A METHODOLOGY FOR DESIGNING ROBUST NONLINEAR CONTROL SYSTEMS

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ABSTRACT: This paper presents an outline of a methodology for the design of nonlinear dynamic compensator for nonlinear multivariable systems which provides guarantees of closed-loop stability, robustness and performance. The method is an extension of the Linear-Quadratic-Gaussian with Loop-Transfer-Recovery (LQG/LTR) methodology for linear systems, thus hinging upon the idea of constructing an approximate inverse operator for the plant. A major feature of the method is an attempted unification of both the state-space and Input-Output formulations. We show that recovery at the plant input can be done as in the linear case, while recovery at the output is very restrictive.

Keywords: State estimation, Nonlinear control systems, Control system synthesis, state-space methods, Observability.

INTRODUCTION

This paper presents a methodology for designing nonlinear dynamic compensators for nonlinear multivariable systems. Our basic philosophy is to extend a linear design methodology, the Linear-Quadratic-Gaussian with Loop-Transfer-Recovery (LQG/LTR) methodology, that has been recently devdeveloped (Doyle and Stein, 1981; Stein and Athans, 1984), in which one develops a "target" loop that is desirable, and then attempts to achieve, or "recover" this loop shape in the actual closed-loop system. We show how loop shaping at both the plant input and plant output can be extended to nonlinear systems, although the plant output case can only be performed under restrictive conditions.

One contribution of this proposed methodology is an attempted unification of state-space formulations with Input-Output (I/O) operator descriptions. Such a unification (as in LQG/LTR) allows both computations of gains (in the state-space) and the handling of plant uncertainty and unmodeled dynamics (with the I/O description). We utilize a result from (Desoer and Wang, 1980) that says, "high loop gains produce small errors," in a precise manner. We also present some new I/O robustness tests, similar in spirit to the singular value robustness tests in the frequency domain as described in (Lehtomaki, 1980).

Thus I/O operators are extremely useful for calculating performance and robustness to unmodeled dynamics, but unfortunately they are extremely difficult to calculate explicitly. We get around this by performing all required synthesis operations (i.e. finding gains, etc.) using a state-space formulation and tying the results back to the I/O domain.

Our main results are the Loop-Operator-Recovery theorem of section 5 and the Extended Kalman Filter (EKF) nondivergence theorem of section 6.

BASIC DEFINITIONS

This paper will freely mix state-space notation with operator notation, so we must first set out our definitions. Our plant model will be

$$\dot{x}(t) = f(x(t)) + Bu(t)$$

$$y(t) = Cx(t) \tag{1}$$

with $x(t)\varepsilon R^n$, $u(t)$, $y(t)\varepsilon R^m$, B an nxm matrix, and C mxn matrix. The non-linearity $f:R^n \to R^n$ is assumed at least twice differentiable. Some of the results presented here can be generalized to nonlinear B and C mappings, see (Grunberg 1986).

Since models are never exact, we will need to include this fact in our control system design. We account for the discrepancy by unstructured unmodeled dynamics, for which we will assume that we have an I/O bound of some type. This will be discussed more fully in section 4.

We now consider the Input-Output (I/O) viewpoint for systems. Let L be the space of all vector-valued functions $u:[0,\infty)\to R^n$ which are square-integrable over finite time intervals (i.e. the space L_{2e}). We will abuse notation slightly by stating $x\varepsilon L$, $u\varepsilon L$, even though u and x are of different dimensions.

Definitions: The truncated norm of x L is

$$||x||_T \triangleq \left(\int_0^T x^T(t)x(t)dt\right)^{1/2} \tag{2}$$

where T denotes transpose. The operator description of a nonlinear system is simply a mapping $P: L\to L$. For example, we write $y=Pu$ to mean the input u produces the output y.

Definitions: (Gain and Stability).

(i) The gain of an operator is

$$||P|| \triangleq \sup_{\substack{u\varepsilon L\\T}} \frac{||Pu||_T}{||u||_T} \tag{3}$$

(ii) The incremental gain of an operator is

$$||P||_\Delta \triangleq \sup_{\substack{u,v\varepsilon L\\T}} \frac{||Pu-Pv||_T}{||u-v||_T} \tag{4}$$

(iii) An operator P is stable if $||P||< +\infty$.

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(iv) An operator P is incrementally stable if $||P||_{\Delta} < +\infty$.

For linear systems, the usual notion of stability coincides with both stability and incremental stability as defined here.

