February 1988 LIDS-P-1755

L^1 Sensitivity Minimization for Plants with Commensurate delays

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Abstract

In this paper, we will consider the problem of L^1-sensitivity minimization for plants with commensurate input delays. We will describe a procedure for computing the minimum performance and characterize optimal solutions. Explicit solutions will be presented in some special cases.

Notation

X^*	Dual space of a normed linear space X
BS	all elements in S with norm ≤ 1
$S^\perp$	the annihilator subspace defined as

$$\{x^* \in X^* | < s, x^* >= 0 \forall s \in S \subset X\}$$

${}^\perp S$	the annihilator subspace defined as

$$\{x \in X | < x, s >= 0 \forall s \in S \subset X^*\}$$

R_+	the positive real line $[0, \infty)$
$L^p(R_+)$	classical Banach spaces on R_+
$BV(X)$	functions of bounded variation on X
$C_0(X)$	continuous function on a locally compact space$X$ such that

$$\forall \epsilon > 0, \ \{x \in X || f(x)| \geq \epsilon\} \ is \ compact$$

1. Introduction

The problem of L^1-sensitivity minimization in the case of an arbitrary rational plant with a pure input delay was considered in [3] where the problem was shown to be equivalent to the rational plant case solved in [2]. In this paper, we consider plants of the form $P = UP_0$ where P_0 has a stable inverse, and U is a polynomial of delays. The method of solution is quite different from the one presented in [3] for the case of pure delay, which cannot be generalized.

The interest in this problem stems from the fact that many systems have delays in various parts of the control loop. Even though one might not want to build an optimal controller based on modeling these delays, it is crucial to identify the limitations of design within such an environment. The theory in this paper provides such information. It mimics in spirit similar results reported in the H_∞ problem such as [4],[5],[8].

2. Mathematical Preliminaries

In this paper will follow the notation introduced in [2]. Let AM denoted the space of all atomic measures of bounded variation on R_+, i.e,

$$AM = \{h | h(t) = \sum_{i=0}^{\infty} h_i \delta(t - t_i), \{h_i\} \in l^1, t_i \geq 0\}$$

with norm defined as

$$||h||_{AM} = \sum_{i=0}^{\infty} |h_i|$$

Let $A = AM \otimes L^1(R_+)$. Then A consists of all distributions of the form

$$h(t) = h_{AM}(t) + h_M(t)$$

where $h_{AM} \in AM$ and $h_M \in L^1(R_+)$. The norm on A is defined as

$$\|h\|_A = \|h_{AM}\|_{AM} + \|h_M\|_{L^1(R_+)}.$$

We will denote by $\hat{A}$ the space of Laplace transform of distributions in A with the same norm. It is well known that A is a Banach algebra and defines a set of BIBO-stable, linear, time-invariant impulse responses with the given operator norm. For the rest of this paper, H will denote the Laplace transform of h, i.e. $H \in \hat{A}$ and $h \in A$.

It can be shown that A can be embedded in the space of function of bounded variation $BV(R_+)$, (This embedding is not a surjection). This injection is done in the obvious way

$$h \to \tilde{h} = \int_0^t h(\tau)d\tau$$

and we have

$$\|h\| = TV(\tilde{h}) = Total\ Variation\ of\ \tilde{h}$$

We will use this notation for the rest of the paper. Denote by $C_0(R_+)$ the space of all continuous function that converge to zero at infinity, with norm

$$\|x\|_\infty = \sup_t |x(t)|$$

It is well known [1], that

$$C_0{}^*(X) = BV(X)$$

for any locally convex linear space X, i.e. for every F bounded linear functional on $C_0(X)$, there exists a function $\tilde{f} \in BV(X)$ such that

$$F(x) = \int_0^\infty x(t)d\tilde{f}(t)$$

with $\|F\| = TV(\tilde{f})$. In the case of $X = R_+$ and $d\tilde{f} = fdt$ with $f \in A$, we denote the above operation as $< x, f >$.

