

LINEAR ESTIMATION FOR 2-D NEAREST-NEIGHBOR MODELS[†]

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Abstract

This paper considers the smoothing problem for 2-D random fields described by stochastic nearest-neighbor models (NNMs). The class of 2-D estimation problems that can be modeled in this way is quite large since NNMs arise whenever partial differential equations are discretized with finite difference methods. The NNM smoother is obtained by using a general smoothing technique developed in [1]-[3] for boundary-value processes in one or several dimensions. In this approach, the smoother is described by a Hamiltonian system of twice the dimension of the original system. For the problem considered here, the smoother is itself in NNM form. By converting this 2-D NNM system into an equivalent 1-D two-point boundary-value descriptor system (TPBVDS) of large dimension, a recursive and stable solution technique is obtained. Under slightly restrictive assumptions, an even faster procedure can be obtained by using the FFT with respect to one of the space dimensions to convert the 1-D TPBVDS mentioned above into a set of decoupled TPBVDSs of low-order which can be solved in parallel. This fast implementation of the smoother is illustrated by two examples, corresponding respectively to the discretized Poisson and heat equations.

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1. Introduction

In two dimensions, a large class of physical processes can be described by nearest neighbor models (NNMs). This is due to the fact when finite difference methods are used to discretize linear 2-D partial differential equations of arbitrary type (hyperbolic, parabolic or elliptic), and of any order, the resulting finite-difference approximation can usually be expressed in the form of a vector NNM. Thus, although the NNM dynamics appear at first sight to be inherently non-causal, they can also be used to model 2-D space-time dynamics, which are causal with respect to the time index, and noncausal with respect to space. On the basis of these observations, it is not surprising that NNMs have been employed widely to model 2-D stochastic images [4]-[6], and in particular to develop algorithms for image retoration and enhancement, as well as for the control and estimation of distributed parameter systems.

This paper is concerned with the development of efficient estimation algorithms for 2-D random fields described by stochastic nearest-neighbor models over a rectangular domain, when local boundary conditions, which include as special cases periodic, Dirichlet and Neumann conditions, are imposed on the domain boundaries. Since NNMs have an acausal structure, we shall focus our attention on the NNM smoothing problem, since this problem is also acausal, in the sense that the measurements need not be produced according to a specific order in 2-D space. Thus, both the class of 2-D estimation problems that we examine, and the NNMs that are used to formulate these problems are *purely noncausal*. This is in contrast with early attempts at deriving 2-D estimation algorithms, which were mimicking the structure of 1-D Kalman filters by introducing artificial 2-D causality concepts, such as quarter-plane or asymmetric half-plane causality (see the discussion appearing in Chapter 4 of [7]). On the other hand, since our goal is to obtain efficient estimation procedures, the algorithms that will be developed for the NNM smoothing problem will be *recursive*, and will be obtained by breaking down noncausal processing steps into parts which are causal. However, since the original problem is noncausal, there is in general a large amount of flexibility in the choice of recursion directions for the algorithms that we propose, and causality appears here as a computational artifice, not as a modeling assumption.

The approach that will be used here to formulate the NNM smoothing problem relies on the general results developed in [1]-[3] for the solution of estimation problems for boundary-value stochastic processes. From a historical point of view, 1-D boundary-value systems and processes were first introduced by Krener [8]-[10] in order to study the internal structure of acausal systems, and to formulate the stochastic realization problem for nonMarkov processes such as reciprocal processes. In [1]-[2], a general solution technique was developed for the estimation of boundary value stochastic processes in one or several dimensions. This approach is extremely general, and relies on the so-called method of complementary models introduced by Weinert and Desai [11] for the study of the smoothing problem for 1-D causal systems. Specifically, it is shown that given an internal model with appropriate boundary conditions for a boundary-value process, the smoothed estimate satisfies a Hamiltonian system of twice the size, and therefore of twice the order, of the original system. The reason why the size is doubled is that it is necessary to estimate not only the state of the internal model of interest, but also the state of the complementary model. This approach was used to study the smoothing problem for 1-D continuous boundary-value processes in [3], and for boundary-value 1-D descriptor systems in [12]. Some rough results for the 2-D NNM smoothing problem were presented in Chapter 6 of [1], and the present paper is in fact an improved version of this earlier work. Subsequently, the complementary model technique was also used by Riddle and Weinert [13]-[15] to study the 2-D smoothing problem for the Helmholtz equation and for 2-D hyperbolic systems. Together with the present paper, these contributions illustrate the wide applicability of the boundary-value process smoothing solution proposed in [1]-[2].

An interesting feature of the NNM smoother is that it is itself in NNM form. Thus, the class of NNM systems is closed under the smoothing operation. This property is rather satisfactory, since it indicates that NNMs are "natural" models for the study of noncausal estimation problems. From a practical point of view, since we seek to develop efficient estimation algorithms, this implies that it is important to obtain efficient NNM solution techniques. The solution proposed in this paper consists in solving the 2-D model in 1-D fashion by writing the 2-D

NNM dynamics columnwise in the form of a 1-D boundary-value system of very large dimension. This 1-D system has second-order dynamics, but can be rewritten as a 1-D two-point boundary-value descriptor system (TPBVDS) of the type examined in [16]-[19], for which a number of recursive solution techniques involving different concepts of causality can be employed. Under slightly more restrictive conditions, this 1-D system can be decoupled into a family of low-order 1-D subsystems by an FFT-based transformation. This decoupling technique is an extension of a method used by Hockney [20] to obtain fast Poisson solvers, and later applied by Jain and Angel [21] to a 2-D estimation problem.

This paper is organized as follows. In Section 2, we describe 2-D NNMs, as well as the local boundary conditions which are used to specify the solution of these models. These conditions include as special cases periodic, Dirichlet and Neumann boundary conditions. The transformation of a 2-D NNM into a 1-D TPBVDS is discussed in Section 3, and a general solution technique is obtained for the transformed system. The FFT solver is presented in Section 4 for the case where the NNM satisfies either periodic boundary conditions, or has vertically symmetric dynamics with Dirichlet or Neumann conditions. The smoothing problem for stochastic 2-D NNMs is formulated in Section 5, and the Hamiltonian system satisfied by the smoothed estimate is described and shown to be in NNM form. Section 6 discusses two examples of 2-D NNM smoothers, corresponding respectively to the discretized 2-D Poisson and heat equations. It turns out that the FFT decoupling technique of Section 4 is applicable to both of these examples. Finally, Section 7 contains several concluding remarks.

2. 2-D Nearest-Neighbor Models

The 2-D nearest-neighbor models (NNMs) that will be considered in this paper are of the form

$$x_{i,j} = A_1 x_{i-1,j} + A_2 x_{i+1,j} + A_3 \dot{x}_{i,j-1} + A_4 x_{i,j+1} + B u_{i,j} \quad (2.1)$$

$$z_{i,j} = C x_{i,j} , \quad (2.2)$$

where the state x , input u , and output z are vectors of dimension n , m , and p respectively, and A_k with $1 \leq k \leq 4$, B , and C are matrices of corresponding

dimensions. Equation (2.1) indicates that the state at point (i, j) is specified by $u_{i,j}$, and by the states at points immediately to the left, to the right, above and below point (i, j) . This explains why (2.1) is called a nearest neighbor model.

Models such as (2.1)-(2.2) arise naturally from the discretization of 2-D partial differential equations with finite difference methods, as can be seen from the following examples.

Examples: NNM form of finite-difference discretizations of PDEs. For each of the 2-D examples discussed below, the continuous space variables are denoted as t and s , and the corresponding discretized variables are i and j , respectively. Furthermore, except for the heat equation, it is assumed that the same mesh size h is used to discretize t and s .

a) *Poisson equation:* The discretized form of

$$\nabla^2 x(t, s) = u(t, s) \quad (2.3)$$

is given by

$$x_{i,j} = \frac{1}{4}(x_{i-1,j} + x_{i+1,j} + x_{i,j-1} + x_{i,j+1}) - \frac{h^2}{4}u_{i,j}, \quad (2.4)$$

which is exactly in the form (2.1).

b) *Heat equation:* Let

$$\frac{\partial}{\partial t}x(t, s) = \alpha \frac{\partial^2}{\partial s^2}x(t, s) + u(t, s) \quad (2.5)$$

where $\alpha > 0$. Then, if t and s are discretized with mesh sizes h and k , i.e., $t = ih$ and $s = jk$, and if backwards and central difference schemes [22] are used respectively to discretize $\partial x / \partial t$ and $\partial^2 x / \partial s^2$, we obtain

$$mx_{i,j} = x_{i-1,j} + n(x_{i,j-1} + x_{i,j+1}) + bu_{i,j} \quad (2.6)$$

where $m = 1 + 2\alpha h/k^2$, $n = \alpha h/k^2$ and $b = h$. This model is almost in NNM form. It can be rewritten in NNM form by dividing by $m > 0$, which gives

$$x_{i,j} = m^{-1}x_{i-1,j} + m^{-1}n(x_{i,j-1} + x_{i,j+1}) + m^{-1}bu_{i,j}. \quad (2.7)$$

Note however that, from a practical point of view, there is no difference between (2.6) and (2.7). These two models correspond to an implicit discretization of the

heat equation (2.5), where to compute $x_{i,j}$ for increasing values of i , it is necessary for each value of i to solve a linear system of equations for the coupled variables $x_{i,j}$, where j varies over all index values. It is shown in [22], p. 69 that this discretization scheme is unconditionally stable, i.e., it is stable for all choices of mesh sizes h and k . The motivation for selecting different meshes h and k to discretize t and s is that, to approximate the first order derivative of x with respect to t and the second order derivative with respect to s with the same degree of accuracy, one must have $h = O(k^2)$.

c) *Biharmonic equation*: Vector NNMs can arise in a variety of ways. One of them is of course from the discretization of higher-order PDEs, such as

$$\nabla^4 x(t,s) = u(t,s) . \quad (2.8)$$

This equation can be decomposed as

$$\nabla^2 x(t,s) = \xi(t,s) , \quad \nabla^2 \xi(t,s) = u(t,s) . \quad (2.9)$$

Then, using the discretization (2.4) of the Laplacian, and denoting

$$X_{i,j} = \begin{bmatrix} x_{i,j} \\ \xi_{i,j} \end{bmatrix},$$

we obtain

$$\begin{bmatrix} 1 & \frac{h^2}{4} \\ 0 & 1 \end{bmatrix} X_{i,j} = \frac{1}{4}(X_{i-1,j} + X_{i+1,j} + X_{i,j-1} + X_{i,j+1}) + \begin{bmatrix} 0 \\ \frac{h^2}{4} \end{bmatrix} u_{i,j} , \quad (2.10)$$

which after inversion of the matrix multiplying $X_{i,j}$, is in NNM form.