In order to simplify equations, we will now define a nonlinear operator, ϕ, by

$$\phi \triangleq [S^{-1}-F]^{-1} \tag{5}$$

where S is the integral operator and F is the non-dynamical operator defined by $(Fx)(t)=f(x(t))$. The nonlinear operator ϕ is shown in block-diagram form in Fig. 1. We can now see the usefulness of ϕ; our plant (1) can now be written in compact form

$$P = C\phi B \tag{6}$$

in complete analogy with the linear transfer function $\phi(s)=(sI-A)$ used in linear control theory. The operator representation (6) will be very useful in the sequel.

We will be concerned with the tracking regulator configuration of Fig. 2, where K is the operator describing the nonlinear compensator. We have a reference command, r, input disturbance, w, and an output disturbance, d.

The loop equations are

$$\begin{aligned} y &= d + P\,(w+u) \\ u &= K\,(r-y) \end{aligned} \tag{7}$$

The rest of the paper will be concerned with

(i) Given a P,K, how "good" a closed-loop system do we have?

(ii) Given a P, how do we design a "good" K?

Sections 3 and 4 deal with (i), while the later sections deal with (ii). Due to page limitations, proofs of results have been omitted and will be published in a future paper. For full details, see Grunberg (1986).

PERFORMANCE

This section will analyze the command-following performance of the complete closed-loop system, as shown in Fig. 2. Let H be the map from r to y in the closed-loop system, with d=0, w=0. Then,

$$H = PK[I+PK]^{-1}. \tag{8}$$

When we have a nonzero disturbance d, we have (with w=0)

$$y = H(r-d) + d \tag{9}$$

Theorem: (Desoer and Wang, 1980)

If, for all r in some set of commands $R \subset L$ and for all d in some set of disturbances $D \subset L$

$$||[I+PK]^{-1}(r-d)||_T << ||r-d||_T \quad \text{for all } T \tag{10}$$

then H≈I on R and D in the sense that

$$||e||_T = ||r-y||_T << ||r||_T + ||d||_T \quad \text{for all } r \in R, d \in D, T. \tag{11}$$

This theorem shows the linearizing effect of high gain feedback; a high loop gain makes the input-output map close to unity. Even though PK may be nonlinear, we still achieve a desirable closed-loop I/O map.

ROBUSTNESS TESTS

Suppose we are given a nominal system, G, that is closed-loop stable, i.e. $[I+G]^{-1}$ is stable. Then robustness tests give us bounds on the amount of deviation from the nominal plant we can allow and still guarantee that the perturbed plant is closed-loop stable. We give one such test here, more exist, see (Grunberg and Athans, 1985; Grunberg, 1986).

Theorem (Division error)

Let $\tilde{G} = G[I+E]^{-1}$ be the actual plant, and let G be closed-loop stable. Then $\tilde{G}$ will be closed-loop stable if there exists $\delta<1$ such that

$$||Ex||_T \leq \delta||(I+G)x||_T \quad \text{for all } x \in L, T \tag{12}$$

and

$$||E|| < \infty$$

This test shows us that the important quantity in evaluating the robustness of a control system is the loop operator, here G=PK. Section 3 shows us that PK is also the important quantity for determining performance, in both a command following and output disturbance rejection context. These results can also be modified so that it can be seen that the quantity (-K)(-P) can also express the robustness and performance in an input disturbance context.

Note that these robustness tests give conditions which must be checked for all signals in some signal space. Needless to say, this could be quite tiresome. Current research is aimed at easing the computational burden associated with checking these conditions.

DESIGN PHILOSOPHY

This section will give an overview of the Nonlinear-Model-Based-Compensator with Loop-Operator-Recovery (NMBC/LOR) procedure. We propose to use an observer based controller in order to guarantee the closed-loop stability of our system. We do that by virtue of the following theorem (Safonov, 1980).

Definition: $\hat{x} = F(y,u)$ is a nondivergent estimate of the state x of

$$\begin{aligned} \dot{x} &= f(x) + Bu + Bw \\ y &= Cx + d \end{aligned} \tag{14}$$

if the map $(w,d) \rightarrow e=x-\hat{x}$ is stable uniformly in u. Here F is a dynamic operator (the estimator) with inputs y and u.

Theorem (Separation)

If $g(\cdot)$ is a state-feedback function such that

$$\dot{x} = f(x) - Bg(x) + Bw \tag{15}$$

is stable from (w,d) to x,

and

$$\sup_x |\nabla g(x)| < \infty \tag{16}$$

and if $\hat{x} = F(y,u)$ is any nondivergent estimate of x, then

$$\dot{x} = f(x) - Bg(\hat{x}) + Bw \tag{17}$$

is stable from (w,d) to x.

