Zariski Structures and Simple Theories

by

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Abstract

In this thesis, I consider generalisations of geometric stability theory to minimal Lascar Strong Types definable in simple theories. Positively, we show that the conditions of linearity and 1-basedness are equivalent for such types. Negatively, we construct an example which is locally modular but not affine using a generalistion of the generic predicate. We obtain reducibility results leading to a proof that in any $\omega-$categorical, 1-based non-trivial simple theory a vector space over a finite field is interpretable and I prove natural generalisations of some of the above results for regular types. I then consider some of these ideas in the context of the conjectured non-finite axiomatisability of any ω-categorical simple theory. In the non-linear Zariski structure context, I consider Zilber's axiomatization in stable examples, and then in the case of the simple theory given by an algebraically closed field with a generic predicate. Comparing Zariski structure methods with corresponding techniques in algebraic geometry, I show the notions of etale morphism and unramified Zariski cover essentially coincide for smooth algebraic varieties, show the equivalence of branching number and multiplicity in the case of smooth projective curves and give a proof of defining tangency for curves using multiplicities. Finally, I give a partial results in the model theory of fields which supports extending the Zariski structure method to simple theories.

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Contents

Chapter 1

Lascar Strong Types and Canonical Basess

In this section, we give a brief overview of Lascar strong types and canonical bases in simple theories which will be used repeatedly in what follows. Much of this material can be found in [14], [20], [23], [21], [24] and [5].

Definition 1. *A formula $\phi(\bar{x}, \bar{b}_0)$ divides over A if there exists an indiscernible sequence $\{\bar{b}_i : 0 \leq i < \omega\}$ over A such that $\{\phi(\bar{x}, \bar{b}_i) : 0 \leq i < \omega\}$ is inconsistent. A type (possibly partial) forks over A if it implies a finite disjunction of formulae dividing over A.*

For $A, B, C \subset \mathcal{M}$, with $\mathcal{M}$ a very saturated model, we take

$A \downarrow_B C$

to mean that $tp(A/BC)$ is a non forking extension of $tp(A/B)$

Kim proved in [20] that if T is a simple theory then forking inside $\mathcal{M}$ satisfies ;

1. Symmetry: Given $\bar{a}, \bar{b}, A$;

$\bar{a} \downarrow_A \bar{b}$ iff $\bar{b} \downarrow_A \bar{a}$

2. Transitivity: Given $A \subset B \subset C$,

$\bar{a} \downarrow_A C$ iff $\bar{a} \downarrow_A B$ and $\bar{a} \downarrow_B C$

3. Extension: Given $A \subset B$, if $\rho \in S^n(A)$, then there exists a realisation $\bar{a}$ with $\bar{a} \downarrow_A B$.

4. Local Character: If $\rho \in S^n(B)$, there exists $A \subset B$ with $|A| \leq |T|$ such that ρ doesn't fork over A.

5. Finite Character: Given $A \subset B$, then $\bar{a} \downarrow_A B$ iff $\bar{a} \downarrow_A \bar{b}$ for each finite tuple $\bar{b} \subset B$.

6. Independence Theorem over a model: If

$\bar{a} \downarrow_{\mathcal{M}} \bar{b}$, $\bar{c} \equiv_{\mathcal{M}} \bar{d}$ and $\bar{c} \downarrow_{\mathcal{M}} \bar{a}$, $\bar{d} \downarrow_{\mathcal{M}} \bar{b}$,

then there exists $\bar{e}$ with

$\bar{e}\bar{a} \equiv_{\mathcal{M}} \bar{c}\bar{a}$, $\bar{e}\bar{b} \equiv_{\mathcal{M}} \bar{d}\bar{b}$ and $\bar{e} \downarrow_{\mathcal{M}} \bar{a}\bar{b}$

We also note the following other trivial consequences of forking.

7. Automorphism Invariance: Non-Forking is preseved under automorphism.

8. Closure: Non-Forking is invariant under closure acl inside $\mathcal{M}^{eq}$, that is

$A \downarrow_B C$ iff $acl(A) \downarrow_{acl(B)} acl(C)$

For complete types p and q over sets $A \subset B$, we say that $p < q$ if q is a forking extension of p. Then $SU(p)$ (Simple U-rank) is defined to be the foundation rank of p with respect to this ordering. (One can also define the analogous rank on formulae (Shelah Rank) usually denoted by R^∞ or D in the simple context). We will be concerned exclusively with supersimple theories for which there are no infinite forking chains, that is sets $A_0 \subset A_1 \subset \ldots A_i \subset \ldots$ and types $\rho_i \in S^n(A_i)$ such that p_{i+1} is a forking extension of ρ_i. For such theories, it can easily be shown that the Local Character axiom can be simplified to take A as a finite set. Moreover, $SU(\rho) < \infty$ for all complete types.

We will say that a complete type ρ in a simple theory is minimal if $SU(\rho) = 1$; this is easily shown to be equivalent to the property that every forking extension of ρ is algebraic. In the stable context, it will be convenient to allow for a slightly broader definition of a minimal formula or a minimal partial type; namely a formula ϕ is strongly minimal if $RM(\phi) = dM(\phi) = 1$ (Morley rank/Morley degree) and a partial type ρ over A is minimal if for any $A \subset B$ it has a unique extension to a a non-algebraic complete type ρ'. (in the case of a complete type in a stable theory this is equivalent to ρ being stationary and having U-rank 1).

The notion of Lascar strong type, generalising the notion of strong type in stable theories, will be central to what follows, so it is worth giving a brief summary of its properties. The idea is to find a rather broader class of sets for which the Independence Theorem holds. In stable theories, it can be shown that if ρ is a complete type over a set A, algebraically closed in $\mathcal{M}^{eq}$, then it is stationary, that is has a unique non-forking extension to any $A \subseteq B$. Then the Independence Theorem must hold for such types, as for any tuples $\bar{a}$ and $\bar{b}$, if $\rho_1 \in S^n(A\bar{a})$ and $\rho_2 \in S^n(A\bar{b})$ are non-forking extensions of ρ, then there must be a unique global non-forking extension $\rho_3 \in S^n(\mathcal{M})$ over a model $\mathcal{M}$ containg $A\bar{a}\bar{b}$. Unfortunately, the proof that strong types are stationary relies on two facts unique to stable theories. The first is the existence of a local rank R_Δ which is defined exactly as for RM but restricting to instances of Boolean combinations of a given formula; this allows us to develop a precise notion of multiplicity and therefore to show that locally a type ρ over A can only have finitely many non forking extensions. The second is the fact that parallelism with respect to Δ formulae is definable, that is given Δ formulae $\phi(\bar{x}\bar{a})$, $\psi(\bar{x},\bar{y})$, $\{\bar{b} : R_\Delta(\phi(\bar{x}\bar{a}) \wedge \psi(\bar{x},\bar{b})) = R_\Delta(\phi(\bar{x}\bar{a}))$ is definable. This allows the construction of defining schemas for Δ formulae inside any non-forking extension of ρ to a model $\mathcal{M}$, which, using the first property, must be defined over $acl(A)$. See [30] and [3] for details. For simple theories, stationarity fails but one still wants to apply the independence theorem, so the notion of Lascar strong type, Lstp, is introduced. Namely one defines $Lstp(\bar{a}/A) = Lstp(\bar{b}/A)$ if there exists $g \in Autf(\mathcal{M})$ with $g(\bar{a}) = \bar{b}$, where $Autf(\mathcal{M})$ is the subgroup of $Aut(\mathcal{M})$ generated by

$$\{g \in Aut(\mathcal{M}) : g \in Aut_{\mathcal{N}}(\mathcal{M}), \text{ for } some\ A \subseteq \mathcal{N} \subseteq \mathcal{M}\}.$$

As is shown in [24], the Independence Theorem holds for $Lstps$. However, the notion of $Lstp$ as defined above is rather inconvenient to work with. To overcome this difficulty, one introduces $\mathcal{M}^{heq}$ containing $\mathcal{M}^{eq}$ consisting of names for classes of type definable equivalence relations on $\mathcal{M}$. As is shown in [23], the following are equivalent,

1. $Lstp(\bar{a}/A) = Lstp(\bar{b}/A)$

2. $E(\bar{a}, \bar{b})$ for any A invariant bounded equivalence relation on A

where a bounded equivalence relation is one having strictly less than $Card(\mathcal{M})$ classes. Moreover, as in [23], equality of $Lstps$ is type definable, therefore

$$Lstp(\bar{a}/A) = Lstp(\bar{b}/A) \text{ iff } tp(\bar{a}/bdd(A)) = tp(\bar{b}/bdd(A))$$

where $bdd(A)$ denotes the bounded closure of A in $\mathcal{M}^{heq}$.

If T is small, Kim shows in [23] that any type definable equivalence relation is equivalent to an intersection of definable ones, from which it easily follows that

$$Lstp(\bar{a}/A) = Lstp(\bar{b}/A) \text{ iff } tp(\bar{a}/acl(A)) = tp(\bar{b}/acl(A)) \ (*)$$

where $acl(A)$ denotes the algebraic closure of A in $\mathcal{M}^{eq}$. This result was later improved in [5] with the assumption that T is supersimple. As everything we consider here only requires this, from now on we will take $(*)$ as the definition of $Lstp$.

For the rest of this section, I will make a few remarks about canonical bases in simple theories, as they are also used on several occasions. In [14], a notion of canonical bases is developed for *Lstps*. The idea is to define a relation R_1 on $tp(a)$, where $p(\bar{x}, a)$ is a complete type having the amalgamation property, given by

$$R_1(a, b) \text{ iff } p(x, a) \text{ and } p(x, b) \text{ have a common non-forking extension}$$

Unlike the stable case, this is not an equivalence relation, however it is type definable and its transitive closure is shown to be type definable by

$$E(a, b) \text{ iff } \exists z(R(a, z) \wedge R(z, b)) \ (**)$$

where R is the relation given by

$$R(a, b) \text{ iff } R_1(a, b) \wedge Generic(b, a)$$

and $Generic(b, a)$ is a type definable relation saying that b has maximal SU-rank among realisations of $R_1(x, a)$, at least in the case that T is supersimple. Note that a may stand for a sequence of infinite length and for *Lstps* in supersimple theories will generally denote $acl^{eq}(\bar{a})$ for some finite tuple!

The parallelism class $\mathfrak{P}$ of $p(\bar{x}, a)$ is the E class of p where E is the transitive closure of the parallelism relation on complete types with the amalgation property. If I define the canonical base of p to be $c = a/E$, then it follows that an automorphism fixes c iff it fixes $\mathfrak{P}$ setwise. In fact, if α is any automorphism fixing c, then it follows by (**) that I can find b having the same type as a such that $p(x,a)||p(x,b)$ and $p(x,b)||p(x,\alpha(a))$, that is I can amalgamate $p(x,a)$ and its image in 1-step.

We need 3 other properties of canonical bases. The first, as shown in [14], is that the Independence theorem still holds for the restriction of a *Lstp* over a to its base $c \subset a$. Therefore, as a is algebraically closed, if b is a congugate over c such that $a \downarrow_c b$ then $p(x,a)$ and $p(x,b)$ are parallel types.

The second is the relation of a canonical base to other sets in our structure $\mathcal{M}$. The result is the following, found in [21];

If $A \subseteq B$ are sets and $\bar{a}$ is a tuple, then $\bar{a} \downarrow_A B$ iff $Cb(Lstp(\bar{a}/B)) \subseteq acl(A)$.(***)

As an immediate consequence we have that if $c = Cb(Lstp(\bar{a}/A))$, then $\bar{a} \downarrow_c A$ and of course $\bar{a} \downarrow_A c$. Moreover, a simple application of the rules of forking shows that if $d = Cb(Lstp(\bar{a}/B))$ and $\bar{a} \downarrow_A B$, then c and d are interalgebraic.

Given a type ρ with domain A, we define $\rho^{eq} = dcl(A \cup \rho)$. Suppose $\bar{a} \in \rho$ and B is an arbitrary set of parameters. The third property is that, under the assumption of T being supersimple, $C = Cb(Lstp(\bar{a}/B)) \subset p^{eq}$. This follows as we can find a finite Morley sequence $\bar{a}_1, \dots \bar{a}_n$ realising ρ with $C \subset dcl(\bar{a}_1, \dots \bar{a}_n)$. In general C will be an infinite tuple of elements, but using this fact we can always take C to be a finite tuple $\bar{c}$ in p^{eq} up to interalgebraicity.

Chapter 2

Pregeometries

A pregeometry is a set S with a closure operation $cl : P(S) \to P(S)$ satisfying the following axioms found in [30];

1. If $A \subseteq S$, then $A \subseteq cl(A), cl(A) = cl(cl(A))$.
2. If $A \subseteq B \subseteq S$, then $cl(A) \subseteq cl(B)$.
3. If $A \subseteq S$, $a, b \in S$, then $a \in cl(Ab) \setminus cl(A)$ implies $b \in cl(Aa)$.
4. If $a \in S$ and $a \in cl(A)$, then there is some finite $A_0 \subset A$ with $a \in cl(A_0)$.

We will give a number of examples relevant to what follows, each one generalising the preceeding one!

Example 1:

One of the simplest example of a pregeometry is vector space over a field F. The closure operation cl on V is given by $cl(A) = span(A) = \{v \in V : v \in span(\bar{a})\}$ where $\bar{a}$ is a finite tuple of elements from A. The axioms 1,2 and 4 are trivial to verify, and axiom 3 follows from the well known Steinitz Exchange Lemma for vector spaces.

Example 2:

More generally, suppose D is a strongly minimal set inside a structure $\mathcal{M}$, defined over a parameter $\bar{c}$, then (D, cl) is a pregeometry with cl defined by $cl(A) = \{x \in D : x \in acl(A\bar{c})\}$, where acl denotes algebraic closure inside the structure $\mathcal{M}$. Axioms 2 and 4 are again immediate, axiom 1 is just transitivity of algebraic closure and the only work is to verify axiom 3; the following is a rather straightforward proof of this fact requiring only the definition of a strongly minimal set;

Proof. Without loss of generality, assume $A\bar{c} = \emptyset$, and let $a \in acl(b) \setminus acl(\emptyset)$. Then there is some formula $\phi(xy)$ such that $\exists^{x=k}\phi(xb)$ and $\phi(ab)$ holds. Now consider the formula

$$\psi(ay) \equiv \exists^{x=k}\phi(xy) \wedge \phi(ay)$$

Then clearly $\psi(ab)$ holds and we may therefore suppose that $\exists^{\infty}y\psi(ay)$, otherwise we are done. By strong minimality, $\psi(ay)$ is cofinite in D, that is $\exists^{y=m}\neg\psi(ay)$. As $a \notin acl(\phi)$, we can find an infinite sequence $(a_1, \ldots a_i \ldots) \subset D$ such that $\exists^{y=m}\neg\psi(a_i y)$ for each i. By compactness, we can then find $b' \in D$ such that $\psi(a_i b')$ holds for all i. Then on the one hand we have that $\exists^{x=k}\phi(xb')$ while on the other $\phi(a_i b')$ holds for infinite i. This is a contradiction.

□

Example 3:

With the discovery of simple theories, generalising stable theories, we can find an even more plentiful supply of pregeometries. This relies crucially on property 1 of forking inside simple theories (see section 1);

Proof. We recall the definition of a minimal type ρ inside a simple theory from section 1. Then the realisations D of ρ form a pregeometry under the closure operation cl given by $cl(A) = \{x \in \rho : x \in acl(A\bar{c})\}$. Here again acl denotes usual model theoretic algebraic closure and $\bar{c}$ denotes the domain of ρ. We need to check the axioms. 1,2 and 4 are trivial to verify. For 3, assuming that $A\bar{c} = \emptyset$, suppose that $a \in cl(b) \setminus cl(\emptyset)$, then $a \not\downarrow b$ and by forking symmetry $b \not\downarrow a$. Then b realises a forking extension of ρ over a and therefore $b \in acl(a)$.

□

Example 4:

It is in fact possible to go one step further! We will say that a non algebraic complete type ρ is regular if it is orthogonal to all its forking extensions. Then the realisations of D of ρ form a pregeometry with the the closure operation cl given by $cl(A) = \{x \in \rho : x \not\downarrow A\}$, where I have supressed the defining parameter of ρ.

Proof. Again we check the axioms, 2 is trivial and 4 follows from the finite character of forking. 3 follows immediately from forking symmetry and all the work is in showing that 1 holds, namely we have to see that if $A \subset \rho$, $a, b_1 \ldots b_n$ is a tuple in ρ such that $b_i \not\downarrow A$ for each i and $a \not\downarrow b_1 \ldots b_n$ then in fact $a \not\downarrow A$. Suppose not, so a realises a non forking extension of ρ to A. Each b_i realises a forking extension of ρ to A so by definition of regularity, we must have that $a \downarrow_A b_1$. Now we just repeat the argument with Ab_1 replacing A, clearly $b_i \not\downarrow Ab_1$ for $i \geq 2$ and again using regularity $a \downarrow_{Ab_1} b_2$, so we get $a \downarrow_A b_1 b_2$. After n steps, using transitivity, we have that $a \downarrow_A b_1 \ldots b_n$ and so as $a \downarrow A$ we get $a \downarrow b_1 \ldots b_n$. This contradicts the original hypothesis.

□

Having found plenty of examples, we will analyse properties of pregeometries in more detail. Given any pregeometry (S, cl), we can associate a canonical geometry (S', cl'). In order to do this, we define an equivalence relation E on $S \setminus cl(\emptyset)$, by

$E(x,y)$ iff $cl(x) = cl(y) \quad x,y \in S$

Then S' is given as a set by $\overline{S \setminus cl(\emptyset)}$ where ,for $x \in S$, $\bar{x}$ denotes the equivalence class of x with respect to E.

Given $A \subset S'$, we let

$A' = \{x \in S : \bar{x} \in A\}$

and we define cl' on S' by setting;

$cl'(A) = \{\bar{x} : x \in cl(A')\}$.

As is easily checked, (S', cl') is still a pregeometry and moreover has the desirable additional properties that $cl'(a) = a$ for every $a \in S$ and $cl'(\emptyset) = \emptyset$. If we consider Example 1 above of a vector space V over a field F, then the corresponding geometry is exactly projective space $P(V)$ over F.

Given (S, cl) and $A \subset S$ we can also localise S at A to obtain a pregeometry (S_A, cl_A). Namely, one takes $S_A = \{x \in S : x \notin cl(A)\}$ and given $B \subset S_A$, we define $cl_A(B) = \{x \in S_A : x \in cl(A \cup B)\}$.

If $A \subset S$ is a closed subset, we define a basis of A to be a a maximal subset $A_0 \subset A$ such that the elements of A_0 are independent, that is $a \notin cl(A_0 \setminus a)$ for every $a \in A_0$. By Zorn's Lemma, using axiom 4, every closed set has a basis. Moreover, if A_0 is a basis for A, then, given $x \in A$, $\{A_0, x\}$ must form a dependent set. Using 3, we easily conclude that $x \in cl(A_0)$ and so A_0 spans A. More importantly, any two bases A_0 and A_1 for A have the same cardinality; this follows easily by repeated application of axiom 3 to interchange elements of A_0 and A_1. We then have a well defined notion of dimension for closed sets $A \subset S$ given by $dim(A) = Card(A_0)$, A_0 a basis for A. For closed sets $B \subset A \subset S$, we may also define $dim(A/B) = dim(A) - dim(B)$ and for arbitrary sets $A, B \subset S$, we define $dim(A/B) = dim(cl(A \cup B)/cl(B))$. As is easily verified, we then have the following additive property of dimension;

$$dim(A \cup B) = dim(A/B) + dim(B)$$

Moreover, if we work inside a strongly minimal set or an SU-rank 1 complete type as above, the notion of dimension on the corresponding pregeometry S coincides with MR or SU-rank.

We consider the case for SU-rank first. By the laws of forking inside simple theories, in particular transitivity, it is a straightforward exercise (using induction!), to check that if $a, b \in \mathcal{M}$ and $A \subset \mathcal{M}$, then $SU(ab/A) = SU(a/bA) + SU(b/A)$, provided both sides of the equation are finite. Then if $dim(a_1, \ldots a_n/A = n$ in S, to show that $SU(a_1 \ldots a_n/A) = n$, we just need to check that SU rank is preserved under non forking extension, again this is an easy exercise, in fact implicit in showing the additivity of SU-rank. For arbitrary tuples $\bar{a}$ from S, observing that algebraic types have SU rank 0, we conclude easily that $dim(\bar{a}/A) = SU(\bar{a}/A)$ for $\bar{a}$ in S. The case for MR is slightly complicated by the fact that MR is not in general additive. However, in this case, n independent elements $a_1 \ldots a_n$ from S over A will determine a unique n type p^n, over A, as S is the solution sets of a strongly minimal formula. Using this, it is reasonably straightforward to deduce that $MR(a_1 \ldots a_n/A) = dim(a_1 \ldots a_n/A) = n$. The general result then follows from the fact that if $\bar{a} \subset acl(A\bar{b})$, $MR(\bar{a}/A) \leq MR(\bar{b}/A)$, so MR is preseved by interalgebraicity. See [3] for details. In general, I will use dim and the model theoretic ranks interchangably.

We now examine possible behaviours of closure inside pregeometries. Let (S, cl) be a pregeometry, then

Definition 2. 1.*(S, cl) is trivial if for $A \subset S$, $cl(A) = \{\cup cl(a) : a \in A\}$*

2.(S, cl) *is modular if for* A, B *finite dimensional closed subsets of* S, $dim(A \cup B) = dim(A) + dim(B) - dim(A \cap B)$

3.(S, cl) *is locally modular if it is modular after localising at a point in* S

All the above properies are preserved under localisation, in particular modularity implies local modularity.

We will now look at these 3 cases in more detail.

1. Trivial Case.

Trivial pregeometries are in a sense as degenerate as possible. Examples are an infinite set with no extra structure, or a model of the theory of the random graph. In the latter case, the random relation makes no contribution to the model theoretic closure.

2. Modular Case.

The canonical example of a modular pregeometry is projective space over a field $P(F)$. Here the closure operation cl is defined by taking $cl(A) = span(A) = \{v \in P(F) : v \in span(\bar{a})\}$ with $span(\bar{a})$ denoting the projective plane spanned by a finite tuple $\bar{a}$ from A. The example is canonical by the following classical fact found in [1];

Fact 1. *If* (S, cl) *is a non-trivial, modular geometry of dimension* ≥ 4 *in which each closed set of dimension* 2 *contains at least* 3 *elements, then* (S, cl) *is isomorphic to projective geometry over a divison ring.*

We now need to analyse the notion of modularity further. First note that we can rewrite the modularity formula in a more digestible form as follows. We have,

$$dim(A \cup B) = dim(A/B) + dim(B) = dim(A) + dim(B) - dim(A \cap B)$$

therefore

$$dim(A/B) = dim(A) - dim(A \cap B) = dim(A/A \cap B) \ (*)$$

As this argument is reversible, we can use $(*)$ as a criterion for modularity, and in fact we can even reduce $(*)$ to the following easier condition given by the lemma

Lemma 2. *(S, cl) is modular iff whenever $a, b \in S$, $B \subset S$, $dim(ab) = 2$ and $dim(ab/B) \leq 1$, then there is $c \in cl(ab) \cap cl(B)$ with $c \notin cl(\emptyset)$ (**)*

Proof. Clearly $(*)$ implies $(**)$. To prove the converse, first note that applying $(**)$ to $(a_{n+1}x)$ and $cl(a_1 \ldots a_n))$ gives us that if $x \in cl(a_1 \ldots a_n a_{n+1}) \setminus cl(a_{n+1})$, then $x \in cl(a_{n+1}b)$ with $b \in cl(a_1 \ldots a_n)$, call this condition $(***)$. Now use induction on $dim(A)$. So suppose that $dim(A) = n + 1$, and let $a_1, \ldots a_{n+1}$ be a basis for A. Let $B \subset S$ with $dim(A/B) = m \leq n + 1$. We may suppose that $m > 0$ and $a_{n+1} \notin cl(B)$ otherwise the result is trivial. Then by additivity we must have that $dim(a_1 \ldots a_n/a_{n+1}B) = m - 1$. By the induction hypothesis, we have that $dim(cl(a_1 \ldots a_n) \cap cl(a_{n+1}B)) = n - (m - 1)$. Call this intersection C and consider $cl(Ca_{n+1}) \subset cl(Ba_{n+1})$. If $m \geq 2$, we may apply the induction hypothesis to calculate $dim(cl(Ca_{n+1}) \cap B) = dim(cl(Ca_{n+1})) - dim(cl(Ca_{n+1})/B) = ((n-(m-1))+1)-1 = n-(m-1)$ (as clearly $a_{n+1} \notin C$). However, $cl(Ca_{n+1}) \cap B = cl(a_1 \ldots a_{n+1}) \cap B$, (using $(***)$ to verify the right to left direction). This gives us that $dim(cl(a_1 \ldots a_{n+1}) \cap B) = dim(cl(a_1 \ldots a_{n+1})) - dim(cl(a_1 \ldots a_{n+1}/B)) = n + 1 - m$ as required. The case when $m = 1$ can be handled seperately, the simplest method is as follows; given $a_1 \ldots a_{n+1}$ with $dim(a_1 \ldots a_{n+1}/B) = 1$, we may assume that $a_1 \notin B$ and then for $i \geq 2$ we have $a_i \in cl(a_1 B)$. Using $(***)$ we pick up points $c_i \in cl(a_1 a_i) \cap B$, and one easily sees that the c_i are independent points in $cl(a_1 \ldots a_{n+1})$. This proves

$dim(cl(a_1 \ldots a_{n+1}) \cap B) \geq n$ which is clearly sufficient.

□

We can use modularity to find simpler conditions to decide when (S, cl) is trivial. We have the following lemma.

