3x_3 - \frac{b^2d}{c^2}x_1x_1x_4 - bdx_2x_3x_3 - \frac{bd}{c}x_1x_2x_4 + x_2x_3x_1 && (f_6)
\end{aligned}$$

At this point only the last term remains unreduced. Restricting attention to it we have:

$$\begin{aligned}
&x_2x_3x_1\\
=&dx_2x_2x_4 - \frac{bd}{c}x_2x_1x_4 + x_2x_1x_3 && (f_5)\\
=&dx_2x_2x_4 + \frac{bd}{c}x_1x_2x_4 - x_1x_2x_3 && (f_6)
\end{aligned}$$

And so we see that the first expression reduces to:

$$\frac{b^2d}{c}x_1x_3x_3 - \frac{b^2d}{c^2}x_1x_1x_4 - bdx_2x_3x_3 + dx_2x_2x_4 - x_1x_2x_3$$

For the second expression we have:

$$\begin{aligned}
&- x_3x_1x_2 \\
=&\frac{bd}{c}x_1x_4x_2 - dx_2x_4x_2 - x_1x_3x_2 && (f_5) \\
=&(\frac{bd}{c}x_1 - dx_2)x_4x_2 - x_1x_3x_2 \\
=&(\frac{bd}{c}x_1 - dx_2)(bx_3x_3 - x_2x_4) - x_1x_3x_2 && (f_2) \\
=&\frac{b^2d}{c}x_1x_3x_3 - \frac{bd}{c}x_1x_2x_4 - bdx_2x_3x_3 + dx_2x_2x_4 - x_1x_3x_2
\end{aligned}$$

Again we have an expression where only the last term remains unreduced. Considering this term, we have:

$$\begin{aligned}
&- x_1x_3x_2 \\
=&\frac{bd}{c}x_1x_2x_4 - x_1x_2x_3 - \frac{b^2d}{c^2}x_1x_1x_4 && (f_4)
\end{aligned}$$

Thus the second expression reduces to:

$$\frac{b^2d}{c}x_1x_3x_3 - bdx_2x_3x_3 + dx_2x_2x_4 - x_1x_2x_3 - \frac{b^2d}{c^2}x_1x_1x_4$$

And as these two reductions are equal, we see that the reduction for the monomial $x_3x_2x_1$ is unambiguous. ■

The forgoing proof should also make clear the process by which the algebra A was discovered, but in any case we shall now outline our method. Starting with the most general set of replacements

$$x_4x_3 = a_1x_3x_4 + a_2x_3x_3 + a_3x_2x_4 + a_4x_2x_3 + a_5x_2x_2 + a_6x_1x_4 + a_7x_1x_3 + a_8x_1x_2 + a_9x_1x_1$$

$$x_4x_2 = b_1x_3x_4 + b_2x_3x_3 + b_3x_2x_4 + b_4x_2x_3 + b_5x_2x_2 + b_6x_1x_4 + b_7x_1x_3 + b_8x_1x_2 + b_9x_1x_1$$

$$x_4x_1 = c_1x_3x_4 + c_2x_3x_3 + c_3x_2x_4 + c_4x_2x_3 + c_5x_2x_2 + c_6x_1x_4 + c_7x_1x_3 + c_8x_1x_2 + c_9x_1x_1$$

$$x_3x_2 = d_1x_2x_4 + d_2x_2x_3 + d_3x_2x_2 + d_4x_1x_4 + d_5x_1x_3 + d_6x_1x_2 + d_7x_1x_1$$

$$x_3x_1 = e_1x_2x_4 + e_2x_2x_3 + e_3x_2x_2 + e_4x_1x_4 + e_5x_1x_3 + e_6x_1x_2 + e_7x_1x_1$$

$$x_2x_1 = f_1x_1x_4 + f_2x_1x_3 + f_3x_1x_2 + f_4x_1x_1$$

we may reduce each of the four 2-chains ($x_4x_3x_2$, $x_4x_3x_1$, $x_4x_2x_1$, $x_3x_2x_1$) in two possible ways, starting on the left side or on the right side. In order that the replacements be the replacements of a skew polynomial ring, it is necessary and sufficient that in each case the two reductions are equal. Equating coefficients, this gives a number of polynomial conditions on the variables a_i,b_i,c_i, d_i,e_i, and f_i. In the most general situation, the resulting system is quite complicated, and cannot easily be solved. However, by restricting to a subspace, for example by setting some of the variables to zero, it is possible to shrink the system enough to solve.

In practice this procedure allows one to produce a large number of families of skew polynomial rings. Using, for example, the algorithm of 3.2.5 one may filter out the regular algebras, and impose other conditions as desired.

3.3 Some Algebraic Properties of A

We will now prove some of the further properties of A.

Proposition 3.3.1 *For generic choices of the parameters, the algebra A is not an iterated ore extension. In particular, A' is not an iterated Ore extension.*

Proof. An iterated Ore extension contains, by the very nature of being an iterated extension, subalgebras with the Hilbert series of polynomial rings on fewer variables. We will show that the algebra A' does not contain a subalgebra on three degree 1 generators with sufficiently small growth for this to be the case. Since the rank of a continuously varying set of vectors is lower semicontinuous, it will follow that A is generically not an Ore extension.

To see that there is not a sufficiently slow growing subalgebra of A', it will suffice to show that there does not exist a three dimensional subspace $U \subset V$ such that $U \otimes U \to A_2$ spans less than 7 dimensions. We verify this on each of a set of affine subschemes covering the Grassmanian of 3 dimensional subspaces of V, making use of the skew polynomial property of A to reduce every element we consider to a canonical form.

The four affine charts which we will examine are $U_1 =< x_1 + ax_4, x_2 + bx_4, x_3 + cx_4 >$, $U_2 =< x_1 + ax_3, x_2 + bx_3, x_4 >$, $U_3 < x_1 + ax_2, x_3, x_4 >$ and $U_4 =< x_2, x_3, x_4 >$, where a, b and c are the affine coordinates.