d) *Poisson equation with a crossover term*: Vector NNMs can also arise if higher-order schemes are used to discretize second-order PDEs. Sometimes the use of a higher-order scheme is dictated by the structure of the PDE itself. Consider for example

$$\left[\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2} + a \frac{\partial^2}{\partial t \partial s} \right] x(t,s) = u(t,s) , \quad (2.11)$$

which is elliptic, provided that parameter a is such that $|a| < 2$. Then, when a

first-order finite-difference discretization scheme is used to approximate the above equation, we obtain the following 9-point stencil model

$$x_{i,j} = \frac{1}{4}(x_{i-1,j} + x_{i+1,j} + x_{i,j-1} + x_{i,j+1}) + \frac{a}{16}(x_{i-1,j-1} + x_{i+1,j+1} - x_{i-1,j+1} - x_{i+1,j-1}) - \frac{h^2}{4}u_{i,j} \quad (2.12)$$

where $x_{i,j}$ depends not only on its four nearest neighbors, but also on values of x at the four corners $(i-1,j-1)$, $(i+1,j+1)$, $(i-1,j+1)$, and $(i+1,j-1)$. It can be transformed to NNM form by state augmentation. Thus, if

$$X_{i,j} = \begin{bmatrix} x_{i,j-1} \\ x_{i,j} \\ x_{i,j+1} \end{bmatrix},$$

the model (2.12) can be rewritten as

$$X_{i,j} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{a}{16} & \frac{1}{4} & -\frac{a}{16} \\ 0 & 0 & 0 \end{bmatrix} X_{i-1,j} + \begin{bmatrix} 0 & 0 & 0 \\ -\frac{a}{16} & \frac{1}{4} & \frac{a}{16} \\ 0 & 0 & 0 \end{bmatrix} X_{i+1,j} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} X_{i,j-1} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 \end{bmatrix} X_{i,j+1} + \begin{bmatrix} 0 \\ \frac{h^2}{4} \\ 0 \end{bmatrix}, \quad (2.13)$$

which is now in NNM form. Note that even though the second-order PDE (2.11) is scalar, the state $X_{i,j}$ has dimension 3. This is due to the presence of the crossover term $a \partial^2 x(t,s) / \partial t \partial s$ in (2.11).

For simplicity, it will be assumed below that model (2.1) is defined over the rectangular domain $1 \leq i \leq I-1$, $1 \leq j \leq J-1$. Then, in addition to model (2.1), some *boundary conditions* need to be specified. What constitutes a proper set of boundary conditions depends on the exact type of the partial difference operator (2.1) or the underlying PDE from which it comes from. For example, if this operator is elliptic (noncausal), initial-value problems are ill-posed. A general

framework for specifying boundary conditions, which can accommodate operators of all types, and which can be used to model a wide class of PDE boundary conditions, consists in assuming that the boundary conditions on the edges of the rectangle $0 \leq i \leq I$, $0 \leq j \leq J$ are local in the sense that they involve only neighboring points along the boundary, but where some coupling is allowed between points on opposite sides of the rectangle. This last feature will enable us to model periodic PDE boundary conditions. We consider therefore the following NNM boundary conditions.

Horizontal conditions:

$$V_L x_{0,j} + W_L x_{1,j} + V_R x_{I,j} + W_R x_{I-1,j} = d_{H,j} \quad (2.14a)$$

with $0 \leq j \leq J$.

Vertical conditions:

$$V_B x_{i,0} + W_B x_{i,1} + V_T x_{i,J} + W_T x_{i,J-1} = d_{V,i} \quad (2.14b)$$

with $1 \leq i \leq I-1$.

In (2.14a) and (2.14b), it is assumed that the boundary matrices V_E and W_E , with $E = L, R, B, T$ have size $2n \times n$. Thus, in conjunction with NNM model (2.1), the horizontal boundary conditions (2.14a) provide enough constraints to specify the states $x_{0,j}$ and $x_{I,j}$ with $0 \leq j \leq J$ on the left and right edges of the rectangle $\Omega = [0, I] \times [0, J]$. Similarly, the vertical conditions (2.14b) introduce sufficiently many constraints to enable the specification of $x_{i,0}$ and $x_{i,J}$ with $1 \leq i \leq I-1$ on the bottom and top edges of Ω . Note that there is a slight asymmetry in the above specification, in the sense that the horizontal boundary condition (2.14a) holds for $j = 0, J$, which has the effect of adding enough constraints to specify the *corner states* $x_{0,0}$, $x_{0,J}$, $x_{I,0}$ and $x_{I,J}$. However, this is clearly an arbitrary convention, and we can just as well use the vertical condition (2.14b) to specify the corner states.

The conditions (2.14) are *local* since they involve only pairs of points located on opposite sides of the rectangle Ω . Specifically, the horizontal condition (2.14a) couples points $(0, j)$, $(1, j)$ located along the left edge of Ω with points (I, j) and $(I-1, j)$ on the right edge, where all these points have the same row index j .

Similarly, the vertical condition (2.14b) couples two pairs of points along the bottom and top edges of rectangle Ω , respectively, and with the same column index i .

The motivation for coupling points located on opposite edges of Ω , is that we want to be able impose *periodic* boundary conditions, which would have the effect of identifying the left and right edges, or the bottom and top edges of rectangle Ω . For example, if the horizontal condition (2.14a) takes the form

$$x_{0,j} = x_{I-1,j} \quad , \quad x_{1,j} = x_{I,j} \quad \text{for } 0 \leq j \leq J \quad , \quad (2.15)$$

the NNM system (2.1) can be viewed as being defined over a *discretized cylinder* with index set $\Omega_C = [1, I-1] \times [0, J]$. Then, after imposing periodic horizontal conditions, if we select also periodic vertical boundary conditions, i.e.,

$$x_{i,0} = x_{i,J-1} \quad , \quad x_{i,1} = x_{i,J} \quad \text{for } 1 \leq i \leq I-1 \quad , \quad (2.16)$$

the NNM is now defined over a *discretized torus*, with index set $\Omega_T = [1, I-1] \times [1, J-1]$.

Another interesting subclass of boundary conditions (2.14) corresponds to the case when the boundary conditions on the left and right, and bottom and top edges of Ω are *separable*, in the sense that independent boundary conditions are specified on each edge of Ω . In this case, the boundary conditions (2.14) take the form

$$\tilde{V}_L x_{0,j} + \tilde{W}_L x_{1,j} = d_{L,j} \quad 0 \leq j \leq J \quad (2.17a)$$

$$\tilde{V}_R x_{I,j} + \tilde{W}_R x_{I-1,j} = d_{R,j} \quad 0 \leq j \leq J \quad (2.17b)$$

$$\tilde{V}_B x_{i,0} + \tilde{W}_B x_{i,1} = d_{B,i} \quad 1 \leq i \leq I-1 \quad (2.17c)$$

$$\tilde{V}_T x_{i,J} + \tilde{W}_T x_{i-1,J} = d_{T,i} \quad 1 \leq i \leq I-1 \quad , \quad (2.17d)$$

where the boundary matrices \tilde{V}_E and \tilde{W}_E with $E = L, R, B, T$ have size $n \times n$. Boundary conditions of this type arise extremely frequently in the study of PDEs, and in particular can be used to model Dirichlet or Neumann boundary conditions, as is shown by considering several examples.

Examples: Boundary conditions for discretized PDEs in NNM form. The PDEs considered in the following examples are assumed to be defined over the rectangle $[0, T] \times [0, S]$, where if h and k are the mesh sizes used to discretize the continuous

variables t and s , we have $T = Ih$ and $S = Jk$. Also, as for the PDE discretization examples considered earlier in this section, it will be assumed that $h = k$, except for the discretization of the heat equation.

a) Consider the Poisson equation (2.3) with the *mixed* boundary conditions

$$-m_L \frac{\partial}{\partial t} x(0,s) + n_L x(0,s) = d_L(s) \quad (2.18a)$$

$$m_R \frac{\partial}{\partial t} x(T,s) + n_R x(T,s) = d_R(s) \quad (2.18b)$$

$$-m_B \frac{\partial}{\partial s} x(t,0) + n_B x(t,0) = d_B(t) \quad (2.18c)$$

$$m_T \frac{\partial}{\partial s} x(t,S) + n_T x(t,S) = d_T(t) . \quad (2.18d)$$

These boundary conditions reduce to Dirichlet conditions when $m_E = 0$ and $n_E = 1$ for $E = L, R, B, T$, and to Neumann conditions when $m_E = 1$ and $n_E = 0$ for all values of index E . Then, a straightforward discretization yields

$$V_E = m_E + n_E h \quad , \quad W_E = -m_E \quad (2.19)$$

for $E = L, R, B, T$, and the boundary vectors appearing in (2.17) are given by $d_{E,l} = h d_E(lh)$, where the index l varies over $[0, J]$ for $E = L, R$, and over $[0, I]$ for $E = B, T$.

b) Consider now the heat equation (2.5) with initial condition

$$x(0,s) = f(s) \quad (2.20a)$$

and boundary conditions

$$x(t,0) = g_B(t) \quad , \quad x(t,S) = g_T(t) . \quad (2.20b)$$

After discretization, we find

$$V_E = 1 \quad , \quad W_E = 0 \quad \text{for } E = L, B, T \quad (2.21a)$$

and

$$d_{L,j} = f(jk) \quad , \quad d_{B,i} = g_B(ih) \quad , \quad d_{T,i} = g_T(ih) . \quad (2.21b)$$

However, in the above formulation, no boundary condition is specified on the right edge of Ω . This is unsatisfactory, since the NNM formulation of this paper

requires absolutely that there should be as many constraints as there are variables to be computed. The trick here is to note that since the discretized equation (2.6) is *causal* with respect to time, which is represented by index i , the variables $x_{i,j}$ for $i \leq I-1$ do not depend on the values for $i = I$, which can therefore be assigned arbitrarily, so that the boundary condition on the right edge is given by

$$x_{I,j} = d_{R,j} \quad 0 \leq j \leq J, \quad (2.21c)$$

where $d_{R,j}$ is arbitrary.

c) Examine the Poisson equation (2.11) with a crossover term, and with Dirichlet boundary conditions obtained by setting $m_E = 0$ and $n_E = 1$ in (2.18). Then, a simple discretization of these conditions is not sufficient to specify the NNM boundary conditions, since as was observed above, we must consider the vector NNM system (2.13). Furthermore, due to the state augmentation procedure used to construct $X_{i,j}$, if the scalar discretized PDE (2.12) is defined over the domain $[0,I] \times [0,J]$, the domain of definition of NNM (2.13) is only $[0,I] \times [1,J-1]$. Over this domain, the discretized Dirichlet boundary conditions for the scalar equation can be rewritten in the NNM form (2.17) as

$$X_{0,j} = \begin{bmatrix} d_L((j-1)h) \\ d_L(jh) \\ d_L((j+1)h) \end{bmatrix}, \quad X_{I,j} = \begin{bmatrix} d_R((j-1)h) \\ d_R(jh) \\ d_{((j+1)h)} \end{bmatrix} \quad (2.22a)$$

$$X_{i,1} + \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} X_{i,2} = \begin{bmatrix} d_B(ih) \\ 0 \\ 0 \end{bmatrix} \quad (2.22b)$$

$$X_{i,J-1} + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} X_{i,J-2} = \begin{bmatrix} 0 \\ 0 \\ d_T(ih) \end{bmatrix}. \quad (2.22c)$$

3. Solution of Boundary Value Nearest Neighbor Models

In this section, we describe a method for computing the solution of the boundary-value problem specified by the NNM dynamics (2.1) and boundary conditions (2.14). The method that we employ relies on a column stacking operation.

whereby the variables $x_{i,j}$ along the i^{th} column of the rectangular domain Ω are combined to form a large state vector \mathbf{x}_i . This procedure is used in Section 3.1 to transform the 2-D NNM dynamics, as well as the boundary and corner conditions, into an equivalent 1-D two-point boundary value system of very large size with second order dynamics. Since the well-posedness of this system is equivalent to that of the original NNM, by writing the equations describing this 1-D system as a single matrix equation, a well-posedness test is obtained for the NNM specified by (2.1) and (2.14). Then, in Section 3.2 the 1-D dynamical system of Section 3.1 is formulated as a 1-D two-point boundary value descriptor system (TPBVDS). A complete study of the properties of these systems and of their solution is presented in [16]-[19]. These results are used to obtain a well-posedness test for NNM (2.1), (2.14) which is simpler than the one obtained in Section 3.1. Then, by using a TPBVDS solution technique proposed in [17], Appendix B and [12], a recursive procedure is obtained for solving NNM models. It relies on decoupling the TPBVDS dynamics into forward and backward stable filters with zero initial and final conditions, respectively. The true boundary conditions are then taken into account by adding a correction term to the solution obtained for zero boundary conditions. This solution can be viewed as an extension in a more general setting of the Mayne-Fraser [23]-[24] two-filter formula for the smoother associated with a 1-D discrete causal system.