Lemma 3. *If (S, cl) is a modular geometry, and for any 2 distinct points $cl(ab) = \{a, b\}$, then (S, cl) is trivial.*

Proof. We first note that as (S, cl) is modular then if $A \subset S$, $x \in cl(Ay)$, we can find $z \in cl(A)$ such that $x \in cl(zy)$ (this is (***) above). Now suppose inductively that we have verified triviality for closed sets of dimension $\leq n$. Let $B \subset S$ closed have dimension $n+1$ with basis $a_1 \ldots a_{n+1}$. If $x \in B$, then I can find $z \in cl(a_1 \ldots a_n)$ with $x \in cl(za_{n+1})$. By the induction hypothesis, $z \in cl(a_1 \cup \ldots \cup cl(a_n))$, so $x \in cl(a_1 \cup \ldots \cup cl(a_{n+1}))$, which proves triviality for B

□

3. Locally Modular Case.

The classical example of a locally modular, non modular pregeometry is affine spane over a field F denoted by $Aff(F)$ where cl is defined by taking $cl(A) = \{v \in span(\bar{a})\}$ and $span(\bar{a})$ denotes the affine plane spanned by $\bar{a}$. Modularity fails by considering 2 parallel lines generated by (ab) and (cd) respectively. In this case, we have $dim(ab) = 2$, $dim(ab/cd) = 1$ but $cl(ab) \cap cl(cd) = \emptyset$, violating the condition (**). If we localise Aff_F at a point, we obtain a vector space $V(F)$, which is of course a modular pregeometry.

Again the example is in a sense canonical due to the following theorem of Hrushowski, which makes essential use of the group configuration for stable theories. I quote the result for complete minimal types in stable theories ,as in [30], but an analogous result holds for minimal types;

Theorem 4. *Let $p \in S(\emptyset)$ be a complete non trivial minimal locally modular type inside a stable theory. Then p is modular or the geometry associated to p is affine geometry over a division ring.*

One would naturally expect this to hold in simple theories, but this turns out to be false! We will see why in the next section.

Finally, we need to mention the following classical result due to Doyen and Hubaut and found in [11];

Fact 5. *If (S, cl) is a non trivial locally modular, locally finite geometry of dimension > 4, in which all closed sets of dimension 2 have the same size, then (S, cl) is affine or projective geometry over a finite field.*

Chapter 3

Linearity and 1-Basedness

We now want to undertake a more thorough analysis of minimal types inside simple theories. For this we will require two new notions, 1-basedness and linearity. See [4] and [30] for more details

Definition 3. *We say that a simple theory T with elimination of hyperimaginaries is* 1-*based if the following condition holds in a big model* $\mathcal{M}$;

For any sets A and B, $A \downarrow_{acl(A)\cap acl(B)} B$, where acl is taken inside $\mathcal{M}^{eq}$!

Lemma 6. *The following are equivalent for simple T;*

1.*T is* 1*-based.*

2. *For any $B \subset M^{eq}$ and tuple $\bar{a}$, $Cb(Lstp(\bar{a}/B)) \subset acl(\bar{a})$.*

3. *If $I =< \bar{a}_i : 0 \leq i < \omega >$ is an indiscernible sequence, then $I \setminus \{\bar{a}_0\}$ is a Morley sequence over $\bar{a}_0$*

Proof. $1 \Rightarrow 2$;

We clearly have that $\bar{a} \downarrow_{acl(\bar{a}) \cap acl(B)} acl(B)$. By properties of canonical bases, given in Section 1, we have $Cb(\bar{a}/B) \subset acl(\bar{a}) \cap acl(B)$, in particular $Cb(\bar{a}/B) \subset acl(\bar{a})$.

□

Proof. $2 \Rightarrow 1$;

Again by facts on canonical bases, we must have that $\bar{a} \downarrow_{acl(\bar{a}) \cap acl(B)} B$ for finite tuples $\bar{a}$ and $B \subset \mathcal{M}^{eq}$. Now 1 follows by the finite character of forking.

□

Proof. $1 \Rightarrow 3$;

Let $< \bar{a}_i : 0 \leq i < \omega >$ be an indiscernible sequence. Then by 1-basedness $\bar{a}_0 \downarrow_{acl(\bar{a}_0) \cap acl(\bar{a}_1 \ldots \bar{a}_n)} \bar{a}_1 \ldots \bar{a}_n$ for $n \geq 1$. The sequence $< \bar{a}_i : 1 \leq i < \omega >$ is indiscernible over $\bar{a}_0$, hence $tp(\bar{a}_1 \ldots \bar{a}_n / acl(\bar{a}_0)) = tp(\bar{a}_2 \ldots \bar{a}_{n+1} / acl(\bar{a}_0))$, therefore, by automorphism, $\bar{a}_0 \downarrow_{acl(\bar{a}_0) \cap acl(\bar{a}_1 \ldots \bar{a}_n)} \bar{a}_2 \ldots \bar{a}_{n+1}$ and in particular $\bar{a}_0 \downarrow_{\bar{a}_1 \ldots \bar{a}_n} \bar{a}_{n+1} (*)$. Now a straightforward SU rank calculation, using indiscernibility to give $SU(\bar{a}_{n+1}/\bar{a}_0 \ldots \bar{a}_{n-1}) = SU(\bar{a}_{n+1}/\bar{a}_1 \ldots \bar{a}_n)$, shows that we can swap $\bar{a}_0$ and $\bar{a}_n$ in $(*)$ to give that $\bar{a}_n \downarrow_{\bar{a}_0 \ldots \bar{a}_{n-1}} \bar{a}_{n+1}$. This shows directly that $< \bar{a}_i >$ is a Morley sequence over $\bar{a}_0$.

□

Proof. $3 \Rightarrow 2$.

Choose a Morley sequence $(\bar{a}_0 \bar{a}_1 \ldots \bar{a}_n \ldots)$ for $Lstp(\bar{a}/B)$ such that $c \in acl(\bar{a}_0 \bar{a}_1 \ldots a_{n+1})$. We can assume the sequence is indiscernibile, hence based on $\bar{a}_0$. Continuing the sequence, we have that $\bar{a}_{n+1} \downarrow_{\bar{a}_0} \bar{a}_1 \ldots \bar{a}_n$ and therefore $\bar{a}_{n+1} \downarrow_{\bar{a}_0} c$. Clearly $\bar{a}_{n+1} \downarrow_c \bar{a}_0$ as part of a Morley sequence, which gives that $c \in acl(\bar{a}_0)$ by facts on canonical bases.

□

We now work inside the solution set D of a minimal type over $\emptyset$. The notion of 1-basedness still makes sense for D by considering D as a structure in its owm right and working in D^{eq}.

Definition 4. *We say that D is linear if for all parameter sets $A \subset D$ and pairs $ab \in D$ with $SU(ab/A) = 1$, then $SU(c) \leq 1$ where $c = Cb(Lstp(ab/A))$.*

We can easily show the following connecting 1-basedness, linearity and local modularity of D.

Theorem 7. *If D is the solution set of a minimal type, then;*

1.D locally modular $\Rightarrow$ 2.D is 1-based $\Rightarrow$ 3.D is linear.

Proof. $1 \Rightarrow 2$

To show the first part of the implication, we first prove the rather strong result that if $c \in D^{eq}$, then c is interalgebraic with a tuple $\bar{a}$ in D over a fixed $d \in D$. To see this, observe trivially that if $d_1 \neq d_2$ are in D, then by automorphism the localised pregeometries D_{d_1} and D_{d_2} are isomorphic. Hence we can assume that D is modular after adding any $d \in D$. Now fix points $c \in D^{eq}$ and $d \in D$. Assume for convenience that $c \downarrow d$. Then we can find an independent sequence $a_1 \ldots a_n$ such that $c \in acl(a_1 \ldots a_n)$ and we may assume that $a_1 \ldots a_n \downarrow_c d$, so $d \downarrow a_1 \ldots a_n$. Now let $b_1 \ldots b_n$ realise $tp(a_1 \ldots a_n/cd)$ such that $b_1 \ldots b_n \downarrow_c da_1 \ldots a_n$. By a rank calculation,

$$SU(b_1 \ldots b_n d/a_1 \ldots a_n d) = SU(b_1 \ldots b_n/a_1 \ldots a_n d)$$

$$= SU(b_1 \ldots b_n/a_1 \ldots a_n dc)$$

$$= SU(b_1 \ldots b_n/c)$$

$$= SU(b_1 \ldots b_n) - SU(c) = n - SU(c)(*) \ (as\ c \in acl(b_1 \ldots b_n))$$

Now as D is modular after adding d, we have that,

$$dim_d(cl(b_1 \dots b_n) \cap cl(a_1 \dots a_n))$$

$$= dim_d(cl(b_1 \dots b_n)) - dim_d(cl(b_1 \dots b_n / cl(a_1 \dots a_n))$$

$$= n - (n - SU(c)) = SU(c)$$

using $(*)$. Let $c_1 \dots c_k$ be a basis for this intersection over d, so $SU(c) = k$. Then I claim that $c \in acl(dc_1 \dots c_k)$. If not, then clearly $b_1 \dots b_n d \not\downarrow_{c_1 \dots c_k d} c$ and so $b_1 \dots b_n d \not\downarrow_{c_1 \dots c_k d} a_1 \dots a_n d$ contradicting local modularity. Now by straightforward rank calculation

$$SU(c_1 \dots c_k / cd) = SU(c / c_1 \dots c_k d) + SU(c_1 \dots c_k / d) - SU(c/d) = 0$$

so $c_1 \dots c_k$ and c are interalgebraic over d as required.

It follows easily that (D, d) is 1-based for any $d \in D$. To see that D itself must be 1-based, we just need to check condition 2 above. So let $\bar{a}$ be a tuple and $B \subset D^{eq}$. Without loss of generality assume B is algebraically closed. Let B' realise $tp(B)$ with $B' \downarrow d$ and $\bar{a}'$ be the conjugate of $\bar{a}$ over B'. Let $c = Cb(Lstp(\bar{a}'/B'))$, so $c \downarrow d$, and $\bar{a}''$ realise $Lstp(\bar{a}'/B')$ with $\bar{a}'' \downarrow_{B'} d$. By elementary properties, c is interalgebraic with the canonical base of $Lstp(\bar{a}''/B'd)$. Then as (D, d) is 1-based, $c \in acl(\bar{a}''d)$. However, $d \downarrow c$ and $d \downarrow_c \bar{a}''$, so in particular $c \downarrow_{\bar{a}''} d$, so in fact $c \in acl(\bar{a}'')$. By automorphisms, it follows that $c \in acl(\bar{a}')$ and then that $c' = Cb(Lstp(\bar{a}/B)) \in acl(\bar{a})$ as required.

□

Alternatively, one can check this using criteria 3 in Lemma 13; in practise, this seems to be the most effective method for testing 1-basedness of a given theory.

Remark 1. *In the stable case, the above argument can be reversed to show that for D the solution set of a minimal type, if D is 1-based then D is in fact locally modular.*

This relies on the fact that it is always possible to find a set of parameters $I \subset D$ such that any element $c \in D^{eq}$ is interalgebraic with a tuple $\bar{a}$ in D over I. In the case of simple theories, this fails completely; a counterexample is given by adding a generic predicate $P(x)$ to a vector space over a finite field and taking the reduct $P(V)$, see Theorem 22.

Proof. $2 \Rightarrow 3$

We still need to prove the last implication, that if D is 1-based then in fact D is linear. This is a trivial rank calculation. Let $(ab) \in D$, $A \subset D$, with $SU(ab/A) = 1$ and $c = Cb(Lstp(ab/A))$, then

$$SU(abc) = SU(ab/c) + SU(c) = 1 + SU(c)$$

$$= SU(c/ab) + SU(ab) = SU(ab) \leq 2$$

so $SU(c) \leq 1$ as required.

□

The rest of this section will be devoted to recovering an analogue of the converse implications which hold only with the assumption of stability. It is rather extraordinary that such an analogue exists in the simple case.

Definition 5. *We set $G(D) = \{c \in D^{eq} : SU(c) = 1\}$*

$G(D)$ is not in general a definable object but being the union of complete rank 1-types forms a pregeometry under the obvious closure operation. We have that $D \subset G(D)$ and all the above notions of (local) modularity, 1-basedness and linearity make sense in $G(D)$. We will be busy proving the following theorem;

Theorem 8. *If D is the solution set of a minimal type, then*

1.D linear$\Rightarrow$ 2.$G(D)$ linear $\Rightarrow$ 3.$G(D)$ modular $\Rightarrow$ 4.D 1-based.

Proof. $1 \Rightarrow 2$

So assume that D is linear. Let $(xy) \in G(D)$ be a pair and $A \subset G(D)$ with $SU(xy/A) = 1$. We may as well assume that (xy) is independent as if $SU(xy) = 1$, then letting $c = Cb(Lstp(xy/A))$, we have $xy \downarrow c$, so $c \in acl(\emptyset)$. Suppose $(xy) \in acl(\bar{a})$ with $\bar{a}$ an independent tuple from D. Let F be a basis for $cl(\bar{a})$ in the localised pre-geometry $D_{(xy)}$, then $F \downarrow xy$ and moreover if we complete F to a basis Fab of $cl(\bar{a})$ in D, then (ab) and (xy) are interalgebraic over F. Let F' realise $tp(F/xy)$ with $F' \downarrow_{xy} abAF$, so

$F' \downarrow_{\emptyset} xyabAF$. (1)

Then by automorphism we can find a further pair $(a'b')$ such that $(a'b')$ and (xy) are interalgebraic over F'. Now we have that;

$SU(xy/A) = SU(xy/F'A) = SU(a'b'/F'A) = 1$ (2)

using the facts that $F' \downarrow_A xy$ and $(xy), (a'b')$ are interalgebraic over $F'A$.

Using linearity of D, we have that

$SU(a'b'/acl(F'A) \cap acl(a'b')) = 1$

As $acl(F'A) \cap acl(a'b') \subset acl(F'A) \cap acl(F'a'b')$ and clearly $(a'b') \notin acl(F'A)$, we must then have;

$SU(a'b'/acl(F'A) \cap acl(F'a'b') = 1$ (3)

Then, using the fact that $F' \subset acl(F'A) \cap acl(F'a'b')$ and again that $(a'b'), (xy)$ are interalgebraic over F', we can replace both occurrences of $(a'b')$ by (xy) in (3) to give;

$SU(xy/acl(F'A) \cap acl(F'xy)) = 1$ (4)

For convenience, let W denote $acl(F'A) \cap acl(F'xy)$. I claim that $xy \downarrow_W c$. If not, then $(xy) \in acl(Wc) \subset acl(F'Ac)$. Hence, using (1), $(xy) \in acl(A)$ which is not the case. It follows by elementary properties of canonical bases that $c \in acl(W)$ and so $c \in acl(F'xy)$. Using (1) again gives $c \in acl(xy)$. This proves that $G(D)$ is linear.

□

Proof. $2 \Rightarrow 3$

To show that $G(D)$ linear implies that $G(D)$ is modular, we use the criterion (**) given in Lemma 2. So let $(ab) \in G(D)$ be an independent pair and $A \subset G(D)$ with $SU(ab/A) = 1$. We already know that $c = Cb(Lstp(ab/A)) \in acl(ab) \cap acl(A)$ and $SU(c/\emptyset) = 1$. By facts on canonical bases, see Section 1,we may assume that c is a single element in D^{eq} up to interalgebraicity, hence c may be taken inside $G(D)$.

□

Proof. $3 \Rightarrow 4$

Finally, we want to show that $G(D)$ modular implies that D is 1-based. The first step of the proof is almost exactly the same as above. Namely one uses modularity of $G(D)$ to show that any $A \subset D^{eq}$ is interalgebraic with $A' \subset G(D)$. For variety, we can show this using a different method which will be used repeatedly later. Let $A \subset D^{eq}$ be a closed set and choose $B_0 \subset D$ closed such that $A \subset dcl(B_0)$. Let F_0 realise $tp(B_0/A)$ with $F_0 \downarrow_A B_0$ and $C = Cb(Lstp(F_0/B_0))$, then I claim that C is interalgebraic with A. By facts on canonical bases, we clearly have that $C \subset acl(A)$. For the converse, suppose that $A \subsetneq acl(C)$ and let $\{B_i : i < \omega\}$ be a Morley sequence realising $tp(B_0/C)$. Then the corresponding $\{A_i : i < \omega\}$ is a Morley sequence for the non-algebraic $tp(A/C)$. The congugate types $p_i = Lstp(F_i/B_i)$ to $p_0 = Lstp(F_0/B_0)$ are all in the same parellelism class and morever as the B_i are independent over C I can find a single F such that $tp(FB_i) = tp(F_iB_i)$ By automorphism, $A_i \subset dcl(F)$ and this is witnessed by a fixed set of formulae. As $tp(A/C)$ is non algebraic, I can clearly find infinite $\bar{a}_i$ distinct tuples in A_i witnessing a single algebraic formula , this is a contradiction. Now as $G(D)$ is modular I have that $F_0 \downarrow_{acl(F_0)\cap acl(B_0)} B_0$, where acl is taken inside $G(D)$; by the same argument, ,replacing A above with $acl(F_0)\cap acl(B_0)$ and noting the change from acl to dcl effects nothing, I have $acl(F_0 \cap B_0) = acl(C) = acl(A)$ as required.

Now suppose that $A, B \subset D^{eq}$ are algebraically closed and $A \not\downarrow_{A\cap B} B$. Let A', B' and C be corresponding interalgebraic closed sets in $G(D)$ to A, B and $A \cap B$ respectively. As $A' \cap B' \subset A \cap B \cap G(D)$, we must have that $A' \cap B' \subset C$, hence as $A' \not\downarrow_C B'$ by transitivity of forking we must have $A' \not\downarrow_{A'\cap B'} B'$, contradicting modularity. This proves that D is 1 based

□

Theorem 14 and Theorem 15 combine to give the following result

Theorem 9. *The following are equivalent;*

1. D is 1-based. 2. D is linear. 3. $G(D)$ is linear. 4. $G(D)$ is modular.

The proof that D linear implies that D is 1-based was also proved using a generic pair argument by Vassiliev in [35].

Chapter 4

Reducibility Questions

Having introduced the non-definable object $G(D)$, we now turn to the question of how $G(D)$ is related to D. Throughout this section, we assume that D is a 1-based minimal Lstp.

Definition 6. *We say that $G(D)$ is reducible into D^k if for any $c \in G(D)$, there is a k-tuple $\bar{b}$ from D such that $a \in acl(\bar{b})$.*

Definition 7. *We say that $G(D)$ is strongly 2-reducible if it is reducible into D^2 and satisfies the right hand side of Lemma 17.*

Lemma 10. *D is locally modular iff for any $c \in G(D)$ any any $d \in D$, there exists $b \in D$ such that $c \in acl(db)$.*

Proof. The proof of left to right is similar to the above. Suppose D is locally modular, let $c \in G(D)$ and $d \in D$. Assume that $c \notin acl(d)$ otherwises we are done. Now repeat the argument in Theorem 14 to find b with $c \in acl(db)$.

For the converse, we use the criterion (**) from Lemma 2. So let $d \in D$, (ab) a pair with $dim_d(ab) = 2$ and $A \subset D$ with $dim_d(ab/A) = 1$. As D is linear, letting

$c = Cb(Lstp(ab/Ad))$, we have that $SU(c) = 1$ and $c \in acl(ab) \cap acl(Ad)$. Then as c lies in $G(D)$, I can find $e \in D$ such that $c \in acl(de)$. As $c \downarrow d$, we must have that $e \notin acl(d)$. Finally, $e \in acl(dc)$ and hence $e \in acl(abd) \cap acl(Ad)$, that is $e \in cl_d(ab \cap A)$. This shows that D_d is modular.

□

We also have the following results connecting the geometries of D and $G(D)$.

Lemma 11. *If D is locally modular and $d \in D$, then the geometry of D_d and the geometry of $G(D)$ are isomorphic as projective geometries.*

Proof. We first show that given $A \subset D^{eq}$, the pregeometries D_d and $G(D)_d$ localised at A are non weakly orthogonal. Suppose $c \in D \setminus acl(dA) \cap D$, then clearly $c \in G(D)$, as $D \subset G(D)$. If $c \in G(D) \setminus acl(Ad) \cap G(D)$, then by strong 2-reducibility I can find $e \in D$ with $c \in acl(de)$ and clearly $e \notin acl(dA)$ as well. Then clearly, taking $A = \emptyset$, the above property determines a bijection f between the geometries D'_d and $G(D)'_d$. To see that f is in fact an isomorphism it is sufficient to check that f preserves lines, which is trivial by interalgebraicity.

□

We can use the above result on strong 2- reducibility to prove the following positive result. Here we take cl to be closure inside $G(D)'$

Lemma 12. *Suppose D is locally modular and the geometry $G(D)'$ is projective over a field F with $card(F) \leq 3$, then D' the geometry of D is modular or affine.*

Proof. We first prove the result for $Card(F) = 3$. We have $D' \subset G(D)'$. Let $m = min\{card\ cl(ab) : a, b \in D'\}$. If $m = 4$, then by Fact 5, we have D' is projective over finite field F with $Card(F) = 3$, as D' cannot be affine otherwise we would have lines in $G(D)'$ of length at least 5. Hence, we assume that $m \leq 3$.

Claim 1

There is no independent (ab) in D' such that $Card(cl(ab) \cap D') = 2$

If so, then amalgamating types, we can find $c \in D'$ such that $Card(cl(ab) \cap D') = Card(cl(ac) \cap D') = Card(cl(bc) \cap D') = 2$. Let $b' \in cl(bc) \cap G(D)' \setminus D'$, then by strong 2-reducibility we can find $a' \in D'$ such that $b' \in cl(aa')$. Now $dim(ac/a'b) = 1$ as $a' \notin cl(bc)$. Hence, as $G(D)'$ is modular, we can find $c' \in cl(ac) \cap cl(a'b)$ and clearly $c' \neq a, c$ as otherwise $b \in cl(aa')$ or $a' \in cl(bc)$. Now using the fact that lines in $G(D)'$ have size 4, let $c'' \in cl(ac)$ with $c'' \neq a, c, c'$, so $c'' \notin D'$. Then $dim(c''a'/bc) = dim(c''a'/ab) = 1$ and moreover $a, b, c \notin cl(a'c'')$ as otherwise $a' \in cl(ac)$ or $a' \in cl(c''b)$. It follows that $cl(a'c'') \cap cl(bc) \notin D'$ and $cl(a'c'') \cap cl(ab) \notin D'$. Hence $Card(cl(a'c'') \cap D') = 1$ which contradicts strong $2-$reducibility.

Claim 2

There is no independent (ab) in D' such that $Card(cl(ab) \cap D') = 4$

If so, again by amalgamating types we can find (abc) such that $Card(cl(ab) \cap D') = 4$, $Card(cl(ac) \cap D') = 3$ and $Card(cl(bc) \cap D') = 3$. Let $e \in G(D)' \setminus cl(ac)$ and $f \in G(D)' \setminus cl(bc)$. Then as $G(D)'$ is modular and $dim(ef/ab) = 1$ we can find $g \in cl(ef) \cap D'$, so $Card(cl(ef) \cap D') \geq 1$. By strong 2 reducibility of D', we have in fact that $Card(cl(ef) \cap D') = 2$ which is impossible by Claim 1.

We conclude that $Card(cl(ab) \cap D') = 3$ for all independent pairs ab in D'. By Fact 5, D' is affine over F with $Card(F) = 3$ or projective over F with $Card(F) = 2$. Clearly the latter case cannot happen as by the above lemma we would have an isomorphism between projective geometries over fields F of size 2 and 3.

The case when $G(D)'$ is projective over a field F with $Card(F) = 2$ is similar and easier.

□

In fact we can use some combinatorial arguments to describe closure fairly explicitly in the case when D is locally modular. The following fact, usually know as Ramsey's colouring theorem, was proved in [12]

Fact 13. *If Aff_F is affine space over a finite field with q elements and an m-colouring on Aff_F is given, then there exists an integer (the Ramsey number) $R(m,q,n)$ such that for any affine space of dimension R inside Aff_F, there exists an affine subspace of dimension n having all 1 colour.*

There is also a corresponding version with Aff_F replaced by $P(F)$, projective space over a finite field.

We now use this result to prove the following;

Theorem 14. *If D is locally modular and $G(D)'$ is projective geometry over a finite field F of size q, then, for all n, there exist $(a_1 \ldots a_n) \in D'$ such that $cl(a_1, \ldots, a_n) \cap D' = q^n$.*

Proof. To see this, pick any point $d \in D'$. Strong 2-reducibility, using the fact that D is locally modular, implies that for any line l from $G(D)'$ passing through d, I can find $d' \neq d$ in $D' \cap l$. Now let $P(N)$ and $P(N-1)$ be projective planes of dimension N and $N-1$ passing through d. $P(N) \setminus P(N-1)$ is then isomorphic to affine space $Aff(N)$ of dimension N, and on each line l in $Aff(N)$, I can find $d' \in D$. Now consider a projection $\pi : Aff(N) \to Aff(N-1)$ onto an affine subspace of codimension 1. If I fix coordinates on the 1 dimensional fibre, there can be at most 2^q possible distinct ways of arranging elements from D', and I colour the base of the projection

$Aff(N-1)$ according to these possibilities. Now if I pick $N-1 \geq R(2^q, q, n)$, by the above fact I am guaranteed to find a monochramatic subspace M of dimension n inside $Aff(N-1)$. Now, I have at least 1 point from D' on the fibres of π restricted to M and the coordinates of D' are the same. It then trivially follows that I can find a linear section σ of π such that $\sigma(M) \subset D'$, and so D' contains an affine space of dimension n.