Thus we can substitute $\hat{x}$ for x in a state-feedback system without loss of stability, just as in the linear case. While we do not have any guarantee of optimality as in the linear case, we can guarantee closed-loop stability with a nondivergent estimator and stabilizing state feedback.

Definition: A model based estimator has the form

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) + Bu + H(t)[y-C\hat{x}] \\ H(t) &= H(t, y(\tau), u(\tau),\ 0 \leq \tau \leq t) \end{aligned} \tag{18}$$

where $H(\cdot)$ can depend on the past of y and u in anyway. We can use this in a compensator

$$K = -G[\phi^{-1}+BG+HC]^{-1}H(-I) \tag{19}$$

in Fig. 2. Note that this is a model-based compensator, where we have used a model-based estimator with gain H, and selected $u=-G\hat{x} = -g(\hat{x})$. If the estimator is nondivergent, then the above compensator will stabilize on plant $P=C\phi B$, by the separation theorem. We now present our LOR theorems. For proofs, see (Grunberg, 1986). For the next two theorems, let z be the compensator state in (18-19).

Theorem (LOR) at plant input).

Let H_μ, the filter gain in the compensator, be a linear operator, parameterized by μ such that

$$\lim_{\mu\to 0} H_\mu\sqrt{\mu} = BW \tag{20}$$

Where W is any invertible operator. Then if B is linear

$$z\to x \quad \text{as} \quad \mu\to 0, \quad \text{if} \quad d,r=0 \tag{21}$$

and

$$\lim_{\mu\to 0} (-K)(-P) = G\phi B. \tag{22}$$

Theorem (LOR at plant output). Let G_ρ be parameterized by $\rho>0$. Let assumptions (a-f) be

(a) $\lim_{\rho\to 0} \sqrt{\rho}\, G=W_1C$; W_1 invertible

(b) $\lim_{\rho\to 0} G\sqrt{\rho} = W_2C$; W_2 invertible

(c) B linear

(d) C linear

(e) $G_\rho[x_1-x_2]\to G_\rho x_1-G_\rho x_2$ if $C(x_1-x_2)\to 0$

(f) $[\phi^{-1}+BG_\rho]^{-1}$ linear $\forall\rho>0$

Then

(i) (a) and (c) imply that

$$\lim_{\rho\to 0} Cz_\rho=0 \quad \text{if } w=0 \tag{23}$$

(ii) (a-f) imply

$$\lim_{\rho\to 0} PK = C\phi H \tag{24}$$

Remark: The LOR at the plant input shows that the loop broken at the plant input can be made to approach the target loop $G\phi B$. In (Grunberg and Athans, 1985; Grunberg, 1986) it is shown that $G\phi B$ has some very desirable properties if G is chosen as the solution to certain nonlinear optimal control problems. Also related are the results about optimal regulators obtained by Glad(1984, 1985).

THE EXTENDED KALMAN FILTER

We now discuss the Extended Kalman Filter (EKF) and its properties. For a basic exposition of the EKF, see Jazwinski (1970).

Definition: A nonlinear system (14) is M-detectable if there exists a model-based estimator of the form (18) that is nondivergent.

Note that this is the most fundamental form of a definition for observability that one can make in a control context. It does not say that the state can be uniquely determined from the measurements-only that the state can be estimated so that the estimation error is no more than proportional to the size of the noises (w,d). We now state our EKF theorem.

Theorem (EKF)

Let $|\nabla f(x)|\le M$ and $|\nabla^2 f(x)|\le N$ for some $M,N<\infty$ and consider the EKF

$$\dot{\hat{x}} = f(\hat{x}) + Bu + H(t)[y-C\hat{x}] \tag{25}$$

$$H(t) = \Sigma(t)C^T \tag{26}$$

$$\dot{\Sigma}(t) = \nabla f(x(t))\Sigma(t) + \Sigma(t)\nabla f^T(x(t)) + \Xi + \Sigma(t)C^TC\Sigma(t) \tag{27}$$

$$\Sigma(t_o) = \Sigma_o,\ t_o\le 0. \tag{28}$$

where $\Xi=\Xi^T>0$.

Then the EKF is nondivergent for some $t_o\le 0$ if the system is M-detectable.

Remark: This theorem shows that if any observer (including the infinite-dimensional optimal filter) is non-divergent, then the EKF will be nondivergent globally. The condition of $t_o\le 0$ simply means that the EKF should be initialized correctly -this however does not mean that large noises or disturbances will cause non-divergence or require the filter to be reinitialized.