3.1. Column Stacking and Well Posedness

As indicated above, the first step of our solution is to perform a column-stacking operation, where the state, input and output vectors along the i^{th} column of rectangle $\Omega = [0,I] \times [0,J]$ are represented by

$$\mathbf{x}_i = \begin{bmatrix} x_{i,0} \\ x_{i,1} \\ \cdot \\ \cdot \\ x_{i,J-1} \\ x_{i,J} \end{bmatrix}, \quad \mathbf{u}_i = \begin{bmatrix} u_{i,1} \\ \cdot \\ \cdot \\ u_{i,J-1} \end{bmatrix} \quad (3.1a)$$

and

$$\mathbf{z}_i = \begin{bmatrix} z_{i,0} \\ z_{i,1} \\ \cdot \\ \cdot \\ z_{i,J-1} \\ z_{i,J} \end{bmatrix}. \quad (3.1b)$$

Here \mathbf{x}_i , \mathbf{u}_i , and \mathbf{z}_i have dimensions $n(J+1)$, $m(J-1)$, and $p(J+1)$, respectively. Note that \mathbf{x}_i and \mathbf{z}_i have two more block entries than \mathbf{u}_i , since $x_{i,j}$ and $z_{i,j}$ are defined on the edges of the rectangular domain Ω , whereas $u_{i,j}$ is only defined in the interior. Then, by combining the NNM relations (2.1) for a fixed value of i and $1 \leq j \leq J-1$ with the vertical boundary conditions (2.14b) for the same value of i , we obtain the 1-D dynamics

$$\Phi_+ \mathbf{x}_{i+1} + \Phi_0 \mathbf{x}_i + \Phi_- \mathbf{x}_{i-1} = \mathbf{n}_i \quad 1 \leq i \leq I-1 \quad (3.2)$$

$$\mathbf{z}_i = (I \otimes C) \mathbf{x}_i \quad (3.3)$$

where \otimes denotes the Kronecker product of two matrices [25], with

$$\Phi_0 = \begin{bmatrix} V_B & W_B & & & & W_T & V_T \\ -A_3 & I & -A_4 & & & 0 & \\ & -A_3 & I & -A_4 & & & \\ & & \cdot & \cdot & \cdot & & \\ & 0 & & \cdot & \cdot & \cdot & \\ & & & & -A_3 & I & -A_4 \end{bmatrix} \quad (3.4a)$$

$$\Phi_+ = \begin{bmatrix} 0 & & & & & & \\ & -A_2 & & 0 & & & \\ & & -A_2 & & & & \\ & 0 & & \cdot & & & \\ & & & & -A_2 & & \end{bmatrix}, \quad \Phi_- = \begin{bmatrix} 0 & & & & & & \\ & -A_1 & & 0 & & & \\ & & -A_1 & & & & \\ & 0 & & \cdot & & & \\ & & & & -A_1 & & \end{bmatrix} \quad (3.4b)$$

and

$$\mathbf{n}_i = \begin{bmatrix} d_{V,i} \\ (I \otimes B)\mathbf{u}_i \end{bmatrix}. \quad (3.4c)$$

Since the boundary matrices V_B , V_T , W_B , and W_T have size $2n \times n$, it is easy to check that the matrices Φ_l with $l = 0, -, +$ are square and have dimension $n(J+1)$. The relation (3.2) defines therefore a 1-D system with second-order dynamics evolving over interval $[0, I]$ and driven by inputs \mathbf{n}_i which are expressed in terms of inputs $u_{i,j}$ of the NNM and of the boundary vector $d_{V,i}$ associated to the vertical conditions on the bottom and top edges of rectangle Ω .

By considering also the horizontal NNM boundary condition (2.14a) on the left and right edges of Ω , we obtain the boundary condition

$$\Gamma_L \mathbf{x}_0 + \Delta_L \mathbf{x}_1 + \Gamma_R \mathbf{x}_J + \Delta \mathbf{x}_{J-1} = \mathbf{d}_H \quad (3.5)$$

for system (3.2), where

$$\Gamma_L = I \otimes V_L \quad , \quad \Gamma_R = I \otimes V_R \quad (3.6a)$$

$$\Delta_L = I \otimes W_L \quad , \quad \Delta_R = I \otimes W_R \quad (3.6b)$$

and

$$\mathbf{d}_H = \begin{bmatrix} d_{H,0} \\ d_{H,1} \\ \cdot \\ \cdot \\ d_{H,J-1} \\ d_{H,J} \end{bmatrix}. \quad (3.6c)$$

Noting again that the boundary matrices V_L , V_R , and W_L , W_T have size $2n \times n$, it is easy to check that Γ_L , Γ_R , Δ_L and Δ_R have size $2(J+1)n \times (J+1)n$, and that vector \mathbf{d}_H has dimension $2(J+1)n$. Thus, the boundary conditions (3.5) and dynamics (3.2) define a boundary value system over $[0, I]$, where the number of constraints imposed by (3.2) and (3.5) equals the total number of variables that need to be computed, namely vectors \mathbf{x}_i for $0 \leq i \leq I$. One possible method of solving this system consists in combining all the equations that define it into a

single matrix equation of very large dimension of the form

$$\Sigma \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \cdot \\ \cdot \\ \mathbf{x}_{I-1} \\ \mathbf{x}_I \end{bmatrix} = \begin{bmatrix} \mathbf{d}_H \\ \mathbf{n}_1 \\ \mathbf{n}_2 \\ \cdot \\ \cdot \\ \mathbf{n}_{I-1} \end{bmatrix}, \quad (3.7)$$

where

$$\Sigma = \begin{bmatrix} \Gamma_L & \Delta_L & & & & \Delta_R & \Gamma_R \\ \Phi_- & \Phi_0 & \Phi_+ & & & \mathbf{0} & \\ & \Phi_- & \Phi_0 & \Phi_+ & & & \\ & & \cdot & \cdot & \cdot & & \\ & \mathbf{0} & & \cdot & \cdot & \cdot & \\ & & & & \Phi_- & \Phi_0 & \Phi_+ \end{bmatrix} \quad (3.8)$$

is a matrix of size $(I+1)(J+1)n$. Then, the 1-D boundary value system (3.2), (3.5) is well posed over interval $[0, I]$, i.e., there exists a unique solution \mathbf{x}_i with $0 \leq i \leq I$ for all possible choices of inputs \mathbf{n}_i and boundary vector \mathbf{d}_H , if and only if Σ is invertible. Since system (3.2), (3.5) was obtained from the original NNM by column stacking, the invertibility of Σ is therefore a necessary and sufficient condition for the well-posedness of the NNM (2.1), (2.14). By using an argument similar to the one appearing in Theorem 1 of [], it is also easy to check that the invertibility of Σ implies that the second-order dynamics (3.2) must be *regular*, i.e., the determinant of the polynomial matrix

$$\Phi(z) = \Phi_+ z^2 + \Phi_0 z + \Phi_- \quad (3.9)$$

is not identically zero for all z .

In practice, the matrix Σ has such a huge dimension that it is not possible nor desirable to invert it directly. In the special case when Σ is obtained by discretizing an elliptic PDE, iterative inversion methods, such as the successive

overrelaxation (SOR), preconditioned conjugate gradient, or multigrid methods can be employed. However, these solutions are limited in scope, and the solution technique that will be described here is totally general, i.e., it applies to finite difference NNM operators of all types. On the other hand, this solution technique is usually not as efficient as the above mentioned methods for solving elliptic PDEs.

3.2. Stable Two-Filter Solution

The solution that we propose relies on transforming the 1-D dynamics (3.2) in such a way that stable forwards and backwards recursions can be used to compute \mathbf{x}_i . In some sense, this method falls within the class of stable marching methods [27]-[29]. Marching methods were originally developed when it was realized that, by column stacking, noncausal 2-D models such as (2.1) could be transformed into 1-D dynamical systems such as (3.2). Then, in the special case when Φ_+ is invertible, (3.2) can be expressed as

$$\mathbf{x}_{i+1} = -\Phi_+^{-1}[\Phi_0\mathbf{x}_i + \Phi_-\mathbf{x}_{i-1} - \mathbf{n}_i], \quad (3.10)$$

which is now a causal system that can be used to compute \mathbf{x}_i recursively, provided that the boundary condition (3.5) is properly taken into account. In addition to requiring that either Φ_+ or Φ_- should be invertible, one major drawback of this approach is that there is no guarantee that the causal system (3.10) is stable. An important criticism of marching methods, at least in this simplistic form, has therefore been that they are numerically unstable, and are not appropriate for solving NNMs on large lattices. The solution which is presented here can be viewed as a *stabilized marching method*, where instead of attempting to propagate the whole system (3.2) in the forwards (or backwards) direction, we break it into smaller parts which are stable when propagated in the forwards and backwards direction, respectively.