□

Remark 2. *Considering Lemma 17, one might expect that in fact only* 2 *reducibility of* $G(D)$ *is needed to characterise local modularity of* D*. However, this is not the case as we can see from considering the case of the generic predicate. Explicitly, let* D *be a non-trivial locally modular Lstp over* $\emptyset$*. As* D *is* 1*-based, it follows by [16] that* D *admits elimination of* $\exists^\infty$*. We can therefore add a predicate* P *to* D *satisfying the axioms of Pillay/Chadzidakis given in [7]. Now the new structure* (D, P) *still has SU-rank* 1 *and is non-trivial locally modular as algebraic closure for* T_D *and* $T_{(D,P)}$ *coincide. Consider the reduct* $P(D) = \{x \in D : P(x)\}$*. Suppose that* I *is an indescernible sequence in* $P(D)$*, then clearly* I *is indiscernible in* (D, P) *and hence is a Morley sequence over the first point inside* (D, P)*. As independence is preserved passing to the reduct* $P(D)$*,* I *is still Morley over the first point inside* $P(D)$*. By the criterion in Lemma 13,* $P(D)$ *must be* 1*- based. We want to show that* $P(D)$ *has* 2 *reducibility. As* $P(D)$ *a reduct of* (D, P)*, we can freely consider elements of* $P(D)^{eq}$ *as living inside* $(D, P)^{eq}$*. By strong* 2*-reducibility of* (D, P)*, if* $c \in G(P(D)))$*, I can find a pair* $a \in P(D)$ *and* $b \in D$*, with* $c \in acl(ab)$*. As* (D, P) *is not trivial, I can also suppose that* $c \notin acl(a) \cup acl(b)$*. Now let* (ef) *realise* $tp(ab/c)$ *with* $ef \downarrow_c ab$*. Then if* $g = Cb(Lstp(ab/ef))$*, as we saw above* g *is interalgebraic with* c*. Now considering* $tp^-(ab/ef)$ *in the language without* P*, I still have that* $acl^-(ef) \cap ab = \emptyset$*. By compactness, and using the axioms for genericity, I can find* $(a'b')$ *realising* $tp^-(ab/ef)$ *such that* $(a'b') \in D(P)$*. Finally, if* $g' = Cb(Lstp^-(a'b'/ef))$*, then clearly* $g' \in dcl(g)$*, so* $g' \not\downarrow g$ *and therefore* g, g' *are interalgebraic. Hence,* $g \in acl(a'b')$ *and so* $c \in acl(a'b')$ *as required. We also need to check that* $P(D)$ *is not locally modular. Let* $d \in P(D)$ *and pick elements* $a_1a_2a_3a_4$ *from* D *such that* (a_1a_2) *is parallel to*

(a_3a_4). *Now consider the following formulae in the language of* (D, P) *saying that* $tp^-(y_1y_2y_3y_4) = tp^-(a_1a_2a_3a_4)$, $(y_1y_2y_3y_4) \subset P$ *and* $acl_D(dy_1y_2) \cap acl_D(dy_3y_4) \cap P \subset acl_D(d)$. *By compactness and using the genericity axioms, it follows easily that this is in fact a partial type with respect to* $T_{(D,P)}$. *Now taking realisations* $a_1a_2a_3a_4$ *in* $P(D)$, *one easily checks that* $dim_d(a_1a_2/a_3a_4) = 1$ *but* $dim_d(a_1a_2/cl_d(a_1a_2) \cap cl_d(a_3a_4)) = 2$ *inside the localised pregeometry* $P(D)_d$. *This contradicts the criteria for modularity in Lemma 2.*

So we have that

Theorem 15. *If* $P(D)$ *is a generic subset of a non trivial minimal Lstp type* D, *then* $P(D)$ *is* 1*-based, but not locally modular. If* D *is locally modular, then moreover* $G(P(D))$ *is* 2*-reducible.*

We now consider the question of reducibility for arbitrary 1-based D. We will prove the following theorem;

Theorem 16. *If* D *is a* 1*-based minimal Lstp and* $c \in G(D)$. *Then,*

1. c *has a reduction in* D^3

2. *If* (xy) *is a fixed independent pair from* D, $c \in acl(x'y'z)$ *with* $x', y', z \in D$ *and* $tp(xy) = tp(x'y')$.

3. *Given fixed* $d \in D$, *there exists a a pair* $ef \in D$ *such that* $c \in acl(def)$.

4. *There exists* $u \in G(D)$ *with* u *and* c *interalgebraic such that* $u = \bar{a}/E$, *where* $\bar{a}$ *is a tuple from* D^6 *and* E *is a definable equivalence relation.*

Proof. 1, we use induction on n, where $c \in G(D)$ has a reduction in D^n. Suppose $c \in acl(a_1 \ldots a_{n+1})$ with $n \geq 3$. We have $SU(a_1c/a_2 \ldots a_{n+1}) = 1$, hence ,using linearity of $G(D)$ and the assumption that $c \notin acl(a_1)$, if $f = Cb(Lstp(a_1c/a_2\dots a_{n+1}))$, then $f \in G(D) \cap acl(a_2\dots a_{n+1}) \cap acl(a_1c)$ and $c \in acl(a_1f)$. By the induction hypothesis, as f is captured by n elements in D, we can suppose that f is captured by 3 elements in D, so we can assume that $f \in acl(a_2a_3a_4)$. We now find a "parallel plane" passing through a_1 capturing f. We have that $SU(fa_4/a_2a_3) = 1$ and we may assume that $f \notin acl(a_4)$ otherwise $c \in acl(a_1a_4)$ and we are done. Therefore, again by linearity of $G(D)$, if $g = Cb(Lstp(fa_4/a_2a_3)$ then $g \in acl(fa_4) \cap acl(a_2a_3) \cap G(D)$ and $f \in acl(ga_4)$. Now I may suppose that $f \notin acl(g)$ and $g \notin acl(a_2)$, otherwise again f is captured by 2 elements from D. Let g' be a congugate to g by an automorphism sending a_2 to a_1, then I can assume g and g' realise the same $Lstp$ and applying the independence theorem I can find g'' amalgamating $tp(g'/a_1)$ and $tp(g/f)$. Now let a_5 be the image of a_3 by an automorphism taking (a_2g) to (a_1g'') and a_6 the image of a_4 by an automorphism taking (fg) to (fg''). Then $f \in acl(a_1a_5a_6)$ and as $c \in acl(a_1f)$, we have $c \in acl(a_1a_5a_6)$ as well. This proves 1.

□

Proof. 2, suppose $c \in G(D)$, then we can find $(a_1a_2a_3) \in D^3$ such that $c \in acl(a_1a_2a_3)$ by 1. We may suppose that $c \notin acl(a_1a_2) \cup acl(a_1a_3) \cup acl(a_2a_3)$ otherwise the result is trivial. Then, let $f = Cb(Lstp(a_1c/a_2a_3))$, so $f \in G(D)$ and $(a_1a_2a_3f)$ are pairwise independent. Now choose $a_4 \in D$ such that $tp(a_1a_4) = tp(xy)$. Applying the independence theorem, we can amalgamate $tp(a_4/a_1)$ and $tp(a_2/f)$ and find a_5a_6 such that $f \in acl(a_5a_6)$ and $tp(a_1a_5) = tp(xy)$. Then $c \in acl(a_1f)$ and so $c \in acl(a_1a_5a_6)$ with $tp(a_1a_5) = tp(xy)$ as required.

□

The proof of 3 is implicit in the proof of 1

Proof. 4, let $c \in G(D)$ and choose $(a_1a_2a_3)) \in D^3$ with $c \in acl(a_1a_2a_3)$. Let $(a_4a_5a_6)$ realise $tp(a_1a_2a_3)$ with $a_4a_5a_6 \downarrow_c a_1a_2a_3$. Then letting $u = Cb(Lstp(a_4a_5a_6/a_1a_2a_3))$, we have that u is interalgebraic with c. Now suppose that an automorphism α fixes

$(a_1a_2a_3a_4a_5a_6)$. If we have that $p = Lstp((a_4a_5a_6/a_1a_2a_3)$, then clearly $(a_4a_5a_6)$ amalgamates p and $\alpha(p)$, hence u is fixed. Therefore, $u \in dcl(a_1a_2a_3a_4a_5a_6)$. Clearly this allows us to define an equivalence relation E on D^6 such that $u = \bar{a}/E$ as required.

$\square$

Let us now examine the consequences of 4 when D is ω-categorical. In this case, there exists finitely many definable equivalence relations on D^6. Enumerate the equivalence relations for which there exists $c \in G(D)$ and a tuple $\bar{a} \in D^6$ such that c and $\bar{a}/E$ are interalgebraic as elements of D^{eq}. Let $\Gamma(D)$ denote the finite union of sorts corresponding to these equivalence relations, so $\Gamma(D)$ is a definable subset of D^{eq}. Clearly $\Gamma(D)$ is a union of rank 1 complete types over $dom(D)$ so $\Gamma(D) \subset G(D)$. Moreover, by 4, every element of $G(D)$ is interalgebraic with an element from $\Gamma(D)$ so the geometries of $\Gamma(D)$ and $G(D)$ coincide. As we saw before, the geometry $\Gamma(D)' = G(D)'$ is then definable as a subset of D^{eq}. If D is non-trivial, then closure on $G(D)'$ is that of projective geometry over a finite field F, hence $G(D)$ as a definable object has a non-trivial strongly minimal stable reduct. Again by the group configuration for stable theories, it follows that a vector space over a finite field F is definable in D^{eq} over a finite parameter.

The result can be summarised in the following theorem which was conjectured by Vassiliev in [35] and also shown by Wagner and Tomasic in [34], using the group configuration theorem for simple theories.

Theorem 17. *If D is 1-based, ω-categorical, non-trivial minimal Lstp, then the geometry $G(D)'$ is definable over $dom(D)$ and has a non-trivial strongly minimal stable reduct preserving projective geometry over a finite field F. Then over a finite parameter in D^{eq}, a vector space over a finite field F is interpretable.*

Chapter 5

Non-Trivial Theories

In [35], Vassiliev also conjectures that in any non-trivial 1-based ω-categorical theory T, a vector space V over a finite field is interpretable. Using results so far proved, we are able to show this.

Definition 8. *We say that a theory T is trivial if for $a, b, c, A \subset \mathcal{M}^{eq}$, if $\{a, b, c\}$ is pairwise independent over A, then $\{a, b, c\}$ is independent over A,*

We now aim to prove the following lemma

Lemma 18. *If T is 1-based, then T is trivial if and only if all SU-rank 1 types are trivial.*

One direction is obvious, we will be concerned with showing right to left.

Proof. Suppose not, then there exist a, b, c and A such that a, b, c are pairwise independent over A and a, b, c are dependent over A. Letting $d = Cb(Lstp(a/bc, A))$,then, as T 1$-$based, $d \in acl(a)$. Therefore, as $a \downarrow_A b$, we have that $d \downarrow_A b$ and similarily, $d \downarrow_A c$. Moreover, as $a \not\downarrow_A b, c$, we must have that $d \notin acl(A)$. Let $e = Cb(Lstp(b/caA))$, then again $e \in acl(b)$, so $e \downarrow_A a$ and hence $e \downarrow_A d$. Similarily, $e \downarrow_A c$. Finally, $e \downarrow_{Acd} a$ as $a \downarrow_{Acd} b$ implies $a \downarrow_{Acd} e$, which gives us that $e \in acl(Acd)$ as $e \in acl(Aac)$ by

$1-$basedness. Let $f = Cb(Lstp(c/abA))$. Repeating the above arguments, we find d, e, f such that $d \in acl(efA)$, $e \in acl(dfA)$ and $f \in acl(deA)$. Moreover, d, e, f are pairwise independent over A and $d, e, f \notin acl(A)$. For, notational convenience, assume the above properties hold for our original a, b, c, and that $A = \emptyset$

As $a \notin acl(\emptyset)$ there exists a set C such that $SU(\bar{a}/C) = 1$. Replacing C by $d = Cb(Lstp(a/C))$ gives $SU(a/d) = 1$ and $d \in acl(a)$ by 1-basedness. Now again by $1-$basedness, we have that $bd \downarrow_{acl(bd)\cap acl(cd)} cd$ and so in particular $b \downarrow_F c$ where $F = acl(bd) \cap acl(cd)$. We must have that $SU(a/F) = 1$, otherwise $a \in acl(bd)$ which is ridiculous as $ad \downarrow b$. We also must have that $a \downarrow_F b$, otherwise $a \in acl(bd)$ again. Similarily, $a \downarrow_F c$. Finally, we have that $SU(b/cF) \leq SU(a/cF) = 1$, as $b \in acl(acF)$. Hence, as $b \downarrow_F c$ and $b \notin acl(F)$, we have $SU(b/F) = 1$. Similarily, $SU(c/F) = 1$. So we have found a, b, c, F such that $SU(a/F) = SU(b/F) = SU(c/F) = 1$, a, b, c are pairwise independent over F and in the algebraic closure of the other two.

Now as $c \downarrow_F b$ we can find d realising $Lstp(c/bF)$ such that $d \downarrow_F abc$. As $b \in acl(Fac)$, by an automorphism we can find b' realising $Lstp(a/F)$ such that b and b' are interalgebraic over Fd. As $b \downarrow_F c$ we can find e realising $Lstp(b/cF)$ such that $e \downarrow_F abcd$. Again, as $c \in acl(Fab)$, we find c' realising $Lstp(a/F)$ such that c and c' are interalgebraic over Fe. Moreover, we have that $de \downarrow_F abc$.

We now want to show that a, b', c' are pairwise independent over Fde. I will just show that $b' \downarrow_{Fde} c'$, the other cases follow similarily. We have $b \downarrow_F c$ and hence $b \downarrow_{Fde} c$ as $de \downarrow_F abc$, so $b' \downarrow_{Fde} c'$ as $b'c'$ is interalgebraic with bc over Fde. As $a \in acl(Fbcde)$, we must have that $a \in acl(Fb'c'de)$. So a, b', c' are algebraic with the other two over Fde.

Now let $f = Cb(stp(ab'c'/Fde))$, so $f \in acl(ab'c') \cap acl(Fde)$ by 1-basedness. As $f \in acl(Fde)$, we still have pairwise independence of a, b', c' over Ff, and as $SU(ab'c'/Ff) = SU(ab'c'/Fde) = 2$, we have still preserved the conditions above except now f is internal to $Lstp(a/F)$. Choose $a''b''$ realising $Lstp(ab'/Ff)$ such that $a''b'' \downarrow_F ab'c'f$. Then we find c'' realising $Lstp(a/F)$ such that $f \in acl(a''b''c'')$ and f is interalgebraic with c'' over $Fa''b''$. We want to show that a, b', c' are pairwise independent over $Fa''b''c''$, again I will just show that $b' \downarrow_{Fa''b''c''} c'$. As $b' \downarrow_{Ff} c'$ and $a''b'' \downarrow_F b'c'f$, we have $b' \downarrow_F a''b''c'f$ and so $b' \downarrow_F a''b''c''c'f$ which gives $b' \downarrow_{Fa''b''c''} c'$. Finally $SU(ab'c'/Fa''b''c'') = 2$ iff $SU(ab'c'/Fa''b''f) = 2$ which is clearly the case.

So if we denote the triple $a''b''c''$ by $I \subset Lstp(a/F)$, we have that a, b', c' realising $Lstp(a/F)$ are pairwise independent and dependent over FI. This means precisely that the localisation of $Lstp(a/F)$ to FI is non trivial as a pregeometry which implies that $Lstp(a/F)$ is non trivial as pregeometry.

□

As is well known, if T is simple and ω-categorical then T has finite SU-rank. Lemma 25 and Theorem 24 in the last section combine to give the following theorem;

Theorem 19. *If T is a non-trivial, 1-based, ω-categorical simple theory, then an infinite dimensional vector space over a finite field is definable in $\mathcal{M}^{eq}$ over a finite parameter.*

Chapter 6

Extension to Regular types

We would like to generalise some of the previous results to the case of regular types. We first need to generalise some of the basic notions around regularity to simple theories. As always we assume that T is supersimple and so has e.h.i. Let p_1 and p_2 be 2 Lstps over possibly different sets. We say that p_2 is hereditarily orthogonal to p_1 if every extension of p_2 is orthogonal to p_1. We now fix a regular complete Lstp p over a domain A, and set p_F to be the localisation of p at $F \supset A$ and D to be the solution set of p. We say that a Lstp q over a domain B is p-simple if there exists F with $A \cup B \subset F$ and sets $I_\alpha \subset p_F$ for each nonforking extension q_α of q over F such that the extensions of q_α to FI_α are all hereditarily orthogonal to the non forking extensions of p to FI_α

It seems plausible that one can choose I to be a single set, independently of α, or at least that if such F, I_α exist then we can rechoose I_α such that $dim(I_\alpha)$, in the sense of the localised pregeometry p_F, is independent of α. In this case, we can define $w_p(q) = min\{\kappa$: there is $F \supset A \cup B$ and I depending on F as above such that $dim(I) = \kappa$ in $p_F\}$. If T is supersimple, then w_p is always finite. We assume the following properties of w_p which can be found in [30]. As w_p is always defined relative to A we assume that all our sets contain A

1. Additivity: If $Lstp(a/B)$ and $Lstp(b/B)$ are p-simple, then so is $Lstp(ab/B)$

and $w_p(ab/B) = w_p(a/Bb) + w_p(b/B)$.

2. Extension: w_p is invariant under non forking extension, and if $Lstp(a/B)$ is p-simple and $B \subset C$ then $Lstp(a/C)$ is p-simple and $w_p(a/C) \leq w_p(a/B)$

3. Algebraicity: If $Lstp(a/B)$ is p simple and $b \in acl(aB)$ then $Lstp(b/B)$ is p-simple and $w_p(b/B) \leq w_p(a/B)$

4. Finite Character: If $Lstp(a/B)$ is p-simple and $B \subset C$ then there exists a finite $\bar{c} \subset C$ such that $w_p(a/C) = w_p(a/B\bar{c})$

By a Morley sequence argument and using 3, we have that if $Lstp(a/B)$ is p-simple, $B \subset C$ and $c = Cb(Lstp(a/C))$, then $Lstp(c/B)$ is also p-simple.

Now we want to define a suitable notion notion of linearity for D. For this we require one more notion. We say that $Lstp(a/B)$ is p-semi regular if for every $B \subset C$, $w_p(a/B) = w_p(a/C)$ iff $a \downarrow_B C$. The fundamental result on p semi-regular types is the following in [30], which I assume generalises to simple theories;

Lemma 20. *Suppose $Lstp(a/B)$ is p-simple and $w_p(a/B) = n$.*

Let $B^{reg} = \{b \in acl(aB) : w_p(b/B) = 0\}$, so $B \subset B^{reg}$, then $Lstp(a/B^{reg})$ is p-semi regular and $w_p(a/B^{reg}) = n$

Definition 9. *We will say that D is linear if the following holds;*

If ab is a pair in D and $B \supset A$ with $Lstp(ab/B)$ semi-regular and p-weight 1, then $w_p(c/A) \leq 1$ where $c = Cb(Lstp(ab/B))$.

We also introduce the following 2 objects.

Definition 10. $G(D) = \{c : Lstp(c/A) \textit{ is p-simple of p-weight } 1\}$

and

$G(D)^{large} = \{c : Lstp(c/A) \textit{ is p-simple of finite p-weight}\}$

The closure operator cl_p on $G(D)^{large}$ is given by $cl_p(B) = \{c \in G(D) : w_p(c/A \cup B) = 0$ and we have a corresponding operator by restriction to $G(D)$.

We have the following desirable properties for cl_p.

1. cl_p is transitive for $G(D)$ and $G(D)^{large}$.

For suppose that $\bar{a} \in cl_p(\bar{b})$ and $\bar{b} \in cl_p(\bar{c})$ then $w_p(\bar{a}\bar{b}\bar{c}/A) = w_p(\bar{c}/A)$ by additivity and $w_p(\bar{b}\bar{a}\bar{c}/A) = w_p(\bar{a}\bar{c}/A) = w_p(\bar{a}/\bar{c}A) + w_p(\bar{c}/A)$, so $\bar{a} \in cl_p(\bar{c})$.

2. cl_p is finite for $G(D)$ and $G(D)^{large}$.

For suppose $B \subset G(D)^{large}$or $G(D)$ and $\bar{a} \in cl_p(B)$ then ,by property 4 of w_p, there is a finite $\bar{b} \subset acl(B)$ such that $\bar{a} \in cl_p(\bar{b})$. By transitivity of cl_p and the fact that algebraic types have p-weight 0, we can assume that $\bar{b} \in B$.

3. cl_p satisfies exchange on $G(D)$.

For suppose that $a \in cl_p(Bc) \setminus cl_p(B)$. Replacing B by B^{reg}, and using transitivity of p-closure we may assume that $w_p(a/B^{reg}) = 1$ and $Lstp(c/B^{reg})$ is p-semi regular. Then, as $w_p(a/B^{reg}c) = 0$, by the extension property we must have that $c \not\perp_{B^{reg}} a$ and so as $c \in G(D)$ then $w_p(c/B^{reg}a) = 0$, that is $c \in cl_p(B^{reg}a)$ and then by transitivity $c \in cl_p(Ba)$.

The above shows that in fact $G(D)$ forms a pregeometry under p-closure, but of course exchange fails in general for $G(D)^{large}$.

We will say that $G(D)$ is linear if (ab) is a pair from $G(D)$ and $B \subset G(D)$ such that $Lstp(ab/B)$ is p-semi regular with $w_p(ab/B) = 1$, then $w_p(c) \leq 1$ where $c = Cb(Lstp(ab/B))$.

We now aim to prove the following lemma;

Lemma 21. *If D is linear then $G(D)$ is linear.*

Proof. We will proceed by following the steps for the finite rank case. For ease of notation assume that $acl(A) = \emptyset$. Now let (ab) be a pair from $G(D)$ with $w_p(ab) = 2$, the case for $w_p(ab) = 1$ is easier, and suppose that $B \subset G(D)$ with $w_p(ab/B) = 1$. Then by definition of w_p we can find a set F as above such that every non forking extension of $Lstp(ab)$ to F can be reduced in p_F. We may choose F such that $F \downarrow ab$ and by automorphism we can find cd in p_F such that $w_p(ab/Fcd) = 0$. By definition of weight $w_p(cd/F) = 2$ Repeating this argument we can rechoose F with $F \downarrow_{ab} B$, so $F \downarrow abB$, and by automorphism find cd as above with the same properties. As $F \downarrow ab$, and w_p is invariant under nonforking extension we have $w_p(ab/F) = 2$. Then by additivity of p-weight we must have $w_p(cd/abF) = 0$ as well.

Claim 1: $w_p(cd/FB) = 1$.

As $w_p(ab/B) = 1$, $F \downarrow_B ab$ and w_p is invariant under non forking extension then $w_p(ab/FB) = 1$. Then

$w_p(abcd/FB) = w_p(ab/cdFB) + w_p(cd/FB) = w_p(cd/FB) =$

$w_p(cd/abFB) + w_p(ab/FB) = 0 + 1 = 1$

giving the claim.

Letting $FB^{reg} = \{b \in acl(cdFB) : w_p(b/FB) = 0\}$

By the above, $Lstp(cd/FB^{reg})$ is p-semi regular with weight 1 and by linearity of D, $w_p(C) \leq 1$ where $C = Cb(Lstp(cd/FB^{reg}))$. Then

Claim 2: $w_p(cd/cl_p(cd) \cap acl(FB^{reg})) = 1$

We have that $cd \downarrow_C FB$, hence $w_p(cd/C) = 1$. Then by additivity and linearity of D we calculate $w_p(C/cd) = w_p(C) - 1 = 0$. Therefore $C \in cl_p(cd)$ and as $C \in cl_p(cd) \cap acl(FB^{reg})$ and $cl_p(cd) \cap acl(FB^{reg}) \subset acl(FB^{reg})$ the claim is shown.

Claim 3: $w_p(ab/W) = 1$, where $W = cl_p(Fab) \cap acl(FB^{reg})$

We clearly still have that $w_p(cd/cl_p(Fcd) \cap acl(FB^{reg})) = 1$. As $F \subset cl_p(Fcd)$, using additivity, $w_p(ab/cl_p(Fcd) \cap acl(FB^{reg})) = 1$. By transitivity of p-closure we must have that $cl_p(Fcd) = cl_p(Fab)$, hence $w_p(ab/W) = 1$ as required.

Now let $C' = Cb(Lstp(ab/B))$. Then

Claim 4: $w_p(ab/WC') = 1$

If not, then as $C' \in acl(B)$, $ab \in cl_p(FB^{reg})$. Again by transitivity of p closure and the definition of FB^{reg} we must have $ab \in cl_p(FB)$. Then as $ab \downarrow_B F$, $ab \in cl_p(B)$, contradicting the fact $w_p(ab/B) = 1$ and giving the claim.

Now $ab \downarrow_{C'} B$ so still $w_p(ab/C') = 1$ and moreover $Lstp(ab/C')$ is still semi-regular. Then by definition of p-semi regularity we must have that $ab \downarrow_{C'} W$ and so $C' \in acl(W)$ Then $C' \in cl_p(Fab)$ and as $C' \downarrow_{ab} F$, we must have $C' \in cl_p(ab)$. Now by a simple weight calculation we have that $w_p(C') = 1$ as required

□

Even though $G(D)^{large}$ is not a pregeometry it still makes sense to talk of the dimension of a closed set. Given $X, Y \subset G(D)^{large}$ closed we define
$dim(X/Y) = max\{w_p(\bar{a}/Y) : \bar{a} \in X\}$

Definition 11. *We will say that $G(D)^{large}$ is modular if the following holds ;*

For finite dimensional closed $X, Y \subset G(D)^{large}$ $dim(X/Y) = dim(X/X \cap Y)$.

We first aim to prove the following lemma;

Lemma 22. *If $G(D)$ is linear then $G(D)$ is modular.*

Proof. As $G(D)$ forms a pregeometry, it is sufficient to check the criterion (**) in Lemma 2. So choose x_1x_2 with $w_p(x_1x_2) = 2$ and Y closed such that $w_p(x_1x_2/Y) = 1$. By finiteness, we can find $\bar{y} \subset Y$ such that $w_p(x_1x_2/\bar{y}) = 1$ and $cl_p(\bar{y}) = Y$. Replacing $\bar{y}$ by $\bar{y}^{reg}$ we can even assume that $Lstp(x_1x_2/\bar{y})$ is p semi regular. By linearity of $G(D)$, we have that $c = Cb(Lstp(x_1x_2/\bar{y})) \in cl_p(x_1x_2) \cap cl_p(\bar{y})$. As $w_p(c) = 1$, then in fact $c \in G(D)$ as required. □

The 2 lemmas combine to give the following theorem.