Let us consider the subalgebra of $B \subset A$ generated by the elements of U_1. Taking all possible degree 2 monomials in the given basis of U_1, and reducing them each to a unique sum of irreducible monomials (via the skew-polynomial relations), we find that B_2 is spanned by the following elements:

$$
\begin{aligned}
v_1 &= a^2x_4x_4 + ax_3x_3 + x_1x_1\\
v_2 &= abx_4x_4 + ax_3x_3 - ax_2x_4 + bx_1x_4 + x_1x_2\\
v_3 &= acx_4x_4 + ax_3x_4 + cx_1x_4 + x_1x_3 + ax_1x_2\\
v_4 &= abx_4x_4 + bx_3x_3 + ax_2x_4 - bx_1x_4 - x_1x_2\\
v_5 &= b^2x_4x_4 + bx_3x_3 + x_2x_2\\
v_6 &= bcx_4x_4 + bx_3x_4 + cx_2x_4 + x_2x_3 + bx_1x_2\\
v_7 &= acx_4x_4 + ax_3x_4 + cx_3x_3 + x_2x_4 - (1+c)x_1x_4 + x_1x_3\\
v_8 &= bcx_4x_4 + bx_3x_4 + cx_3x_3 - (1+c)x_2x_4 + x_2x_3 + x_1x_4\\
v_9 &= c^2x_4x_4 + 2cx_3x_4 + x_3x_3 + cx_1x_2
\end{aligned}
$$

We wish to see that for no choice of the coordinates can these elements span less than a seven dimensional space. With respect to the obvious basis, we can represent these elements as rows in the following matrix, which we must now show always has rank at least seven:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & a^2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & a & 0 & b & -a & 0 & ba \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & c & 0 & a & ca \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & b & 0 & -b & a & 0 & ba \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & b^2 \\
0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & c & b & cb \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & c & 0 & -1-c & 1 & a & ca \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & c & 0 & 1 & -1-c & b & cb \\
0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2c & c^2
\end{pmatrix} \tag{3.3.1.1}$$

Restricting attention to seven of the rows, and ignoring the zero columns, we have:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & a^2 \\
0 & 1 & 0 & 0 & 0 & a & b & -a & 0 & ba \\
0 & 0 & 1 & 0 & 0 & b & 0 & 0 & 0 & b^2 \\
0 & a & 0 & 1 & 0 & 0 & c & 0 & a & ca \\
0 & 0 & 0 & 0 & 1 & c & 1 & -1-c & b & cb \\
0 & c & 0 & 0 & 0 & 1 & 0 & 0 & 2c & c^2 \\
0 & 0 & 0 & 1 & 0 & c & -1-c & 1 & a & ca
\end{pmatrix} \tag{3.3.1.2}$$

By a few row operations we are reduced to the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & a^2 \\ 0 & 1 & 0 & 0 & 0 & a & b & -a & 0 & ba \\ 0 & 0 & 1 & 0 & 0 & b & 0 & 0 & 0 & b^2 \\ 0 & 0 & 0 & 1 & 0 & -a^2 & c-ba & a^2 & a & ca-a^2b \\ 0 & 0 & 0 & 0 & 1 & c & 1 & -1-c & b & cb \\ 0 & 0 & 0 & 0 & 0 & 1-ca & -cb & ca & 2c & c^2-cba \\ 0 & 0 & 0 & 0 & 0 & c+a^2 & -1-2c+ba & 1-a^2 & 0 & a^2b \end{pmatrix} \tag{3.3.1.3}$$

Now it is clear that the first 5 rows are linearly independent, and that their span will not intersect the span of the last two rows. Thus it will be enough to check that the following matrix has maximal rank:

$$\begin{pmatrix} 1-ca & -cb & ca & 2c & c^2-cba \\ c+a^2 & -1-2c+ba & 1-a^2 & 0 & a^2b \end{pmatrix} \tag{3.3.1.4}$$

Say that this matrix were not maximal rank. Then the last two columns tell us that $abc = 0$, and so one of a, b or c is zero. We see by inspection that in each of these cases the matrix must have rank 2.

On the second chart we have (ignoring zero columns, and permuting rows) the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 2b & 0 & a^2 & -a & a & 0 & 0 \\ 0 & 1 & 0 & b & a & ab & a & -a & 0 & 0 \\ 0 & 0 & 1 & 0 & 2b & b^2 & & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & b & a & ab & -b & b & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 1 & -1 & 0 & a & 0 \\ 0 & b & 0 & 0 & 0 & 1 & 0 & -1 & b & 0 \end{pmatrix} \tag{3.3.1.5}$$

We can see that the the first six rows span a six dimensional space. If either a or b is zero then the eighth or ninth row (respectively) has a leading 1 in the 6th column and the matrix has rank at least 7. Otherwise, the sum of the second and seventh rows has a leading non-zero entry in the 6th column, and the matrix again has rank ≥ 7.

On the third chart we can pick seven of the rows to give us:

$$\begin{pmatrix} 1 & 0 & a^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.3.1.6}$$

which is clearly full rank. The final chart consists of a single point, at which we can see by inspection that the matrix is rank 8. ■

Proposition 3.3.2 *The algebra A' is an AS-regular algebra of dimension* 4.

Proof. We must verify the three conditions of AS-regularity: that the algebra has correct GK-dimension, that k has a projective resolution of the appropriate length, and that the algebra is Gorenstein. We will address these conditions in turn.

As remarked above, the condition that A is a skew polynomial ring on four generators automatically implies that it has the same Hilbert series as a a commutative polynomial ring on four generators, hence the GK-dimension condition is satisfied.

As we have already noted, in the case of a skew polynomial ring, Theorem 1.4 of [1], provides us with a minimal free resolution of k_A, thus showing that A has the correct global dimension. Moreover, for any specific algebra we could use this explicitly given resolution to verify that the Ext groups take the correct values for the algebra to be AS-regular. The calculation is, however, omitted, as it is subsumed by the following lemma. ■

Lemma 3.3.3 *Let B be an algebra on the generators $\{x_i\}_{i=1}^4$, which is both a left skew polynomial ring, and a right skew polynomial ring. Suppose in addition that the replacement for x_2x_1 has the form $x_2x_1 = ax_1x_2 + bx_1x_1$ for some scalars a and b (a necessarily being non-zero). Then B is AS-regular.*

As noted above, the conditions on projective and GK dimension follow immediately from the skewness of B. Thus we need only prove the Gorenstein condition. We will prove this by considering the resolutions provided by 3.2.5.

Since B is a left and right skew polynomial ring, we get immediately two resolutions:

$$L^* : 0 \longleftarrow k_B \longleftarrow B \xleftarrow{M_0} W^{(0)} \otimes B \xleftarrow{M_1} W^{(1)} B \xleftarrow{M_2} W^{(2)} \otimes B \xleftarrow{M_3} W^{(3)} \otimes B \longleftarrow 0 \qquad (3.3.3.1)$$

(the M_j being the matrices corresponding to the δ_j in 3.2.5) and

$$R^* : 0 \to W^{(3)} \otimes B \stackrel{N_3}{\to} W^{(2)} \otimes B \stackrel{N_2}{\to} W^{(1)} \otimes B \stackrel{N_1}{\to} W^{(0)} \otimes B \stackrel{N_0}{\to} B \to_B k \to 0$$

If we could provide an isomorphism of complexes between one of these and the dual of the other, then we would clearly be done. If we knew a priori that B was regular, then we could produce such an isomorphism from Lemma 2.0.3, since in that case we would know that the entries of $M_i \dots M_3$ and $N_{3-i} \dots N_0$, for $i = 0, \dots, 3$, were bases of the same space, and the change of basis matrix could give the required map. Thus we shall prove that the conclusion of the lemma holds in this case.