However, instead of considering directly the second-order system (3.2), we transform it into a two-point boundary value descriptor system (TPBVDS) of the type examined by Nikoukhah, Willsky and Levy [16]-[19]. To do so, consider the augmented state

$$\mathbf{q}_i = \begin{bmatrix} \mathbf{x}_{i-1} \\ \mathbf{x}_i \end{bmatrix}. \quad (3.11)$$

Then, the dynamics (3.2)-(3.3) and boundary condition (3.5) can be expressed as

$$E \mathbf{q}_{i+1} = F \mathbf{q}_i + G \mathbf{n}_i \quad 1 \leq i \leq I \quad (3.12)$$

$$\mathbf{z}_i = H \mathbf{q}_i \quad (3.13)$$

and

$$U_L \mathbf{q}_1 + U_R \mathbf{q}_I = \mathbf{d}_H, \quad (3.14)$$

where

$$E = \begin{bmatrix} \Phi_0 & \Phi_+ \\ I & 0 \end{bmatrix}, \quad F = \begin{bmatrix} -\Phi_- & 0 \\ 0 & I \end{bmatrix} \quad (3.15a)$$

$$G = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & I \otimes C \end{bmatrix} \quad (3.15b)$$

$$U_L = \begin{bmatrix} \Delta_L & \Gamma_L \end{bmatrix}, \quad U_R = \begin{bmatrix} \Gamma_R & \Delta_R \end{bmatrix}. \quad (3.15c)$$

The relations (3.12)-(3.15) define a TPBVDS over interval $[1, I]$. This system has first-order dynamics, and it is easy to check that

$$|zE - F| = |\Phi(z)|, \quad (3.16)$$

where $\Phi(z)$ is the second-order matrix polynomial defined in (3.9), so that no new dynamics have been introduced by going from (3.2) to (3.12). Owing to the simple nature of the augmentation procedure (3.11), we can also conclude that the TPBVDS (3.12)-(3.15) is well-posed over the interval $[1, I]$ if and only if the second-order system (3.2) with boundary condition (3.5) is well posed over $[0, I]$, which in turn was shown to be equivalent to the well-posedness of the original NNM system. But it was shown in [17] that an arbitrary TPBVDS of the form (3.12), (3.14) is well-posed if and only if the matrix

$$S = U_L E^{I-1} + U_R F^{I-1} \quad (3.17)$$

is invertible. The invertibility of S in (3.17) can therefore be used to characterize the well-posedness of the NNM (2.1), (2.14). Since the size of this matrix is "only"

$2(J+1)n$, the invertibility of S is much easier to test than that of the matrix Σ which was used to characterize NNM well-posedness in Section 3.1.

At this point, the NNM problem has been reduced to the solution of a TPBVDS over a finite interval. Several solution techniques for TPBVDSs have been proposed in [17], Appendix B and [12]. As was mentioned above, the solution which is described here relies on breaking the descriptor dynamics (3.12) into smaller parts which are causal and stable in the forwards and backwards directions, respectively. Specifically, since the NNM that we consider is assumed to be well-posed, the matrix pencil $zE - F$ is regular, and according to Weierstrass's canonical decomposition of a regular pencil [30], there exists some invertible matrices M and T such that

$$M(zE - F)T = \begin{bmatrix} zI - F_f & 0 \\ 0 & zF_b - I \end{bmatrix}, \quad (3.18)$$

where the eigenvalues of matrices F_f and F_b have magnitude less or equal to 1. Furthermore, if $|zE - F|$ has no zero on the unit circle, then all the eigenvalues of F_f and F_b are strictly inside the unit circle. Then, if

$$MB = \begin{bmatrix} B_f \\ B_b \end{bmatrix}, \quad (3.19)$$

the transformed state variables

$$\begin{bmatrix} \mathbf{q}_{f,i} \\ \mathbf{q}_{b,i} \end{bmatrix} = T \mathbf{q}_i \quad (3.20)$$

satisfy the forwards and backwards recursions

$$\mathbf{q}_{f,i+1} = F_f \mathbf{q}_{f,i} + B_f \mathbf{n}_i \quad (3.21a)$$

$$\mathbf{q}_{b,i} = F_b \mathbf{q}_{b,i+1} - B_b \mathbf{n}_i \quad (3.21b)$$

These recursions are asymptotically stable if $zE - F$ has no zero on the unit circle. Under the transformation (3.20), the boundary condition (3.14) takes the form

$$\begin{bmatrix} U_{L,f} & U_{L,b} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{f,1} \\ \mathbf{q}_{b,1} \end{bmatrix} + \begin{bmatrix} U_{R,f} & U_{R,b} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{f,I} \\ \mathbf{q}_{b,I} \end{bmatrix} = \mathbf{d}_H \quad (3.22)$$

where

$$\begin{bmatrix} U_{L,f} & U_{L,b} \end{bmatrix} = U_L T^{-1}, \quad \begin{bmatrix} U_{R,f} & U_{R,b} \end{bmatrix} = U_R T^{-1}. \quad (3.23)$$

Note that although the forwards and backwards dynamics (3.21a) and (3.21b) for \mathbf{q}_f and \mathbf{q}_b are decoupled, the boundary conditions remain coupled, so that \mathbf{q}_f and \mathbf{q}_b cannot be computed separately. Let $\mathbf{q}_{f,i}^0$ and $\mathbf{q}_{b,i}^0$ be the solutions of (3.21a) and (3.21b) with zero initial and final conditions, respectively. Then

$$\mathbf{q}_{f,i} = F_f^{i-1} \mathbf{q}_{f,1} + \mathbf{q}_{f,i}^0 \quad (3.24a)$$

$$\mathbf{q}_{b,i} = F_b^{I-i} \mathbf{q}_{b,I} + \mathbf{q}_{b,i}^0. \quad (3.24b)$$

Substituting (3.24) inside (3.22), and solving for $\mathbf{q}_{f,1}$ and $\mathbf{q}_{b,I}$ gives

$$\begin{bmatrix} \mathbf{q}_{f,1} \\ \mathbf{q}_{b,I} \end{bmatrix} = K^{-1}(\mathbf{d}_H - U_{R,f} \mathbf{q}_{f,I}^0 - U_{L,b} \mathbf{q}_{b,1}^0), \quad (3.25)$$

where

$$K = \begin{bmatrix} U_{L,f} + U_{R,f} F_f^{I-1} & U_{R,b} + U_{L,b} F_b^{I-1} \end{bmatrix}. \quad (3.26)$$

Finally, substituting (3.25) inside (3.24), we find

$$\begin{bmatrix} \mathbf{q}_{f,i} \\ \mathbf{q}_{b,i} \end{bmatrix} = \begin{bmatrix} F_f^{i-1} & 0 \\ 0 & F_b^{I-i} \end{bmatrix} K^{-1}(\mathbf{q} - U_{R,f} \mathbf{q}_{f,I}^0 - U_{L,b} \mathbf{q}_{b,1}^0) + \begin{bmatrix} \mathbf{q}_{f,i}^0 \\ \mathbf{q}_{b,i}^0 \end{bmatrix}. \quad (3.27)$$

The solution in the original basis can then be obtained by inverting (3.20).

From a practical point of view, the solution technique described above consists in propagating the forwards and backwards filters (3.21a) and (3.21b) for $\mathbf{q}_{f,i}^0$ and $\mathbf{q}_{b,i}^0$, and then combining the resulting values with boundary condition (3.22) to obtain $\mathbf{q}_{f,i}$ and $\mathbf{q}_{b,i}$ via (3.27). The most computationally demanding part of this algorithm is the computation of $\mathbf{q}_{f,i}^0$ and $\mathbf{q}_{b,i}^0$.

The above TPBVDS solution is similar to the Mayne-Fraser [23], [24] two-filter formula for the 1-D fixed-interval smoothing problem. At first sight, there seems to be little relation between the fixed-interval smoothing problem for discrete-time causal systems and the solution of TPBVDSs, but it turns out that

the 1-D discrete-time smoother can be expressed as a TPBVDS (see [1], Section 5.3), which explains why the same solution technique can be used for these two problems.

The TPBVDS solution described here is not the only one that can be developed. In [17] an alternative solution method is proposed which relies on recursions propagating inwards and outwards with respect to the center of the interval where the TPBVDS is defined. This choice is a manifestation of the fact that since causality appears here only as a computational device, we are not restricted to process the 2-D NNM data in any particular order.

4. Efficient FFT Solver

One drawback of the NNM solution described in Section 3 is that the vectors \mathbf{x}_i obtained by column stacking have very large size. The matrices E and F appearing in the TPBVDS (3.12)-(3.15) have size $2(J+1)n$, and therefore the matrices F_f and F_b obtained by pencil decomposition have a very large dimension. In addition, even if E and F are sparse, there is no guarantee that F_f and F_b will also be sparse, so that the forwards and backwards recursions (3.21) require in general a large amount of computation. In this section, we consider several special cases where some additional structure is present, which can be exploited to obtain fast NNM solvers. Specifically, in Section 4.1, we consider the case where the NNM is defined over a discretized cylinder, and in Section 4.2, it is assumed that the NNM dynamics (2.1) satisfy the symmetry condition $A_3 = A_4$, and that the boundary conditions on the bottom and top edges are either (i) Dirichlet or (ii) Neumann conditions. For all these cases, it turns out that the FFT, or the discrete sine or cosine transforms (DST, DCT) can be used to transform the high-order TPBVDS obtained in Section 3.2 into decoupled low-order 1-D TPBVDSs which can be solved in parallel. Since fast algorithms can be used to implement the FFT, DST and DCT and their inverses, this solution technique is very efficient. It is worth noting that the use of the FFT was first proposed by Hockney [20] to obtain a fast Poisson solver. Later Jain and Angel [21] (see also [31]) also employed the FFT to obtain an efficient solution for a 2-D estimation problem expressed in terms of the Poisson equation. The NNM solution described

here can be viewed as an extension of these earlier results.

4.1. NNM Over a Discretized Cylinder

In the first case, it is assumed that the vertical boundary conditions (2.14a) are *periodic*, i.e.,

$$x_{i,0} = x_{i,J-1} \quad , \quad x_{i,1} = x_{i,J} \quad \text{for } 1 \leq i \leq J-1 \quad , \quad (4.1)$$

in which case the domain Ω corresponds to a discretized cylinder. Then, it is easy to check that the components $x_{i,0}$ and $x_{i,J}$ need not be included in the stacked vector \mathbf{x}_i , whose dimension is therefore only $n(J-1)$, and in equation (3.2), we can identify

$$\Phi_0 = I \otimes I - Z_c^T \otimes A_3 - Z_c \otimes A_4 \quad (4.2a)$$

$$\Phi_- = -I \otimes A_1 \quad , \quad \Phi_+ = -I \otimes A_2 \quad , \quad \mathbf{n}_i = (I \otimes B) \mathbf{u}_i \quad , \quad (4.2b)$$

where Z_c is the $(J-1) \times (J-1)$ circular shift matrix

$$Z_c = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & 0 & \cdot & \\ & 0 & & \cdot & \cdot \\ & & & \cdot & 1 \\ 1 & & & & 0 \end{bmatrix} . \quad (4.3)$$

The special structure of the 1-D system specified by (3.2), (3.5) (3.6) and (4.2) can be exploited by performing a state transformation on \mathbf{x}_i which decouples this system into $J-1$ subsystems of dimension n . To do so, let D be the $(J-1) \times (J-1)$ discrete Fourier transform (DFT) matrix with entries

$$d_{l,j} = \frac{1}{(J-1)^{1/2}} \omega^{(l-1)(j-1)} \quad 1 \leq l, j \leq J-1 \quad , \quad (4.4a)$$

where

$$\omega = e^{-i2\pi/(J-1)} . \quad (4.4b)$$

The matrix D has the property that it is unitary, i.e., $DD^H = D^H D = I$, and it diagonalizes Z_c , so that

$$Z_c = D \Lambda D^H \quad \text{with} \quad \Lambda = \text{diag}\{\omega^{j-1}\}. \quad (4.5)$$