3. Problem Definition

Let the plant be given by

$$P = UP_0$$

where $P_0^{-1} \in \hat{A}$ and

$$U = \sum_{n=0}^{N} u_n e^{-s(\delta n)}, \quad u_N \neq 0$$

The problem is to minimize the weighted sensitivity function over all stabilizing compensators. The problem is equivalent to [2]

$$\mu_0 = \inf_{Q \in \hat{A}} \|W - UQ\|_{\hat{A}} \qquad (OPT)$$

Without loss of generality, we will assume that the polynomial

$$U(z) = \sum_{n=0}^{N} u_n z^n$$

has all its zeros inside the open unit disc D. The factors with zeros outside the unit disc are invertible in $\hat{A}$ and can be lumped in Q. Also, we will assume that W is a strictly proper, stable rational function. Even though this assumption is not necessary, it makes the presentation more transparent.

4. Problem Solution:

Let the subspace $S \subset A$ be defined as

$$S = \{k \in A | k = u * q, q \in A\}$$

The following theorem characterizes ${}^{\perp}S$.

Theorem 1

$${}^{\perp}S = \{f \in C_0(R_+) | \sum_{i=0}^{N} u_i f(t + \delta i) = 0\}$$

Proof :
Let $f \in {}^{\perp}S$, then $< f, k >= 0 \; \forall \; k \in S$. However

$$\begin{aligned} < f, k > &= \int_0^\infty f(t) \sum_{i=0}^{N} u_i q(t - \delta i) dt \\ &= \sum_{i=0}^{N} u_i \int_0^\infty f(t) q(t - \delta i) dt \\ &= \sum_{i=0}^{N} u_i \int_0^\infty f(t + \delta i) q(t) dt = 0 \end{aligned}$$

$$\Longleftrightarrow \sum_{i=0}^{N} u_i f(t + \delta i) = 0 \qquad Q.E.D$$

Consider $[{}^{\perp}S]^{\perp}$. Then, from [7], $[{}^{\perp}S]^{\perp}$ = weak*-closure of $S = \bar{S}^{w} \subset BV(R_{+})$. Let $\tilde{w} = \int_0^t w(t)dt$ and consider the problem:

$$\mu_1 = \min_{\tilde{k}\in\bar{S}^{w}} TV(\tilde{w} - \tilde{k}) \qquad (OPT1)$$

The above problem (OPT1) will be trivially equivalent to (OPT) if $\bar{S}^{w} = S$. This not necessarily the case, however by considering the dual problems, we will verify that $\mu_1 = \mu_0$ and hence (OPT1) is equivalent to (OPT). Dualizing (OPT1) [6], we get

$$\begin{aligned}\min_{\tilde{k}\in\bar{S}^{w}} TV(\tilde{w} - \tilde{k}) &= \sup_{f\in B^{\perp}S} \int_0^{\infty} f(t)d\tilde{w}(t) \\ &= \sup_{f\in B^{\perp}S} < f, w >\end{aligned}$$

To compute the above problem, we will find explicit representation of $f \in {}^{\perp}S$. The one parameter family of difference equation characterizing ${}^{\perp}S$ gives rise to a stable semi-group by which all elements in ${}^{\perp}S$ can be generated given any initial vector $\xi(\tau), \tau \in [0, \delta)$. Define a subspace M inside $C^N[0, \delta)$ such that $\xi \in C^N[0, \delta)$ if

$$\lim_{t\to\delta} \xi_{i-1}(t) = \xi_i(0), \; i = 2, ..., N$$

$$\lim_{t\to\delta} \xi_N(t) = -\sum_{n=0}^{N-1} \frac{u_n}{u_N}\xi_{n+1}(0)$$

. Then the following theorem is true.