Note that by the way the resolution is constructed in 3.2.5, the entry of $M_0M_1M_2$ corresponding to the 2-chain $x_ix_jx_k$, with $i > j > k$, has greatest monomial $x_ix_jx_k$. Thus in particular, we see that the entries of this matrix are all linearly independent. We will let U be their span, and u_{ijk} the element of with greatest monomial the 2-chain $x_ix_jx_k$, and let u_{ijk}^* be the dual basis of U^*.

We will first show that the span of the entries of M_0 equals the span of the entries of M_3. By construction, the span of the entries of M_0 is all of V, so we must show that the span of the entries of M_3 contains a basis for V. Also, since then entries of $M_0M_1M_2$ span U, the span of the entries of M_3 must equal $U^* \vdash w(B)$.

Now consider the map given by M_3. By the construction of the resolution, and our hypothesis on the replacement for x_2x_1, we have

$$\delta_3([x_4x_3x_2x_1]) = [x_4x_3x_2]\,x_1 - i_2([x_4x_3]\,ax_1x_2 + [x_4x_3]\,bx_1x_1 + g)$$

where g consists of monomials smaller than $x_4x_3x_1x_1$. In particular, g cannot contain monomials of the form $x_4x_3x_2*$ or $x_4x_3x_1*$. Thus two of the entries of M_3 are x_1 and $ax_2 + bx_1$.

Next, we may assume that the replacement for x_4x_3 involves x_3x_4. If not, then in any case, since B is a left and right skew polynomial ring, x_3x_4 must appear in one of the replacements. Due to the constraints imposed by the monomial order, it must appear in either the replacement for x_4x_1 or x_4x_2. But then, by a change of variables of the form $x_3 \mapsto x_3 + \alpha x_2 + \beta x_1$ (which preserves skewness), we may modify the replacements such that the replacement for x_4x_3 does involve the monomial x_3x_4.

Now, having made this assumption, we will consider the resolution of ${}_Bk$ mentioned in the remark after 3.2.5. This resolution has the form

$$0 \to B \otimes W^{(3)} \to B \otimes W^{(2)} \to B \otimes W^{(1)} \to B \otimes W^{(0)} \to B \to_B k \qquad (3.3.3.2)$$

and the leftmost map is defined by

$$[x_4x_3x_2x_1] \mapsto x_4[x_3x_2x_1] - i_2(x_3x_4[x_2x_1] + g)$$

where g consists of monomials smaller than $x_3x_4x_2x_1$. Thus we have that $w(B) \dashv u^*_{321}$ is non-zero scalar times a element of V of the form $x_4 + c_3x_3 + c_2x_2 + c_1x_1$ and $w(B) \dashv u^*_{421}$ is a scalar times an element of the form $x_3 + d_2x_2 + d_1x_1$.

Since this result is true for an arbitrary skew polynomial ring, the corresponding result must be true for B considered as a right skew polynomial ring. And so, in particular, $U^* \dashv w(B)$ contains elements of the form $x_4 + c_3x_3 + c_2x_2 + c_1x_1$ and $x_3 + d_2x_2 + d_1x_1$. Thus the entries of M_3 span V, as desired. From this is immediately follows that $V^* \vdash w(B) = U$.

Finally we must show that the entries of M_2M_3 span I. To do this it will suffice, from what we already know of U, to show that the elements of $V^*\vdash U$ span I.

In order to see this we will consider the maps defined in Anick's resolution applied to various 2-chains. Looking at the images of $[x_4x_3x_1]$, $[x_4x_2x_1]$, and $[x_3x_2x_1]$ by the map δ_2 in 3.3.3.1 we get, respectively:

$$\delta_2([x_4x_3x_1]) = [x_4x_3]x_1 + g_{431}$$
$$\delta_2([x_4x_2x_1]) = [x_4x_2]x_1 + g_{421}$$
$$\delta_2([x_3x_2x_1]) = [x_3x_2]x_1 + g_{321}$$

where g_{ijk} consists of monomials smaller than $x_ix_jx_k$. Hence $U\dashv x_1^*$ contains elements with maximum monomials x_4x_3, x_4x_2 and x_3x_2. Considering the images of $[x_4x_3x_1]$, $[x_4x_2x_1]$ by the map δ_2 in 3.3.3.2, we have:

$$\delta_2([x_4x_3x_1]) = x_4[x_3x_1] - i_1(x_3x_4[x_1] + g_{341})$$
$$\delta_2([x_4x_2x_1]) = x_4[x_2x_1] + g_{421}$$

Thus $x_4^*\vdash U$ contains the relations with greatest monomials x_3x_1 and x_2x_1, and $x_3^*\vdash U$ contains an element with greather monomial x_4x_1. Finally, since both $V^*\vdash U$ and $U\dashv V^*$ are equal to the span of the entries of M_1M_2, we have $V^*\vdash U = U\dashv V^* = I$.

Thus B is Gorenstein, and so AS-regular, as claimed. ■

3.4 Geometry of A

We now turn to the question of calculating some of the associated schemes of modules for this algebra. This information about A is interesting in its own right, but will also be useful when we calculate the automorphism group of A'.

Proposition 3.4.1 *The algebra A has finite scheme of truncated points of length 2, for general choices of the parameters. In particular, the point scheme of A' consists of the* *points*

$$(1,0,0,0) \qquad (0,1,0,0) \qquad (0,0,0,1)$$
$$(\lambda, 1, \tfrac{-8\lambda^3}{(\lambda+1)^3(\lambda-1)^2}, \tfrac{-4\lambda^2}{(\lambda+1)(\lambda-1)^2})$$

where λ *is a root of the polynomial* $x^{10} - 5x^8 + 74x^6 + 54x^4 + 5x^2 - 1$ *and the associated automorphism of the point scheme acts by the identity on the first three points, and by taking the point associated to the root* λ *to the point associated to the root* $-\lambda$ *on the other 10 points.*

Proof.

Since the dimension of the point scheme can only jump on a closed subspace, it will be sufficient to prove the claim for the particular algebra A'.

As mentioned before, the scheme of truncated, right point modules is the scheme, Γ, cut out of $\mathbf{P}(V^*) \times \mathbf{P}(V^*) = \mathbf{P}^3 \times \mathbf{P}^3$ by the above relations, considered as elements of $\mathcal{O}_{\mathbf{P}\times\mathbf{P}}(1,1)$. If we write the relations as $M\mathbf{x}$, where:

$$M = \begin{pmatrix} 0 & -x_1 & x_4 & -x_3 \\ 0 & x_4 & -x_3 & x_2 \\ x_4 & 0 & -x_3 & x_1 \\ 0 & x_3 & -x_2 & x_2 - x_1 \\ x_3 & 0 & -x_1 & x_1 - x_2 \\ x_2 & x_1 & 0 & 0 \end{pmatrix} \tag{3.4.1.1}$$

then the first projection of the above set will be the locus where M has less than maximal rank. So we start by calculating this locus.