Then, consider the state transformation

$$(D^H \otimes I)\mathbf{x}_i = \xi_i = \begin{bmatrix} \xi_{i,1} \\ \cdot \\ \xi_{i,j} \\ \cdot \\ \xi_{i,J-1} \end{bmatrix}, \quad (4.6a)$$

where the new state vector ξ_i is partitioned into subvectors $\xi_{i,j}$ of size n . Similarly, let

$$(D^H \otimes I)\mathbf{u}_i = \mathbf{v}_i \quad , \quad (D^H \otimes I)\mathbf{d}_H = \delta, \quad (4.6b)$$

where \mathbf{v}_i and δ are also partitioned into vector entries $v_{i,j}$ and δ_j . Using the transformation (4.6), and taking into account (4.2), (4.5), as well as the Kronecker product identities

$$(A \otimes B)(C \otimes D) = AC \otimes BD \quad , \quad (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}, \quad (4.7)$$

the 1-D system (3.2), (3.5) is transformed into $J-1$ decoupled subsystems of the form

$$(I - \omega^{-(j-1)}A_3 - \omega^{j-1}A_4)\xi_{i,j} = A_1\xi_{i-1,j} + A_2\xi_{i+1,j} + Bv_{i,j} \quad (4.8)$$

where $1 \leq j \leq J-1$, and with boundary conditions

$$V_L \xi_{0,j} + W_L \xi_{1,j} + V_R \xi_{I,j} + W_R \xi_{I-1,j} = \delta_j \quad (4.9)$$

The dynamics (4.8) and boundary conditions (4.9) have exactly the same structure as (3.2), (3.5) and consequently, by state augmentation each of the above subsystems can be written in TPBVDS form as

$$\begin{bmatrix} -A_2 & I - \omega^{-(j-1)}A_3 - \omega^{j-1}A_4 \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi_{i+1,j} \\ \xi_{i,j} \end{bmatrix} = \begin{bmatrix} 0 & A_1 \\ I & 0 \end{bmatrix} \begin{bmatrix} \xi_{i,j} \\ \xi_{i-1,j} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v_{i,j} \quad (4.10)$$

with

$$\begin{bmatrix} W_L & V_L \end{bmatrix} \begin{bmatrix} \xi_{1,j} \\ \xi_{0,j} \end{bmatrix} + \begin{bmatrix} V_R & W_R \end{bmatrix} \begin{bmatrix} \xi_{I,j} \\ \xi_{I-1,j} \end{bmatrix} = \delta_j . \quad (4.11)$$

The stable two-filter solution technique described in Section 3.2 can then be used to solve each of these individual TPBVDSs. The advantage of this approach over the general procedure of Section 3 is that the decoupled TPBVDSs (4.10)-(4.11) have size $2n$, whereas TPBVDS (3.12)-(3.15) has dimension $2(J+1)n$. Thus, the number of operations required to solve the above TPBVDSs over interval $[1, J]$ is $O(IJ)$, whereas the complexity of the algorithm presented in Section 3 is $O(IJ^2)$. In fact, the most computationally demanding step of the fast NNM solver described above is not the solution of the TPBVDSs (4.10)-(4.11). It is the implementation of the transformations (4.6b) which relate the original inputs and boundary vectors to their transformed counterparts, and of the inverse transformation

$$\mathbf{x}_i = (D \otimes I) \xi_i \quad (4.12)$$

which relates the solution of the decoupled TPBVDSs to the original coordinate system. Because of its Kronecker product form, the transform (4.12) consists in n decoupled FFTs of length $J-1$, represented here by D . the number of operations required by (4.12) is therefore $O(J \log J)$, and since this transformation, as well as transformations (4.6b) must be performed for every value of i , the complexity of the fast NNM solver described above is $O(IJ \log J)$.

4.2. Vertically Symmetric NNMs

NNMs which are defined over a discretized cylinder are not the only ones that give rise to fast solvers. When the NNM dynamics (2.1) have the vertical symmetry $A_3 = A_4$ (which is the case for example for the Poisson and heat equations, as well as the biharmonic equation described in Section 2), and when the boundary conditions on the bottom and top edges are of Dirichlet or Neumann type, it is possible to obtain fast solvers.

We consider first the case of *Dirichlet conditions*. In this case, we have

$$x_{i,0} = d_{B,i} \quad , \quad x_{i,J} = d_{T,i} \quad (4.13)$$

so that it is not necessary to include $x_{i,0}$ and $x_{i,J}$ in the stacked vector \mathbf{x}_i introduced in (3.1a). This vector has therefore dimension $n(J-1)$. With this observation, the dynamics (3.2) take the form

$$\Phi_0 = I \otimes I - \Pi \otimes A_3 \quad , \quad \Phi_- = -I \otimes A_1 \quad , \quad \Phi_+ = -I \otimes A_2 \quad (4.14a)$$

$$\mathbf{n}_i = (I \otimes B)\mathbf{u}_i + \begin{bmatrix} A_3 d_{B,i} \\ 0 \\ \cdot \\ \cdot \\ 0 \\ A_3 d_{T,i} \end{bmatrix} , \quad (4.14b)$$

with

$$\Pi = Z + Z^T , \quad (4.15a)$$

where Z denotes here the $(J-1) \times (J-1)$ truncated shift matrix

$$Z = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & 0 & \\ & & \cdot & & \\ & 0 & & 0 & 1 \\ & & & & 0 \end{bmatrix} . \quad (4.15b)$$

Then, let S be the $(J-1) \times (J-1)$ discrete sine transform (DST) matrix with entries

$$s_{l,j} = \left(\frac{2}{J}\right)^{1/2} \sin(lj\pi/J) \quad 1 \leq l, j \leq J-1 . \quad (4.16)$$

The matrix S is symmetric and orthonormal, i.e., $S = S^T$ and $S^2 = I$, and it diagonalizes Π , so that

$$S \Pi S^T = \Lambda = \text{diag}\{\lambda_j\} , \quad (4.17)$$

where $\lambda_j = 2\cos(j\pi/J)$ with $1 \leq j \leq J-1$. Thus, if we replace D^H by S in the state transformation (4.6a) and definition (4.6b) of δ , and if

$$\nu_i = (S \otimes I)\mathbf{n}_i \quad (4.18)$$

with $1 \leq l \leq J-1$. The matrix K is orthonormal, i.e., $KK^T = K^T K = I$, and it diagonalizes Π , so that

$$K^T \Pi K = \Lambda = \text{diag}\{\lambda_j\} \quad (4.23a)$$

with

$$\lambda_j = 2\cos\left[(j-1)\frac{\pi}{J-1}\right] \quad 1 \leq j \leq J-1. \quad (4.23b)$$

Consequently, if K plays the same role as D and S in the state, input and boundary vector transformations considered earlier in this section, the 1-D system (3.2), (3.5) with dynamics and boundary matrices given by (4.14) and (3.6) is transformed into $J-1$ decoupled subsystems specified by (4.19), where the only difference is that the eigenvalues λ_j appearing in these systems are now given by (4.23b). These subsystems can be solved in parallel, and since the FFT can also be used to implement the DCT, the complexity of the resulting algorithm is $O(IJ \log J)$.

5. NNM Smoother

In this section we examine the smoothing problem for 2-D random fields described by a NNM driven by white Gaussian noise. Note that since NNMs are intrinsically acausal, the only linear estimation problem that preserves the acausality of the system formulation is the smoothing problem. Given noisy NNM observations over the rectangle Ω , the general approach developed in [1]-[2] for estimating boundary value processes is used to show that the smoother dynamics and boundary conditions are themselves in the form of a NNM of twice the size of the original NNM. Thus the class of NNMs, unlike say the class of 1-D causal systems, is closed under the smoothing operation. A consequence of this observation is of course that the two-filter solution techniques described in Section 3 and 4 can be used to compute the NNM smoothed estimate. Since the smoother for boundary value processes derived in [1]-[2] is expressed in operator form, we first obtain in Section 5.1 an operator characterization of the NNM smoother. The Green's identity for NNMs is then used in Section 5.2 to convert this operator description into equivalent NNM dynamics and boundary conditions for the smoother.

5.1. Operator Characterization of the NNM Smoother

The NNM smoothing problem can be described as follows. First, assume that the input sequence $u_{i,j}$ driving the NNM (2.1) is a zero-mean white Gaussian noise sequence defined over the interior $\tilde{\Omega} = [1, I-1] \times [1, J-1]$ of rectangle Ω , and with intensity

$$E[u_{i,j} u_{k,l}^T] = Q \delta_{ik} \delta_{jl} . \quad (5.1)$$

The boundary vectors $d_{H,j}$ and $d_{V,i}$ appearing in boundary conditions (2.14) are also assumed to be zero-mean white Gaussian noise sequences which are mutually uncorrelated and uncorrelated with the noise $u_{i,j}$, and with intensities

$$E[d_{H,j} d_{H,l}^T] = \Pi_H \delta_{jl} \quad , \quad E[d_{V,i} d_{V,k}^T] = \Pi_V \delta_{ik} . \quad (5.2)$$

Then, the state $x_{i,j}$ of NNM (2.1) is a zero-mean 2-D Gaussian random field, and we are given some noisy observations

$$y_{i,j} = C x_{i,j} + r_{i,j} \quad (i,j) \in \tilde{\Omega} \quad (5.3)$$

of this field over the interior domain $\tilde{\Omega}$. Here $r_{i,j}$ is a zero-mean white Gaussian noise sequence uncorrelated with the driving noise $u_{i,j}$ and the boundary and corner vectors, and with intensity

$$E[r_{i,j} r_{k,l}^T] = R \delta_{i,k} \delta_{j,l} , \quad (5.4)$$

where $R > 0$. In addition to the above interior measurements, we may also be given some boundary measurements which have a structure to the boundary and corner conditions described in Section 2, i.e.,

$$y_{H,j} = H_L x_{0,j} + G_L x_{1,j} + H_R x_{I,j} + G_R x_{I-1,j} + r_{H,j} \quad (5.5a)$$

$$y_{V,i} = H_B x_{i,0} + G_B x_{i,1} + H_T x_{i,J} + G_T x_{i,J-1} + r_{V,i} . \quad (5.5b)$$

In the above measurements, $r_{H,j}$ and $r_{V,i}$ are assumed to be zero-mean white Gaussian noises, which are mutually uncorrelated, and uncorrelated with u , r , and the boundary vectors, and with intensity

$$E[r_{H,j} r_{H,l}^T] = R_H \delta_{jl} \quad , \quad E[r_{V,i} r_{V,k}^T] = R_V \delta_{ik} . \quad (5.6)$$

The motivation for considering boundary observations which have a form different

from the interior observations is that equations (5.5) can be used to model the case where we observe the discretized normal derivative of a PDE along the boundary of a domain. For example, when the normal derivative is observed along the left and right edges of Ω , if h is the discretization mesh size, the measurements can be expressed as

$$y_{L,j} = \frac{1}{h}(x_{0,j} - x_{1,j}) + r_{L,j} \quad , \quad y_{R,j} = \frac{1}{h}(x_{I,j} - x_{I-1,j}) + r_{R,j} \quad (5.7)$$

where $r_{L,j}$ and $r_{R,j}$ are uncorrelated white Gaussian noises. These boundary measurements clearly correspond to a special case of (5.5a). An example of this type appears in the inverse resistivity problem considered in [32], where a potential distribution is imposed on the boundary of a resistive medium, and the resulting current density, which is proportional to the normal derivative of the potential, is measured on the boundary.