COMPLETE PROCEDURE

We now outline the complete procedure to use these previous results to design a nonlinear dynamic compensator.

LOR at Plant Input

STEP 1: Obtain the plant model in form (1), augmenting the plant with integrators if desired.

STEP 2: Design a state-feedback $g(\hat{x})=G\hat{x}$ so that $G\phi B$ is desirable as a loop operator at the plant input in Fig. 2. We can judge the desirability of this target loop by its disturbance rejection capability and its robustness. We will need to worry that the bandwidth is not too high in relation to the unmodeled dynamcis. This can be checked via the robustness tests of section 4. One way to guarantee robustness properties of $G\phi B$ is to use optimal control theory.

STEP 3: Now design a nondivergent estimator, with gain H_μ, so that (20) is satisfied. One way to do this is to use an EKF.

STEP 4: We now invoke the LOR Theorem, and select a value of $\mu>0$ small enough so that

$$(-K_\mu)(-P) \cong G\phi B \tag{29}$$

to our required degree of accuracy. The degree of matching can be determined by simulation. We simply use K_μ as our final compensator.

LOR at Plant Output

STEP 1: Obtain the plant model in form (1) with the special restriction that the conditions for LOR at the plant output are satisfied. This means that our plant must be in both controller and observer form, see Krener (1986).

STEP 2: Design a filter so that $C\phi H$ is a desirable as a loop operator in Fig. 2. Note that the loop $C\phi H$ is fictitious in that we cannot implement it since we only have control of the plant by u(through the B matrix). We judge the desirability of this target loop by its command-following capability, its disturbance rejection capability, and its robustness. One way to produce nondivergent estimators with a modifiable loop operator is to use an EKF.

STEP 3: We now compute a stabilizing state-feedback gain so that (a-f), of the LQR at plant output theorem are satisfied.

STEP 4: We invoke the LQR at the plant output theorem and select a value of $\rho>0$ small enough so that

$$PK_\rho \cong C\Phi H \tag{30}$$

to our required degree of accuracy. We use K_ρ as our final compensator.

SIMULATION

In order to demonstrate the NMBC/LQR procedure, we selected a simple damped pendulum example:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2\\ -\sin(x_1) \quad -0.5\, x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} u \tag{31}$$

$$y = (1 \quad 0)\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{32}$$

$$u = -g_\rho(x) = \sin(x_1) + 0.5\, x_2 + \frac{1}{\sqrt{\rho}} x_1 + \frac{0.5}{\sqrt[4]{\rho}} x_2 \tag{33}$$

where $g_\rho(x)$ has the desired asymptotic property (23). For the EKF, a value of $\Xi=(100)BB^T$ was chosen. The response could have been sped up by scaling Ξ upward. Figure 3 shows the open-loop step response for $C\Phi H$ and PK for various values of ρ. The convergence can be easily seen. The solid lines represent the response for $C\Phi H$ (the target loop) and the dashed lines represent the response of PK (the actual loop). For values of $\rho=.00001$ or less, recovery is quite good. Figures 4 and 5 show the closed-loop response to steps for $C\Phi H$ and PK for various values of ρ, r. In other words we are plotting $C\Phi H[I+C\Phi H]^{-1}r$ and $PK[I+PK]^{-1}r$ for different values of r. Note that the closed loop is always stable (in contradistinction to the open-loop case). Figure 6 represents an attempt to plot a sort of frequency response for the nonlinear system. In Figure 6(a) we plot the input signal that we are using: a "sweep" signal. Figure 6(b) plots the sensitivity functions $[I+C\Phi H]^{-1}r$ and and $[I+PK]^{-1}r$. Note that the solid line ($[I+C\Phi H]^{-1}r$) has no overshoot (magnitude greater than 3.14=magnitude of input). This demonstrates a robustness property of the EKF, see Grunberg (1986). Although this example is very simple it demonstrates the basic idea behind the NMBC/LQR methodology and shows the steps involved.

CONCLUSIONS

This paper has outlined a new design methodology for nonlinear compensator design and shown the various guaranteed properties. While the method is obviously still in its infancy, it appears quite possible that a practical methodology can be developed, guaranteeing

(i) closed-loop stability,
(ii) good robustness margins
(iii) design parameters to adjust performance.