We see by inspection that the the matrix is singular at the points $(1,0,0,0)$, $(0,1,0,0)$, and $(0,0,0,1)$. Moreover, it is easy to see by checking the cases that if any one of x_1, x_2, x_3 or x_4 is zero, then the matrix can be singular only if three of them are. For example, in the case that $x_1 = 0$, the matrix is:

$$\begin{pmatrix} 0 & 0 & x_4 & x_3 \\ 0 & x_4 & -x_3 & x_2 \\ x_4 & 0 & -x_3 & 0 \\ 0 & x_3 & -x_2 & x_2 \\ x_3 & 0 & 0 & -x_2 \\ x_2 & 0 & 0 & 0 \end{pmatrix}$$

which if, say, $x_2 \neq 0$, is singular iff and only if

$$\begin{pmatrix} 0 & x_4x_3 & 0 & 0 \\ 0 & x_2x_4 - {x_3}^2 & 0 & 0 \\ 0 & {x_3}^2 & 0 & 0 \\ 0 & x_3 & -x_2 & 0 \\ 0 & 0 & 0 & x_2 \\ x_2 & 0 & 0 & 0 \end{pmatrix}$$

is. And this last matrix can clearly only be singular if $x_3 = x_4 = 0$. The other eleven cases are similar.

Thus we see that the point scheme can consist of only the points listed above, as well as points for which all of the coordinates are non-zero. We end by enumerating these points.

Under the assumption that all of the variables are non-zero we are free to row reduce the matrix, and we arrive at:

$$\begin{pmatrix} x_4 & 0 & -x_3 & x_1 \\ 0 & x_4 & -x_3 & x_2 \\ 0 & 0 & {x_4}^2 - x_3x_1 & x_3x_4 + x_2x_1 \\ 0 & 0 & {x_3}^2 - x_2x_4 & x_2x_4 - x_1x_4 - x_2x_3 \\ 0 & 0 & {x_3}^2 - x_1x_4 & x_1x_4 - x_2x_4 - x_3x_1 \\ 0 & 0 & x_2x_3 + x_3x_1 & -2x_2x_1 \end{pmatrix}$$

Now since we know that $x_4 \neq 0$, it will suffice to find the points where the submatrix

$$\begin{pmatrix} {x_4}^2 - x_3x_1 & x_3x_4 + x_2x_1 \\ {x_3}^2 - x_2x_4 & x_2x_4 - x_1x_4 - x_2x_3 \\ {x_3}^2 - x_1x_4 & x_1x_4 - x_2x_4 - x_3x_1 \\ x_2x_3 + x_3x_1 & -2x_2x_1 \end{pmatrix}$$

does not have full rank. We can simplify the matrix slightly by further row operations to:

$$\begin{pmatrix} 2{x_4}^2 - x_1x_3 + x_2x_3 & -2x_3x_4 \\ 2{x_3}^2 - x_4x_2 - x_1x_4 & -x_2x_3 - x_1x_3 \\ -x_1x_4 + x_4x_2 & 2x_1x_4 - 2x_4x_2 - x_1x_3 + x_2x_3 \\ x_1x_3 + x_2x_3 & -2x_1x_2 \end{pmatrix} \tag{3.4.1.2}$$

The minors of this matrix are (ignoring non-zero factors):

$$\begin{gathered} 4x_2{x_4}^2 + 4x_1{x_4}^2 - {x_1}^2x_3 + {x_2}^2x_3 - 4{x_3}^2x_4 \\ (x_1 - x_2)(2x_2x_3x_4 - x_2{x_3}^2 + 4{x_4}^3 - 4x_3{x_4}^2 - 2x_1x_3x_4 + x_1{x_3}^2) \\ 2{x_1}^2x_2x_3 - 4x_1x_2{x_4}^2 - 2x_1{x_2}^2x_3 + 2x_1{x_3}^2x_4 + 2x_2{x_3}^2x_4 \\ 2(x_2 - x_1)(x_2{x_4}^4 - 2{x_3}^2x_4 + {x_3}^3 + x_1{x_4}^2) \\ 2x_1{x_2}^2x_4 - 2x_1x_2{x_3}^2 + 2{x_1}^2x_2x_4 + {x_2}^2{x_3}^2 + {x_1}^2{x_3}^2 \\ (x_2 - x_1)(2x_2x_3x_4 - 2x_1x_2x_4 - x_2{x_3}^2 + 2x_1x_3x_4 - x_1{x_3}^2) \end{gathered} \tag{3.4.1.3}$$

We can eliminate x_3 and x_4 from these equations to be left with

$$\begin{gathered} x_4 = \frac{-8{x_1}^3{x_2}^3}{(x_1 + x_2)^3(x_2 - x_1)^2} \\ x_3 = \frac{-4{x_1}^2{x_2}^2}{(x_1 + x_2)(x_2 - x_1)^2} \\ {x_1}^{10} - 5{x_1}^8{x_2}^2 + 74{x_1}^6{x_2}^4 + 54{x_1}^4{x_2}^6 + 5{x_1}^2{x_2}^8 - {x_2}^{10} \end{gathered}$$

from which we see that the first projection is zero dimensional, and that the point scheme consists of the claimed points.

Since we know that A is isomorphic to A^{op}, it is immediately clear that the second projection of the scheme of truncated point modules is also zero-dimensional, and is in fact the same.

Now, as the two projections of Γ are zero dimensional, Γ must be as well, and so it follows from the proof of [9, theorem 1.4] that Γ is actually the graph of an automorphism.

To see that the automorphism of this scheme is as described we can, for the first three points, easily verify by inspection that 3.4.1.1 has the correct kernel at the three specializations. For the remaining points, we note that if the automorphism acted as the identity, then by [11, theorem 4.1], we would have $\wedge^2 V \subset I$, since the point scheme would be contained in the diagonal, and this is obviously not the case.

Finally,since the galois group associated to $x^5 - 5x^4 + 74x^3 + 54x^2 + 5x - 1$ is all of S_5, we see that the only remaining possibility is the one described, since no non-trivial permutation of the roots can be equivariant for the action of the Galois group. ■

Corollary 3.4.2 *The point scheme of A is zero dimensional.*

Proof. This follows from the previous result and the fact that since Γ is the graph of an isomorphism, every truncated point module can be extended to a point module. ■

Using our knowledge of the point variety, we can determine the automorphism group of A'.