Then, the *NNM smoothing problem* consists in computing the conditional mean

$$\hat{x}_{i,j} = E[x_{i,j} | \mathbf{Y}] \quad (5.8)$$

where \mathbf{Y} denotes the Hilbert space of zero-mean random variables spanned by the interior observations $y_{i,j}$ for $(i,j) \in \tilde{\Omega}$, and by the boundary observations $y_{H,j}$ with $0 \leq j \leq J$, and $y_{V,i}$ with $1 \leq i \leq I-1$. To solve this problem, we will use the general results obtained in [1], [2] for the estimation of boundary value processes. However, since these results are expressed in abstract operator form, our first step will be to rewrite the NNM (2.1), (2.14) and observations (5.3) and (5.5) in operator form.

In this framework, the NNM dynamics (2.1) take the form

$$(L\mathbf{x})_{i,j} = Bu_{i,j} \quad (5.9)$$

where, if D_1 and D_2 denote respectively the backward horizontal and vertical shift operators, i.e.,

$$D_1 x_{i,j} = x_{i-1,j} \quad , \quad D_2 x_{i,j} = x_{i,j-1} \quad , \quad (5.10a)$$

we have

$$L = I - A_1 D_1 - A_2 D_1^{-1} - A_3 D_2 - A_4 D_2^{-1} . \quad (5.10b)$$

Note that in (5.9) x and Lx are defined respectively over the domains Ω and $\tilde{\Omega}$. Let also Δ_b be the restriction operator such that

$$\mathbf{x}_b = \Delta_b \mathbf{x} \quad (5.11)$$

is the restriction of \mathbf{x} to the first and last two columns and rows of Ω . Define

$$\mathbf{x}_L = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix}, \quad \mathbf{x}_R = \begin{bmatrix} \mathbf{x}_I \\ \mathbf{x}_{I-1} \end{bmatrix}, \quad (5.12a)$$

where the vectors \mathbf{x}_i are defined as in (3.1a), and let

$$\mathbf{x}_B = \begin{bmatrix} \mathbf{x}'_0 \\ \mathbf{x}'_1 \end{bmatrix}, \quad \mathbf{x}_T = \begin{bmatrix} \mathbf{x}'_J \\ \mathbf{x}'_{J-1} \end{bmatrix}, \quad (5.12b)$$

where

$$\mathbf{x}'_j = \begin{bmatrix} x_{1,j} \\ x_{2,j} \\ \cdot \\ \cdot \\ x_{I-1,j} \end{bmatrix} \quad (5.13)$$

is the vector obtained by scanning the states $x_{i,j}$ along the j^{th} row of Ω , where we omit the first and last elements of each row. Then, the restriction \mathbf{x}_b can be represented in vector form as

$$\mathbf{x}_b = \begin{bmatrix} \mathbf{x}_L \\ \mathbf{x}_R \\ \mathbf{x}_B \\ \mathbf{x}_T \end{bmatrix}, \quad (5.14)$$

and the boundary conditions (2.14) can be written in operator form as

$$V \mathbf{x}_b = \mathbf{d}_b \quad (5.15a)$$

with

$$V = \begin{bmatrix} \Gamma_L & \Delta_L & \Gamma_R & \Delta_R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Gamma_B & \Delta_B & \Gamma_T & \Delta_T \end{bmatrix} \quad (5.15b)$$

and

$$\mathbf{d}_b = \begin{bmatrix} \mathbf{d}_H \\ \mathbf{d}_V \end{bmatrix}. \quad (5.15c)$$

The matrices Γ_L , Γ_R , Δ_L and Δ_R , and vector \mathbf{d}_H appearing in the above expressions are defined in (3.6), and

$$\Gamma_B = I \otimes V_B \quad , \quad \Gamma_T = I \otimes V_T \quad (5.16a)$$

$$\Delta_B = I \otimes W_B \quad , \quad \Delta_T = I \otimes W_T \quad (5.16b)$$

$$\mathbf{d}_V = \begin{bmatrix} d_{V,1} \\ d_{V,2} \\ \cdot \\ \cdot \\ d_{V,J-1} \end{bmatrix}, \quad (5.16c)$$

where the matrices Γ_B , Γ_T , Δ_B , and Δ_T have size $2(I-1)n \times (I-1)n$ and the vector \mathbf{d}_V has dimension $2(I-1)n$. Finally, the vector \mathbf{d}_b given by (5.15c) is a zero-mean Gaussian vector with variance

$$\Pi_b = E[\mathbf{d}_b \mathbf{d}_b^T] = \begin{bmatrix} I \otimes \Pi_H & 0 \\ 0 & I \otimes \Pi_V \end{bmatrix}. \quad (5.17)$$

Similarly, the interior and boundary observations (5.3) and (5.5) can be denoted in operator form as

$$\mathbf{y} = C\mathbf{x} + \mathbf{r} \quad (5.18)$$

$$\mathbf{y}_b = H\mathbf{x}_b + \mathbf{r}_b, \quad (5.19a)$$

where

$$\mathbf{y}_b = \begin{bmatrix} \mathbf{y}_H \\ \mathbf{y}_V \end{bmatrix}, \quad \mathbf{r}_b = \begin{bmatrix} \mathbf{r}_H \\ \mathbf{r}_V \end{bmatrix} \quad (5.19b)$$

are obtained by scanning the horizontal and vertical boundary observations and

noises, and the matrix H has a structure identical to that of V , i.e.,

$$H = \begin{bmatrix} \Theta_L & \Psi_L & \Theta_R & \Psi_R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Theta_B & \Psi_B & \Theta_T & \Psi_T \end{bmatrix}, \quad (5.19c)$$

with

$$\Theta_E = I \otimes H_E, \quad \Psi_E = I \otimes G_E \quad \text{for } E = L, R, B, T. \quad (5.19d)$$

The covariance of the zero-mean Gaussian vector \mathbf{r}_b is given by

$$R_b = E[\mathbf{r}_b \mathbf{r}_b^T] = \begin{bmatrix} I \otimes R_H & 0 \\ 0 & I \otimes R_V \end{bmatrix}. \quad (5.20)$$

Then, it was shown in [1],[2] that the smoother dynamics and boundary conditions could be expressed in operator form as

$$\begin{bmatrix} L & -BQB^* \\ C^*R^{-1}C & L^\dagger \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ C^*R^{-1}\mathbf{y} \end{bmatrix} \quad (5.21)$$

$$\left[\begin{array}{cc} V^* \Pi_b^{-1} V + H^* R_b^{-1} H & E \end{array} \right] \begin{bmatrix} \hat{\mathbf{x}}_b \\ \hat{\lambda}_b \end{bmatrix} = H^* R_b^{-1} \mathbf{y}_b, \quad (5.22)$$

where B^* , C^* , V^* and H^* are the adjoint operators of B , C , V and H , respectively, and where L^\dagger denotes the formal adjoint of difference operator L . L^\dagger and the boundary operator E are defined through the Green's identity

$$\langle Lx, \lambda \rangle_{S(\tilde{\Omega})} = \langle x, L^\dagger \lambda \rangle_{S(\tilde{\Omega})} + \langle x_b, E \lambda_b \rangle_{S_b} \quad (5.23)$$

where $S(\tilde{\Omega})$ and S_b are the vector spaces of n -vector functions indexed over the domain $\tilde{\Omega}$, and over the first and last two rows and columns of Ω , respectively, and where $\langle \cdot, \cdot \rangle_S$ denotes the inner product over these spaces. The variable $\hat{\lambda}_{i,j}$ appearing in (5.) is the conditional mean of $\lambda_{i,j}$ with respect to the space \mathbf{Y} spanned by the observations, where $\lambda_{i,j}$ is the state of the complementary model associated to $x_{i,j}$. The concept of complementary model was originally introduced by Weinert and Desai [11], and it was the key element used in [1],[2] to derive the smoothing equations (5.21)-(5.22). Note also that (5.21) has a Hamiltonian structure similar to that of the smoother for 1-D causal processes [33].

5.2. NNM Characterization of the Smoother

As such, the operator characterization (5.21)-(5.22) describes completely the NNM smoother. However, this characterization can be made more explicit by noting that in Green's identity (5.23), we have

$$(L^{\dagger}\lambda)_{i,j} = \lambda_{i,j} - A_2^T \lambda_{i-1,j} - A_1^T \lambda_{i+1,j} - A_4^T \lambda_{i,j-1} - A_3^T \lambda_{i,j+1} \quad (5.24)$$

and

$$\begin{aligned} \langle \mathbf{x}_b, E \lambda_b \rangle_{S_b} &= \sum_{j=1}^{J-1} \left\{ \begin{bmatrix} x_{0,j}^T & x_{1,j}^T \end{bmatrix} E_L \begin{bmatrix} \lambda_{0,j} \\ \lambda_{1,j} \end{bmatrix} + \begin{bmatrix} x_{I,j}^T & x_{I-1,j}^T \end{bmatrix} E_R \begin{bmatrix} \lambda_{I,j} \\ \lambda_{I-1,j} \end{bmatrix} \right\} \\ &+ \sum_{i=1}^{I-1} \left\{ \begin{bmatrix} x_{i,0}^T & x_{i,1}^T \end{bmatrix} E_B \begin{bmatrix} \lambda_{i,0} \\ \lambda_{i,1} \end{bmatrix} + \begin{bmatrix} x_{i,J}^T & x_{i,J-1}^T \end{bmatrix} E_T \begin{bmatrix} \lambda_{i,J} \\ \lambda_{i,J-1} \end{bmatrix} \right\} \quad (5.25) \end{aligned}$$

with

$$E_L = \begin{bmatrix} 0 & -A_1^T \\ A_2^T & 0 \end{bmatrix}, \quad E_R = \begin{bmatrix} 0 & -A_2^T \\ A_1^T & 0 \end{bmatrix} \quad (5.26a)$$

$$E_B = \begin{bmatrix} 0 & -A_3^T \\ A_4^T & 0 \end{bmatrix}, \quad E_T = \begin{bmatrix} 0 & -A_4^T \\ A_3^T & 0 \end{bmatrix}. \quad (5.26b)$$