Theorem 23. *If D is linear then $G(D)$ is modular.*

We now aim to prove the following;

Theorem 24. *If D is linear then $G(D)^{large}$ is modular*

Here the problem is made more difficuly by the fact that $G(D)$ is not a pregeometry.

Proof. We first reduce the problem to a finite one, as in in general $cl_p(X)$ will be a very large set! Suppose $G(D)^{large}$ is not modular, then there exists closed sets X and Y such that $dim(X/Y) < dim(X/X \cap Y)$. Taking $\bar{x} \in X$ so that $w_p(\bar{x}/Y)$ is maximal, by definition we have that $w_p(\bar{x}/X \cap Y) < w_p(\bar{x}/Y)$. By finiteness, I can find $\bar{c} \subset X \cap Y$ and $\bar{y} \subset Y$ such that $w_p(\bar{x}/\bar{c}) < w_p(\bar{x}/\bar{y})$ and moreover as weight is preserved on both sides I can take $\bar{c}$ and $\bar{y}$ such $cl_p(\bar{c}) = X \cap Y$ and $cl_p(\bar{y}) = Y$. Therefore, it is sufficient to prove that

$w_p(\bar{x}/\bar{y}) = w_p(\bar{x}/\bar{c})$ where $cl_p(\bar{c}) = cl_p(\bar{x}) \cap cl_p(\bar{y})$ (*)

We show $(*)$ by induction on $w_p(\bar{x}/\bar{y})$ for $\bar{x}$ and $\bar{y}$ finite tuples from $G(D)^{large}$.

Base Case. $w_p(\bar{x}/\bar{y}) = 1$.

Suppose $w_p(\bar{x}) = n$, then I can find $F \downarrow \bar{x}\bar{y}$ and $z_1 \ldots z_n \in p_F$ such that $\bar{x}$ and $z_1 \ldots z_n$ are weight equivalent over F. As before, one checks that $w_p(z_1 \ldots z_n/F\bar{y}) = 1$. Without loss of generality, we can assume that $w_p(z_i/F\bar{y}) = 1$ for each i. Now adding parameters $e_1 \ldots e_n \subset cl_p(\emptyset)$ we may assume that $Lstp(z_i/e_1 \ldots e_n)$ is semi regular for all i and all the conditions are preseved with $F\bar{y}e_1 \ldots e_n$ replacing $F\bar{y}$. We must have that $w_p(z_1 z_i/F\bar{y}e_1 \ldots e_n) = 1$ for all i, hence by linearity of D, we can find $c_i \in G(D)$ for $i \geq 2$ with $cl_p(c_i) = cl_p(z_1 z_i) \cap cl_p(F\bar{y}e_1 \ldots e_n)$. Clearly, $e_i \subset cl_p(c_i)$, so without loss of generality $e_i \subset c_i$. Now $w_p(c_i/z_1 z_i) = 0$ and $w_p(c_i/z_1) = 1$, otherwise $z_1 \not\downarrow_{e_1} c_i$ and $z_1 \subset cl_p(F\bar{y})$. Hence $z_i \not\downarrow_{z_1 e_i} c_i$. As $w_p(z_i/e_i z_1) = 1$ and $Lstp(z_i/e_i z_1)$ is semi regular we have that $w_p(z_i/z_1 c_i) = 0$ so $z_i \subset cl_p(z_1 c_i)$. We want to show that $w_p(c_2 \ldots c_n) = n - 1$ from which, taking $\bar{c} = c_2 \ldots c_n$, we clearly have that $w_p(z_1 \ldots z_n/\bar{c}) = w_p(z_1 \ldots z_n/F\bar{y})$ and $cl_p(\bar{c}) = cl_p(z_1 \ldots z_n) \cap cl_p(F\bar{y})$. Suppose not, say $c_n \subset cl_p(c_2 \ldots c_{n-1})$, then as $z_n \subset cl_p(z_1 c_n)$ and $c_2 \ldots c_{n-1} \subset cl_p(z_1 \ldots z_{n-1})$, we have that $z_n \subset cl_p(z_1 \ldots z_{n-1})$ contradicting the fact that $z_1 \ldots z_n$ are independent realisations of p_F. Now it follows that we can find a tuple $\bar{c}'$ such that $w_p(z_1 \ldots z_n/\bar{c}') = w_p(z_1 \ldots z_n/F\bar{y})$ and $cl_p(\bar{c}') = cl_p(Fz_1 \ldots z_n) \cap cl_p(F\bar{y})$. By the usual arguments we have that $w_p(\bar{x}/\bar{c}') = 1$ and $cl_p(\bar{c}') = cl_p(F\bar{x}) \cap cl_p(F\bar{y})$. Finally, we can assume that $Lstp(\bar{x}/\bar{y})$ is semi regular and one checks that $w_p(\bar{x}/\bar{c}'C) = 1$, where $C \in G(D)^{large}$ is $Cb(Lstp(\bar{x}/\bar{y}))$. As in the previous lemma, this forces $C \in cl_p(\bar{c}')$ and then $C \in cl_p(\bar{x})$, which gives the result.

Induction Step.

We now inductively assume the result for $\bar{x}$ and $\bar{y}$ with $w_p(\bar{x}/\bar{y}) = m$ and suppose that $w_p(\bar{x}/\bar{y}) = m + 1$. Now again we can find $F \downarrow \bar{x}\bar{y}$ and $z_1 \ldots z_n \in p_F$ such that $z_1 \ldots z_n$ is weight equivalent to $\bar{x}$ over F. Then still $w_p(\bar{x}/F\bar{y}) = m + 1$ and we may assume $z_i \notin cl_p(F\bar{y})$ for some i, otherwise $\bar{x} \in cl_p(F\bar{y})$ which is not the case. Using the fact that $w_p(z_1/F\bar{y}) = 1$ say, then by a weight calculation we have that $w_p(\bar{x}/z_1F\bar{y}) = m$. We now temporarily add F to the language, and take p-closure to include F. Then, working in $G(D)^{large}_F$, we have that $w_p(\bar{x}/\bar{y}) = m + 1$ and $w_p(\bar{x}/z_1\bar{y}) = m$. Applying the induction hypothesis to $G(D)^{large}_F$, we can find c in $G(D)^{large}_F$ such that $cl_p(c) = cl_p(\bar{x}) \cap cl_p(z_1\bar{y})$. Then $w_p(cz_1/\bar{y}) = 1$ as $c \in cl_p(z_1\bar{y})$ and $z_1 \notin cl_p(\bar{y})$. Therefore we can find $d \in G(D)^{large}_F$ such that $cl_p(d) = cl_p(cz_1) \cap cl_p(\bar{y})$ and moreover $w_p(d) = w_p(cz_1) - 1 = w_p(c) - 1 = w_p(\bar{x}) - m - 1$. As $cl_p(cz_1) \cap cl_p(\bar{y}) = cl_p(\bar{x}) \cap cl_p(\bar{y})$, this tells us exactly that $w_p(\bar{x}/F\bar{y}) = w_p(\bar{x}/d)$ where $cl_p(Fd) = cl_p(F\bar{x}) \cap cl_p(F\bar{y})$. Now letting $C' = Cb(Lstp(\bar{x}/\bar{y})$ and assuming as usual that $Lstp(\bar{x}/\bar{y})$ is semi regular, we have that $w_p(\bar{x}/FdC') = m$ otherwise as $C' \in acl(\bar{y})$ then $w_p(\bar{x}/F\bar{y}) < m$ which is not the case. Hence, by semi regularity we have that $C' \in cl_p(F\bar{x})$ and then as $F \downarrow_{\bar{x}} C'$, $C' \in cl_p(\bar{x})$. This proves the result.

□

So we have,

Theorem 25. *If D is linear then $G(D)$ and $G(D)^{large}$ are both modular.*

Naturally one would also expect further generalisations from [9] to the case of regular types. Namely, one conjectures the 2 following propositions

1. If $G(D)$ or $G(D)^{large}$ is modular then D is linear

We will say that $c \in G(D)$ has a reduction in D^k if there exists a tuple $(a_1 \dots a_k) \in D^k$ such that $c \in cl_p(a_1 \dots a_k)$. Then;

2. If $G(D)$ is modular, every element in $G(D)$ has a reduction in D^3

Chapter 7

Non-Finite Axiomatisability

In this section we give some results towards the conjectured non-finite axiomatisability of a complete, 1-based, w-categorical simple theory T.

The proof of the above when $\mathcal{M} \models T$ is itself the solution set of a minimal trivial Lstp ρ is given in [10]

Working in a saturated $\mathcal{M} \models T$, we will for convenience assume that $\mathcal{M}$ is the solution set of a complete type, though this is not essential. We must have that $\mathcal{M}$ has finite SU rank n.

Definition 12. *We call a finite tuple $a_1 \dots a_m \in \mathcal{M}$ ascending if $SU(a_1) \geq 1$ and $SU(a_{i+1}/a_i \dots a_1) \geq 1$ for $1 \leq i \leq m-1$.*

As is easily checked, any tuple $\bar{a}$, after reordering, may be written in the form $\bar{a}_1\bar{a}_2$ with $\bar{a}_1$ ascending and $\bar{a}_2$ algebraic over $\bar{a}_1$.

Let $\mathcal{N} \subset \mathcal{M}$. be a substructure

Definition 13. *We say that $\mathcal{N}$ is k-generic if*

1. $\mathcal{N}$ is algebraically closed

2. For all ascending tuples $\bar{a} \in \mathcal{N}$ with $SU(\bar{a}) \leq k$, every type $\rho \in S^1(\bar{a})$ is realised in $\mathcal{N}$

First we show that condition 2. is first order definable by a sentence Gen_k.

Proof. An ascending tuple $a_1 \dots a_m$ has SU-rank at least m, and hence ascending tuples $\bar{a}$ with $SU(\bar{a}) \leq k$ have length at most k. Enumerate the finite number of types $p_1, \dots, p_{n_k}$ realised by ascending tuples $\bar{a}$ with $SU(\bar{a}) \leq k$, and for each p_i let p_i^j for $1 \leq j \leq r_i$ enumerate the possible extensions to a type of $length(p_i) + 1$. Then Gen_k will be the sentence;

$$Gen_k \equiv \forall \bar{y}(\vee_{1 \leq i \leq n_k} p_i(\bar{y}) \rightarrow \wedge_{1 \leq j \leq r_i} \exists x p_i^j(x, \bar{y}))$$

□

Secondly, we show that if σ is a sentence in T with quantifier length m and $\mathcal{N}$ is an $nm + N(m)$ generic substructure of $\mathcal{M}$ with $N(m)$ a constant depending on m to be determined then $\mathcal{N}$ satisfies σ.

Proof. For this it is sufficient to find $N(m)$ such that if $a_1 \dots a_k$ is a tuple in $\mathcal{N}$ with $k \leq m$ then all 1-types over $a_1 \dots a_k$ are realised in $\mathcal{N}$. Reorder $a_1 \dots a_k$ so it is of the form $\bar{a}_1 \bar{a}_2$ with $\bar{a}_1$ ascending and $\bar{a}_2$ algebraic over $\bar{a}_1$. As $length(\bar{a}_1) \leq k \leq m$, $SU(\bar{a}_1) \leq nm$. Consider the number of congugates of $\bar{a}_2$ over $\bar{a}_1$ and choose $N(m)$ larger than any number which could appear here. By the technique in [10], any 1-type over $a_1 \dots a_k$ can be realised inside $\mathcal{N}$.

□

We now have the following lemma.

Lemma 26. *Suppose that for any k, we can find l with $l >> k$ and $\mathcal{N} \subset \mathcal{M}$ k-generic but not l-generic, then T cannot be finitely axiomatisable.*

Proof. Suppose for contradiction that $\sigma \vdash T$. If σ has quantifier length m, then taking $k = nm + N(m)$, any k-generic structure $\mathcal{N}$ will satisfy σ and therefore $\mathcal{N} \models T$. However, clearly the sentence $Gen_l \in T$ for any l and in particular for such $l >> k$ as given in the hypothesis of the lemma.

□

By the lemma, we just need to find a way of building such structures $\mathcal{N}$ inside $\mathcal{M}$. In order to do this, we first need the following coordinatisation lemma

Lemma 27. *For any $a \in \mathcal{M}$ we can find a set of coordinates $e_1 \ldots e_{n-1} \subset \mathcal{M}^{eq}$ with $SU(a/e_1 \ldots e_{n-1}) = 1$, $e_1 \ldots e_{n-1} \subset dcl(a)$ and $SU(e_{i+1}/e_1 \ldots e_i) = 1$.*

Proof. Let $a \in \mathcal{M}$ with $SU(a/\emptyset) = n$. By definition of SU rank we can find an extension $q \supset p = Lstp(a)$ over $A \subset M$ such that $SU(q) = n-1$, that is $SU(a/A) = n-1$. We may take A to be a model of T and hence assume that q is still a $Lstp$. Let $e_1 = Cb(Lstp(a/A))$, then $e_1 \in acl(a) \cap A$ by 1−basedness and $SU(a/e_1) = n-1$. By straightforward rank calculation $SU(a/e_1) + SU(e_1) = SU(e_1/a) + SU(a)$, so $SU(e_1) = 0 + n - (n-1) = 1$, and realises a minimal type in $\mathcal{M}^{eq}$. Replacing e_1 by its conjugates over a we may even assume that $e_1 \in dcl(a)$. Now we fix the type p_1 of e_1 and repeat the argument for $tp(a/e_1)$. Again we find e_2 such that $SU(e_2/e_1) = 1$, $SU(a/e_1e_2) = n-2$ and $e_2 \in dcl(a)$. Continuing in this way, we find $e_1, \ldots e_{n-1}$ such that $SU(a/e_1 \ldots e_{n-1}) = 1$, $SU(e_{i+1}/e_ie_{i-1} \ldots e_1) = 1$ for $1 \leq i \leq n-2$ and $e_1 \ldots e_{n-1} \in dcl(a)$.

□

We now fix the minimal types $p_i = tp(e_i/e_{i-1} \ldots e_1)$ for $1 \leq i \leq n-1$. We have a canonical choice of coordinates $e'_1 \ldots e'_{n-1}$ for any $a' \in \mathcal{M}$, namely take the image of $e_1 \ldots e_{n-1}$ under an automorphism taking a to a'. This is well defined as $e_1 \ldots e_{n-1}$ was assumed to be in $dcl(a)$.

We say that a theory T is unidimensional if in $\mathcal{M}$ any 2 Lstps are non orthogonal.

We now split the proof into 2 cases;

Case 1. T is trivial and unidimensional ;

Proof. Let p_1 be the type of SU-rank 1as found above. After replacing p_1 by some extension over $acl(\emptyset)$ we may suppose that p_1 is $Lstp$. Now I claim that every $a \in \mathcal{M}$ is interalgebraic with a tuple $a_1 \dots a_n$ realising p_1. We work with the elements $e_1, \dots e_{n-1}$ above. As $SU(e_2/e_1) = 1$, and $Lstp(e_2/e_1)$ is non orthogonal to a non forking extension p_{1,e_1} of p_1 to e_1, we can find a parameter $\bar{c}$, $e_2' \models p_{1,e_1}$, $e_2' \downarrow_{e_1} \bar{c}$ and $e_2'' \models tp(e_2/e_1)$, $e_2'' \downarrow_{e_1} \bar{c}$ such that $e_2'' \in acl_{e_1}(\bar{c}e_2')$. By triviality, we must have in fact that $e_2'' \in acl_{e_1}(e_2')$. By automorphism we may freely suppose thet $e_2 \in acl_{e_1}(e_2')$. Then the pair e_1e_2' realises p_1 and we still have that $SU(a/e_1e_2') = n-2$. Repeating the process, we can replace the $e_1 \dots e_{n-1}$ with $e_1e_2' \dots e_{n-1}'$realising p_1 such that $SU(a/e_1 \dots e_{n-1}') = 1$. Finally using the fact that the minimal type $tp(a/e_1 \dots e_{n-1}')$ is non orthogonal to p_1 gives a tuple $e_1 \dots e_n'$ in p_1 interalgebraic with a as required. As T is trivial, the pregeometry of p_1 is trivial. Replace p_1 by its geometry p_1' in p_1^{eq}, so again p_1' is trivial and a $Lstp$. Clearly, every element a is then interalgebraic with elements $a_1' \dots a_n'$ realising p_1'

Now by results in [10] we can build $\mathcal{C}$ a $k+M$-generic but not l-generic structure for $l >> k+M$ inside p_1'. We consider the set $\mathcal{N} \subset \mathcal{M}$ given by $acl(\mathcal{C}) \cap \mathcal{M}$ and proceed to show that for M to be determined $\mathcal{N}$ is k-generic but not m-generic for $m >> k$. This is almost exactly as in the proof from [10]. So let $\bar{a} \in \mathcal{N}$ be an ascending tuple with $SU(\bar{a}) \leq k$. Then there exists a corresponding tuple $\bar{a}'$ inside p_1' such that $\bar{a}$ and $\bar{a}'$ are interalgebraic. As $\mathcal{C}$ is algebraically closed, we must have that $\bar{a}' \in \mathcal{C}$. Let $\bar{a}_1 \dots \bar{a}_s$ with $\bar{a}_1 = \bar{a}$ be the conjugates of $\bar{a}$ over $\bar{a}'$. As our construction can be carried out uniformly for ascending tuples of bounded rank k, we have a bound for $t(k)$ for s independent of the particular tuple $\bar{a}$ and we need to choose $M > t(k)n$. Let $p \in S^1(\bar{a})$ be a 1-type which we want to realise in $\mathcal{N}$. As $\mathcal{N}$ is algebraically closed, we can assume that $SU(p) \geq 1$. Then if b is a realisation of p in $\mathcal{M}$, we can find

$b_1 = b, b_2, \ldots b_s$ independent over $\bar{a}$ such that $tp(\bar{a}_1 b_1 \bar{a}') = tp(\bar{a}_i b_i \bar{a}')$ for $1 \leq i \leq s$. Replace the b_i by interalgebraic tuples b_i' in p_1 and using $k + t(k)n$ genericity, we can find $b_1'' \ldots b_i'' \ldots b_s'' = \bar{b}'' \in \mathcal{C}$ such that $tp(\bar{b}''/\bar{a}') = tp(\bar{b}'/\bar{a}')$. Finally, we can find $\bar{c} \in \mathcal{N}$ such that $tp(\bar{c}\bar{b}''\bar{a}') = tp(\bar{b}\bar{b}'\bar{a}')$ as $\mathcal{N}$ is algebraically closed and $\bar{b}, \bar{b}'$ are interalgebraic. We must then pick up some c_i such that $tp(c_i\bar{a}) = tp(b\bar{a})$. This shows that $\mathcal{N}$ is k-generic. As $\mathcal{C}$ is algebraically closed and using unidimensionality, for any ascending $\bar{c} \subset \mathcal{C}$ with $SU(\bar{c}) \leq l$ we can find $\bar{d} \subset \mathcal{N}$ interalgebraic with an ascending $\bar{c}' \supset \bar{c}$ in $\mathcal{C}$ and $SU(\bar{d}) \leq nl$. Letting $t'(l)$ be a uniform bound on the conjugates of such $\bar{d}$ over $\bar{c}'$, by the same argument one shows that if $\mathcal{N}$ is $nl + nt'(l)$ generic then $\mathcal{C}$ is $l-$generic. Setting $m = nl + nt'(l)$ gives the following theorem. □

Theorem 28. *If T is a trivial unidimensional 1-based ω-categorical simple theory, then T is not finitely axiomatisable.*

Case 2. T is Non-Trivial, Unidimensional with a Stably Embedded Minimal Type.

Proof. We assume that we can show non-finite axiomatisability for minimal modular types with the amalgamation property. (*)

. Let T be a non-trivial w-categorical 1-based simple theory. By Lemma 25, and unidimensionality ,we can find a non-trivial minimal $Lstp$ p defined over $\emptyset$ inside $\mathcal{M}$. Working inside p^{eq} and using w-categoricity we can even find a modular non trivial partial type p', and we can assume the solution set of p' forms a geometry. We cannot immediately conclude that p' has the amalgamation property, that is there exists non-trivial bounded equivalence relations on the completions of p'. However, we have that each completion p_i' is 6-reducible from a fixed complete type $q_i \subset p^6$. Any bounded equivalence relation on p_i' induces a bounded equivalence relation on q_i, and by w-categoricity there can only be finitely many such distinct equivalence relations. Hence, we can decompose each p_i' into a union of finitely many $Lstps$, and so p' decomposes into a finite union of $Lstps$ as well. By hypothesis we can find a stably embedded minimal type q and easily the modular partial type $q' \supset q$ must be stably embedded as well. Now using unidimensionality and modularity arguments, we may also assume that p' is stably embedded.

By hypothesis, we can construct $\mathcal{C} \subset p'$ which is k-generic but not l-generic for $l >> k$. Let $\mathcal{N} \subset \mathcal{M}^{eq}$ be maximal with the property that $acl^{eq}(N) \cap p' \subset C$, by Zorn's Lemma such a $\mathcal{N}$ exists. We claim that for a constant $n(k)$ to be determined, if C is $n(k)$-generic then $\mathcal{N}$ restricted to $\mathcal{M}$ is k-generic. So let $\bar{a}$ be an ascending tuple in $\mathcal{N}$ of rank $\leq k$ and $q \in S^1(\bar{a})$. As $\mathcal{N}$ is algebraically closed, we can suppose that q is not algebraic. Let b realise q so $SU(b/\bar{a}) \geq 1$. I claim that if all q of rank 1 over ascending tuples of rank $\leq k+n-1$ are realised then all q over ascending tuples of rank $\leq k$ are realised in $\mathcal{N}$. Let q be such realised by b in $\mathcal{M}$, so $SU(b/\bar{a}) \leq n$. Assume $SU(b/\bar{a}) = 2$, then we can find e_1 such that $SU(b/\bar{a}e_1) = 1$ and $SU(e_1/\bar{a}) = 1$, note that e_1 needn't live inside $\mathcal{M}$. By assumption we may assume that $e_1 \in \mathcal{N}$, after automorphism fixing $\bar{a}$. Then $\bar{a}e_1$ has SU-rank $\leq k+1$, hence we can realise $tp(b/\bar{a}e_1)$ inside $\mathcal{N}$. In general if $SU(b/\bar{a}) = n$, repeating this process m times, I find $e_1e_2\ldots e_m$ such that $SU(b/e_m\ldots e_1\bar{a}) = n-m$ and $SU(e_m/e_{m-1}\ldots e_1\bar{a})$=1. As $SU(e_{m-1}\ldots e_1\bar{a} = SU(\bar{a}) + m - 1$, so I can suppose that $e_m \in \mathcal{N}$. Finally, I find $b' \in \mathcal{N}$ realising $tp(b/e_{m-1}\ldots e_1\bar{a})$ as $SU(e_{m-1}\ldots e_1\bar{a}) \leq k+(n-1)$, and in particular b' realises $tp(/\bar{a})$. So I need only prove the result for SU-rank 1 types over ascending tuples of rank $\leq n+k$.

So suppose $SU(q) = 1$ and is defined over an ascending tuple $\bar{a}$ in $\mathcal{N}$. Then as there are only finitely many 2 types over $\bar{a}$, there can only be finitely many distinct finite equivalence relations on q, so after adding a finite parameter $\bar{a}'$ to $\bar{a}$ with $\bar{a}' \subset acl(\bar{a})$, we can decompose q into *Lstps*. As $\mathcal{N}$ is algebraically closed, we can suppose that $\bar{a}'$ subset $\mathcal{N}$. Let m be the bound on the number of distinct 2 types over ascending tuples of rank $\leq n+k$, and s the bound on the number of conjugates of such tuples then if $\mathcal{N}$ realises all minimal *Lstps* over tuples of the form $\bar{a}_1\bar{a}_2$ with $\bar{a}_1$ ascending and of rank $\leq n+k$ and $\bar{a}_2$ having $\leq s$ conjugates over $\bar{a}_1$ then clearly it must realise all minimal types of rank $\leq n+k$ and we have an explicit bound s on the number of conjugates.

So let q be a minimal $Lstp$ over an ascending tuple $\bar{a}$. By unidimensionality the geometry of q is non trivial, and q embeds into a modular partial type q', q' may also be defined over the parameters for q. Now let b realise q. If $b \in \mathcal{N}$ then we are done. Otherwise, by definition of the envelope, we can find $\bar{n} \in \mathcal{N}$, and $c \in p'$ such that $c \in acl(b\bar{n}) \setminus acl(\bar{n})$. As N is algebraically closed, we may suppose that c and b are interalgebraic over $\bar{a}\bar{n}$. By 1-basedness or modularity of q', we can reduce the parameter $\bar{n}$. Namely, let $f = Cb(Lstp(ab/\bar{a}\bar{n}))$, then still we have interalgebraicity of c, b over $f\bar{a}$ and we can assume that $SU(f/\bar{a}) = 1$. (By arguments using non orthogonality of modular types, it is in fact possible to remove f altogether, but in this case b may no longer realise q only q'.) So we need to find a realisation p' inside C having the same type as c over $f\bar{a}$ and we are done. Note that we can bound the dimension of $f\bar{a}$ but we cannot assume that $f\bar{a}$ lies inside p'. As p' is stably embedded, it is sufficient to find a realisation inside C of $tp(c/acl(f\bar{a}) \cap p')$. As $acl(f\bar{a}) \subset \mathcal{C}$ and we can bound the rank of $f\bar{a}$, we have this by sufficient genericity of $\mathcal{C}$. Finally, we need to show that $\mathcal{N}$ is not l-generic for $l >> k$. As the envelope $\mathcal{N}$ covers $\mathcal{C}$, that is $\mathcal{C} = acl^{eq}(\mathcal{N}) \cap p'$, we can use similar arguments to the above.

□

Summarisng we have;

Theorem 29. *Under the assumption* $(*)$*, if* T *is a non-trivial, unidimensional* 1*-based* ω*-categorical simple theory with a stably embedded minimal type, then* T *is not finitely axiomatisable.*

Chapter 8

Differentials in Algebraic Geometry

In this section, we collect some basic results in algebraic geometry needed for sections 10,11,12 and 15. The main references are [19],[15] and [28].