Proposition 3.4.3 *The automorphism group of A' is trivial.*

Proof. Let $\sigma : A' \to A'$ be an automorphism of A'. Then the dual action of σ^* on $\mathbf{P}(V^*) \times \mathbf{P}(V^*)$ must preserve Γ, and in particular permute the points $(1,0,0,0) \times (1,0,0,0)$, $(0,1,0,0) \times (0,1,0,0)$ and $(0,0,0,1) \times (0,0,0,1)$. Thus the action of σ on V must be a permutation, ρ, of the three variables x_1, x_2, and x_4 times a nonsingular matrix of the form:

$$\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ b_1 & b_2 & a_3 & b_4 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

First say $b_i \neq 0$ for some i, and let $j \in \{1,2,4\}$ be a number other than $\rho^{-1}(i)$. Then either $x_{\rho^{-1}(i)}x_j + x_jx_{\rho^{-1}(i)}$ or $x_{\rho^{-1}(i)}x_j + x_jx_{\rho^{-1}(i)} - x_3x_3$ is one of the relations. In either case, applying σ gives a polynomial for which the coefficients of the monomials x_3x_j and x_jx_3 are both equal to the non-zero value a_jb_i. Since for every element of I the coefficients of x_jx_3 is the negative of the coefficient of x_3x_j, σ cannot preserve I in this case. Thus for σ to be an automorphism of A' we must have $b_i = 0$ for all i.

Now, since σ takes $x_1x_2 + x_2x_1$ to an element not involving x_3, it must go to a multiple of itself, thus ρ is either the identity or swaps x_1 and x_2. Consider the first case. Since σ preserves the span of each monomial, it must also preserve the span of each of the relations $\{f_i\}_{i=1}^{6}$ in our complete set of replacements, that is, they must each be eigenvectors for σ. Applying σ to these eigenvectors yields that the following monomials in the a_i are all equal: a_1a_3, a_2a_4, a_1a_4, a_2a_3 and a_3a_3. Since the a_i are all non-zero, they must all be equal. In the case where ρ exchanges x_1 and x_2, a similar argument again shows that all the the a_i must be equal, however this does not provide an automorphism of the algebra, since the relation $x_4x_3 - x_3x_4 - x_1x_2$ is mapped to an element not in I.

Thus the automorphisms of A' are scalars. ■

We note that the foregoing result actually holds generically in the family of algebras A.

We see from the preceding proposition, and 3.3.1, that A' is in fact not even a twist by an automorphism of an iterated Ore extension, as we desired. Moreover, the fact that A' has no non-trivial automorphism also implies that the only normal elements of A' are in fact central. The next result calculates the central subring of A'.

Proposition 3.4.4 *The center of A' is isomorphic to a commutative polynomial ring in two variables, with generators in degree two.*

Proof. We will consider the following two elements of A'_2:

$$f_7 = x_2x_2 + x_1x_1$$

$$f_8 = x_4x_4 - x_2x_4 + x_2x_3 + x_1x_4 - x_1x_3 + x_1x_1$$

It is easy to see that these two elements are central in A', merely by verifying that each commutes with the generators x_i. Additionally, by verifying the consistent reduction of the new overlaps, we see that combining these two elements with the relations of A' gives us a new set $S = \{f_i\}_{i=1}^{8}$ which is again a complete set of replacements.

Let $B = A/(f_7, f_8)$ be the quotient of A by these two new elements. Since the f_i are a complete set of replacements, we know that a basis for B as a k-algebra is given by the irreducible monomials, that is, those monomials of the form $x_{j_1} \dots x_{j_d}$ where the j_i are increasing, with at most one equal to 2 and at most one equal to 4. Counting, we find that

$$\dim_k B_i = \begin{cases} 1 & \text{if i=0} \\ 4i & \text{otherwise.} \end{cases}$$

And, in particular $\dim_k B_i = \binom{i+3}{3} - 2\binom{i+1}{3} + \binom{i-1}{3}$, which tells us that the elements f_7 and f_8 were regular.

Let C be the subalgebra generated by f_7 and f_8. To see that C is in fact the entire center of A', we will show that B has no central elements and use the following lemma (which is presumably well known).

Lemma 3.4.5 *Let R be a graded k-algebra, and S a finitely generated central subalgebra generated by a regular sequence of elements of positive degree. Then, if there exists a central element not in S, the quotient $R/(S_{>0})$ contains a non-zero central element.*

Proof. We will prove this by induction on the number of generators of S. Let $s_1, \dots, s_n$ be the generators of S, and let c be a homogeneous central element of R outside of S, of minimal possible degree. If we show that $R/(s_1)$ contains a central element outside of $S/(s_1)$, then the result will follow. The only way such an element can fail to exist is if c goes to $S/(s_1)$ in the quotient, that is, if c can be written $c = ds_1 + f$ for some elements $d \in R$ and $f \in S$. But for every $x \in R$ we have $xds_1 = ds_1x = dxs_1$, and as s_1 is regular $dx = xd$. Since d is lower degree than c it must be zero, and so $c \in S$ contradicting the assumption that c is not in this subalgebra. Thus c cannot map to the image of S, and the central element we required exist. ■

Continuing with the proof of the proposition, we will now show that B has no central elements. First let us calculate which elements commute with x_1.

For every monomial m of degree d we will calculate the leading monomial in the reduction of mx_1. Since left multiplying by x_1 will never make a reduced monomial unreduced, it is enough to consider monomials of the form $x_3 \dots x_3 x_4$, $x_3 \dots x_3$, $x_2 x_3 \dots x_3 x_4$ and $x_2 x_3 \dots x_3$. We consider the four cases in turn.

(i) In the case ${x_3}^n x_4 x_1$, the leading monomial in the reduction must clearly be ${x_3}^{n+2}$, since this is the greatest monomial remaining after one reduction step, and it is already reduced.

(ii) In the reduction of $x_2 {x_3}^n x_4 x_1$ the leading term in the reduction will be $x_2 {x_3}^{n+2}$, by the same reasoning as in the first case.

(iii) In the case ${x_3}^{n-1} x_1$ we first note that every reduced monomial appearing in the reduction must be less than or equal to $x_2 {x_3}^{n-2} x_4$. To see this note that it follows by induction on the power of x_3 that the leading variable must be less than x_3, and the specified monomial is the greatest irreducible one with this property. To see that the coefficient on $x_2 {x_3}^{n-2} x_4$ in the reduction is actually non-zero, we will show by induction that the reduction of ${x_3}^{n-1} x_1$ has the form $n x_2 {x_3}^{n-2} x_4 - n x_1 {x_3}^{n-2} x_4 + x_1 {x_3}^{n-1} + {x_1}^2(\dots)$. The result is clearly true for $n = 0$, now assume it holds for $n-1$. Then we have ${x_3}^n x_1 = (n-1) x_3 x_2 {x_3}^{n-2} x_4 - (n-1) x_3 x_1 {x_3}^{n-2} x_4 + x_3 x_1 {x_3}^{n-1} + {x_1}^2(\dots)$. Since $x_4 {x_3}^n$ reduces to ${x_3}^n x_4 + x_2(\dots) + x_1(\dots)$, we see that the sum of the first two terms above reduces to $(n-1) x_2 {x_3}^{n-1} x_4 - (n-1) x_1 {x_3}^{n-1} x_4 + x_1 x_1(\dots)$, and for the remaining term we have

$$\begin{aligned} x_3 x_1 {x_3}^{n-2} &= x_1 {x_3}^{n-1} + x_2 x_4 {x_3}^{n-2} - x_1 x_4 {x_3}^{n-2} \\ &= x_1 {x_3}^{n-1} + x_2 {x_3}^{n-2} x_4 - x_1 {x_3}^{n-2} x_4 + {x_1}^2(\dots). \end{aligned}$$

Adding these together, we find that the reduction of ${x_3}^{n-2} x_1$ has the required form.

