A NECESSARY AND SUFFICIENT CONDITION FOR ROBUST BIBO STABILITY

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ABSTRACT

This paper is concerned with obtaining necessary and sufficient conditions for robustly stabilizing a class of plants characterized by a known linear shift invariant plant with additive perturbations of the form of possibly nonlinear, time varying ℓ^∞-stable operators. We will show that, in some sense, the small gain theorem is necessary and sufficient even when the perturbations are restricted to a class of linear operators.

Key Words: Robust BIBO stability, time varying perturbations, small gain theorem.

Notation

R	real numbers
$\ell^p(1 \le p < \infty)$	one-sided sequence of real numbers $u = (u_0, u_1, u_2,...)$ with the norm $\|u\|_p = (\Sigma|u_i|^p)^{1/p} < \infty$.
ℓ^∞	one-sided sequence of real numbers $u = (u_0, u_1, u_2, ...)$ with the norm $\|u\|_\infty = \sup|u_i| < \infty$.
$\ell^\infty e$	extension space of ℓ^∞.
$\mathcal{A}$	algebra of BIBO linear, shift-invariant, causal operators on ℓ^∞.

H_∞	bounded analytic functions on the unit disc with the norm $\|f\|_\infty = \text{ess sup} \|f(e^{i\theta})\| < \infty$.
$g_p (1 \le p \le \infty)$	gain of an operator $\Delta : \ell^P \to \ell^P$.

$$g_p(\Delta) = \sup_{f \in \ell^P \backslash \{0\}} \frac{\|\Delta(f)\|_p}{\|f\|_p}$$

P_N	truncation operator
	$P_N: (u_0, u_1, u_2,...) \to (u_0, u_1,...,u_{N-1}, 0,0,...)$.

An operator $F:\ell^\infty_e \to \ell^\infty_e$ is called proper (causal) if $P_N F = P_N F P_N$ and strictly proper if $P_N F = P_N F P_{N-1}$. Note that the set of strictly proper operators is an ideal of the ring of proper operators. Note also that if F is a strictly proper operator then $(1+F)$ is invertible as an operator $\ell^\infty_e \to \ell^\infty_e$.

1. Introduction

The basic motivation of this work is to incorporate robustness in the ℓ^1-based design methodology [1], and highlight its conservatism.

Define the class of plants

$$\Omega = \{P = P_0 + W\Delta\}$$

where P_0 is LSI (linear shift invariant), W is a stable LSI filter and Δ is an arbitrary ℓ^∞-stable, strictly proper operator (i.e. possibly nonlinear time varying) with gain

$$g_\infty (\Delta) = \sup_{f \in \ell^\infty \backslash \{0\}} \frac{\|\Delta(f)\|_\infty}{\|f\|_\infty} < 1 .$$

It was shown in [2] that a LSI compensator C that stabilizes P_0, will stablize all plants $P \epsilon \Omega$ if

$$\|C(1+CP_0)^{-1}W\|_{\mathcal{A}} \le 1 \tag{C1}$$

where $||H||_{\mathcal{A}} = ||h||_1$ and h is the pulse response of H and $\mathcal{A}$ denotes the algebra of BIBO linear, shift-invariant, causal operators on ℓ^∞ (see [1], [2]). We note that the conservatism of (C1) was not discussed.

The main result of this paper is to show that C1 is also necessary. By this we mean that if C1 is not satisfied, then there exists an admissible Δ such that the closed loop system is not BIBO stable.

At this point it is interesting to note that results parallel to the above have been derived for the H_∞-problem [4]. There it was shown that

$$||C(1+CP_0)^{-1}W||_{H^\infty} \leq 1 \qquad \qquad \text{(C2)}$$

is necessary and sufficient for robustly stabilizing the class

$$\Omega' = \{P = P_0 + W\Delta, \ g_2(\Delta) < 1\}.$$

Also, it was shown that if C2 is not satisfied then there exists a LSI Δ, $||\Delta||_\infty < 1$ such that the closed loop system is not ℓ^2-stable.

2. Main Result

The main result of this paper is the following Theorem concerning the small gain Theorem:

Theorem: Let $Q \varepsilon \mathcal{A}$ and suppose Δ is an ℓ^∞ - stable, strictly proper operator with $g_\infty(\Delta) < 1$, then the operator $(1+Q\Delta)$ has an ℓ^∞ - stable inverse with bounded gain for all Δ if and only if $||Q||_{\mathcal{A}} = ||q||_1 \leq 1$.