We begin with the "naive" definition of the tangent space to an affine variety $X \subset A^n$. Geometrically, the tangent space $Tan_x(X)$ consists of

$$\{\bar{v} \in A^n : df_x(\bar{v}) = 0 : f \in I(X)\}$$

where the differential df_x at $x \in X$ is given by

$$df_x(\bar{v}) = \Sigma \frac{\partial f}{\partial x_i}(x) v_i$$

An element of the Zariski tangent space $Tan_{x,X}$ can then be reconsidered as a derivation of the germs of functions at that point under a map δ. Namely, if $\bar{v} \in Tan_{x,X}$, then $\delta v.g = dg_x(\bar{v})$, for $g \in \bar{k}[\bar{x}]$, and by definition this descends to a derivation of the quotient ring $R(X)$. Using the chain rule, this extends uniquely to a derivation of the local ring $O_{x,X}$. In fact, letting m_x be the maximal ideal of germs vanishing at x, and m_x^2 the ideal of germs "vanishing to second order" the differential map d dualising δ is in fact an isomorphism from the vector space $\frac{m_x}{m_x^2}$ to $Tan^*_{x,X}$;

$$d : \frac{m_x}{m_x^2} \rightarrow Tan^*_{x,X}$$

$$d(f + m_x^2)(\bar{v}) = df_x(\bar{v})$$

As d is a derivation, one sees easily that this is well defined, and the rest is a straightforward algebraic check.

When we pass to arbitrary varieties X by patching together affine varieties, we keep track of our local rings by introducing a structure streaf O_X on X, and so we can make sense of the tangent space at a point purely algebraically. However, we do not lose track of what is going on geometrically as we have the following result for affine varieties;

If X and Y are irreducible affine varieties with function rings $R(X)$ and $R(Y)$, then a ring map $f^* : R(Y) \rightarrow R(X)$ determines a unique morphism $f : X \rightarrow Y$ in the Zariski toplogy and maps $f^* : O_Y(U) \rightarrow O_X(f^{-1}(U))$, and conversely a morphism in the Zariski topology determines a unique ring map $R(Y) \rightarrow R(X)$.

Now suppose we have a set of affine varieties U_i and glueing morphisms f_{ij} from U_{ij} to U_{ji}, which are compatible in the sense that $f_{ij}f_{jk} = f_{ik}$ as maps in the Zariski topology from $U_{ij} \cap U_{ik}$ to $U_{ki} \cap U_{kj}$. We obtain X as a topological space from the f_i using the quotient and disjoint union topolgies and take as our structure sheaf $O_X(U)$ on an open set U to consist of functions $g_i \in O_{U_i}(U \cap U_i)$ which are compatible in the sense that $f_{ij}^*(g_j) = g_i$. On each U_i we still preserve our original affine sheaf, as given a function g_i in $O_{U_i}(U)$, we obtain a bunch of functions $f_{ji}^* g_i$ which are automatically compatible by the uniqueness result above, so passing to the germ at a point $x \in X$ is equivalent to taking the corresponding local ring in any affine variety U_i containing its representative x_i. The maps

$f_{ij}^{*} : O_{x_j,U_j} \to O_{x_i,U_i}$

identifying these local rings, then induce isomorphisms

$df_{ij} : \frac{m_{x_i}}{m_{x_i^2}}^{*} \to \frac{m_{x_j}}{m_{x_j^2}}^{*},$

and on the level of affine varieties we recover our original "chart definition" of a tangent vector as

$df_{ij}(\delta\bar{v}.g) = \delta\bar{v}(f_{ij}^{*}g) = dg_{x_j}(df_{ij}\bar{v})(chainrule) = \delta(df_{ij}\bar{v}).g$

so the tangent space is just a collection of vectors $\bar{v}_i$ in Tan_{x,U_i} with $df_{ij}\bar{v}_i = \bar{v}_j$.

When we pass to schemes, we lose this geometric picture as our structure sheaves may not be reduced and therefore each tangent space may be "fatter" than the underlying geometric object, however there is still some geometric sense in this notion as we will see.

For an arbitrary scheme, our geometric intuition is recovered primarily through a generalised notion of vector bundle and base change.

Suppose that we have ring maps

$f^{*} : C \to A$

$g^{*} : C \to B$

making A and B into C-modules, then we can form the tensor product $A \otimes_C B$, which is still naturally a $C-$ module, and carries a natural ring structure, giving the commutative diagram,

$$\begin{array}{ccc} C & \xrightarrow{f^*} & A \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle i_1} \\ B & \xrightarrow{i_2} & A \otimes_C B \end{array}$$

which ,passing to schemes, becomes

$$\begin{array}{ccc} Spec(A \otimes_C B) & \xrightarrow{pr_1} & Spec(A) \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle f} \\ Spec(B) & \xrightarrow{g} & Spec(C) \end{array}$$

The tensor product has the universal property that if we are given ring maps

$h^* : A \to D$

$j^* : B \to D$

such that $h^* f^* = j^* g^*$, then we have a unique extension

$(h, j)^* : A \otimes_C B \to D$

$\Sigma a_i \otimes b_i \mapsto \Sigma h(a_i) j(b_i)$

giving the commuting diagram

$$
\begin{array}{ccc}
Spec(D) & & \\
\big\downarrow{\scriptstyle (h,j)} & & \\
Spec(A \otimes_C B) & \xrightarrow{pr_1} & Spec(A) \\
\big\downarrow{\scriptstyle pr_2} & & \big\downarrow{\scriptstyle f} \\
Spec(B) & \xrightarrow{g} & Spec(C)
\end{array}
$$

Using this universal property, it is a relatively straightforward matter to prove that given schemes X and Y over Z, that the fibre product $X \times_Z Y$ exists uniquely as a scheme over Z, to give the following diagram for arbitrary schemes

$$
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{pr_1} & X \\
\big\downarrow{\scriptstyle pr_2} & & \big\downarrow{\scriptstyle f} \\
Y & \xrightarrow{g} & Z
\end{array}
$$

The scheme $X \times_Z Y$ considered as a scheme over Y is usually referred to as the base change of X from Z to Y. The intuition behind the construction is that the fibres of X over Z are pulled back to a set of fibres over Y using the map g, while retaining both the algebraic structure of the ambient scheme as well as branching properties of the map (see Section 10). To see this more clearly, consider the case of a curve C_f in A^2 cut out by irreducible polynomial $f(xy)$ and the non-reduced scheme consisting of C_f with multiplicity n, that is $Spec(\frac{A[xy]}{f^n})$. The projection map of C_f onto A^1 is the canonical map $Spec(\frac{A[xy]}{f^n}) \rightarrow Spec(A[x])$, and topologically the fibre over a point α just consists of the finite set of points $\{y : f(\alpha, y) = 0$, however applying base change gives us the scheme

$$Spec(\bar{k} \otimes_{A[x]} \frac{A[xy]}{f^n}) = Spec(A[x]/(x-\alpha) \otimes_{A[x]} \frac{A[xy]}{f^n}) = Spec(\frac{A[y]}{f^n(\alpha,y)})$$

which not only consists of the right points (α, y) over α, but also counts them with the correct multiplicity $n.mult_f(\alpha, y)$.

Our other main tool in understanding schemes geometrically is the use of vector bundles. Recall that a coherent module F on a scheme X is just an O_X module with an open covering by affines U_i such that $F|U_i \cong \bar{M}$ for $\bar{M}$ a finitely generated $O_X(U_i)$ module. Again, we can carry out the base change construction for modules as follows.

Suppose that $Spec(B)$ and $Spec(C)$ are affine schemes with $g^* : C \rightarrow B$, and M is a coherent module on $Spec(C)$, so M is just an O_C-module. Then we can form the tensor product $B \otimes_C M$ to give an O_B-module, which corresponds to pulling back the module over $Spec(B)$;

$$\begin{array}{ccc} B \otimes_C M & \xrightarrow{g^*} & M \\ \downarrow & & \downarrow \\ Spec(B) & \xrightarrow{g} & Spec(C) \end{array}$$

Again this construction is easy to globalise for arbitrary schemes Y, Z and $g : Y \rightarrow Z$. Formally, we define the pullpack of a coherent module F on Y to be the sheafification of

$$g^*F = O_Z \otimes_{g^{-1}O_Y} g^{-1}F$$

where $g^{-1}F(U) = lim_{\rightarrow, g(U) \subset V} F(V)$. This allows us to define a map locally, on affine sets $Spec(B)$ mapping into affines $Spec(C)$, from $B \otimes_C M$ to $g^*F|Spec(B)$, and it is straightforward to see on the stalks that this is an isomorphism, the stalk just being the base change of F to the corresponding local ring; $O_{x,Z} \otimes_{O_{g(x),Y}} F$, so we can get an isomorphism on any affine in Z.

One important case of this construction is when we take the field $k(y)$ associated to a point $y \in Y$. This gives a map $Spec(k(y)) \to Y$ and the base change of F over $k(y)$ given by $k(y) \otimes_{O_{y,Y}} F$ is then just a vector space over $k(y)$ corresponding to the fibre of the module at y.

A locally free module F is a coherent module with the additional property that,over any affine, M is freely generated by $f_1, \ldots f_n$. At least working over $\bar{k}$, a locally free module of rank n corresponds exactly to a vector bundle of dimension n. In order to see this, use the fact that over a set of affines U_i for Y, we have, by definition, trivialising sections for $F|U_i$. On the intersections U_{ij}, these determine an invertible O_Y module map from $O_Y(U_i)^n$ to itself which is just given by an invertible $n \times n$ matrix M_{ij} with coefficients in $O_Y(U_i)$. On triple overlaps U_{ijk}, we must have that $M_{ij}M_{jk} = M_{ik}$ which is exactly the patching data required to define a vector bundle on Y; in the case of algebraic varieties over $\bar{k}$, the M_{ij} determine the glueing morphisms between $U_i \times_k A^n$ and $U_j \times_k A^n$.

We now want to use the machinery above to develop a notion of tangent spaces for arbitrary schemes. This is done using the sheaf of differentials. For arbitrary rings $S \subset R$, we can form the $R-$module $\Omega_{R/S}$, as the free module generated by the elements $\{dr : r \in R\}$ quotiented by the following relations

$$d(r_1r_2) = r_1dr_2 + r_2dr_1$$

$$d(r_1 + r_2) = dr_1 + dr_2$$

$$ds = 0 : s \in S \ (*)$$

If we are given a morphism $f : X \to Y$ between arbitrary schemes, then on the level of affines $U_i \subset X$ with $f(U_i) \subset U_j \subset Y$, we have a map $f^* : R(U_j) \to R(U_i)$

which allows us to form the local modules $\Omega_{R(U_i)/R(U_j)}$ on X and we want to patch this modules together to get the sheaf of relative differentials $\Omega_{X/Y}$ on X.

In the special case of algebraic varieties Y over $\bar{k}$, this is easy to globalise, namely we can take the function field $\bar{k}(Y)$ of Y and form the module of meromorphic differentials on Y given by $\Omega_{\bar{k}(Y)/\bar{k}}$. At the level of local rings, $\Omega_{O_{y,Y}/\bar{k}}$,consisting of meromorphic differentials without poles at Y, is then just an $O_{y,Y}$ submodule of $\Omega_{\bar{k}(Y)/\bar{k}}$. We can then define

$$\Omega_{Y/\bar{k}}(U) = \cap_{y \in U} \Omega_{O_{y,Y}/\bar{k}}$$

which clearly gives and O_Y module on Y. The dual of this module

$$\Omega^*_{Y/\bar{k}} = Hom(\Omega_{Y/\bar{k}}, O_Y)$$

can then be interpreted as the sheaf of meromorphic vector fields on Y and $\Omega^*_{\bar{k}(Y)/\bar{k}}$ is then $Der_{\bar{k}}(\bar{k}(Y))$, the derivations of $\bar{k}(Y)$ over $\bar{k}$.

Alternatively, we can use the patching interpretation of varieties given above and observe that the f_{ij} allow us to identify $\Omega_{U_i/\bar{k}}(U_{ij})$ and $\Omega_{U_j/\bar{k}}(U_{ji})$ via;

$$f^*_{ij} dg = d(f^*_{ij} g) \ , g \in O_{U_j}(U_{ji})$$

For arbitrary schemes X and Y, there is a remarkable map which allows us to globalise the construction (*). Namely the map

$$\Omega_{R/S} \rightarrow R \otimes_S R$$

$$dr \mapsto r \otimes 1 - 1 \otimes r \ (**)$$

gives an isomorphism between $\Omega_{R/S}$ and J/J^2 where J is the kernel of the canonical map

$f : R \otimes_S R \to R$

$f : \Sigma g_i \otimes h_i \mapsto \Sigma g_i h_i$ (***)

Geometrically, if X is a closed subscheme of Y with ideal sheaf given by J, then J/J^2 has the natural structure of an O_X-module, as locally over an affine $U_i \subset X$, $O_X(U_i) = O_Y(U_i)/J(U_i)$. By analogy with the definition of the tangent space, J/J^2 is the normal sheaf $N_{X/Y}$ of X in Y. The map $(***)$ then identifies J/J^2 locally with the normal bundle on $\Delta(X)$ considered as a subscheme of $X \times_Y X$ via the diagonal morphism;

$\Delta : X \to X \times_Y X$

The pullback of J/J^2 on X is a bundle on X which by $(**)$ is locally isomorphic to $\Omega_{R(U_i)/R(U_j))}$ as required.

We can now see how the sheaf $\Omega_{X/\bar{k}}$ is related to the tangent space of a closed point x for a scheme over $\bar{k}$. This is again given by the map d:

$d : \frac{m_x}{m_x^2} \to \Omega_{X/\bar{k}} \otimes k(x)$

$d(f + m_x^2) = df$

d is well defined as if $f = g_1 g_2$ with $g_1, g_2 \in m_x$, then

$df = d(g_1 g_2) = g_1 dg_2 + g_2 dg_1 \ (in\ \Omega_{X/\bar{k}})$

$= g_1(x) dg_2 + g_2(x) dg_1 \ (in\ \Omega_{X/\bar{k}} \otimes k(x)) = 0$

It is rather straightforward now to see that d is in fact an isomorphism, as there is an obvious inverse to d given by;

$$d^{-1} : \Omega_{X/\bar{k}} \to \frac{m_x}{m_x^2}$$

$$df \mapsto f - f(x)$$

which descends to $\Omega_{X/\bar{k}} \otimes k(x)$ as it kills elements in the submodule $m_x\Omega_{X/\bar{k}}$.

This proves that for all closed points x, $\Omega_{X/\bar{k}} \otimes k(x) \cong \frac{m_x}{m_x^2}$, and recovers our intuition of the cotangent space to an arbitrary scheme X over $\bar{k}$ as a fibre of the sheaf of differentials.

One of the main reasons for using the sheaf of differentials $\Omega_{X/Y}$ to encode properties of tangent spces for arbitrary schemes X over Y , is that there is a strong relationship between the behavior of a coherent module base changed at a point and its behavior in an open neighborhood of that point. This is provided by the geometric form of Nakayma's Lemma;

Lemma 30. *Let F be a coherent sheaf on a scheme X such that $F \otimes_{O_{x,X}} k(x) = 0$, then there exists an open neighborhood U around x such that $F|U = 0$.*

Proof. In order to see this, as F is coherent, we can find $f_1, \dots f_n$ generating F_x. By hypothesis, $\bar{f}_1 = \bar{f}_2 =, \dots, \bar{f}_n = 0$ in $F_x \otimes_{O_{x,X}} k(x)$, and hence $f_1, \dots f_n \in m_xF_x$. This just means that $F_x = m_xF_x$, whcih by the normal form of Nakayama's Lemma implies we can find $g \in O_{x,X} \setminus m_x$ annhilating F_x. As g is a unit, this gives $F_x = 0$ and hence ,taking the intersection of neighborhoods $U_1 \cap \dots U_n$ on which the f_i vanish, gives a neighborhood U such that $F|U = 0$.

□

We can now use this lemma to prove a number of important properties for the sheaf of differentials. The most important of these are the following results;

Theorem 31. *If X is an algebraic variety over $\bar{k}$ of dimension n, then there exists an open dense subset U of X such that X is nonsingular and $\Omega_{X/\bar{k}}|U$ is a locally free module of rank n.*

Proof. On an affine open set, X is isomorphic to a variety X_i in A^n cut ouy by polynomials of the form $f_1, \ldots f_m$. Then the singular locus is just the $\{x \in X_i : rank((\frac{\partial f_i}{\partial x_j})_{1\leq i\leq m, 1\leq j\leq n}) < n$ which is a proper closed set of X_i, so we may assume that X is non singular. At $x \in X$, we can choose a basis $g_1, \ldots g_n$ for $\Omega_{X/\bar{k}} \otimes k(x)$ and use the $g_1, \ldots g_n$ to define a map from $O_X(U)^n$ to $\Omega_{X/\bar{k}}|U$. Taking the quotient sheaf F of $\Omega_{X/\bar{k}}|U$ by $O_X(U)^n$ on U gives that $F_x \otimes k(x) = 0$, hence by Lemma 37, we may assume that $F|U = 0$, this shows at least that the $f_1, \ldots f_n$ generate F on some open U containing x. To prove freeness, let K be the kernel of the map from $O_X(U)^n$ to $\Omega_{X/\bar{k}}|U$, then as X is reduced and $K \neq 0$, we can find a section s of K and a point $y \in U$ such that $s_y \neq 0$. Applying $\otimes k(y)$ to the exact sequence

$$0 \rightarrow K_y \rightarrow O_{X,y}^n \rightarrow \Omega_{X/\bar{k},y} \rightarrow 0$$

and noting that $s_y \otimes k(y) \neq o$, gives $dim_y \Omega_{X/\bar{k},y} \otimes k(y) \geq n+1$, contradicting the hypothesis. So we conclude that $\Omega_{X/\bar{k}}$ is a free module on the non singular locus of X.

□

By previous remarks, we recover the intuition of the cotangent space as a vector bundle on the nonsingular locus. We call elements $g_1, \ldots, g_n$ trivialising $\Omega_{X/\bar{k}}|U$ a set of uniformizing parameters for X over U.

Theorem 32. *If X is a non-singular algebraic variety of dimension n, and Y, Z are* *irreducible closed subsets. Then if W is a component of $Y \cap Z$, we have,*

$$dim(W) \geq dim(Y) + dim(Z) - n$$

or equivalently

$$codim(W) \leq codim(Y) + codim(Z)$$

Proof. We have that $Y \cap Z \cong Y \times Z \cap \Delta(X)$ inside $X \times_{\bar{k}} X$. Let $g_1, \ldots, g_n$ be uniformizers on an open subset U inside X. Then we saw above that $\Omega_{X/\bar{k}}$ is just the pullback of the conormal sheaf J/J^2 for the inclusion of $\Delta(X)$ inside $X \times_{\bar{k}} X$. As $\Omega_{X/\bar{k}}$ is locally free, so is J/J^2, and in particular generated freely on $\Delta(U)$ by the functions $g_1 \otimes 1 - 1 \otimes g_1, \ldots, g_n \otimes 1 - 1 \otimes g_n$. At a point $x \in \Delta(U)$, we have that $g_1 \otimes 1 - 1 \otimes g_1, \ldots, g_n \otimes 1 - 1 \otimes g_n$ generate J_x/J_x^2 and therefore form a basis for the vector space $J_x/m_x J_x$ as clearly any function belonging to J_x lies in m_x the ideal of functions in $O_{X \times X, x}$ vanishing at x. Then, as $J_x/m_x J_x$ is just the base change $J \otimes k(x)$ of the ideal sheaf J at the point x, it follows by Nakayama's lemma that these functions generate J on an open neighborhood U containing x (not freely!). It follows that $Y \times Z \cap \Delta(X)$ is cut out by exactly n equations inside $Y \times Z$, so by standard dimension theory we have the result.

□

This theorem is the basis for the pre-smoothness axiom PS in both the Hrushovski and Zilber formulation of Zariski structures. (see below)

The "piece de resistance" of these uniformity arguments is the following, which generalises the obvious result for subvarieties of A^n;

Theorem 33. *Suppose X is a nonsingular algebraic variety over $\bar{k}$, and $f_1, \ldots f_k \in O_{x,X}$ have the property that the differentials $df_1, \ldots df_k$ are independent in $\Omega_{X/\bar{k}} \otimes k(x)$, then on some open subset U containing x, the ideal J generated by $f_1, \ldots f_k$ defines* *a nonsingular subscheme Y, and we get an exact splitting of locally free modules on $Y \cap U$ given by;*

$$0 \to J/J^2 \to i^*\Omega_{X/\bar{k}} \to \Omega_{Y/\bar{k}} \to 0 \text{ (*)}$$

Conversely, if Y is a nonsingular subscheme of X, then the exact sequence $(*)$ holds on Y, and all all the modules are locally free.

Proof. As the differentials $f_1, \ldots f_k$ are independent in $\Omega_{X/\bar{k}} \otimes k(x)$, we may complete them to a set of uniformisers for $\Omega_{X/\bar{k}}$ on an open subset U containing x. Let Y be the subscheme of U defined by the ideal J generated by $f_1, \ldots f_m$. Then we have the exact sequence on the right given by;

$$J/J^2 \to_d i^*\Omega_{X/\bar{k}} \to \Omega_{Y/\bar{k}} \to 0 \text{ (**)}$$

However, by construction, the map from J/J^2 given by d must be injective on U, as the differentials $df_1, \ldots df_k$ remain independent at each point of U, hence, we have the exact splitting given by $(*)$. It only remains to see that Y is nonsingular. By dimension theory, we must have that $dim(Y) \geq n-k$, on the other hand, the splitting $(*)$ becames an exact sequence of vector spaces when we base change to a point $y \in Y \cap U$, and hence $dim(T_{y,Y}) = dim(\Omega_{Y/\bar{k}} \otimes k(y)) = n-k$, which forces Y to be non-singular as a subscheme of U, and in particular irreducible and reduced. Finally, $\Omega_{Y/\bar{k}}$ is locally free on $Y \cap U$.

For the converse, we again have the exact sequence on the right given by $(**)$. We know the the kernel of the central mapping is a locally free module generated by uniformisers $dg_1, \ldots dg_k$ on a neighborhood U of a fixed $y \in Y$. Now take the subscheme Y' defined by the ideal J' generated by these uniformisers, and repeat the first part of the argument to show that Y' is nonsingular in U. However $Y \subset Y'$ and their dimensions agree, so being both non singular on U. we must have that $Y = Y'$

on U and the sequence $(*)$ is therefore exact on U. Repeating the argument for any point in Y, the sequence is exact everywhere.

□

Note that the above arguments do not show that Y is a complete intersection, as even if we can find $f_1, \dots f_k$ vanishing on Y with independent differentials, there may still be an "excess intersection", though the argument shows that this must be disjoint from Y.

These arguments fail completely for non-reduced schemes over $\bar{k}$. To take the example given earlier of the curve C_f of multiplicity n, we have by the axioms for differentials that $d(f^n) = nf^{n-1}df = 0$, so df is a torsion point for $\Omega_{X/\bar{k}}$ everywhere, in particular $\Omega_{X/\bar{k}}$ is nowhere locally free! (and C_f is singular everywhere!!).

As an example of the use of differentials for arbitrary schemes X over Y, consider an extension $K \subset L$ of number fields. Then O_K and O_L carry the structure of Dedekind domains which may be considered as schemes over $\mathcal{Z}$. The inclusion $O_K \subset O_L$ corresponds to a finite cover $f : Spec(O_L) \rightarrow Spec(O_K)$, so we consider $Spec(O_L)$ as a scheme over $Spec(O_K)$. If $\mathcal{P}$ is a prime in O_K, then by general theory $\mathcal{P}$ splits as a product $\mathcal{P}_1^{m_1}, \dots, \mathcal{P}_n^{m_n}$ of primes in O_L. If $m_i \geq 2$, we say that $\mathcal{P}_i$ is ramified over $\mathcal{P}$.

Theorem 34. *An extension of number fields is ramified at finitely many primes or at all primes.*

We need to compute the sheaf of differentials Ω_{O_L/O_K}. As is easily checked, the localisation $(\Omega_{O_L/O_K})_{\mathcal{P}_i}$ is just the sheaf of differentials $\Omega_{O_{L,\mathcal{P}_i}/O_{L,\mathcal{P}}}$ of the local ring $O_{L,\mathcal{P}_i}$ over $O_{K,\mathcal{P}}$.

Chapter 9

Etale Morphisms

Etale morphisms are central to the development of more advanced notions in algebraic geometry such as deformation theory and form the basis for etale cohomology. In the next section, I will show a strong link exists between such morphisms and the Zariski notion of unramified cover. This not only means that Zariski structures might be interesting for algebraic geometers but also opens up the possibility of developing algebraic geometry in a wider context. Much of the material in this section can be found in [28],[27],[15] and [13]

Definition 14. *A morphism of finite type f between schemes X and Y is said to be etale if for all $x \in X$ there are open affine neighborhoods U of x and V of $f(x)$ with $f(V) \subset U$ such that restricted to these neighborhoods the pull back on functions is given by the inclusion;*

$$f^* : R(V) \to R(V)\frac{[x_1, \cdots, x_n]}{f_1, \cdots, f_n}$$

and $det(\frac{\partial f_i}{\partial x_j})(x) \neq 0$, $(*)$

A straightforward example is the projection pr of $y = x^2$ onto the $x-$ axis which is etale at the origin, as $pr^* : k[x] \to k[x][\frac{y}{y-x^2}]$ and $d/dy(y-x^2)(0) = 1$. The corresponding calculation for the $y-$axis gives $pr^* : k[y] \to k[y][\frac{x}{y-x^2}]$ and $d/dx(y-x^2)(0) = 0$ proving that it is not etale at the origin.