(iv) The leading term in the reduction of $x_2 {x_3}^{n-2} x_1$ is $x_1 x_2 x_3 {x_3}^{n-3} x_4$. We can prove this in a manner similar to the previous case.

Monomial	Leading Term in Reduction
${x_3}^{n-2}x_4x_1$	${x_3}^n$
${x_3}^{n-1}x_1$	$x_2{x_3}^{n-2}x_4$
$x_2{x_3}^{n-3}x_4x_1$	$x_2{x_3}^{n-1}$
$x_2{x_3}^{n-2}x_1$	$x_1x_2{x_3}^{n-3}x_4$

Table 3.1: Leading terms of reductions

The results are summarized in table 3.1.

Note that, for a distinct monomials, m and m', appearing in the table we have distinct maximum monomials appearing appearing in the reductions of mx_1 and $m'x_1$. And also that the largest monomial appearing in the reduction of mx_1 is strictly larger than x_1m. We will denote the maximal monomial appearing in the reduction of mx_1 by $\mathrm{lm}_1\, m$.

Now suppose we have an element $c = \Sigma\alpha_i m_i$ in the center of B, where the α_i are in k, and the m_i are monomials. We may assume that m_1 is the monomial with maximum $\mathrm{lm}_1\, m$ out of all the monomials appearing in the sum. If m_1 is a monomial from the table, or a monomial from the table multiplied on the left by ${x_1}^m$, then by what we have just said $m_1x_1 - x_1m_1$, and hence $cx_1 - x_1c$, will involve a non-zero multiple of some irreducible monomial greater than x_1c. This contradicts the assumption that c is central. Thus it must be the case that all of the monomials in c involve only the variables x_1 and x_2. In particular, they are each either of the from $xi \ldots x_1x_2$ or $xi \ldots x_1$. We can verify by inspection that the first of these does not commute with x_1, and that the second one commutes with x_2 only when it is of even degree.

To rule out the possibility that any of the remaining monomials, ${x_1}^n$, are central, we will show that none of them commute with x_3. In particular, we will show that for $n \geq 3$ we have

$$x_3{x_1}^{n-1} = {x_1}^{n-3}(a_1(n)x_2{x_3}^2 + a_2(n)x_1{x_3}^2 + a_3(n)x_1x_2x_4 + a_4(n){x_1}^2x_4 + a_5(n){x_1}^2x_3) \tag{3.4.5.1}$$

where

$$
\begin{aligned}
a_1(n) = -a_2 &= \lfloor n-1 \rfloor \\
a_3(n) &= n-1 \\
a_4(n) &= \begin{cases} -1 & \text{if n is even} \\ 0 & \text{otherwise} \end{cases} \\
a_5(n) &= 1
\end{aligned} \tag{3.4.5.2}
$$

By inspection this equation holds for $n = 3$ and we have the following reductions

$$
\begin{aligned}
x_2{x_3}^2x_1 &= 2x_1x_2x_3x_4 - x_1x_2{x_3}^2 + {x_1}^3x_2 - x_14 \\
x_1x_3x_3x_1 &= 2x_1x_2x_3x_4 - 2{x_1}^2x_3x_4 + {x_1}^2{x_3}^3 + {x_1}^3x_2 - {x_1}^4 \\
x_1x_2x_4x_1 &= x_1x_2{x_3}^2 + {x_1}^2x_2x_4 \\
{x_1}^2x_4x_1 &= {x_1}^2{x_3}^2 - {x_1}^3x_4 \\
{x_1}^2x_3x_1 &= {x_1}^2x_2x_4 - {x_1}^3x_4 + {x_1}^3x_3.
\end{aligned}
$$

Thus if 3.4.5.1 holds in degree $n-1$, then the monomials ${x_1}^{n-3}x_2x_3x_4$, ${x_1}^{n-2}x_3x_4$, ${x_1}^{n-1}x_2$ and ${x_1}^n$ cancel out and do not appear in degree n. For the other monomials we have the recursive formulae

$$
\begin{aligned}
a_1(n) &= a_3(n-1) - a_1(n-1) \\
a_2(n) &= a_4(n-1) + a_2(n-1) \\
a_3(n) &= a_5(n-1) + a_3(n-1) \\
a_4(n) &= -a_5(n-1) - a_4(n-1) \\
a_5(n) &= a_5(n-1)
\end{aligned}
$$

and substituting the previous values, we see that $a_i(n)$ take the values claimed. Thus, in particular, since $a_4(n)$ is always positive, we see that it is never the case that $x_3{x_1}^{n-1} = {x_1}^{n-1}x_3$, and so ${x_1}^{n-1}$ cannot be central. ■

It is interesting to note that, though it goes unremarked, the algebra presented in [13] also possesses two regular central elements (in the presentation of the algebra given in the paper, they are $ax_2x_2 + cx_1x_1$ and $x_3x_3 - cx_2x_2$). A similar argument can be used in that case to establish that the algebra is not finite over its center. This approach has the added benefit of revealing what the center actually is.

Proposition 3.4.6 *The algebra A has a one dimensional truncated line scheme, for general choices of the parameters.*

Proof. For this calculation we will consider a slightly different specialization of the parameters. In particular, we will take $a = b = c = 1$ and $d = 0$, giving us the relations:

$$x_4x_3 - x_3x_4 - x_1x_2$$

$$x_4x_2 + x_2x_4 - x_3x_3$$

$$x_4x_1 + x_1x_4 - x_3x_3$$

$$x_3x_2 - x_2x_3$$

$$x_3x_1 - x_1x_3$$

$$x_2x_1 + x_1x_2$$

. Note in particular that this specialization is the ore extension by (σ, δ) of the AS-regular subalgebra generated by $\langle x_1, x_2, x_3 \rangle$, where σ and δ are defined by $\sigma(x_1) = -x_1$, $\sigma(x_2) = -x_2$, $\sigma(x_3) = x_3$, $\delta(x_1) = x_3x_3$, $\delta(x_2) = x_3x_3$ and $\delta(x_3) = x_1x_2$. Hence by well known results, it is in fact an Auslander-regular domain.

By the main theorem of [11], for such an algebra, the scheme of line modules is isomorphic to the locus where the relations, considered as elements of $V \otimes V$ intersect the subspace of tensors of rank ≤ 2. Writing an arbitrary element of I as $\Sigma_i^6 t_i f_i$ (where, we recall, the f_i are generators of I), this translates into finding where the matrix

$$\begin{pmatrix} 0 & t_6 - t_1 & -t_5 & t_3 \\ t_6 & 0 & -t_4 & t_2 \\ t_5 & t_4 & -t_3 - t_2 & -t_1 \\ t_3 & t_2 & t_1 & 0 \end{pmatrix} \tag{3.4.6.1}$$

has rank ≤ 2.