Then, substituting (5.24) inside the operator description (5.21) of the NNM smoother dynamics, we can rewrite these dynamics as

$$\alpha_0 \begin{bmatrix} \hat{x}_{i,j} \\ \hat{\lambda}_{i,j} \end{bmatrix} = \alpha_1 \begin{bmatrix} \hat{x}_{i-1,j} \\ \hat{\lambda}_{i-1,j} \end{bmatrix} + \alpha_2 \begin{bmatrix} \hat{x}_{i+1,j} \\ \hat{\lambda}_{i+1,j} \end{bmatrix} + \alpha_3 \begin{bmatrix} \hat{x}_{i,j-1} \\ \hat{\lambda}_{i,j-1} \end{bmatrix} + \alpha_4 \begin{bmatrix} \hat{x}_{i,j+1} \\ \hat{\lambda}_{i,j+1} \end{bmatrix} + \beta y_{i,j} \quad (5.27)$$

with

$$\alpha_0 = \begin{bmatrix} I & -BQB^T \\ C^T R^{-1} C & I \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 \\ C^T R^{-1} \end{bmatrix} \quad (5.28a)$$

$$\alpha_1 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2^T \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} A_2 & 0 \\ 0 & A_1^T \end{bmatrix} \quad (5.28b)$$

$$\alpha_3 = \begin{bmatrix} A_3 & 0 \\ 0 & A_4^T \end{bmatrix}, \quad \alpha_4 = \begin{bmatrix} A_4 & 0 \\ 0 & A_3^T \end{bmatrix}, \quad (5.28c)$$

where (5.27) is almost in NNM form. This relation can be brought to NNM form by noting that α_0 is invertible with

$$\alpha_0^{-1} = \begin{bmatrix} I & BQB^T \\ -C^T R^{-1} C & I \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \quad (5.29a)$$

where

$$D_1 = (I + BQB^T C^T R^{-1} C)^{-1}, \quad D_2 = (I + C^T R^{-1} CBQB^T)^{-1}. \quad (5.29b)$$

This yields

$$\begin{bmatrix} \hat{x}_{i,j} \\ \hat{\lambda}_{i,j} \end{bmatrix} = \tilde{\alpha}_1 \begin{bmatrix} \hat{x}_{i-1,j} \\ \hat{\lambda}_{i-1,j} \end{bmatrix} + \tilde{\alpha}_2 \begin{bmatrix} \hat{x}_{i+1,j} \\ \hat{\lambda}_{i+1,j} \end{bmatrix} + \tilde{\alpha}_3 \begin{bmatrix} \hat{x}_{i,j-1} \\ \hat{\lambda}_{i,j-1} \end{bmatrix} + \tilde{\alpha}_4 \begin{bmatrix} \hat{x}_{i,j+1} \\ \hat{\lambda}_{i,j+1} \end{bmatrix} + \tilde{\beta} y_{i,j} \quad (5.30a)$$

with

$$\tilde{\alpha}_k = \alpha_0^{-1} \alpha_k \quad 1 \leq k \leq 4, \quad \tilde{\beta} = \alpha_0^{-1} \beta, \quad (5.30b)$$

which is now in NNM form.

Similarly, by using (5.25)-(5.26) and taking into account the structure (5.15b) and (5.19c) of boundary matrices V and H , the boundary conditions (5.22) for the NNM smoother can be rewritten more explicitly as

$$\begin{aligned}
 & \left\{ \begin{array}{c} V_L^T \\ W_L^T \\ V_R^T \\ W_R^T \end{array} \right\} \Pi_{\bar{H}}^{-1} \begin{bmatrix} V_L & W_L & V_R & W_R \end{bmatrix} + \left\{ \begin{array}{c} H_L^T \\ G_L^T \\ H_R^T \\ G_R^T \end{array} \right\} R_{\bar{H}}^{-1} \begin{bmatrix} H_L & G_L & H_R & G_R \end{bmatrix} \left\{ \begin{array}{c} \hat{x}_{0,j} \\ \hat{x}_{1,j} \\ \hat{x}_{I,j} \\ \hat{x}_{I-1,j} \end{array} \right\} \\
 & + \begin{bmatrix} E_L & 0 \\ 0 & E_R \end{bmatrix} \begin{bmatrix} \hat{\lambda}_{0,j} \\ \hat{\lambda}_{1,j} \\ \hat{\lambda}_{I,j} \\ \hat{\lambda}_{I-1,j} \end{bmatrix} = \begin{bmatrix} H_L^T \\ G_L^T \\ H_R^T \\ G_R^T \end{bmatrix} R_{\bar{H}}^{-1} y_{H,j} \quad (5.31a)
 \end{aligned}$$

$$\begin{aligned}
 & \left\{ \begin{array}{c} V_B^T \\ W_B^T \\ V_T^T \\ W_T^T \end{array} \right\} \Pi_{\bar{V}}^{-1} \begin{bmatrix} V_B & W_B & V_T & W_T \end{bmatrix} + \left\{ \begin{array}{c} H_B^T \\ G_B^T \\ H_T^T \\ G_T^T \end{array} \right\} R_{\bar{V}}^{-1} \begin{bmatrix} H_B & G_B & H_T & G_T \end{bmatrix} \left\{ \begin{array}{c} \hat{x}_{i,0} \\ \hat{x}_{i,1} \\ \hat{x}_{i,J} \\ \hat{x}_{i,J-1} \end{array} \right\} \\
 & + \begin{bmatrix} E_B & 0 \\ 0 & E_T \end{bmatrix} \begin{bmatrix} \hat{\lambda}_{i,0} \\ \hat{\lambda}_{i,1} \\ \hat{\lambda}_{i,J} \\ \hat{\lambda}_{i,J-1} \end{bmatrix} = \begin{bmatrix} H_B^T \\ G_B^T \\ H_T^T \\ G_T^T \end{bmatrix} R_{\bar{V}}^{-1} y_{V,i} \quad (5.31b)
 \end{aligned}$$

But these boundary conditions are precisely in the form (2.14)! Thus, the NNM solution techniques developed in Sections 3 and 4 are directly applicable to the NNM smoother (5.30)-(5.31), since the smoother itself is in NNM form. The fact that the class of NNM models is invariant under the smoothing operation is also quite satisfying, since it indicates that these models are perfectly adapted to the study of noncausal estimation problems.

5.3. Smoothing Error Dynamics

It was also shown in [1]-[2] that the smoothing error $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$ admits the operator characterization

$$\begin{bmatrix} L & -BQB^* \\ C^*R^{-1}C & L^+ \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}} \\ -\hat{\lambda} \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & C^*R^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{r} \end{bmatrix} \quad (5.32)$$

with boundary condition

$$\begin{bmatrix} V^* \Pi_b^{-1} V + H^* R_b^{-1} H & E \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_b \\ -\hat{\lambda}_b \end{bmatrix} = V^* \Pi_b^{-1} \mathbf{d}_b - H^* R_b^{-1} \mathbf{r}_b . \quad (5.33)$$

The 2-D NNM which corresponds to the operator expression (5.32) is identical to (5.30a), except for the input term:

$$\begin{aligned} \begin{bmatrix} \tilde{x}_{i,j} \\ -\hat{\lambda}_{i,j} \end{bmatrix} &= \tilde{\alpha}_1 \begin{bmatrix} \tilde{x}_{i-1,j} \\ -\hat{\lambda}_{i-1,j} \end{bmatrix} + \tilde{\alpha}_2 \begin{bmatrix} \tilde{x}_{i+1,j} \\ -\hat{\lambda}_{i+1,j} \end{bmatrix} + \tilde{\alpha}_3 \begin{bmatrix} \tilde{x}_{i,j-1} \\ -\hat{\lambda}_{i,j-1} \end{bmatrix} + \tilde{\alpha}_4 \begin{bmatrix} \tilde{x}_{i,j+1} \\ -\hat{\lambda}_{i,j+1} \end{bmatrix} \\ &+ \alpha_0^{-1} \begin{bmatrix} B & 0 \\ 0 & C^T R^{-1} \end{bmatrix} \begin{bmatrix} u_{i,j} \\ r_{i,j} \end{bmatrix} . \end{aligned} \quad (5.34)$$

Similarly, the operator representation (5.33) of the boundary conditions yields boundary conditions which are identical to (5.31a,b), but with different right-hand sides.

The NNM system (5.34) can then be written in 1-D form by using the column scanning technique of Section 3, and the resulting 1-D representation can be used to compute the error covariance $P(i,j;k,l) = E[\tilde{x}_{i,j} x_{k,l}^T]$, which is a useful quantity if we want to evaluate the performance of the NNM smoother.

6. Smoothing Examples

In this section, we apply the results of the previous sections to implement the NNM smoother for two examples, corresponding respectively to the discretized stochastic Poisson and heat equations. In particular, it is shown that the fast FFT solver developed in Section 4 can be used to implement the NNM smoother for both of these examples.

6.1. 2-D Poisson Equation

The dynamics of the process to be estimated are given by

$$x_{i,j} = \frac{1}{4}(x_{i-1,j} + x_{i+1,j} + x_{i,j-1} + x_{i,j+1}) + u_{i,j} , \quad (6.1)$$

where the variance of the white Gaussian noise process $u_{i,j}$ is q . The boundary conditions are in Dirichlet form, i.e.,

$$\tilde{V}_E = 1 \quad , \quad \tilde{W}_E = 0 \quad \text{for } E = L, R, B, T \quad (6.2)$$

in (2.17), where the variance of the zero-mean boundary vectors $d_{E,k}$ is π . The interior observations are simply the process itself plus some additive white Gaussian noise process $r_{i,j}$ of unit variance:

$$y_{i,j} = x_{i,j} + r_{i,j} \quad , \quad (i,j) \in \tilde{\Omega} \quad , \quad (6.3)$$

and we assume that the state x is observed exactly on the boundary, i.e.,

$$y_{L,j} = x_{0,j} \quad , \quad y_{R,j} = x_{I,j} \quad , \quad y_{B,i} = x_{i,0} \quad , \quad y_{T,i} = x_{i,J} \quad . \quad (6.4)$$

Therefore, for this problem the matrices A_k with $1 \leq k \leq 4$, B , C , Q and R are all scalars, and in particular,

$$A_1 = A_2 = A_3 = A_4 = 1/4 \quad (6.5a)$$

$$B = C = R = 1 \quad , \quad Q = q \quad . \quad (6.5b)$$

Substituting these values inside expression (5.27) for the NNM smoother, we find

$$\begin{aligned} & \begin{bmatrix} 1 & -q \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_{i,j} \\ \hat{\lambda}_{i,j} \end{bmatrix} \\ &= \frac{1}{4} \left\{ \begin{bmatrix} \hat{x}_{i-1,j} \\ \hat{\lambda}_{i-1,j} \end{bmatrix} + \begin{bmatrix} \hat{x}_{i+1,j} \\ \hat{\lambda}_{i+1,j} \end{bmatrix} + \begin{bmatrix} \hat{x}_{i,j-1} \\ \hat{\lambda}_{i,j-1} \end{bmatrix} + \begin{bmatrix} \hat{x}_{i,j+1} \\ \hat{\lambda}_{i,j+1} \end{bmatrix} \right\} + \begin{bmatrix} 0 \\ y_{i,j} \end{bmatrix} . \quad (6.6) \end{aligned}$$