At first sight, it seems that the definition should depend on the choice of affine cover we take to verify the condition $(*)$, however we can soon see that this is not the case. The condition on partial derivatives tells us exactly that the sheaf of relative differentials $\Omega_{X/Y}$ vanishes on X. Locally, on an affine set U mapping into V, we have $\Omega_{X/Y}|U = \Omega_{R(U)/R(V)}$, which is the free module over $R(V)$ generated by $dx_1, \ldots dx_n$ subject to the relations $df_i = \Sigma \frac{\partial f_i}{\partial x_j} dx_j = 0$. By assumption, the function $det(\frac{\partial f_i}{\partial x_j})$ is a unit in the local ring $O_{x,X}$, which implies that the dx_i vanish in $\Omega_{X/Y,x}$, hence on some open set containing x as required. Now, for any choice of affine cover, we must still have $(*)$, as base changing to the point $x \in X$ the fibre of $\Omega_{X/Y}$ can only be zero if the kernel of the matrix $(\frac{\partial f_i}{\partial x_j}(x))$ vanishes.

We first want to see how the notion of an etale morphism simplifies when we assume that X and Y are non-singular algebraic varieties over $\bar{k}$. We have

Theorem 35. *If X and Y are non-singular algebraic varieties over $\bar{k}$ and $f : X \to Y$ is a morphism, then f is etale iff $df : (m_x/m_x^2)^* \to (m_{f(x)}/m_{f(x)}^2)^*$ is an isomorphism everywhere.*

Proof. We have the exact sequence,

$$f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$$

where the map on the left is just given by pulling back differentials onto X. If f is etale, this becomes

$$f^*\Omega_Y \to \Omega_X \to 0$$

and hence, tensoring with $k(x)$

$$f^*\Omega_Y \otimes k(x) \to \Omega_X \otimes k(x)$$

is an isomorphism of vector spaces. Identifying $\Omega_X \otimes k(x)$ with $T^*_{x,X}$ gives that $df : (m_x/m_x^2)^* \to (m_{f(x)}/m^2_{f(x)})^*$ is an isomorphism of tangent spaces, or dually $f^*(m_{f(x)})$ generates m_x. In fact this holds in general for arbitrary schemes as we clearly didn't use non singularity

Conversely, assume that df is an isomorphism on tangent spaces, then if we take local uniformisers $f_1, \ldots, f_n$ at $f(x) \in Y$, the f^*df_i form a basis for $\Omega_X(x)$. By Nakayama's lemma, they generate Ω_X on an open set U containing x. We have that $f^*\Omega_Y$ is locally free, just by the non singularity of Y, and so $f^*\Omega_Y$ and Ω_X are isomorphic on some U containing x. Repeating for all x and using the exact sequence gives $\Omega_{X/Y} = 0$. We still need to find a local presentaion of the form required in $(*)$, which is acheived by the following trick; namely we may suppose that X and Y are affine and choose an embedding of X into A^n for some n, then X may be considered as a smooth subvariety of the smooth variety $Y \times A^n$. with the originsl f corresponding to the projection $(\pi|X)$. As X is smooth, we can use Theorem 40 to present X locally as a subvariety of the form $Spec(R(Y)[x_1, \ldots x_n]/(f_1, f_2 \ldots f_n))$ with the $f_i \in R(Y)[x_1, \ldots x_n]$ Then, repeating the argument above gives that the condition $(*)$ has to be satisfied.

□

This gives us a convenient test for etaleness given an arbitrary morphism of finite type between X and Y. If we take local uniformisers $g_1, \ldots g_n$ at $x \in X$, the dg_i generate Ω_X freely on an open U' of x. If we pull back a set of uniformisers $f^*f_1, \ldots, f^*f_n$ on Y to X, we can locally define the Jacobian Jac_g^f to be;

$$det(\frac{\partial f^* f_i}{\partial g_j})$$

which means write the 1-forms $f^*df_i = \Sigma_j a_{ij} dg_j$ and take $det(a_{ij})$

If f is etale in a neighborhood of x, the f^*df_i also generate Ω_X freely on an open U'' of x. Taking the intersection $U'' = U \cap U'$, gives us that the Jacobian $Jac_{\bar{g}}^{\bar{f}}|U'' \neq 0$. Conversely, if $Jac_{\bar{g}}^{\bar{f}}(x) \neq 0$, then it is non zero on an open neighborhood U'' of x and by the above theorem we have that f is etale on this neighborhood. We conclude that etaleness is an open condition on X and we can describe the ramification locus of f as the closed set defined locally by the vanishing of $Jac_{\bar{g}}^{\bar{f}}$; if this is a non empty proper subset, then by dimension theory we have that the ramification locus has codimension 1 in X.

We should check that this does not depend on our choice of uniformizers, in other words give a coordinate free description of the ramification locus. Let K_X and K_Y be the canonical line bundles on X and Y. Then f induces a natural map;

$df : f^*K_Y \rightarrow K_X$

$df_1 \wedge \ldots \wedge df_n \mapsto df^*f_1 \wedge \ldots \wedge df^*f_n$

Using the rules for alternating products, we have;

$df^*f_1 \wedge \ldots \wedge df^*f_n = (\Sigma_j a_{1j})dg_j \wedge \ldots \wedge (\Sigma_j a_{nj})dg_n = det(a_{ij})dg_1 \wedge \ldots \wedge dg_n$

So the ramification locus is given exactly by the degeneracy of the map df, that is $x \in X : rank_x(df) < n$. As df determines a section of the bundle $Hom(f^*K_Y, K_X) \cong K_X \otimes (f^*K_Y)^*$, we can just write this as $ch_1(K_X \otimes (f^*K_Y)^*) = ch_1(K_X) - ch_1(f^*K_Y)$, by the rules for Chern classes.

The above formulation is especially useful when we consider the more general question of how to describle the "higher order" ramification of a morphism f between nonsingular varieties X and Y of dimension n. Namely, we want to describe the loci

$\Sigma_k = x \in X : rank_x df \leq k$ for $0 \leq k \leq n-1$ where this time df is considered as a map between the vector bundles $f^*\Omega_Y$ and Ω_X on X, henceforth denoted by E and F. For $k = n-1$, we get the usual ramification locus, and we have a sequence $\Sigma_0 \subseteq \Sigma_1 \ldots \subseteq \Sigma_k \subseteq \Sigma_{n-1}$ of higher ramification. Locally df gives us a map from X into $GL(n)$ and the degeneracy locus Σ_k is mapped into $M_k \subset GL(n)$ given by the set of matrices of rank $\leq k$. Hence ,by usual dimension theory, the codimension of Σ_k (if non empty) is at most $(n-k)^2$. In case this is the codimension, the ramification loci Σ_k are locally complete intersections and we can compute them using the Thom-Porteous formula. The proof is so instructive that it is worth including. We first linearise the problem on X by considering the Grassmannian associated to E given by,

$$Grass(n-k, E) = \{(x, A) : x \in X, A_{n-k} \subset E_x\}$$

where A_{n-k} is an $n-k$ dimensional subspace of the vector space E_x. We pull back the bundle E on X, via the natural projection map $\pi : Grass(n-k, E) \to X$, to get a bundle π^*E on $Grass(n-k, E)$. Then we have the following canonical exact sequence of vector bundles on $Grass(n-k, E)$ given by,

$$0 \to S_{n-k} \to \pi^*E \to Q_k \to 0$$

where S_{n-k} and Q_k are the canonical bundles of dimension $n-k$ and k ,associating the spaces A and $\frac{E_x}{A}$ respectively to a point (x, A) in $Grass(n-k, E)$. This sequence allows us to compute the Chow ring of $G = Grass(n-k, E)$ as follows;

Namely if $s_i = ch_i(Q_k)$ for $1 \leq i \leq k$, then s_i is determined by a generic map of the trivial bundle of rank $k-i+1$ to Q_k or equivalently by the zero locus of $k-i+1$ independent sections. Given such sections $\sigma_1, \ldots \sigma_{k-i+1}$ of E on X, we can extend them canonically to sections of Q_k on G by setting $\sigma_j(x, A) = \frac{\sigma_j(x)}{A)}$. Then the common zero locus will just be $(x, A) : A \supset span(\sigma_1(x), \ldots, \sigma_{k-i+1}(x))$. Restricting to a generic fibre $\pi^{-1}(x)$ of G over X, this is the class s_i in $Grass(n,k)$ represented by the set of k-planes containing $k-i+1$ fixed independent vectors. As is easily seen, these classes generate $Grass(n,k)$ and so we have that $A^*(G)$ is generated over $A^*(X)$ by the chern classes $s_1, \ldots s_k$ of Q_k considered as a bundle on G. The exact sequence above gives us the one relation $ch(\pi^*E) = ch(Q_k)ch(S_{n-k})$ on G which gives us that

$$A^*(G) = A^*(X)\frac{[s_1, \ldots s_k]}{[\frac{ch(E)}{1+s_1+\ldots s_k}]_{l>n-k} = 0}$$

Now we have the sequence

$$S_{n-k} \to \pi^*E \to \pi^*F$$

on G lifting the map df on X. The degeneracy Σ_k on X will then be given by the proper pushforward of the degeneracy on G

$$\Sigma_k = \{(x,A) : df_x|A = 0\}$$

Denoting the composite map for the sequence above by ϕ, this will be $\{(x,A) : (rank\phi)_{x,A} = 0$, which is just the zero locus of ϕ considered as a section of $Hom(S, \pi^*F) \cong S^* \otimes \pi^*F$. Our assumptions on the codimension of Σ_k allow us to compute this as the top Chern class $ch_{(n-k)n}(S^* \otimes \pi^*F)$ and fortunately there is a formula for this given by

$$det(c_{ij})_{1 \leq i,j \leq n-k}, c_{ij} = c_{n+(j-i)}, i \leq j$$

$$= c_{n+(i-j)}, j \leq i$$

where $1 + c_1 + \ldots + c_{n+(n-k)} = \frac{ch(\pi^*F)}{ch(S)}$. Now by the first exact sequence, we have that $ch(S) = \frac{ch(Q)}{ch(\pi^*E)}$, so $1 + c_1 + \ldots + c_{n+(n-k)} = ch(Q)\pi^*(\frac{ch(F)}{ch(E)})$. Now we just have to push this forward to a cycle on X, which we do by writing the above determinant in the form;

$$det(e_{ij})_{1\leq i,j\leq n-k}det(s_{kl})_{1\leq k,l\leq n-k}$$

$$, \qquad e_{ij} = e_{n-(n-k)+(j-i)}, i \leq j$$

$$= e_{n-(n-k)+(i-j)}, j \leq i$$

$$s_{kl} = s_{n-k+(l-k)}, k \leq l$$

$$= s_{n-k+(k-l)}, l \leq k$$

where the s_i are as above and $1 + e_1 + e_2 + \ldots = \frac{ch(F)}{ch(E)}$. Now the cycles s_{kl} push forward to trivial cycles on X, and we get the formula for the degeneracy locus ;

$$\Sigma_k = det(e_{ij})_{1\leq i,j\leq n-k}$$

$$e_{ij} = e_{n-k+(j-i)}, i \leq j$$

$$= e_{n-k+(i-j)}, j \leq i$$

Note that the codimension of each element in the determinantal formula is $(n-k)^2$ so this makes sense! Unfortunately, things get considerably more difficult if the degeneracy locus fails to be a locally complete intersection. The general idea is to blow up the variety X along the excess intersection, prove a Porteous formula for the blowup, then push the resulting formula back down to X

In Section 12, we will find that Zariski structure techniques lead naturally to the notion of local isomorphisms between definable sets, which are used essentially in defining tangency. The rest of this section will be devoted to finding an algebraic interpretation in Theorem 43 and 48. We will also use the results of Theorems 45 and 47 in Section 12.

The result of Theorem 42 leads to the following important result, which is an analytic version of the inverse function theorem.

Theorem 36. *If $f : X \to Y$ is an etale cover of non-singular algebraic varieties over $\mathcal{C}$, then f is a covering map of topological spaces in the complex topology.*

Proof. Choose $y \in Y$ and let $f^{-1}(y) = \{x_1, \ldots x_n\}$. Choose local uniformisers $f_1, \ldots f_n$ for $y \in Y$ and $g_1^i, \ldots g_n^i$ for $x_i \in X$. These define etale maps from some open neighborhoods U_y of Y and U_{x_i} of x_i to A^n, and isomorphisms in the complex topology from neighborhoods $V_y \subset U_y$ and $V_{x_i} \subset U_{x_i}$ to open balls $B^n \subset \mathcal{C}^n$. Taking the $\bar{g}^i$ and $\bar{f}$ as local coordinates around x_i and y and $\theta = \bar{g}^{-1} f \bar{f} : B^n \to B^n$, the functions a_{ij} correspond to $\theta^* df_i.\partial g_j = df_i.\theta_*.\partial g_j = \frac{\partial \theta_i}{\partial \theta_j}$, which is just the usual Jacobian of θ. Hence, by the inverse function theorem, Θ is a local isomorphism, and hence so is f. Now, taking our neighborhoods sufficiently small gives that $f^{-1}(U_y) = V_1 \cup \ldots V_i \cup \ldots V_n$ for disjoint V_i and it follows that $f^{-1}(U_y) \cong f^{-1}(y) \times U_y$, that is X is a covering space of Y.

□

This theorem can in fact be shown even if X is just assumed to be of finite type over $\bar{k}$, which gives the extraordinary result that the category of covering spaces and covering maps over X an analytic space (essentialy any scheme of finite type over $\mathcal{C}$ considered in the complex topology)) is equivalent to the set of etale covers of X. For an arbitrary etale morphism, repeating the above argument for $Im(f)$, a dense open subset of Y, gives that X is "generically" a covering space of Y.

The strength of the above result leads naturally to the question of what should be the algebraic analogue of the inverse function theorem for arbitrary varieties over algebraically closed fields $\bar{k}$ in arbitrary characteristic. If we just work in the Zariski topology, the result clearly fails, for example the morphism $f : A^1 \setminus 0 \to A^1 \setminus 0$ given by $z \mapsto z^n$ is an etale cover of $A^1 \setminus 0$ in the Zariski topology, but doesn't split as a product locally around any point $\lambda \in \bar{k} \setminus 0$.

The problem is resolved by taking a finer etale toplogy on Y in which the local rings $O^{\wedge}_{y,Y}$ resemble a completion of the original local rings rings $O_{y,Y}$. Namely, we consider a category Y_{et} whose objects are etale morphisms $U \to Y$ and whose arrows are Y-morphisms from $U \to V$. This category has the following 2 desirable properties. First given $y \in Y$, the set of objects of the form $(U, x) \to (Y, y)$ form a directed system, namely $(U, x) \subset (U', x')$ if there exists a morphism $U \to U'$ taking x to x'. Secondly, we can take "intersections" of open sets U_i and U_j by considering $U_{ij} = U_i \times_Y U_j$; the projection maps are easily show to be etale and the composition of etale maps is etale, so $U_{ij} \to Y$ still lies in Y_{et} (this allows us to formulate Cech cohomology exactly as for arbitrary schemes in the Zariski topology). Note that if Y is an irreducible variety over $\bar{k}$, then all etale morphisms into Y must come from reduced schemes of finite type over $\bar{k}$, though they may well fail to be irreducible considered as algebraic varieties, so we do not need to consider arbitrary schemes. Now we can define the local ring of Y in the etale toplogy to be;

$$O^{\wedge}_{y,Y} = lim_{\to, y \in U} O_U(U)$$

As any open set U of Y clearly induces an etale morphism $U \to_i Y$ of inclusion, we have that $O_{y,Y} \subset O^{\wedge}_{y,Y}$. We want to prove that $O^{\wedge}_{y,Y}$ is a Henselian ring and in fact the smallest Henselian ring containing $O_{y,Y}$. We need the following lemma about Henselian rings;

Lemma 37. *Let R be a local ring with residue field k. Suppose that R satisfies the following condition;*

If $f_1, \ldots f_n \in R[x_1, \ldots x_n]$ and $\bar{f}_1 \ldots \bar{f}_n$ have a common root $\bar{a}$ in k^n, for which

$Jac(\bar{f})(\bar{a}) = (\frac{\partial \bar{f}_i}{\partial x_j})_{ij}(\bar{a}) \neq 0$, then $\bar{a}$ lifts to a common root in R^n (*).

Then R is Henselian.

Proof. One checks that R is Henselian directly, namely suppose $f \in R[x]$ is a monic polynomial such that $\bar{f} = \bar{g}\bar{h}$ splits as a product of monic coprime polynomials of degree r and s in k. Writing f as a product $(x^r + y_1 x^{r-1} + \ldots + y_r)(x^s + y_{r+1} x^{s-1} + \ldots y_{r+s})$ sets up a system of $r + s$ equations in $R[y_1, \ldots y_r y_{r+1} \ldots y_{r+s}]$ of the form;

$y_1 + y_{r+1} = r_1$ (1)

$y_2 + y_1 y_{r+1} + y_{r+2} = r_2$ (2)

...

$y_r y_{r+s} = r_{r+s}$ (r+s)

where the r_i are the coefficients of f considered as a polynomial of degree $r + s$. The factorisation of $\bar{f}$ in k gives a solution $\bar{a}$ to this system when the r_i are reduced to k, and the Jacobian $(\frac{\partial g_i}{\partial y_j})_{1 \leq i,j \leq r+s}$ has non zero determinant at $\bar{a}$ as $det(Jac)$ is given by the product of the resultants of $(x^r + y_1 x^{r-1} + \ldots + y_r)$ and $(x^s + y_{r+1} x^{s-1} + \ldots y_{r+s})$, which is non zero as $\bar{g}$ and $\bar{h}$ are coprime. So the result is a consequence of $(*)$.

□

The condition $(*)$ is usually known as Hensel's Lemma and is true for any complete local ring, so all complete local rings are Henselian.

It remains to show that $O^{\wedge}_{y,Y}$ satisfies $(*)$.

Proof. Given $f_1, \ldots f_n$ satisfying the condition of $(*)$, we can assume the coefficients of the f_i belong to $O_{U_i}(U_i)$ for covers $U_i \to Y$; taking the intersection $U_{1 \ldots i \ldots n}$ we may even assume the coefficients define functions on a single etale cover U of Y. By the remarks above we can consider U as an algebraic variety over $\bar{k}$, and even an affine algebraic variety after taking the corresponding inclusion. We then consider the variety $V \subset U \times A^n$ defined by $Spec(\frac{R(U)[x_1,\dots,x_n]}{f_1,\dots f_n})$. Letting $u \in U$ denote the point in U lying over $y \in Y$, the residue of the coefficients of the f_i at u corresponds to the residue in the local ring R, which tells us exactly that the point $(u, \bar{a})$ lies in V. By the Jacobian condition, we have that the projection $\pi : V \to U$ is etale at the point $(u, \bar{a})$, and hence on some open neighborhood of $(u, \bar{a})$, using Nakayama's Lemma applied to $\Omega_{V/U}$. Therefore, replacing V by the open subset $U' \subset V$ gives an etale cover of U and therefore of Y, lying over y. Now clearly the coordinate functions $x_1, \dots x_n$ restricted to U' lie in $O^{\wedge}_{y,Y}$ and lift the root $\bar{a}$ to a root in $O^{\wedge}_{y,Y}$

□

We define the Henselization of a local ring R to be the smallest Henselian ring $R' \supset R$, with $R' \subset Frac(R)^{alg}$. By the above, we have that

Theorem 38. *Given an algebraic variety Y, $O^{\wedge}_{y,Y}$ is the Henselization of $O_{y,Y}$*

The following fact is due to Artin,

Fact 39. *The Henselization of*

$\bar{k}[x_1, \dots x_n]_{(x_1,\dots x_n)}$

is

$\bar{k}[[x_1, \dots x_n]] \cap \bar{k}(x_1, \dots x_n)^{alg}$

This gives $O^{\wedge}_{\bar{0},A^n}$ for affine space A^n by Theorem 45.

Now given an etale map $f : X \to Y$ between algebraic varieties, f induces an isomorphism between $O^{\wedge}_{f(x),Y}$ and $O^{\wedge}_{x,X}$ for all $x \in X$.

Proof. To see this, suppose $g \in O^{\wedge}_{f(x),Y}$, then g belongs to $O_Z(Z)$ for some Z with Z etale over Y. Then the product $Z \times_Y X$ is etale over Z and X, so pulling back g to $Z \times_Y X$ give an element of $O^{\wedge}_{x,X}$, clearly the map is injective as all etale maps are dominant morphisms and surjectivity from the fact that an etale cover of X is then an etale cover of Y. □

The converse is also true, for arbitrary algebraic varieties X and Y, if f induces an isomorphism between the Henselizations $O^{\wedge}_{f(x),Y}$ and $O^{\wedge}_{x,X}$, then f is etale, see [28]. This gives,

Theorem 40. *A morphism $f : X \rightarrow Y$ is etale iff $f^* : O^{\wedge}_{f(x),Y} \rightarrow O^{\wedge}_{x,X}$ is an isomorphism for every $x \in X$.*

We now have an extraordinary generalisation of the analytic version of the inverse function theoren;

Theorem 41. *Let $f : X \rightarrow Y$ be an etale cover of algebraic varieties over $\bar{k}$, then f is a covering map in the etale topology.*

Proof. Choose $y \in Y$, and consider the fibre product $X \times_{\bar{k}} O^{\wedge}_{y,Y}$ which is etale over $O^{\wedge}_{y,Y}$. Then we may write this locally in the form $Spec(\frac{O^{\wedge}_{y,Y}[x_1,\ldots x_n]}{f_1,\ldots f_n})$ with $det(\frac{\partial f_i}{\partial x_j}) \neq 0$ at each closed point in the fibre over y. This means exactly that the f_i have common roots in the residue field k^n corresponding to the points over y and satisfying the condition of the Lemma. Hence they lift to roots $g_i \in O^{\wedge}_{y,Y}$ and we now have a map $O^{\wedge}_{y,Y}[\frac{x_1,\ldots x_n]}{f_1,\ldots f_n} \rightarrow O^{\wedge}_{y,Y}$ given by sending x_i to g_i Such a map is a section of the original morphism passing through the given point in the fibre. Using these sections and the fact that sections are uniquely determined, gives a splitting of $X \times_{\bar{k}} O^{\wedge}_{y,Y}$ as a product $O^{\wedge}_{y,Y} \times f^{-1}(y)$ as required.

□

Chapter 10

Zariski Structure Axioms and Examples

For T stable, we let p denote a minimal type inside a saturated model $\mathcal{M}$, see section 1 for definitions. In the next section, where we consider the simple case, we will demand that p is a Lascar strong type with $SU(p) = 1$. For this section, we assume stability.

Fact 42. *If T is stable, , the trace of T on p^k is definable with parameters from p.*

Proof. Suppose $\phi(\bar{x}, \bar{b}) \cap p^k$ defines a subset of p^k. Then

$$\bar{a} \models \phi(\bar{x}, \bar{b}) \cap p^k \Leftrightarrow \phi(\bar{a}, \bar{y}) \in tp(\bar{b}/p) \Leftrightarrow d\phi(\bar{a})$$

where $d\phi$, the defining schema for $stp(\bar{b}/p)$, is over $c \in acl^{eq}(p)$. By a straighforward dimension argument, we can find a sequence $a_1, \ldots a_k$ in p with $c \in dcl(a_1, \ldots a_k)$ So, by automorphism, we see that $d\phi$ is defined over $a_1, \ldots a_k$.

□

In the simple case, this may not occur; if so, we say that p is stably embedded.

We now consider the universe of p with induced structure inherited from T. Given some subcollection $\{C\}$ of the induced definable sets, which we will call closed sets, we say that p is Zariski with respect to $\{C\}$ if the following axioms, which may be found in [?], hold;

1. (L) Basic relations are closed:

Conjunction and disjunction of closed sets are closed.
Graph of $=$ is closed.
Singletons are closed and p is closed.
Cartesian products of closed sets are closed.

2. (P) The projections $pr : p^{n+1} \to p^n$ are proper and continous maps.

That is if $C \subset p^{n+1}$ and $C' \subset p^n$ are closed then $\exists x C \subset p^n$, and $p^{-1}(C') \subset p^{n+1}$ are also closed.

3. (DCC) The topology given by the closed sets on p^n is Noetherian and p is irreducible.

The condition (DCC) implies that every closed set C can be written essentially uniquely as a union of irreducibles;

$C = C_1 \cup \ldots \cup C_n$
It is also straightforward to verify by induction that p^n is irreducible for $n \geq 1$.

4. (DIM) We define a dimension notion on closed sets as follows;

$dim(C)$ is the maximum value of m for which there exists a chain $\{C_i\}$ of irreducible closed sets such that $C_0 \subset \ldots \subset C_m$. We then require that $dim(p^n) \leq n$.

The following properties are then easy to verify;

$dim(C_1 \cup C_2) = max\ dim(C_i)$ for $C_1,\ C_2$ closed.

$dim(pt) = 0$

$dim(C_1) < dim(C_2)$ if C_2 is irreducible and $C_1 \subsetneq C_2$

$dim(p^n) \geq n$

5. (PS) For all closed irreducible closed sets $X_1, X_2 \subset p^n$, $dim(S_1 \cap S_2^{comp}) \geq dim(S_1) + dim(S_2) - dim(p^n)$

Examples.

1. A smooth projective algebraic curve C definable in ACF, the theory of algebraically closed fields.

The product C^n has the natural structure of an algebraic variety with the closed sets given by the Zariski topology. We will verify the axioms;

(L) follows by properties of the Zariski topology and the fact that on an affine cover of C^n, the diagonals are cut out by linear polynomial equations.

(P). We say that an algebraic variety X is complete if for all varieties Y, the projection morphism

$$p_2 : X \times Y \rightarrow Y$$

is closed. Clearly, for such a variety X, this implies that that the projection maps $pr : X^{n+1} \rightarrow X^n$ are closed, taking Y to be X^n in the above definition. If $Z \subset X$ is a closed subvariety of X and X is complete, then so is Z, as is easily checked from the definition. Our example C is a closed subvariety of $P^n(\bar{k})$ for some n, therefore to verify (P), we need to know that $P^n(\bar{k})$ is complete. This is a classical theorem due to Grothendieck, see [19] for a proof.