We first note that one of the 3×3 minors of the matrix 3.4.6.1 is $t_2t_3(t_1 - 2t_6)$. We can see by inspection that in each of the three cases $t_2 = 0$, $t_3 = 0$, and $t_1 = 2t_6$, that the other minors imply that either four of the t_i are zero, or $t_1 = t_6 = 0$, $t_2 = -t_3$ and $t_4 = -t_5$. Thus we see that the locus consists of the four components $V(t_1, t_2, t_3, t_4)$, $V(t_1, t_2, t_3, t_5)$, $V(t_1, t_2, t_3, t_6)$, and $V(t_1, t_6, t_2 + t_3, t_4 + t_5)$, and so in any case is one dimensional. ■

Chapter 4

Classification of Q appearing in Regular Algebras

In this section we will undertake to carry out a partial classification of four dimensional regular algebras in the manner of [4].

We start by recalling the general ideas. Consider a regular algebra A such that for a normalized resolution Q is diagonal.

$$Q = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix}$$

Writing $\tau = \phi_{1,4} \circ \rho$, we have that $\tau(w(A)) = w(A)$. If we express $w(A)$ with respect to the basis $\langle x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes x_{i_4} \rangle_{i_1,i_2,i_3,i_4}$, then it is easy to see that the coefficient of $x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes x_{i_4}$ is equal to α_{i_4} times the coefficient of $x_{i_4} \otimes x_{i_1} \otimes x_{i_2} \otimes x_{i_3}$, and iterating this four times, that $\Pi_{k=1}^{4} \alpha_{i_k} = 1$. Hence $w(A)$ is in the span of all elements of $V^{\otimes 4}$ of the form $\delta_{i_1,i_2,i_3,i_4} := \Sigma_{k=0}^{3} \tau^k (x_{i_1} x_{i_2} x_{i_3} x_{i_4})$ where (i_1, i_2, i_3, i_4) is a tuple such that the product $\Pi_{k=1}^{4} \alpha_{i_k} = 1$. Let us denote this span by W_Q.

Thus if $w(A)$ is to be non-zero, as it must be for an AS-regular algebra, then at least one degree four product in the α_i equal to 1. However we can do better than this, for example we have

Lemma 4.0.7 *An algebra is not AS-regular in either of the following two cases:*

(i) There exists integers i, j and k, $j \neq k$, such that none of the monomials $x_i x_j$, $x_j x_i$, $x_i x_k$ and $x_k x_i$ appear in any of the relations.

(ii) The algebra has a relation of the form $x_i x_i$.

Proof.

(i) In this case all of the monomial $x_1 a_1 x_1 a_2 x_1 \ldots x_1$, where each a_i either x_k or x_j, are irreducible, and so the algebra A must grow exponentially.

(ii) In this case $x_i x_i x_i x_i$ is in $I \otimes V \otimes V \cap V \otimes I \otimes V \cap V \otimes V \otimes I$, so this space would be at least two dimensional were A AS-regular, which is a contradiction.■

Moreover, if we assume also that A is noetherian, which we shall for the remainder of this section, then, by the result of [3] that such an A is a domain, it follows that A does not even have a relation of the form $x_i x_j$.

This result easily translates into a statement about the potential existence of AS-regular algebras corresponding to a diagonal matrix Q.

Corollary 4.0.8 *Let Q be a diagonal matrix. Then:*

(i) There are no AS-regular algebras with associated matrix Q if there exists integers i,j and k, $j \neq k$, such that no element of W_Q involves either both of x_i and x_j or x_i and x_k

(ii) If there exists integers i and j such that the monomial $x_i x_j$ appears in exactly one basis element of W_Q, then there does not exists an AS-regular algebrea A with $w(A)$ involving all of the basis elements of W_Q.

Proof. These follow since the statements about the elements of W_Q immediately imply the corresponding statements about I in the conditions of the previous lemma. ■

We may apply the preceding result to find a, possibly too large, list of of the matrices Q for which there is a regular algebra. We may do this by considering the set of subgroups of the free group on the α_i, thinking of a subgroup as a collection of monomials in the α_i which evaluate to 1. Traversing this set we may find the minimal such subgroups for which the associated basis fails to satisfy the above corollary 4.0.8. Having done this we can easily find the set of Q which are consistent with each subgroup.

Having done the above calculation, we arrive at the following finite list of possible cases for Q (up to permuting the dialgonal, and picking the root of unity):

$$(1,1,\zeta_3,\zeta_3),\ (1,\zeta_5,\zeta_5^2,\zeta_5^4),\ (1,1,\zeta_6^2,\zeta_6^5),\ (\zeta_6,\zeta_6^3,\zeta_6^3,\zeta_6^5),\ (\zeta_6,\zeta_6,\zeta_6^3,\zeta_6^3),\ (\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^3),$$
$$(\zeta_8,\zeta_8^2,\zeta_8^3,\zeta_8^7),\ (\zeta_{10},\zeta_{10}^5,\zeta_{10}^7,\zeta_{10}^9),\ (\zeta_{12},\zeta_{12},\zeta_{12}^9,\zeta_{12}^9),\ (\zeta_{12},\zeta_{12}^3,\zeta_{12}^5,\zeta_{12}^{11})\ (\zeta_{12},\zeta_{12}^3,\zeta_{12}^3,\zeta_{12}^7),$$
$$(\zeta_{16},\zeta_{16}^5,\zeta_{16}^{13},\zeta_{16}^{15}),\ (\zeta_{20},\zeta_{20}^5,\zeta_{20}^9,\zeta_{20}^{17}),\ (\zeta_{24},\zeta_{24}^9,\zeta_{24}^{13},\zeta_{24}^{21}),\ (\zeta_{24},\zeta_{24}^{13},\zeta_{24}^{17},\zeta_{24}^{21}),\ (d,\zeta_3 d,\zeta_3^2 d,\tfrac{\zeta_3^2}{d^3}),$$
$$(d,-d,id,\tfrac{1}{d^3}),\ (d,d,-d,\tfrac{1}{d^3}),\ (c,-c,d,\tfrac{1}{c^2 d}),\ \text{and}\ (b,c,d,(bcd)^{-1})$$

where b, c and d are arbitrary. Now we must determine for which of these values of Q there actually are corresponding AS-regular algebras (a priori, we might need to consider further cases for Q if one of the last five families turned out not to have corresponding regular algebras, but it will turn out that this is not case).

We first consider the five non-constant families. In this we will be aided by the following observation.