From this theorem, we obtain the result on robust stability:
Corollary: Condition C1 is both necessary and sufficient.
Proof of Corollary: Immediate from [2] and the above theorem.
Proof of main theorem:

The sufficiency is immediate from the small gain theorem [3]. For necessity, we assume that $||q||_1 > 1$ and construct a Δ such that $(1+Q\Delta)^{-1}$ is not ℓ^∞-stable. The proof will consist of two major steps:

Step I: In the following lemma, we show that if $\|q\|_1 > \delta > 1$, then there exists an input $e \in \ell^\infty{}_e$ such that

$$\frac{\|P_{N-1}(Qe)\|_\infty}{\|P_N e\|_\infty} \geq \delta > 1 \quad \forall N > N_0$$

and

$$\lim_{N \to \infty} \|P_N e\|_\infty = \infty$$

Step II: Use the above e to construct an admissible linear shift varying Δ such that if $u = Qe$, then

$$(1+Q\Delta)u \in \ell^\infty$$

Noting that $u \in \ell^\infty{}_e \backslash \ell^\infty$, this proves that $(1+Q\Delta)^{-1}$ is not ℓ^∞-stable.

Step I
Lemma:

Suppose $q \in \ell^1$ be a convolution operator with $\|q\|_1 > \delta > 1$. Then there exists an input $e \in \ell^\infty{}_e \backslash \ell^\infty$ and an integer N_0 such that for $N > N_0$

$$\frac{\|P_{N-1}(q*e)\|_\infty}{\|P_N e\|_\infty} \geq \delta > 1 .$$

Proof: The proof is by construction. Since $\|q\|_1 > \delta$, there exists an $e_0 \in \ell^\infty$, $e_0(0) = 0$, $\|e_0\|_\infty = 1$, such that $\|q*e_0\|_\infty > \delta$. Because q is causal, given M with $\|q*e_0\|_\infty > M > \delta$, there is an N_0 such that

$$\|P_{N_0}(q*P_{N_0}e_0)\|_\infty \geq M.$$

Given $m > 0$, there is an N_1 such that

$$\sum_{k=N_1+1}^{\infty} |q(k)| < m$$

such a N_1 can be found since $q \in \ell^1$.

We define $e \in \ell^\infty_e$ as follows:

$$e(k(N_0+N_1)+i) = \begin{cases} \alpha(k)e_0(i) & 0 \leq i \leq N_0 \\ \\ 0 & N_0+1 \leq i \leq N_0+N_1 \end{cases}$$

$$(k=0,1,2,...)$$

where

$$\alpha(k+1) = \delta^{-1}\beta(k)$$
$$\beta(k+1) = M\alpha(k+1) - \gamma(k)$$
$$\gamma(k+1) = m\alpha(k+1) + \gamma(k)$$
$$\alpha(0) = 1, \quad \beta(0) = M, \quad \gamma(0) = m.$$

Notice that α, β, γ have the following interpretations.

$\alpha(k)$: the magnitude of the k-th chunk of input

$\beta(k)$: a lower bound of the magnitude of the output during the k-th excitation.

$\gamma(k)$: an upper bound of the accumulation of the fringe effect of the j-th (j=0,...,k) input.

Now we claim that for m small enough and δ close enough to 1, we have $e \in \ell^\infty_e \backslash \ell^\infty$ and

$$\frac{\|P_{N-1}(q*e))\|_\infty}{\|P_N e\|_\infty} \geq \delta > 1, \quad N > N_0.$$

Henceforth we assume that m is small enough and δ is close enough to 1.

(i)   We will show that, $e \in \ell^\infty_e \backslash \ell^\infty$:

Let $\xi(k) = [\beta(k), \gamma(k)]^T$. Then we have the following difference equation:

$$\xi(k+1) = A\xi(k), \quad \xi(0) = [M,m]^T$$
$$\alpha(k+1) = [\delta^{-1}, 0]\xi(k), \quad \alpha(0) = 1$$

$$A = \begin{bmatrix} M\delta^{-1} & -1 \\ m\delta^{-1} & 1 \end{bmatrix}$$

For $\delta=1$, $m=0$, A has eigenvalues, $\{1,M\}$. Furthermore the unstable eigenvalue M is controllable with the input vector $[M,0]^T$ and observable with the output vector $[1,0]$. Hence by the continuity argument we see that A is unstable and the unstable eigenvalue is controllable with the input vector $[M,m]^T$ and observable with the output vector $[1,0]$. Hence it follows that $\alpha(k) \to \infty$ as $k \to \infty$. This proves $e \varepsilon \ell^\infty e \backslash \ell^\infty$