(DCC) Suppose that $\{X_i : i < \omega\}$ is an infinite descending chain of closed subsets of C^n. As C^n may be covered by finitely many affine open subvarieties $Y_1 \ldots Y_n$, this implies that $\{Y_j \cap X_i : i < \omega\}$ defines a descending chain of closed subvarieties of each Y_j. Then by the Nullstellensatz and the fact that the coordinate ring $\bar{k}[x_1 \ldots x_n]$ is Noetherian, each such chain stabilises inside Y_j. Then clearly the chain stabilises inside C^n.

(DIM) The notion of dimension (dim) as given above corresponds to the notion of dimension (dim') in algebraic geometry, which for an irreducible subvariety X of C^n is defined as $tr.deg(\bar{k}(X)/\bar{k})$ for $\bar{k}(X)$ the function field of X. To see this, suppose that $dim(X) \geq n+1$, and X is irreducible, then by definition one can find an irreducible closed subvariety $X' \subset X$ with $dim(X') \geq n$ and so inductively $dim'(X') \geq n$. Taking an affine open of X intersecting X', we can assume that X and X' are affine as the function field is unchanged. Then by straightforward commutative algebra, it follows that $dim'(X') < dim'(X)$ and so $dim'(X) \geq n+1$. Conversely, if $dim'(X) \geq n+1$, then again assuming X is irreducible affine, if we take $f \in R(X)$ to be a non-unit, then each irreducible component of $V(f) \subset X$ has codimension 1 in X, see [19], . Therefore, $dim'(V(f)) \geq n$ and inductively $dim(V(f)) \geq n$. As each component of $V(f)$ is a proper closed subset of X, $dim(X) \geq n+1$. Now clearly we have that dim corresponds to dim' and so in particular we know that $dim(C^n) = n$.

(PS) One checks that the for $(x_1 \ldots x_n) \in C^n$, the maximal ideal $m_{\bar{x}} \subset \mathcal{O}_{\bar{x}}$ is isomorphic to $\Sigma_{i=1}^{n} \mathcal{O}_{x_1 \ldots \hat{x}_i \ldots x_n} \otimes m_{x_i}$. Then by a simple calculation $Tan_{\bar{x}}(C^n) \cong \Sigma_{i=1}^{n} Tan_{x_i}(C)$, so C^n is smooth. Now the result follows from Theorem 39 in Section 9.

In this case, dim corresponds to MR calculated in ACF. To see this note that it is sufficient to assume that $X \subset C^n$ is irreducible affine and is definable in the home sort $\bar{k}^n$. Choose $\bar{a}$ in X with $\bar{a}$ generic over $\bar{k}$. Then $MR(X) = MR(\bar{a}/\bar{k}) = t.deg(\bar{k}(\bar{a})/\bar{k})$.

However, there is a map from $R(X)$ to $\bar{k}(\bar{a})$ given by sending f to $f(\bar{a})$. The map must be injective as if $f(\bar{a}) = 0$, then as $\bar{a}$ is generic over $\bar{k}$, we must have $f|X = 0$ and therefore $f = 0$, Then clearly the map extends to an isomorphism between $Frac(X)$ and $\bar{k}(\bar{a})/\bar{k}$. In fact, as we show below, this correspondence between dim and the model theoretic rank will follow in general from the axioms.

2. Strongly minimal sets or minimal types C in DCF, the theory of differentially closed fields.

This time we equip C^n with the topology generated by the closed sets given by positive boolean combinations of differential formulae.

Again (L) is straightforward to verify.

(P) in general fails. By quantifier elimination for DCF, if $X \subset C^{n+1}$ is closed then $\exists x X \equiv \cup_{i=1}^{n} F_i/E_i$ with F_i and E_i closed. In the case when C is a strongly minimal, this is enough to satisfy the axioms for Zariski structures given in [16]that the projection of a closed set should be constructible.

(DCC) If $\{X_i : i < \omega\}$ is a descending chain of closed sets inside K^n then if we let $I^\delta(X_i) = \{f \in K\{x_1, \ldots x_n\} : f|X_i = 0\}$, $I^\delta(X_i)$ is an ascending chain of differential ideals in the differential coordinate ring of K^n. By Ritt's basis theorem, such a chain terminates, hence so does X_i.

(DIM) This time it is considerably more difficult to verify that $dim(C^n) \leq n$. The simplest way is to show directly that for any closed $X \subset C^n$, $MR(X) = dim(X)$, and use the fact that MR is additive for strongly minimal sets. One direction is obvious, suppose X is irreducible and $MR(X) \geq m + 1$, then we can write X as $\cup_{i<\omega} X_i$ with X_i disjoint definable setsand $MR(X_i) \geq m$. By quantifier elimination we can suppose that each $X_i = F_i/E_i$ with F_i and E_i closed irreducible. If some $F_i \subsetneq X$, then as inductively $dim(F_i) \geq m$, we have that $dim(X) \geq m+1$ and we are done. Otherwise $F_i = X$ for each $i < \omega$, which forces $F_i = E_i \cup E_j$ for any $j \neq i$ contradicting irreducibility. For the other direction, suppose X is irreducible and $dim(X) \geq m+1$, then we can find a proper irreducible closed $X' \subset X$ such that $dim(X') \geq m$. It will be sufficient to show that $MR(X') < MR(X)$. This is done by showing the intermediate step that for $X \subset C^n$ closed, $Krull(X) = eMR(X)(*)$ where $Krull(X)$ is defined to be the $t.deg(Frac(X)/K)$ and $Frac(X)$ is the differential fuction field of X. By a straightforward algebra calculation, $Krull(X') < Krull(X)$ from which the result follows. The proof of $(*)$ can be found in [3], and relies crucially on the following characterisation of forking in DCF, that for tuples $\bar{a}$ and $\bar{b}$ and $k \subset K$, such that $t.degk < \bar{a} >$ is finite

$\bar{a} \downarrow_k \bar{b}$ iff $t.degk < \bar{a} > /k = t.degk < \bar{a}\bar{b} > /k < \bar{b} >$ $(**)$

(PS) The above fact $(*)$ give a direct method of relating dimension theory for strongly minimal sets in DCF and dimension theory for algebraic varieties. If C^* is the corresponding algebraic variety to C, see [31] for the geometrical interpretation, then one can find an open smooth subvariety $U^* \subset C^*$. Then U^* will correspond to a cofinite open subset U of C for which (PS) holds.

3. Strongly minimal sets C in $LDCF$, the theory of Lie differentially closed fields. By analogy with DCF, the verification of the axioms will be almost identical. The main technical obstacle lies in proving $(**)$.

4. Compact riemann surfaces, these may be defined inside a many sorted stable structure introduced by Pillay/Moosa. The Riemann existence theorem essentially reduces the structure of these objects to Example 1 above.

5. $\cap p^n C$ in $T_{SCF,p}$,the theory of a seperably closed field K of characteristic p and finite degree of imperfection, where C is a smooth projective curve defined over F_p,

the finite field with p elements. Here, we have no notion of MR, but the corresponding type is minimal in the sense of stable structures. In [17], Hrushowski uses a more sophisticated version of (P), which allows him to use Zariski structure techniques for arbitrary minimal sets.

In all the above cases, the dimension corresponds to the model theoretic rank, this is a general phenomenon.

Definition 15. *For $X \subset p^n$, let*

$$U(X) = max\{U(\bar{a}) : \bar{a} \in X\}$$

For the simple case, we can similarily define the SU-rank of a definable subset. Moreover, observe that if the ambient theory T is ω-stable, then by the fact that Zariski structures are unidimensional, we must have that U-rank corresponds to Morley rank MR, so the rank U is very suggestive. We now aim to prove the following lemma

Lemma 43. *If $X \subseteq p^n$ is closed then $U(X) = dim(X)$.*

Proof. We may clearly assume that X is irreducible, and proceed by induction on $dim(X) = k$. Now we need the following lemma which generalises Noether normalisation for affine algebraic varieties, and can be found in [18].

Fact 44. *X is irreducible with $dim(X) = k$ iff there exists a generically finite map pr of X onto p^k for some k.*

(This fact uses 2.3-2.5 of [18])

Now given the above, let $E \subset p^k$ be the proper closed set admitting infinite fibres. Then $dim(E) < dim(p^k) = k$ and by induction we have that $U(E) < k$. By property of $U-$rank we have that $U(p^k \backslash E) = k$ and $pr : X \cap pr^{-1}(p^k \backslash E) \to p^k \backslash E$ is finite to one. However, U-rank is preseved by finite maps, so we have that $U(X \cap pr^{-1}(p^k \setminus E) = k$. Now clearly the complement of $X \cap pr^{-1}(p^k \setminus E)$ has lower dimension than X as X is irreducible so has $U - rank < k$. Therefore, we have that $U(X) = k$.

□

We now deduce the following, which is given as an axiom for Zilber's formulation of Zariski structures in [29].

(DF) If $X \subseteq p^{n+m}$ is closed. Then,

$$F(X, k) = \{\bar{a} \in p^n : dim(X \cap pr^{-1}(\bar{a})) > k\}$$

is closed.

Proof. By the above lemma, it is sufficient to show that

$$\{\bar{a} : U(X(\bar{a}) \geq k + 1)\}$$

is closed. By additivity of U-rank, this occurs iff we can find independent elements $b_1, \ldots b_{k+1} \subset \bar{b}$ in p such that $X(\bar{b}\bar{a})$ holds. However, this holds iff

$$\exists x_{\sigma(k+2)} \ldots \exists x_{\sigma(n)} X(x_1, \ldots x_n, \bar{a})$$

has maximal U-rank for some $\sigma \in S_n$, iff

$$d(\exists x_{\sigma(k+2)} \ldots \exists x_{\sigma(n)} X(\bar{x}, \bar{y}))(\bar{a})$$

holds for the defining schema of p^{k+1}, the generic type of $k+1$-elements of p. Finally, using stability, the above is a positive Boolean combination of

$$\exists x_{\sigma(k+2)} \ldots \exists x_{\sigma(n)} X(\bar{b}_i, \bar{y})$$

for $\bar{b}_i \in p$. Clearly, this set is closed.

$\square$

We now formulate a notion of generics and loci for this topology. If $\bar{a} \in p^k$, we define $loc(\bar{a}/A)$ to be the intersection of all closed sets defined over A containing $\bar{a}$. By Noetherianity, such a set clearly exists and is the smallest closed set over A containing $\bar{a}$. We say that $\bar{a}$ is generic in X closed if $locus(\bar{a}/A) = X$. We now claim the following.

Lemma 45. *If $X \subseteq p^{n+m}$ is closed and irreducible, then $\bar{a}\bar{b}$ is generic in X iff $\bar{a}$ is generic in $pr(X)$ and $\bar{b}$ is generic in $X(\bar{a})$.*

Proof. One direction is fairly straighforward. Suppose $\bar{a}$ is not generic in $pr(X)$, then $\bar{a} \in E \subsetneq pr(X)$ and $\bar{a}\bar{b} \in pr^{-1}(E) \cap X \subsetneq X$. If $\bar{b}$ is not generic in $X(\bar{a})$, then there exists L such that $\bar{b} \in L(\bar{a}) \subsetneq X(\bar{a})$. Then $\bar{a}\bar{b} \in L \cap X \subsetneq X$. In both cases, we get a contradiction.

For the other direction, suppose that $\bar{a}\bar{b}$ is not generic in X, then there exists D such that $\bar{a}\bar{b} \in D \subsetneq X$. Then $\bar{a} \in pr(D)$, but $pr(D)$ is closed so $pr(D) = pr(X)$. Similarily, as $\bar{b} \in D(\bar{a})$, we must have that $dim(D(\bar{a})) = dim(X(\bar{a}))$ Now, $U(D(\bar{a})) = U(X(\bar{a})) = k$ say. We have that

$$\{\bar{a} : U(D(\bar{a})) = k\}$$

is open in $pr(X)$ and definable over $acl(\emptyset)$. Now let $\bar{a}'$ be generic in $pr(D)$ in the sense of U-rank, and $\bar{b}'$ generic in $D(\bar{a}')$ in the sense of $U-$rank. Then, we have that

$$U(D) \geq U(\bar{a}', \bar{b}') = U(\bar{b}'/\bar{a}') + U(\bar{a}') = dimD(\bar{a}') + dimpr(D) = k + dimpr(X)$$

as $\bar{a}'$ must lie on every open set defined over $acl(\emptyset)$ in $pr(D)$. Then, if $\bar{a}''\bar{b}''$ is generic in X in the sense of $U-$rank, we have that

$$U(X) = U(\bar{a}''\bar{b}'') = U(\bar{b}''/\bar{a}'') + U(\bar{a}'') \leq k + dimpr(X)$$

as $\bar{a}''$ is generic in $pr(X)$ in the weaker sense. Hence,

$$U(D) \geq k + dimpr(X) \geq U(X)$$

so $dim(D) \geq dim(X)$ which contradicts the fact that $D \subsetneq X$ is proper.

□

We now want to show the following generic fibres lemma;

Lemma 46. *(GF) If $X \subseteq p^{n+m}$ is closed irreducible, then if $\bar{a}$ is generic inside $pr(X)$ we have that $dim(X) = dim(pr(X)) + min_{\bar{a} \in pr(X)} dimX(\bar{a}) = dimpr(X) + dimX(\bar{a})$ for $\bar{a}$ generic in $pr(X)$.*

Proof. Let X be irreducible, with $X \subseteq p^{n+m}$. Let $\bar{a} \in pr(X) \subset p^n$ be generic in the weak sense. Then $X(\bar{a})$ is a generic fibre, possibly not irreducible, of dimension k say. We first need the following generalising a result in [18];

Lemma 47. *If C is any closed set of dimension k, then C admits a generically finite map onto p^k.*

Write C as a union of irreducibles,

$$C = C_1 \cup \ldots \cup C_k$$

We may suppose that $dim(C) = dim(C_1)$ and $dim(C_i) \leq k$ for $i \geq 2$. Now C_1 admits a generically finite map onto p^k, so has infinite fibres over a proper closed set $E \subset p^k$ and we consider the map $pr : C_i \to p^k$ for $i \geq 2$. If $pr(C_i) = E_i \subsetneq p^k$, then the map on C is finite outside $E \cup E^i$, so we may as well assume that $pr(C_i) = p^k$ for some $i \geq 2$. Now suppose that $\bar{a} \in p^k$ is generic in the sense of U-rank, then if $C_i(\bar{a})$ is infinite we can find a pair $\bar{a}\bar{b}$ in C_i such that;

$$U(\bar{a}\bar{b}) = U(\bar{b}/\bar{a}) + U(\bar{a}) \geq k + 1$$

which contradicts the fact that $dim(C_i) \leq k$. Hence we may assume that $C_i(\bar{a})$ is finite. Now the result follows by the fact that infinite fibres must occur on a proper closed set E_i.

Now let $\pi : X(\bar{a}) \to p^k$ be a generically finite reduction and $b \in X(\bar{a})$ be a generic. Then, by a previous lemma we have that $\bar{a}\bar{b}$ is generic inside X, and $\pi(\bar{b})$ is generic in $p^k/acl(\bar{a})$. Let $X' = locus(\pi(\bar{b}), \bar{a})) \subseteq p^{n+k}$. Then we claim that X maps generically finitely onto X'. We have that $(\pi(\bar{b}, \bar{a})) \in dcl(\bar{b}, \bar{a})$ using the closed relation $\bar{z} = \bar{a} \wedge \bar{y} = \pi(\bar{b})$. Now consider,

$$\{(\bar{y}, \bar{z}) \in X' \wedge \exists \bar{t} \in X(\bar{z} = \bar{t}) \wedge \exists \bar{t}' \in X(\bar{y} = \pi(\bar{t}'))\}$$

This is a closed set containing $(\pi(\bar{b}, \bar{a}))$, so equals X'. Moreover, the fibre over $(\pi\bar{b}, \bar{a})$ is of the form $\{(\bar{x}, \bar{a}) : \pi(\bar{x}) = \pi(\bar{b}\})$, which is clearly finite. We therefore have that $dim(X) = dim(X')$. We clearly have that $pr(X') = pr(X)$ as $\bar{a}$ is generic in $pr(X)$ and $X'(\bar{a}) = locus(\pi(\bar{b}/\bar{a})) = p^k$ by construction. Now

$$\{\bar{y} : C'(\bar{y}) = p^k\}$$

is closed and contains $\bar{a}$. It follows that $X' = pr(X) \times p^k$, and so we have that

$$dim(X) = dim(X') = dim(pr(X) \times p^k) = U(pr(X) \times p^k) = dim(pr(X)) + dim X(\bar{a})$$

□

Remark 3. *The above results also combine to give us the additivity of dim, namely for a set of parameters A, and a pair $\bar{a}, \bar{b}$, we have that*

$$dim(\bar{a}\bar{b}/A) = dim(\bar{a}/\bar{b}A) + dim(\bar{b}/A)$$

We now work in the restricted universe $\mathcal{M}$ given by the closed sets. In order to apply the technique of specialisations, it is necessary to pass to an elementary extension $\mathcal{M}_*$ of $\mathcal{M}$. For such an extension, we define a closed set to be of the form;

$C(\bar{x}, \bar{a})$ for C closed in $\mathcal{M}$ and $\bar{a}$ a tuple in $\mathcal{M}_*$

We need to check the axioms are preserved. The conditions (L) and (P) are easy to check, (DCC) requires (EU) in [29], namely let;

$$C_1(\bar{a}_1, \mathcal{M}_*) \supset C_2(\bar{a}_2, \mathcal{M}_*) \supset \ldots$$

be a sequence of closed sets. Then using (EU) which gives ω_1-compactness, it is possible to pull the parameters back to $\mathcal{M}$.

The condition (PS) is checked in [29], for completeness I will check the condition for (GF).

Proof. So suppose that $S \subset {\mathcal{M}_*}^{n+s}$ irreducible is of the form $C(\bar{b})$ where $\bar{b} \in {\mathcal{M}_*}^k$ and $C \subset {\mathcal{M}_*}^{n+s+k}$. Assume C is irreducible and $\bar{b}$ is generic in $pr(C) = pr_2 \circ pr_1(C)$ where

$pr_1 : \mathcal{M}_*{}^{n+s+k} \to \mathcal{M}_*{}^{n+k}$

$pr_2 : \mathcal{M}_*{}^{n+k} \to \mathcal{M}_*{}^{k}$

Then we have that,

$dim(C) = dim\ pr(C) + dim\ C(\bar{b})$ (1)

$dim\ pr_1(C) = dim\ pr_2 \circ pr_1(C) + dim\ pr_1(C)(\bar{b})$ (2)

Now let $\bar{a}$ be generic in $pr_1(C)(\bar{b})$, then it follows that $\bar{a}\bar{b}$ is generic in $pr_1(C)$. Hence,

$dim(C) = dim\ pr_1(C) + dim\ C(\bar{a}\bar{b})$ (3)

We therefore have that,

$dim\ C(\bar{b}) = dim(C) - dim\ pr(C)$ by (1)

$= dim\ pr_1(C) + dim\ C(\bar{b})(\bar{a}) - dim\ pr(C)$ by (3)

$= dim\ pr_2 \circ pr_1(C) + dim\ pr_1(C)(\bar{b}) + dim\ C(\bar{b})(\bar{a}) - dim\ pr_2 \circ pr_1(C)$ by (2)

$= dim\ pr_1(C)(\bar{b}) + dim\ C(\bar{b})(\bar{a})$

□

Definition 16. *We say that D is presmooth if for all relatively closed irreducible $C_1, C_2 \subset D^k \times M^l$, and S an irreducible component of $C_1 \cap C_2$, we have;*

$dim(S) \geq dim(C_1) + dim(C_2) - dim(D^k \times M^l)$

So, in particular, by the (PS) axiom, the universe $\mathcal{M}$ is pre-smooth. Pre-smooth sets of dimension 1 behave well with respect to covers, that is if, $S \subset D^k \times \mathcal{M}^l$ is relatively closed and pr is the projection onto D^k, then not only do we have that

$$dim(S) = dim(pr(S)) + dim_{a \in pr(S) generic} S(\bar{a})$$

but also, if r is the dimension of a minimal fibre, then every component of $S(\mathcal{M}, \bar{a})$ for any $\bar{a}$ in $pr(S)$ has dimension at least r.

In order to see this, using the fact that D has dimension 1, by a sequence of generically finite maps , we may assume that S projects onto D^k. Then for $\bar{a}$ in D^k, we have that

$$S(\bar{a}) = S \cap (\bar{a} \times \mathcal{M}^l)$$

Hence, every irreducible component of $S(\bar{a})$, has dimension at least

$$dim(S) + dim(\mathcal{M}^l) - dim(D^k \times \mathcal{M}^l) = dim(S) - dim(D^k) = dim(S) - dimpr(S) = dim_{\bar{a} \in pr(S) generic} S(\bar{a}) = r$$

Chapter 11

Zariski Structures and Algebraic Geometry

The interesting link with algebraic geometry is made possible through the use of specialisations. Let $\mathcal{M}_* \succ \mathcal{M}$ and $A \subset \mathcal{M}_*$. We call $\pi : \mathcal{M}_* \to \mathcal{M}$ a specialisation on A if for all closed n-sets C defined over $\mathcal{M}$ and n-tuples $\bar{a} \in A$ we have that if $S(\bar{a})$, then $S(\pi(\bar{a}))$. Note that such a specialisation must fix $\mathcal{M}$. We have the following lemma, which relies on (P), that the projection of closed sets is closed.

Lemma 48. *If $\mathcal{M}_* \succ \mathcal{M}$, then there exists a specialisation $\pi : \mathcal{M}_* \to \mathcal{M}$ on $\mathcal{M}_*$.*

Proof. Suppose we have consructed a partial specialisation π on a subset $A \subset \mathcal{M}_*$. Let $b' \in \mathcal{M}_*$, then we just need to extend π to $A \cup b'$. For this, consider

$$\{C(x, \pi(\bar{d})) : C \text{ is } \mathcal{M} - \text{closed}, \bar{d} \in A^n, C(b', \bar{d})\}$$

We have that $\bar{d}$ satisfies $\exists x C(x, \mathcal{M}_*)$ but this set is closed, hence so does $\pi(\bar{d})$ in $\mathcal{M}$. This gives us a realisation b of $C(x, \pi(\bar{d}))$ in $\mathcal{M}$, and hence the above set is finitely realised. By (DCC), we can find a realisation b for the full set. One checks immediately that extending π by setting $\pi(b') = b$ gives a specialisation.

□

The following is one concrete way of extending a specialisation in $P(ACF_0)$.

Lemma 49. *Let $\bar{k}$ be an algebraically closed field and $\bar{k}[[t]]$ the ring of formal power series in t with fraction field $\bar{k}((t))$ the field of formal Laurent series. Then there exists a unique specialisation $\pi : P^1(\bar{k}((t))^{alg}) \to P^1(\bar{k})$ extending the residue map $res : \bar{k}[[t]] \to \bar{k}$.*

Proof. The map $\pi : P^1(\bar{k}((t))) \to P^1(\bar{k})$ is given by sending (f, g) to $(res(t^n f), res(t^n g))$ where $n \in \mathcal{Z}$ is chosen such that $\{t^n f, t^n g\} \subset \bar{k}[[t]]$ and not both have residue 0. Clearly this is well defined. To see that this is indeed a specialisation, we will just check it for closed 2-sets C defined over $\bar{k}$. The Segre embedding is defined by

$$P^1(\bar{k}) \times P^1(\bar{k}) \to P^3(\bar{k})$$

$$((x_0, x_1), (y_0, y_1)) \mapsto (x_0y_0, x_0y_1, x_1y_0, x_1y_1) =$$

We can similarily define a specialisation $\pi : P^3(\bar{k}((t)) \to P^3(\bar{k}))$ and the following diagram is easily checked to commute;

$$\begin{array}{ccc} P^1(\bar{k}((t))) \times P^1(\bar{k}((t))) & \xrightarrow{Segre} & P^3(\bar{k}((t))) \\ \downarrow \pi & & \downarrow \pi \\ P^1(\bar{k}) \times P^1(\bar{k}) & \xrightarrow{Segre} & P^3(\bar{k}) \end{array}$$

Therefore it is sufficient to check that π defined on $P^3(\bar{k}((t)))$ gives a specialisation. This is trivial to check using the fact that π is a ring homomorphism on $\bar{k}[[t]]$ and fixes the residue field. By the previous lemma, we know that π must extend to a specialisation on $\bar{k}((t))^{alg}$. To see that it is unique, use the fact that as $\bar{k}[[t]]$ is Henselian, every integral extension is ramified, so the algebraic closure $\bar{k}((t))^{alg} = \cup_{n \geq 0} \bar{k}((t^{1/n}))$. Then the full specialisation is determined on each root $t^{1/n}$ which must be taken to zero.

□

Definition 17. *Let $(\mathcal{M}_*, \pi)$ be a specialisation. For $\bar{a} \in \mathcal{M}^n$, we define the infintesimal neighborhood of $\bar{a}$ to be;*

$$\mathcal{V}_{\bar{a}} = \pi^{-1}(\bar{a})$$

The first property of infintesimal neighborhoods is that we can move inside closed sets.

Lemma 50. *If $D(\bar{y})$ is an irreducible set defined in $\mathcal{M}$, $\bar{b} \in D$ and $dim(D) = r$, then there exists a $\bar{b}' \in \mathcal{V}_{\bar{b}} \cap D_*$ such that $dim(\bar{b}'/\mathcal{M}) = r$*

Proof. Consider

$$D(\bar{y}) \cup \{\neg C(\bar{y}, \bar{d}) : \bar{d} \in \mathcal{M}, dim(D(\bar{y}) \cap C(\bar{y}, \bar{d}) < r)\}$$

Clearly, this set is consistent as D is irreducible of dimension r. Hence, we can find a realisation $\bar{b}'$ in $\mathcal{M}_*$ such that $dim(\bar{b}'/\mathcal{M}_*) = r$. It then follows that we can define a partial specialisation on $\mathcal{M}_*$ by setting $\pi(\bar{b}') = \bar{b}$, for if $C(\bar{y}, \bar{d})$ is a closed set such that $\neg C(\bar{b}, \bar{d})$, then we must have that $dim(D(\bar{y}) \cap C(\bar{y}, \bar{d})) < d$ otherwise, D being irreducible, $D(\bar{y}) \subset C(\bar{y}, \bar{d})$, so by construction $\neg C(\bar{b}', \bar{d})$ also holds. Such a partial specialisation extends to a total specialisation on $\mathcal{M}_*$.