Lemma 4.0.9 *Suppose that there is a three dimensional AS-regular algebra with diagonal Q having eigenvalues (a,b,c). Then for all scalars $d \neq 0$, there is a four dimensional regular algebra with diagonal Q having eigenvalues (ad, bd, cd, d^{-3}).*

Proof. Let the resolution of k_B be:

$$0 \to B \xrightarrow{\mathbf{x}^T} B^3 \xrightarrow{f_1} B^3 \xrightarrow{\mathbf{x}} B \to k_B$$

Then, by definition of Q we have that $(\mathbf{x}^T f_1)^T = Q f_1 \mathbf{x}$.

Now let B' be the algebra given adjoining a variable t to B subject to the relation $\alpha tx + xt = 0$ for every $x \in B_1$. It is not hard to see that B' has a free resolution of the form

$$\begin{tikzcd}
 & 0 \arrow[r] & B' \arrow[r, "f_2"] & B'^3 \arrow[r, "f_1"] & B'^3 \arrow[r, "f_0"] & B' \arrow[r] & k_B \\
0 \arrow[r] & B' \arrow[r, "f_2"] \arrow[ur, "\alpha^3 t"] & B'^3 \arrow[r, "f_1"] \arrow[ur, "\alpha^2 t"] & B'^3 \arrow[r, "f_0"] \arrow[ur, "\alpha t"] & B' \arrow[ur, "t"] & &
\end{tikzcd}$$

Taking the product of the maps in this resolution, we find that $\mathrm{w}(B')$ is of the form:

$$\alpha^3 t f_2 f_1 f_0 + \alpha^2 f_2 t f_1 f_0 + \alpha f_2 f_1 t f_0 + f_2 f_1 f_0 t$$

And consequenetly, we see that the eigenvectors of $\mathrm{Q}(B')$ are as claimed by the lemma. ■

Now, since there are AS-regular algebras of dimension 3 with Q equal to $(\zeta_9, \zeta_9^4, \zeta_9^7)$, $(1, -1, i)$,$(1, 1, -1)$,$(a, b, (ab)^{-1})$ and $(a, -a, a^{-2})$ see [4, Table 3.11], we have that all five non-constant families correspond to AS-regular algebras. For example, taking a global dimension 3 algebra with Q $(cd^{1/3}, -cd^{1/3}, c^{-2}d^{-2/3})$ applying the lemma with $\alpha = d^{-1/3}$, we get a four dimensional AS-regular algebra with the diagonal entries of Q being $(c, -c, \frac{1}{c^2 d}, d)$.

Finally, we can rule out a number of the remaining discrete Q by specific calculations.

For example, consider the case where the entries of Q are $(\zeta_{24}^1, \zeta_{24}^9, \zeta_{24}^{13}, \zeta_{24}^{21})$, and suppose we have a regular algebra A corresponding to this Q. Then examining Q, we see that $w(A)$ is in the span of the vectors

$$x_3x_3x_1x_4 + \dots$$

$$x_3x_3x_4x_1 + \dots$$

$$x_1x_3x_4x_3 + \dots$$

$$x_1x_1x_1x_4 + \dots$$

$$x_3x_3x_3x_2 + \dots$$

$$x_4x_4x_4x_2 + \dots$$

and contracting with $V^* \otimes V^*$ we have that I is spanned by elements

$$
\begin{array}{lll}
Ax_1x_1 + Ex_3x_3 & Cx_1x_1 + I\zeta_{24}^{9}x_3x_3 & \\
Bx_1x_1 + I\zeta_{24}^{22}x_3x_3 & Ax_1x_1 + G\zeta_{24}^{22}x_3x_3 & Dx_1x_2 + Bx_2x_1 + Gx_3x_4 + Fx_4x_3 \\
Dx_1x_3 + C\zeta_{24}^{14}x_3x_1 & Bx_1x_3 + D\zeta_{24}^{23}x_3x_1 & Cx_1x_2 + D\zeta_{24}^{9}x_2x_1 + F\zeta_{24}^{23}x_3x_4 + E\zeta_{24}^{23}x_4x_3 \\
Gx_1x_3 + F\zeta_{24}^{13}x_3x_1 & Fx_1x_3 + E\zeta_{24}^{13}x_3x_1 & Ax_1x_4 + C\zeta_{24}^{23}x_2x_3 + B\zeta_{24}^{23}x_3x_2 + A\zeta_{24}^{23}x_4x_1 \\
Hx_2x_2 + Jx_4x_4 & Hx_2x_2 + J\zeta_{24}^{6}x_4x_4 & Ex_1x_4 + I\zeta_{24}^{8}x_2x_3 + I\zeta_{24}^{21}x_3x_2 + G\zeta_{24}^{21}x_4x_1 \\
Hx_2x_4 + H\zeta_{24}^{15}x_4x_2 & Jx_2x_4 + J\zeta_{24}^{21}x_4x_2 &
\end{array}
$$

for some variables A,B,...,J. Now, from the last four relations, we see that either H or J must be zero. But if either is zero, then the other must be as well, or else there would be a relation of the form x_ix_i. But if $H = J = 0$ then none of the relations involve the monomials x_2,x_2x_4 or x_4x_2, and so, as we pointed out in 4.0.7, the algebra will grow too quickly to be regular. Thus there are no regular algebras with this Q.

Similar, but more involved, arguments can be used to rule out the existance of regular algebras in the cases $(\zeta_{24}, \zeta_{24}^{13}, \zeta_{24}^{17}, \zeta_{24}^{21})$, $(\zeta_{16}, \zeta_{16}^{5}, \zeta_{16}^{13}, \zeta_{16}^{15})$, $(\zeta_{12}, \zeta_{12}^{3}, \zeta_{12}^{5}, \zeta_{12}^{11})$, $(\zeta_{8}, \zeta_{8}^{2}, \zeta_{8}^{3}, \zeta_{8}^{7})$. Thus we have the following.

Proposition 4.0.10 *If A is a AS-regular algebra of dimension four, then $Q(A)$ is one of:*

$$(1,1,\zeta_3,\zeta_3),\ (1,\zeta_5,\zeta_5^2,\zeta_5^4),\ (1,1,\zeta_6^2,\zeta_6^5),\ (\zeta_6,\zeta_6^3,\zeta_6^3,\zeta_6^5),\ (\zeta_6,\zeta_6,\zeta_6^3,\zeta_6^3),\ (\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^3),$$
$$(\zeta_{10},\zeta_{10}^5,\zeta_{10}^7,\zeta_{10}^9),\ (\zeta_{12},\zeta_{12},\zeta_{12}^9,\zeta_{12}^9),\ (\zeta_{12},\zeta_{12}^3,\zeta_{12}^3,\zeta_{12}^7),\ (\zeta_{20},\zeta_{20}^5,\zeta_{20}^9,\zeta_{20}^{17}),\ (d,\zeta_3 d,\zeta_3^2 d,\tfrac{\zeta_3^2}{d^3}),$$
$$(d,-d,id,\tfrac{1}{d^3}),\ (d,d,-d,\tfrac{1}{d^3}),\ (c,-c,d,\tfrac{1}{c^2d}),\ \textit{and}\ (b,c,d,(bcd)^{-1}).$$

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