Taking also into account the form of the boundary conditions and observations (6.4) inside (5.31), it is easy to check that the NNM smoother boundary conditions are of *Dirichlet* type, i.e.,

$$\hat{x}_{0,j} = y_{L,j} \quad , \quad \hat{x}_{I,j} = y_{R,j} \quad , \quad \hat{x}_{i,0} = y_{B,i} \quad , \quad \hat{x}_{i,J} = y_{T,i} \quad (6.7a)$$

$$\hat{\lambda}_{0,j} = 0 \quad , \quad \hat{\lambda}_{I,j} = 0 \quad , \quad \hat{\lambda}_{i,0} = 0 \quad , \quad \hat{\lambda}_{i,J} = 0 \quad . \quad (6.7b)$$

Then, since the NNM smoother dynamics are vertically symmetric, and the boundary conditions are in Dirichlet form, the FFT solver described in Section 4.2 can be used to solve (6.6)-(6.7). Let $\{ \xi_{i,j} \}$, $\{ \mu_{i,j} \}$ and $\{ \eta_{i,j} \}$ be the sequences

obtained by applying the discrete sine transform S given by (4.16) to the estimates $\{ \hat{x}_{i,j} \}$, $\{ \hat{\lambda}_{i,j} \}$, and observations $\{ y_{i,j} \}$ for a fixed index i , i.e.,

$$\xi_{i,j} = \left(\frac{2}{J}\right)^{1/2} \sum_{l=1}^{J-1} \hat{x}_{i,l} \sin(lj\pi/J) \quad 1 \leq j \leq J-1 \quad (6.8a)$$

$$\mu_{i,j} = \left(\frac{2}{J}\right)^{1/2} \sum_{l=1}^{J-1} \hat{\lambda}_{i,l} \sin(lj\pi/J) \quad 1 \leq j \leq J-1 \quad (6.8b)$$

$$\eta_{i,j} = \left(\frac{2}{J}\right)^{1/2} \sum_{l=1}^{J-1} y_{i,l} \sin(lj\pi/J) \quad 1 \leq j \leq J-1. \quad (6.8c)$$

Let also

$$\epsilon_{i,j} = \left(\frac{2}{J}\right)^{1/2} \sin(j\pi/J) (y_{B,i} + (-1)^{j-1} y_{T,i}) \quad (6.8d)$$

be the sequence representing the effect of the DST on the boundary conditions (6.7c) and (6.7d) on the bottom and top edges. Then, by applying the DST to the columns of the NNMS smoother (6.6)-(6.7), we obtain the decoupled subsystems

$$\begin{aligned} & \begin{bmatrix} 1 - \frac{1}{2} \cos(j\pi/J) & -q \\ 1 & 1 - \frac{1}{2} \cos(j\pi/J) \end{bmatrix} \begin{bmatrix} \xi_{i,j} \\ \mu_{i,j} \end{bmatrix} \\ &= \frac{1}{4} \left\{ \begin{bmatrix} \xi_{i-1,j} \\ \mu_{i-1,j} \end{bmatrix} + \begin{bmatrix} \xi_{i+1,j} \\ \mu_{i+1,j} \end{bmatrix} \right\} + \begin{bmatrix} \epsilon_{i,j} \\ \eta_{i,j} \end{bmatrix} \end{aligned} \quad (6.9)$$

where $1 \leq j \leq J-1$, with boundary conditions

$$\xi_{0,j} = \eta_{L,j} \quad , \quad \mu_{0,j} = 0 \quad (6.10a)$$

$$\xi_{I,j} = \eta_{R,j} \quad , \quad \mu_{I,j} = 0 \quad , \quad (6.10b)$$

where $\{ \eta_{L,j} \}$ and $\{ \eta_{R,j} \}$ denote the DST transforms of boundary measurements $\{ y_{L,j} \}$ and $\{ y_{R,j} \}$, respectively. These subsystems can then be written in TPBVDS form and solved by decomposing the TPBVDS model in forwards and backwards stable components. By observing that the modes σ of the system (6.9) are the zeros of the determinant of the matrix

$$\Phi(w) = \begin{bmatrix} 1 - \frac{1}{4}(w + 2\cos(j\pi/J)) & -q \\ 1 & 1 - \frac{1}{4}(w + 2\cos(j\pi/J)) \end{bmatrix} \quad (6.11)$$

where $w = \sigma + \sigma^{-1}$, it is clear that if σ is a mode, so is σ^{-1} , so that in the TPBVDS decomposition, there will be two forwards stable and two backwards stable modes. Unfortunately, even for this simple example, the TPBVDS decomposition cannot be computed in closed form.

6.2. Discretized Heat Equation

Consider now the discrete heat equation

$$mx_{i,j} = x_{i-1,j} + n(x_{i,j-1} + x_{i,j+1}) + u_{i,j} \quad (6.12)$$

where the variance of noise $u_{i,j}$ is q . Assume also that the boundary conditions, interior observations and boundary observations are the same as for the previous example. Then, the NNM smoother takes the form

$$\begin{aligned} \begin{bmatrix} m & -q \\ 1 & m \end{bmatrix} \begin{bmatrix} \hat{x}_{i,j} \\ \hat{\lambda}_{i,j} \end{bmatrix} \\ = \begin{bmatrix} \hat{x}_{i-1,j} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{\lambda}_{i-1,j} \end{bmatrix} + n \left\{ \begin{bmatrix} \hat{x}_{i,j-1} \\ \hat{\lambda}_{i,j-1} \end{bmatrix} + \begin{bmatrix} \hat{x}_{i,j+1} \\ \hat{\lambda}_{i,j+1} \end{bmatrix} \right\} + \begin{bmatrix} 0 \\ y_{i,j} \end{bmatrix} \end{aligned} \quad (6.13)$$

and the boundary conditions are given by (6.7a) and

$$\hat{\lambda}_{I,j} = \hat{\lambda}_{i,0} = \hat{\lambda}_{i,J} = 0; \quad (6.14)$$

with $\hat{\lambda}_{0,j}$ free. This last feature just corresponds to the fact that the $\hat{\lambda}$ dynamics are anticausal in the i direction, so that the values of $\hat{\lambda}_{i,j}$ with $i \geq 1$ are not affected by $\hat{\lambda}_{0,j}$. Again, the NNM smoother dynamics (6.13) are vertically symmetric, and the vertical boundary conditions are in Dirichlet form, so that the FFT solver of Section 4.2 is applicable to this system. Performing the transformations (6.8a-d), the NNM smoother is decoupled into $J-1$ subsystems of the form

$$\begin{bmatrix} m - 2n \cos(j\pi/J) & -q \\ 1 & m - 2n \cos(j\pi/J) \end{bmatrix} \begin{bmatrix} \xi_{i,j} \\ \mu_{i,j} \end{bmatrix} = \begin{bmatrix} \hat{x}_{i-1,j} \\ \hat{\lambda}_{i+1,j} \end{bmatrix} + \begin{bmatrix} \epsilon_{i,j} \\ \eta_{i,j} \end{bmatrix}, \quad (6.15)$$

with $1 \leq j \leq J-1$. But equation (6.15) is equivalent to the TPBVDS system

$$\begin{aligned} \begin{bmatrix} m - 2n \cos(j\pi/J) & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \xi_{i,j} \\ \mu_{i+1,j} \end{bmatrix} \\ = \begin{bmatrix} 1 & q \\ 0 & m - 2n \cos(j\pi/J) \end{bmatrix} \begin{bmatrix} \xi_{i-1,j} \\ \mu_{i,j} \end{bmatrix} + \begin{bmatrix} \epsilon_{i,j} \\ -\eta_{i,j} \end{bmatrix}, \end{aligned} \quad (6.16)$$

where the boundary conditions are given by

$$\xi_{0,j} = \eta_{L,j} \quad , \quad \mu_{I,j} = 0 . \quad (6.17)$$

Thus, in this particular case, no state augmentation is necessary to bring the transformed smoother to TPBVDS form. This is due to the fact that the heat equation is causal in the i direction. Thus, if we apply the DST transform to vertical index j in equation (6.12), the coupling with respect to the j variable is eliminated, and we obtain a standard causal 1-D system, for which the smoother is the standard 1-D smoother, which is given here by (6.16). Another interesting feature of this smoother is that the boundary conditions do not depend on the "fake" boundary conditions and boundary measurements $y_{R,j}$ on the right edge (see Example 2.2b) which were introduced to guarantee that the discrete heat equation was in the general NNM form (2.1), (2.17).

7. Conclusions

A general smoothing method has been obtained for 2-D random fields described by 2-D NNMs with local boundary conditions. This smoothing procedure relies on a general approach to the formulation of noncausal estimation problems developed in [1]-[2]. In this approach, both the state of the system and of its complementary model need to be estimated, and accordingly, the smoother is described by a Hamiltonian system of twice the dimension of the original system. For the NNM case, it turns out that the Hamiltonian is itself in NNM form, with local boundary conditions of the type used to specify the NNM system that we seek to estimate. This property indicates that NNMs capture well the intrinsic noncausality associated with estimation problems in several dimensions. Also, the computation of the NNM smoothed estimates reduces to the solution of a NNM system. A general solution technique has been developed for NNM systems. This

solution consists in writing a 2-D NNM columnwise as a 1-D boundary-value system of large dimension, which can then be solved by using the recursive techniques developed in [16]-[17] for the solution of 1-D TPBVDSs. For the special case where the 2-D NNM has periodic vertical boundary conditions, or has vertically symmetric dynamics, an even more efficient solution technique based on the use of the FFT, DST or DCT as a vertical decoupling transformation was also obtained, whereby the solution of a 2-D NNM reduces to the solution of a set of low-order decoupled 1-D TPBVDSs.

One of the main themes of this paper is that straightforward attempts at extending 1-D Kalman filtering techniques to several dimensions are misguided, since random fields in several dimensions are usually not generated causally, and multidimensional random observations are often not obtained sequentially, but all at one time. This implies that noncausal random field models, such as NNMs, and smoothing problems, provide the most natural way to formulate multidimensional estimation problems. In other words, a *purely noncausal* formulation of multidimensional estimation problems should be employed. However, it is still possible to reintroduce recursiveness at the algorithmic level in order to obtain fast estimation techniques. Since causality is in this case a computational device, many different types of recursions are possible, which reflect the great amount of latitude we have in processing the available data.

An important limitation of the results presented here is that we have assumed that the domain of definition of the 2-D NNMs under consideration was rectangular. For practical applications, random fields are usually defined over very irregular domains, so that at first sight the results developed here have a limited applicability. However, this impression is incorrect, since recently developed *domain decomposition* techniques for PDEs [34]-[35] make it possible to divide an irregular domain in rectangular subdomains, and then to solve the original problem over each subdomain separately, while handling the coupling between subdomains with a preconditioned conjugate gradient algorithm. This approach would lead here to a parallel implementation of 2-D NNM estimation algorithms, where observations over different subdomains could be processed in parallel, and then combined to obtain an overall estimate. Finally, in addition to being parallel,

this approach makes also possible, provided that the conditions of Section 4 are satisfied, to use FFT solvers over the rectangular subdomains. See [34] for a description of a domain decomposition solver of this type for the 2-D Poisson equation. The application of domain decomposition techniques to NNM estimation problems seems therefore to be a promising area for future research.

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