(ii) Now we show that $\dfrac{\|P_{N-1}(q^*e)\|_\infty}{\|P_N e\|_\infty} \geq \delta > 1$, $N > N_0$, which can be

proven as follows:

(ii-1) First recall that the input $P_{N_0} e_0$ has the following property:

$$\|P_{N_0}(q*P_{N_0}e_0)\|_\infty \geq M$$

$$\|(1-P_{N_0+N_1})(q*P_{N_0}e_0)\|_\infty < m$$

(ii-2) $\{\alpha(k)\}$, $\{\beta(k)\}$, $\{\gamma(k)\}$ are monotone increasing sequences. To see that, consider (β,γ) -plane and let K be the cone generated by $\{[1,0]^T, [1, M\delta^{-1}]^T\}$. Notice that $A\xi \geq \xi \geq 0$ if and only if $\xi \varepsilon K$. Since the initial state $\xi(0) = [M,m]^T \varepsilon K$, then it is clear $\{\beta(k)\}$ and $\{\gamma(k)\}$ are increasing if $A^k\xi(0) \varepsilon K$ for any k. Hence, it suffices to show that there is an A-invariant cone in K which contains $\xi(0)$. Let $\xi_e \geq 0$ be an eigenvector of the largest eigenvalue of A. Then a straightforward calculation shows that the cone generated by $\{[1,0]^T, \xi_e\}$ is a desired one.

(ii -3) $\quad \| P_N e\|_\infty \le \alpha(k) \qquad k(N_0+N_1) < N \le (k+1)(N_0+N_1)$

$$\|P_{N-1}(q*e)\|_\infty \ge \beta(k) \qquad N_0 + k(N_0 + N_1) < N \le (k+1)(N_0 + N_1)$$

This is a consequence of (ii-2) and the definition of α, β, γ.

(ii-4) $\qquad \dfrac{\|P_{N-1}(q*e)\|_\infty}{\|P_N e\|_\infty} \ge \delta \qquad (N > N_0)$

Indeed, $\delta = \beta(k)/\alpha(k+1)$ and $\beta(0)/\alpha(0) = M > \delta$. Hence if $k(N_0+N_1) < N \le (k+1)(N_0+N_1)$, $k \ge 1$, then

$$\frac{\|P_{N-1}(q*e)\|_\infty}{\|P_N e\|_\infty} \ge \frac{\beta(k-1)}{\alpha(k)} = \delta$$

If $N_0 < N \le N_0+N_1$, then

$$\frac{\|P_{N-1}(q*e)\|_\infty}{\|P_N e\|_\infty} \ge \frac{\beta(0)}{\alpha(0)} \ge \delta .$$

Q.E.D.

Step II: Construction of the Δ (linear shift varying):
By Lemma there is an $e\varepsilon\, \ell^\infty_e \backslash \ell^\infty$ such that

$$\frac{\|P_{N-1}(Qe)\|_\infty}{\|P_N e\|_\infty} \ge \delta > 1, \ N > N_0 \tag{1}$$

Using e, we define Δ in the following way: let Δ_N be a bounded linear functional defined on $P_{N-1}\ell^\infty = \{u\varepsilon\ell^\infty | u_k = 0 \ \forall \ k \ge N\}$, i.e.,

$$\Delta_N : P_{N-1}\, \ell^\infty \to R$$

for all N, then Δ is defined as

$$[\Delta(w)](N) = \Delta_N \ (P_{N-1} \ w) . \tag{2}$$

where $[\Delta(\omega)](N)$ is the Nth entry of the sequence $\Delta(\omega)$.
Now, we set $\Delta_N = 0$ for any $N \leq N_0$. For $N > N_0$, consider the linear functional f_N defined on the 1-dimensional subspace $\{P_{N-1}(Qe)\}$ as

$$f_N : \{P_{N-1}(Qe)\} \to R$$

$$f_N \ (P_{N-1} \ (Qe)) = - \ e(N)$$

Let Δ_N be a Hahn-Banach extension [5] of f_N for all $N > N_0$.

Then it can be shown that Δ is admissible, i.e., ℓ^{∞}-stable, strictly proper and of the gain less than 1, and that $(1+Q\Delta)^{-1}$ is not ℓ^{∞}-stable. Indeed (1) implies that

$$\|\Delta_N\| = \|f\| = \frac{|e(N)|}{\|P_{N-1}(Qe)\|_{\infty}} \leq \delta^{-1} < 1. \tag{3}$$

Hence we have

$$\|\Delta\| = \sup_{\|w\|_{\infty}=1} \|\Delta(w)\|_{\infty}$$

$$= \sup_{\|w\|_{\infty}=1} \ \sup_N \ |[\Delta(w)](N)|$$

$$\leq \delta^{-1} < 1. \tag{4}$$

Moreover (2) implies Δ is strictly proper.