Theorem 2:
$f \in {}^{\perp}S$ if and only if there exists a vector $\xi(\tau) \in M$ such that

$$f(\tau + \delta i) = \beta^{\top}(i)\xi(\tau) \qquad \tau \in [0, \delta), \; i = 0, 1, 2, \ldots$$

$$\beta^{\top}(i) = e_1 F^i$$

$$F = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -u_0/u_N & \ldots & \ldots & \ldots & -u_{n-1}/u_N \end{pmatrix} \quad e_1 = (1, 0, 0 \ldots 0)$$

Proof
From Theorem 1, $f \in {}^{\perp}S$ if and only if:

$$f(t + \delta N) = -\sum_{n=0}^{N-1} \frac{u_n}{u_N} f(t + \delta n)$$

Let $t = \tau + \delta i,\ \tau \in (0, \delta),\ i = 0, 1, 2, \ldots$ and define:

$$x_\tau^1(i) = f(\tau + \delta i)$$
$$x_\tau^2(i) = f(\tau + \delta(i+1))$$
$$\cdot$$
$$\cdot$$
$$x_\tau^N(i) = f(\tau + \delta(i + N - 1))$$

Let

$$x_\tau(i) = \begin{pmatrix} x_\tau^1(i) \\ \cdot \\ \cdot \\ x_\tau^N(i) \end{pmatrix},$$

then

$$x_\tau(i+1) = F x_\tau(i)$$

Hence,

$$x_\tau(k) = F^k x_\tau(0) = F^k \begin{pmatrix} f(\tau) \\ f(\tau + \delta) \\ \cdot \\ \cdot \\ f(\tau + \delta(N-1)) \end{pmatrix}$$

Finally, we note that f is completely characterized from its values on the interval $[0, \delta N)$. This completes the proof. Note that

$$f(t + \delta i) = e_1 F^k x(0) \qquad Q.E.D$$

To abbreviate the notation, we will denote the association of a vector ξ with f by the linear map T, i.e.

$$T : M \to C_0(R_+).$$

Now we go back to the dual problem for the evaluation of μ_1. Let T^* denote the adjoint of T defined as a map:

$$T^* : BV(R_+) \to M^*$$

It is straight forward to verify that T^* maps A into A with support $[0, \delta)$, in particular:

$$T^* w(\tau) = \sum_{i=0}^{\infty} w(\tau + i)\beta(i) \qquad \tau \in [0, \delta)$$

The dual problem is given by

$$\sup < T^* w, \xi >$$
$$\|T\xi\|_\infty \leq 1$$

$$\xi \in M$$

Note that $\|T\xi\|_\infty \leq 1$ if and only if

$$|\beta^\top(i)\xi(\tau)| \leq 1 \;\; \forall \tau \in [0,\delta) \quad i = 0,1,2,\ldots$$

Since $\beta^\top(i) = e_1 F^i$ with (e_1, F) observable and F_1 has eigenvalues inside the disc (i.e. a stable semi-group), then there exists an integer I such that

$$\|T\xi\|_\infty \leq 1 \iff |\beta^\top(i)\xi(\tau)| \leq 1 \forall \tau \in [0,\delta), \quad i = 0,1,...,I$$

A lower bound of the integer I can be easily derived. Next, we will show that the dual problem can be solved by solving a linear programming problem for each $t \in [0,\delta)$.

Theorem 3

$$\mu_1 = \int_0^\delta \mu_t dt$$

where

$$\mu_t = \max \;\; x^\top T^* w(t)$$
$$|\beta^\top(i)x| \leq 1$$
$$x \in R^N$$
$$i = 0,1,\ldots,I$$

Proof :

$$\sup_\xi \int_0^\delta \xi^\top(t)(T^*w)(t)dt \leq \int_0^\delta \sup_\xi(t)|\xi^\top(t)T^*w(t)|dt$$
$$\leq \int_0^\delta \mu_t dt$$

Denote by x_t, the solution achieving μ_t. Since T^*w is a continuous function of t (w is rational and strictly proper), there exists a continuous function $\xi(t) \in M$ that approximates the solution arbitrarily closely. The details are standard and hence, omitted.

Remark 1:

The dual of (OPT) yield a similar problem with $\xi(\tau) \in L_\infty^N[0,\delta)$. By solving the problem pointwise we get the same x_t and hence $\mu_0 = \mu_1$. This shows that (OPT) is equivalent to (OPT1).