Further work is needed, both in extending the procedure to plants not in controller form and also in making the procedure more feasible as far as computations go. It appears likely that tests involving "for all signals" can be done over some smaller set of signals, provided that they are "dense" in the set of all signals. This denseness may be less restrictive than the standard topological definition, as the plants and compensators involved have certain smoothness properties.

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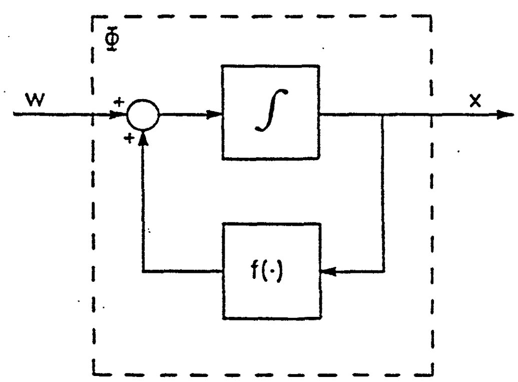

Figure 1: The Φ Operator.

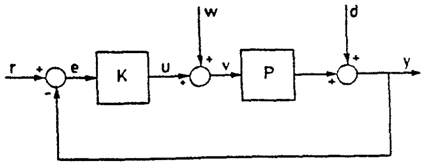

Figure 2: Closed-Loop System

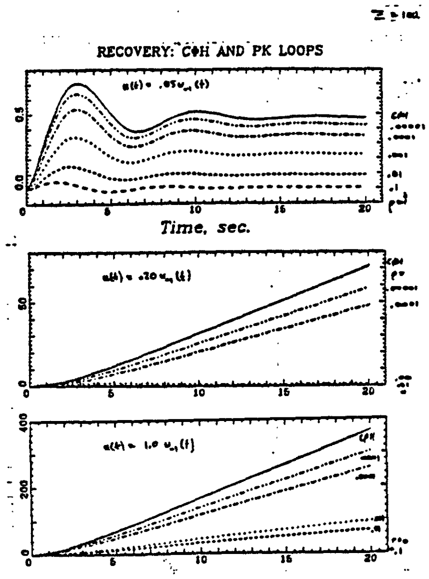

Figure 3: Open Loop Simulations
Recovery: CΦH and PK Loops

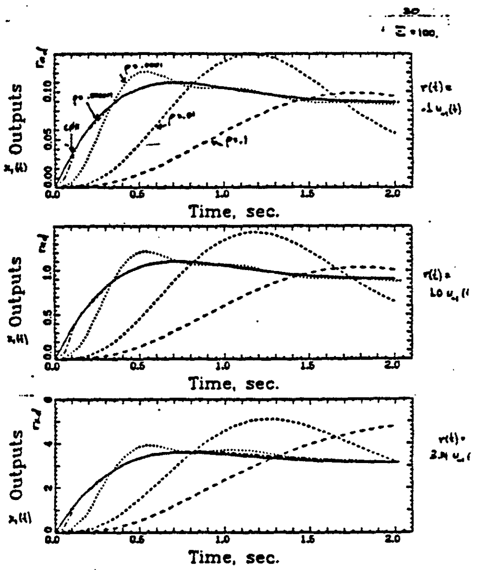

Figure 4: Recovery at Plant Output
CΦH and PK Loops: Closed Loop

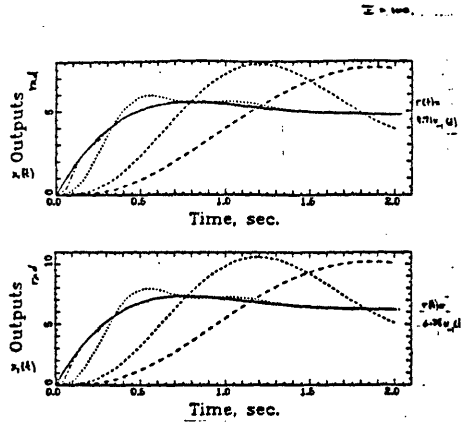

Figure 5: CΦH and PK Loops: Closed Loop.

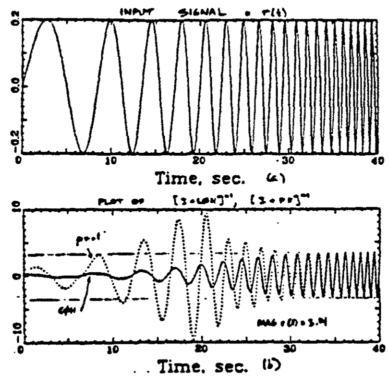

Figure 6: Sensitivity Calculations.