Finally to show that $(1+Q\Delta)^{-1}$ is not ℓ^{∞}-stable, it suffices to show that $(1+Q\Delta)(u) \ \varepsilon \ \ell^{\infty}$ where $u=Qe$, since from (1), $u \ \varepsilon \ \ell^{\infty}_e \backslash \ell^{\infty}$. Note that (1) and (2) imply

$$\Delta(u) = P_{N_0} e\text{-}e. \tag{5}$$

Hence we have

$$(1+Q\Delta)(u) = u+Q[P_{N_0} e\text{-}e]$$

$$= Q(P_{N_0} e) \, \varepsilon \, \ell^{\infty}$$

as desired because $P_N e \, \varepsilon \, \ell^{\infty}$.   Q.E.D.

Remark: One possible Δ satisfying (2) consists of time-varying gain and delay. To see this, let $M(N)$, $N > N_0$, be an integer such that $|Qe(M(N))| = ||P_{N-1}(Qe)||_{\infty}$. It is easy to see that

$$\Delta_N (P_{N-1} w) = -\frac{e(N)}{Qe(M(N))} w(M(N))$$

satisfies (2).

3. **Discussion of the Results:**

The theorem shows that Δ can always be constructed as a linear time (shift) varying operator. It should be noted that it is generally **not** possible to construct a linear shift invariant operator. It is well known that "$\mathcal{A}$" is a subalgebra of H_{∞}, i.e. $\mathcal{A}$ H_{∞}. Also, if $H \varepsilon \mathcal{A}$ then $H^{-1} \varepsilon \mathcal{A}$ iff

$$\inf_{|z|<1} |H(z)| > 0$$

which is precisely the condition for invertibility of an element in H_{∞}. Hence, the operator

$$(1+Q\Delta)$$

is invertible in $\mathcal{A}$ for all LSI Δ, iff $||Q||_\infty \cdot ||\Delta||_\infty \leq 1$. Suppose $||Q||_{\mathcal{A}} > 1$ but $||Q||_\infty < 1$, then the destabilizing LSI Δ has to satisfy $||\Delta||_\infty \geq 1/||Q||_\infty > 1$ and hence $||\Delta||_{\mathcal{A}} > 1$. Thus, we cannot destabilize the system with an admissible LSI Δ.

Also, note that a nonlinear <u>shift invariant</u> Δ can be constructed to make $(1+Q\Delta)^{-1}$ unstable. Let e, u $\epsilon \ell^\infty e$ be as in the proof of main theorem. A possible Δ is

$$\Delta(w) = \begin{cases} P_{N(w)} (P_{N_0} e-e) & \text{if } N(w) \neq 0 \\ S^{K(w)} \Delta(S^{-K(w)} w) & \text{if } (N(w) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $N(w)$ is the smallest integer N such that $w(N) \neq u(N)$, $k(w)$ is the smallest integer K such that $w(k) \neq 0$, S is the shift operator; i.e. $[Sw](N) = w(N-1)$, $[Sw](0) = 0$. It is straightforward to verify that $g_\infty(\Delta) < 1$ and hence Δ is admissible. This construction was used in [6] in a different context.

CONCLUSIONS:

The main objective of this paper is to demonstrate that (C1) is both necessary and sufficient for robustly stabilizing all $P\epsilon\Omega$. This is equivalent to showing that $(I+Q\Delta)$ has a stable inverse for all Δ with gain less than unity if an only if $||Q||_{\mathcal{A}} \leq 1$. Hence, if $||Q|| > 1$, we are able to constract either a linear shift varying Δ, or a nonlinear shift invariant Δ such that $(I+Q\Delta)^{-1}$ is unstable.

Considering only linear perturbations, the result have an interesting interpretation. Given a linear Δ, $(I+Q\Delta)$ has a stable inverse if and only if -1 is not in the spectrum of $Q\Delta$, whose spectral radius $\rho(Q\Delta)$ is always bounded by $||Q|| \, ||\Delta||$. The result of this paper shows that

$$\sup_{||\Delta|| \leq 1} \rho(Q\Delta) = ||Q|| \cdot$$

where Δ is linear, but possibly shift varying. Also, we have shown that in general

$$\sup_{\|\Delta\|\leq 1} \rho(Q\Delta) < \|Q\| \cdot \|\Delta\|$$

if Δ is linear shift invariant. This interpretation highlights the conservation of (C1) as a robustness measure.

Acknowledgement

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