The previous theorem enables us to compute the optimal value μ_0 by computing an infinite number of linear programming problems. However, we note that the constraint set is the same in all of these problems. Since F is a stable matrix and (e_1, F) observable, the constraint set is given by

$$C_\beta = \{x \in R^N | \beta^\top(i)x| \leq 1\,,\; i = 0,1,\ldots,I\}$$

is a bounded polygon. Let $x_1, \dots, x_p$ be the extreme points of C_β. Define subsets $I_+(j), I_-(j)$ of $\{0, 1, \dots, I\}$ such that $k \in I_+(j)$ if and only if $\beta^\top(k)x_j = 1$, and $k \in I_-(j)$ if and only if $\beta^\top(k)x_j = -1$, for $j = 1, \dots, p$. Let P_j be the convex cone generated by $\{\beta(k) | k \in I_+(j)\} \cup \{-\beta(k) | k \in I_-(j)\}$ (note that P_j is a N-dimensional cone) and let $I(j) = I_+(j) \cup I_-(j)$. In the next Lemma, we will show that if the vector $T^*w(t)$ at some fixed t, lies in the cone P_j, then the solution in Theorem 3 is given by x_j. Hence, to solve the problem in Theorem 3, we first compute $T^*w(t)$ for all $t \in [0, \delta)$, which is easily done since W is rational. Then we project this vector on the p different convex cones P_j. x_j will be the maximizing solution in the interval where the vector $T^*w(t)$ lies in the P_j cone. This lemma is true in the generic case of nondegenerate problems, and corresponds to results already known in linear programming problems.

Lemma

$$\max_{x \in C_\beta} x^\top y = {x_j}^\top y$$

if and only if $y \in P_j$ corresponding to x_j.

Proof

If $y \in P_j$, then $y = \sum_{i \in I(j)} \alpha_i \beta(i)$, $\alpha_i \geq 0$ *for* $i \in I_+$, $\alpha_i \leq 0$ *for* $i \in I_-(j)$. Therefore, for any $x \in C_\beta$, we have:

$$\begin{aligned} y^\top x_j - y^\top x &= \sum \alpha_i (\beta^\top(i) x_j - \beta^\top(i) x) \\ &\geq \sum |\alpha_i| (1 - |\beta^\top(i) x|) \\ &\geq 0 \end{aligned}$$

It is well known that the above maximization problem is always achieved at an extreme point. To prove the converse of this theorem, we will show that if y does not belong to the cone P_j then there exists an $x \in C_\beta$ in any neighborhood of x_j such that $x^\top y > {x_j}^\top y$. Hence y must belong to at least one cone. Let $y = \sum_{i \in I(j)} \alpha_i \beta(i)$ where $\alpha_k < 0$ for some $k \in I_+(j)$. Choose an $x \in C_\beta$ in such a way that $\beta^\top(i)x = 1$ for $i \in I_+(j), i \neq k$, $\beta^\top(i)x = -1$ for $i \in I_-(j)$ and $\beta^\top(k)x < 1$. The existence of such a vector is possible if the problem is nondegenerate, i.e $\beta(i), i \in I(j)$ are linearly independent. Then we have

$$\begin{aligned} x^\top y &= \sum \alpha_i \beta^\top(i) x \\ &= \sum_{j \neq k} |\alpha_j| + \alpha_k \beta^\top(k) x \\ &> \sum_{j \neq k} |\alpha_j| + \alpha_k \\ &= {x_j}^\top y \end{aligned}$$

which completes the proof.

Once the extreme points are calculated, the indices $I(j)$ are easily determined. Expressing the vector $T^*w(t)$ as a linear combination of the vectors spanning the convex cones involves solving p systems of linear equations (generically). Then the coefficients $\alpha(t)$ (as in the proof) determine exactly the maximizing solution. This procedure has the advantage of evaluating the infinitely many linear programming problems by a finite number of inversion problems followed by signature functions. For delay polynomials of low order, this procedure is quite attractive. For high order polynomials, the problem is computationally complex. Later on, we will present examples of special cases where these linear programming problems reduce tremendously.

Remark 2:

Recall that we have considered the enlarged problem $(OPT1)$. By following this approach, we avoid looking for a solution to the primal problem inside $(L^\infty)^*$. This way we guarantee existence inside $\bar{S}^w$. The following theorem characterizes these solutions.

Theorem 4

A solution $\tilde{\phi} \in BV(R_+)$ of the problem:

$$\tilde{\phi} = \min_{\tilde{k} \in \bar{S}^*} TV(\tilde{w} - \tilde{k})$$

exists and satisfies the following:

1. $T^*\tilde{\phi} = T^*\tilde{w}$
2. $\tilde{\phi}$ is aligned with $T\xi$
3. $TV(\tilde{\phi}) = \mu_0$

Proof

The proof is a restatement of the alignment conditions [6] and the fact that $\tilde{\phi} - \tilde{w} \in \bar{S}^w$.

5. Examples

In this section we will present three examples that represent important special cases of the theory, and provide complete solution for every case.

Example I:

$$U(s) = e^{-s\delta}$$

then

$$T = \text{identity embedding map}$$
$$T^* = \Pi_\delta = \text{projection onto} \quad [0, \delta)$$

Optimal solution $\phi = \Pi_\delta w$.

Example II:

$$U(s) = u_0 + u_1 e^{-s\delta}$$

Define

$$b = -u_0/u_1 \in D$$

Then

$$\beta(i) = b^i$$

$$T^* w = \sum_{i=0}^{\infty} b^i w(t + i\delta) = w_1(t)$$

$$\mu_0 = \sup_{|b^i \xi(t)| \leq 1} < w_1, \xi > = \sup_{|\xi(t)| \leq 1} < w_1, \xi >$$
$$= \|w_1\|_1$$

The optimal solution is given by

$$\phi = \begin{cases} w_1 & t \in [0, \delta) \\ 0 & t \in [\delta, \infty) \end{cases}$$

Example III: Let

$$W = \frac{1}{s + a}.$$

then

$$T^* w(t) = \sum_{i=0}^{\infty} e^{-at} e^{-a\delta i} \beta(i) \qquad t \in [0, \delta)$$
$$= e^{-at} \sum_{i=0}^{\infty} e^{-a\delta i} \beta(i)$$
$$= e^{-at} \alpha_0$$

Thus

$$\mu_t = e^{-at} \quad \max x^\top \alpha_0$$
$$|e_1 F^i x| \leq 1$$
$$i = 0, 1, 2, \ldots, I$$

and

$$\mu_t = e^{-at} x_0{}^\top \alpha_0$$

for some x_0. Hence, μ_0 is given by

$$\mu_0 = \frac{1}{a}(1 - e^{-a\delta}) x_0{}^\top \alpha_0$$

An optimal ϕ can be constructed to satisfy Theorem 4. We note that the above maximization problem is the dual of:

$$\mu_t = e^{-at} max \sum_{i=0}^{I} |\phi_i|$$

$$\sum_{i=0}^{I} \phi_i \beta(i) = \alpha_0$$

In fact, the optimal $\phi_i \neq 0$ only when $|\beta^\top(i)x_0| = 1$. Let ϕ^0 be an optimal solution and define the function $\phi(t)$ as:

$$\phi(t + \delta i) = \begin{cases} \phi_i^0 e^{-at} & i \leq I,\ t \in [0, \delta) \\ 0 & \text{otherwise} \end{cases}$$

It is straightforward to verify that ϕ satisfies the conditions of Theorem 4, and hence an optimal solution.

Conclusions

In this paper, we present a solution to the L^1 minimization problem for plants with commensurate delays. In the special case where the weight has one pole, the problem can be solved by a single linear programming problem. Other special cases were also presented. A method for computing the optimal performance and a characterization of minimizing solutions are given. Issues such as computational complexity, and existence of solutions in the algebra A will not be discussed in this paper.

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This research was supported by the Army Research Office, Center for Intelligent Control Systems, under grant ARO DAAL03-86-K-0171.