□

For what follows, it is convenient to work with the notion of a λ existentially closed specialisation, which has the universal property that specialisations over $\mathcal{M}$ factor through it. The full force of the importance of using pre-smooth sets is contained in the following theorem;

Theorem 51. *Suppose that $F \subset D \times \mathcal{M}^k$ is an irreducible cover of D with D presmooth, such that $F(a,b)$ and $a \in D$ is a regular point for F. Then for every $a' \in \mathcal{V}_a \cap D_*$, we can find $b' \in \mathcal{V}_b$ such that $(a',b') \in F$ and $dim(b'/a'\mathcal{M}) = r$, the dimension of a generic fibre of F.*

Proof. There are three stages. First, we check for the consistency of the following partial type over $\mathcal{M}_*$, where $a' \in \mathcal{V}_a \cap D$ is generic over M;

$$\{F(a',y)\} \cup \{\neg C(d,y) : d \in \mathcal{M}_*, \neg C(\pi(d),b)\}$$

Clearly, a realisation b' of this type gives a specialisation for b such that $F(a',b')$ holds. If this fails to be consistent, we get a closed set $Q \subset \mathcal{M}^{n+k}$ such that $F(a',y) \subseteq Q(d,y)$ whereas $\neg Q(\pi(d),b)$. The point of pre-smoothness is to show that the space

$$L(x,z) \subset D \times M^n = \{(x,z) : F(x,y) \subset Q(z,y)\}$$

which in general is not relatively closed in $D \times M^n$ at least corresponds to a closed set over a dense open subset of D.

Then applying a specialisation gives $L(a,\pi(d))$ which means that $Q(\pi(d),b)$ holds, a contradiction

The second stage replaces $a' \in \mathcal{V}_a$ generic in D with an arbitrary a'' and follows easily by properties of specialisations. Finally, in the third stage, we take care of the dimensions by moving inside the corresponding fibre.

□

We now let $F \subset D \times M^k$ be an irreducible generically finite cover of D, that is the dimension of the generic fibre over D is 0. Replacing D by the set of regular points in D for F and using the fact that open subsets of presmooth sets are presmooth, we may assume that the cover F of D is finite everywhere.

Definition 18. *Zariski multiplicity*

Given $(a, b) \in F$, *let*

$mult_b(a, F/D) = Card(F(a', \mathcal{M}_*) \cap \mathcal{V}_b)$ *for* $a' \in \mathcal{V}_a \cap D$ *generic over* $\mathcal{M}$

We want to show this is well defined.

Proof. Suppose $a'' \in \mathcal{V}_a$ with $Card(F(a'', \mathcal{M}_*)) \cap \mathcal{V}_b = n$. Consider the relation $N(x, y_1, \ldots, y_n) \subset D \times \mathcal{M}_*^{nk}$, given by

$$N(x, y_1, \ldots, y_n) = F(x, y_1) \wedge \ldots \wedge F(x, y_n)$$

Then we have that N is a finite cover of D and moreover by presmoothness of D, each irreducible component of N has dimension at least

$$n(dim(F) + (n-1)k) - (n-1)(dim(D) + nk) = dim(D) + n(n-1)k - n(n-1)k = dim(D)$$

so clearly each component is a finite cover of D. Now, choose an irreducible component N_i containing $(a'', b''_1, \ldots, b''_n)$, so by specialisation also contains $(a, b, \ldots, b)$ and consider the open set $U \subset N_i$ given by

$$U(x, y_1, \ldots, y_n) = N_i(x, y_1, \ldots, y_n) \wedge y_1 \neq y_2 \neq \ldots \neq y_n$$

Then, for any $a' \in \mathcal{V}_a$ generic in D, it follows we can find a tuple $(b'_1, \ldots, b'_n)$ such that $N_i(a', b'_1, \ldots, b'_n)$, and $(b'_1, \ldots, b'_n) \in \mathcal{V}_{(b, \ldots, b)}$. As, is easily checked the tuple

$(a', b'_1, \ldots, b'_n)$ is generic inside N_i, hence must lie inside U. This proves that the $b'_1, \ldots, b'_n$ are distinct.

□

Definition 19. *We say that a point $(ab) \in F$ is ramified in the sense of Zariski structures if $mult_b(a, F/D) \geq 2$.*

Now suppose $F \subset D \times \mathcal{M}^n$ is an irreducible finite cover of D with D presmooth, then we have the following easily checked lemma

Lemma 52. *$mult(a, F/D) = \Sigma_{b \in F(a,\mathcal{M}^k)} mult_b(a, F/D)$ does not depend on the choice of $a \in D$, and is equal to the size of a generic fibre over D*

This bears a striking similarity with the concept of flatness from algebraic geometry. If $f : X \to Y$ is a finite morphism between algebraic varieties and Y is irreducible then f is flat iff

$$dim_{k(y)}(f_*(O_X) \otimes_{O_y} k(y))$$

is independent of y, see [28]. We will make this analogy more precise below

That multiplicity is definable is the content of the following lemma;

Lemma 53. *The sets*

$$j_k(F/D) = \{(a, b) \in F : mult_{(a,b)}(F/D) \geq k\}$$

are relatively closed inside F.

Proof. Again consider the closed relation $N(x, y_1, \ldots, y_k)$ introduced above. Let N' be the union of all irreducible components meeting the open set U, then we claim that

$$j_k(F/D)(a,b) \text{ iff } N'(a, \underbrace{b, \ldots, b}_{k})$$

Left to right follows by the fact that we can find $a' \in \mathcal{V}_a \cap D$ and distict $b'_1, \ldots, b'_k \in \mathcal{V}_b$ such that $U(a', b'_1, \ldots, b'_k)$. Taking an irreducible component through such a tuple and applying a specialisation gives the result. Right to left follows by taking an irreducible component through $(a, b, \ldots, b)$ and using Theorem 57 to find a generic tuple $(a', b'_1, \ldots, b'_k) \in \mathcal{V}_{(a,b,\ldots,b)}$. As such a component meets U, this tuple lies inside U as required.

□

We need the following simple lemma;

Lemma 54. *If $\bar{a}' \in D$, then $F(\bar{a}')$ contains a point of ramification in the sense of Zariski structures iff $|F(\bar{a}')| < |F(\bar{a})|$ where $\bar{a}$ is generic in D.*

Proof. We have seen that $|F(\bar{a})| = \Sigma_{\bar{b} \in F(\bar{a}', \mathcal{M}^n)} mult_{\bar{a}', \bar{b}}(F/D)$. If $|F(\bar{a}')| < F(\bar{a})|$, then there must exist $\bar{b} \in F(\bar{a}')$ with $mult_{(\bar{a}', \bar{b})}(F/D) \geq 2$ so the result follws by the definition of ramification in Zariski structures. The converse is similar.

□

We now show the following theorem;

Theorem 55. *If $pr : F \to D$ is an irreducible finite cover with F relatively closed inside $D \times \mathcal{M}^n$, $\mathcal{M}$ is $P(ACF_0)$ and F, D smooth, then the notions of Zariski unramified and etale coincide.*

Proof. Now we assume that F and D are both smooth considered as algebraic varieties. As we are working inside projective space, pr is a proper morphism. By Chevalley's criteria, we can also assume that pr is a finite morphism in the sense of algebraic geometry, that is given an affine open $U \subset D$, $pr^{-1}(U)$ is affine and $R(pr^{-1}(U))$ is an integral ring extension of $R(U)$. Now, suppose that pr is etale, then as is shown in Mumford, pr is flat and so $dim_{k(y)}(f_*(O_F) \otimes_{O_y} k(y))$ is locally constant on D. As pr is etale, we have seen that $pr_* : T_{x,F} \to T_{pr(x),D}$ is an isomorphism, and therefore as is easily checked

$$dim_{k(y)}(f_*(O_F) \otimes_{O_y} k(y)) = |F(y)| \text{ for } y \in D$$

This shows that $|F(y)|$ is independent of $y \in D$ which by the above lemma shows that pr is unramified in the sense of Zariski structures.

For the converse, we may assume that $pr : F \to D$ is a finite morphism, and show that for generic $\bar{a} \in D$, that $|F(\bar{a})| = deg(pr) = deg[k(F) : k(D)]$. As $char(k(F)) = 0$, the extension is seperable so we can find a primitive element $g \in k(F)$ such that $k(F) = k(D)(g)$. Clearly the minimum polynomial p of g over $k(D)$ has degree $n = deg[k(F) : k(D)]$. Let $h_1, \ldots h_{n-1} \in k(D)$ be the coefficients of p, then $R(D)(h_1 \ldots h_{n-1})$ determines the function ring of a Zariski open subset U of D. Clearly $R(U)[g]$ is an integral extension of $R(U)$ and corresponds to the projection restricted to $U' = pr^{-1}(U) \cap g \neq 0$. By dimension theory, the zero set $Z(g) \subset D$ cannot intersect with a generic fibre of the original map $pr : F \to D$. Now we consider the discriminant $D(p)$ of the polynomial p as a regular function on U and we have that for generic $\bar{a} \in U$ that $D(p)(\bar{a}) \neq 0$. This implies that for generic $\bar{a} \in U$ $|pr^{-1}(\bar{a})| = n = deg[k(F) : k(D)]$. Now we are in a position to apply Theorem 5, p145, of [32] which requires that D should be smooth, namely that $pr_* : T_{x,F} \to T_{pr(x),D}$ is an isomorphism for $x \in F$. As F and D were assumed to be nonsingular, this is sufficient to show that pr is etale by Theorem 42

□

Remark 4. *It is worth remarking that the above proof depends heavily on the fact that the considered morphism is proper. If we consider the problem of adapting the proof for arbitrary varieties, then ,using the fact that F is smooth, we can by Theorem*

40 (see also Theorem 42) find a locally finite presentation of pr but in doing so we may lose points in some of the fibres. The problem depends on a deeper consideration of the geometry of the varieties.

We now consider the case of a projective model of ACF_p. This time the analogy fails. If we consider the Frobenius map $Fr : P^1 \to P^1$, then $Graph(Fr) \subset P^1 \times P^1$ is a finite cover of P^1 and both $Graph(f)$ and P^1 are smooth. The projection map pr onto the second coordinate is unramified in the sense of Zariski structures as pr is a bijection. However pr fails to be etale in the sense of algebraic geometry as $pr_* : T_{x,Graph(Fr)} \to T_{pr(x),P^1}$ is zero everywhere. We aim to show that this is the only bad example, more precisely we have the following;

Theorem 56. *In $P(ACF_0)$, the notions of Zariski unramified and etale correspond for a finite cover $F \to D$ with F and D smooth. In $P(ACF_p)$, with the same hypotheses on F and D, any Zariski unramified cover pr factors generically as $g \circ h$ with h etale and g is locally of the form $Fr^{n_1} \times \ldots \times Fr_{n_m}$*

Proof. Suppose first that $F \to D$ is a finite morphism with F and D affine. We first find a field L such that $k(F)/L$ is a purely inseperable extension and $L/k(D)$ is seperable. Let R' be the integral closure of $R(D)$ in L and R'' the integral closure of $R(D)$ in $k(F)$. As $R(F)$ is integral over $R(D)$ we have that $R(F) \subset R''$, but F was assumed to be smooth so $R(F)$ is integrally closed in $k(F)$ and therefore $R'' = R(F)$. Now corresponding to the ring inclusions

$$R(D) \to R' \to R(F)$$

we have the sequence of finite morphisms

$$F \to_{pr_1} Spec(R') \to_{pr_2} D$$

We first consider the cover $F \to_{pr_1} Spec(R')$. Let $g_1, \ldots g_m$ generate $R(F)$ over R'. As the extension $k(F)/L$ is purely inseperable, we can write the minimum polynomials p_i of g_i in the form $r_{i,1}g^{p^{n_i}} - r_{i,2} = 0$ where $r_{i,1}$ and $r_{i,2}$ are in R'. Let $U \subset Spec(R')$ be the open subvariety determined by $\cap U_{r_{i,1}}$ where $U_{r_{i,1}} = \{x \in Spec(R) : r_{i,1}(x) \neq 0\}$. Then $pr_1^{-1}(U)$ has coordinate ring $R'_{r_{i_1},\ldots,r_{i_n}}[g_1 \ldots g_m]$ and is easily checked to be a bijection on points with U, in fact can only be a map of the form $(Fr^{n_1}, \ldots Fr^{n_m})$ for an embedding of $Spec(R)$ in A^m. Now restricting pr_2 to U, we may suppose that $pr_2(U)$ is open in D and therefore smooth. Now using the fact that U is unramified in the sense of Zariski structures over $pr_2(U)$, and applying the previous result, we have that $pr_2|U$ is etale. Note that on the complement $C = Spec(R') \setminus U$, we can apply induction on dimension to factor this generically; we just need the following easily checked lemma, the restriction of a Zariski unramified cover is unramified even if the restriction is not irreducible. For the general case of a finite morphims $pr : F \to D$, enumerate the affine pieces U_i such that $pr_2|U_i$ is etale. The aim is now to patch the U_i to form an etale cover V of an open set $W \subset D$. The patching data is given by the integral closure of $R(pr(U_i) \cap pr(U_j))$ in L. By facts on integral closure, it is easily verified the patching data agrees on triple overlaps.

□

When F and D are smooth, $pr : F \to D$ is a finite cover as above, this result in fact shows that, for $P(ACF_0)$, $pr(j_2) = pr(\Sigma_{n-1})$ where j_2 is the ramification locus for Zariski structures and Σ_{n-1} is the degeneracy locus introduced in Section 10. In fact, using part of the next result, it is possible to show that $j_2 = \Sigma_{n-1}$. The natural question is then the following;

For $P(ACF_0)$, (F/D) a cover of dimension n as above, is there an explicit formula relating j_k and Σ_{n-l} for $j, l \geq 2$? (*)

The answer to $(*)$ will clearly invoke the methods used in proving the Thom-Porteous formula in Section 10.

Without the smoothness assumption, strange things can happen. For example, consider the conic given by the equation $z^2 = x^2 + y^2$, which is singular at $(0,0,0)$. Then the projection onto the $xy-$axis is etale everywhere except at the origin which is a closed set of codimension 2. Then $j_2 = \{(0,0,0)\}$ for Zariski structures but $\Sigma_{n-1} = \emptyset$.

Intuitively, if F is a cover of D, then $Mult_{ab}(F/D)$ should count the number of branches of F over D. We shall make this more precise in the case of curves. We first need the following simple lemma that the Zariski notion of multiplicity is multiplicative, namely;

Lemma 57. *Suppose that F_1, F_2 and F_3 are presmooth, irreducible, with $F_2 \subset F_1 \times \mathcal{M}^k$ and $F_3 \subset F_2 \times \mathcal{M}^l$ finite covers. Let $(abc) \in F_3 \subset F_1 \times \mathcal{M}^k \times \mathcal{M}^l$. Then $mult_{abc}(F_3/F_1) = mult_{ab}(F_2/F_1) mult_{bc}(F_3/F_2)$.*

Proof. To see this, let $m = mult_{ab}(F_2/F_1)$ and $n = mult_{bc}(F_3/F_2)$. Choose $a' \in \mathcal{V}_a \cap F_1$ generic over $\mathcal{M}$. By definition, we can find distinct $b_1 \dots b_m$ in $\mathcal{M}^k_* \cap \mathcal{V}_b$ such that $F_2(a', b_i)$ holds. As F_2 is a finite cover of F_1, we have that $dim(a'b_i/\mathcal{M}) = dim(a'/\mathcal{M}) = dim(F_1) = dim(F_2)$, so each $(ab_i) \in \mathcal{V}_{ab} \cap F_2$ is generic over $\mathcal{M}$. Again by definition, we can find distinct $c_{i1} \dots c_{in}$ in $\mathcal{M}^l_* \cap \mathcal{V}_c$ such that $F_3(a'b_ic_{ij})$ holds. Then the mn distinct elements $(a'b_ic_{ij})$ are in $\mathcal{V}_{abc}$, so by definition of multiplicity $mult_{abc}(F_3/F_1) = mn$ as required.

□

Now we work inside a projective model $P(ACF_0)$.

Definition 20. *Given smooth projective curves C_1 and C_2 and a finite morphism $f : C_1 \to C_2$, the index of ramification or branching number at a is $ord_a(f^*h)$ where h is a local uniformiser for C_2 at $f(a)$.*

This is independent of the choice of h, as the quotient of 2 uniformisers h/h' is a unit in $\mathcal{O}_{f(a)}$. Given finite morphisms $f : C_3 \to C_2$ and $g : C_2 \to C_1$, if

$ord_{a,f(a)}(C_3/C_2) = m$ and $ord_{f(a),gf(a)}(C_2/C_1) = n$, then taking a local uniformiser h at $gf(a)$, we have that $g^*h = h_1^n u$ locally at $f(a)$ for a unit u and uniformiser h_1 in $\mathcal{O}_{f(a)}$. Similarily $f^*g^*h = h_2^{mn}u'$ for a unit u' and uniformiserh_2 in $\mathcal{O}_a$. This shows that $ord_{a,gf(a)}(C_3/C_1) = mn$, so the branching number is also multiplicative for smooth projective curves.

We aim to show the following;

Theorem 58. *In $P(ACF_0)$, the notions of Zariski multiplicity and branching number coincide for a finite morphism $f : C_2 \to C_1$ between smooth projective curves.*

Proof. As C_1 has a non-constant meromorphic function, we can write C_1 as a finite cover of $P^1(\bar{k})$. As we have checked both the branching number and Zariski multiplicity are multiplicative over composition, it is straightforward to see that we need only check the notions agree for the cover $\pi : C_1 \to P^1(\bar{k})$. Now considering this cover restricted to A^1, let x be the canonical cooordinate with $ord_a(\pi^*(x)) = m$, so we have that $\pi^*x = h^m u$, for u a unit in $\mathcal{O}_a$ and h a uniformiser at a. We can solve the equation $z^m = u$ in some finite extension of $Frac(C_1)$, which determines an etale map ϕ near a of a new curve C_1' onto C_1. The following fact may be found in [27];

Fact 59. *Any etale morphism can be locally presented in the form*

$$\begin{array}{ccc} V & \xrightarrow{\cong} & Spec((A[T]/f(T))_d) \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ U & \xrightarrow{\cong} & Spec(A) \end{array}$$

where $f(T)$ is a monic polynomial in $A[T]$ and $f'(T)$ is invertible in $(A[T]/f(T))_d$.

This is enough to show that C_1' is unramified over $a \in C_1$ in the sense of Zariski structures. For suppose not and f has degree n. Let $\sigma_1 \ldots \sigma_n$ be the elementary symmetric functions in n variables $T_1, \ldots T_n$. Consider the equations

$$\sigma_1(T_1, \ldots, T_n) = a_1$$

$$\ldots$$

$$\sigma_n(T_1, \ldots, T_n) = a_n \ (*)$$

where $a_1, \ldots a_n$ are the coefficients of f with appropriate sign. These cut out a closed subscheme of $C \subset Spec(A[T_1 \ldots T_N])$. Suppose $(ab) \in Spec(A[T]/f(T))$ is ramified in the sense of Zariski structures, then I can find $(a'b_1b_2) \in \mathcal{V}_{abb}$ with $(a'b_1)$,$(a'b_2) \in Spec(A(T)/f(T))$ and b_1, b_2 distinct. Then complete (b_1b_2) to an n-tuple $(b_1b_2c'_1 \ldots c'_{n-2})$ corresponding to the roots of f over a'. The tuple $(a'b_1b_2c'_1 \ldots c'_{n-2})$ satisfies C, hence so does the specialisation $(abbc_1 \ldots c_{n-2})$. Then the tuple $(bbc_1 \ldots c_{n-2})$ satisfies $(*)$ with the coefficients evaluated at a. However such a solution is unique up to permutation and corresponds to the roots of f over a. This shows that f has a double root at (ab) and therefore $f'(T)|_{ab} = 0$, which implies that C'_1 is ramified over C_1 at (ab).

We may therefore assume that $\pi^* x = h^m$ for h a local uniformiser at a. Now we have the sequence of ring inclusions given by

$$\bar{k}[x] \rightarrow \bar{k}[x, y]/(y^m - x) \rightarrow R$$

where R is the coordinate ring of C_1 in some affine neighborhood of a. It follows that we can factor our original map such that C_1 is etale near a over the projective closure of $y^m - x = 0$. Again, repeating the above argument, we just need to check that the projective closure of $y^m - x$ has multiplicity m at 0 considered as a cover of $P^1(\bar{k})$. This is trival, let $\epsilon \in \mathcal{V}_0$ be generic over $\mathcal{M}$,then as we are working in characteristic 0 we can find distinct $\epsilon_1, \ldots \epsilon_m$ in $\mathcal{M}_*$ solving $y^m = \epsilon$. By specialisation, each $\epsilon_i \in \mathcal{V}_0$. □

Finally, we can use unramified Zariski covers to define the notion of a local function, which we will later generalise to the notion of a germ.

Definition 21. *Given a finite covering $F \subset D \times \mathcal{M}^k$, we say that F defines a local function on D at (ab) if $F|(\mathcal{V}_a \times \mathcal{V}_b)$ is the graph of a function from $\mathcal{V}_a \cap D$ into $\mathcal{V}_b$.*

It is a rather straightforward now to see that if F is a finite covering of D with D pre-smooth, then F is unramified at a point $(ab) \in F$ iff F defines a local function at (ab). Hence, we have the following result which is the inverse function theorem for zariski structures.

Theorem 60. *If F is an irreducible finite covering of D, then there is an open subset D_1 of D such that F defines a local function on D_1.*

To see this, simply take an open subset to ensure D is pre smooth, and a further open subset to remove the ramification locus. The result generalises easily to the case of reducible finite covers.

Comparing this with Theorem 43 and Theorem 48 strongly suggests that one can show the equivalence of etale morphisms and unramified Zariski covers for $P(ACF_0)$ without the restrictive assumption of smoothness, that is for the wider class of pre-smooth sets. The idea is this, suppose a morphism $f : X \to Y$ fails to be etale, then by Theorem 47, we must have that $f^* : O^{\wedge}_{f(x),Y} \to O^{\wedge}_{x,X}$ is non-surjective for some $x \in X$, that is $O^{\wedge}_{f(x),Y} \subset O^{\wedge}_{x,X}$ is a proper inclusion of Henselian rings, using Theorem 45. By general facts on Henselian rings, we can present this inclusion in a straightforward way corresponding to a morphism f^{lift} lifting f to a new pair of etale covers. This sets up a diagram of the form;

$$\begin{array}{ccc} Y' \times_Y X & \xrightarrow{etale} & X \\ \Big\downarrow{f^{lift}} & & \Big\downarrow{f} \\ Y' & \xrightarrow{etale} & Y \end{array}$$

where the horizontal arrows are etale, and therefore by adapting Theorem 62 should be Zariski unramified. By similar methods to Theorem 64 again, f^{lift} should be ramified in the sense of Zariski structures and therefore ,using Lemma 62, we can see that the original morphism f should be Zariski ramified.

Chapter 12

Defining Tangency of Curves

Having developed the basic machinery of Zariski structures, we turn to the problem of interpreting a field inside $\mathcal{M}$. The extra assumption that we need is that $\mathcal{M}$ is non-locally modular, at least for the stable case. In the simple case, we require a stronger condition, namely that $\mathcal{M}$ should not be $1-$based. As has been shown in [9], this turns out to be equivalent to the notion of non-linearity for such structures, namely there exists $\rho \in S_2(\mathcal{M})$ such that $U(Cb(\rho)) \geq 2$. In the case where $\mathcal{M}$ is a stable minimal type with the property that $acl(\emptyset)$ is infinite, the last condition being automatic for Zariski structures, it is a known result that $\mathcal{M}$ has weak elimination of imaginaries. In the case that $\mathcal{M}$ is simple, it is again possible to adapt this result under some assumptions. Namely, assume that

$$c = Cb(stp(\rho)) \text{ and } c = a_1, \dots, a_n/E$$

where is a $\emptyset$ definable equivalence relation on $\mathcal{M}$. Then, without loss of generality we can choose $a_1, \dots, a_n$ such that $a_1, \dots, a_j \in acl(c)$ and j is maximal with this property. Consider the statement;

$$\exists x_{j+2}, \dots, \exists x_n (c = a_1, \dots, a_j, x_{j+1}, x_{j+2}, \dots x_n/E)$$

then under the assumption that this formula is stable and by hypothesis non-algebraic, it follows easily that it must be realised inside $acl(\emptyset)$ which is an elementary substructure in the zariski set up. This contradicts the maximality of j unless $j = n$ in which case the tuple $a_1, \ldots, a_n$ is interalgebraic with c. A simple argument then shows that E can be chosen to be the equivalence relation of permutation on $\mathcal{M}^n$.

It now follows that we can obtain a rank 2 family of curves on $\mathcal{M}$. Namely choose $\bar{a} \in \mathcal{M}^n$ such that $\bar{a}$ is interalgebraic with c. Then let $b_1 b_2$ realise a non forking extension of p to $\bar{a}c$. As $\bar{a}$ and c are interalgebraic, we still have that $U(b_1 b_2/\bar{a}) = 1$ and moreover we can assume that $mult(b_1 b_2/\bar{a}) = 1$ in the sense of the Zariski closed sets, taking irreducible components of the $locus(b_1 b_2/\bar{a})$. Now let $L = locus(b_1 b_2 \bar{a}/acl(\emptyset))$, then this gives a 2 dimensional family of curves, with generically irreducible fibre.

Let L represent a 2 dimensional family of curves in $\mathcal{M}^2$ which are finite to finite and such that the family is generically irreducible. Let $I \subset L \times \mathcal{M}^2$ be the incidence relation defining this family. Let

$$I_k \subset L^k \times \mathcal{M}^{k+1} = \{(g_k, \ldots, g_1, x_0, \ldots, x_k) : (x_0 x_1) \in g_1 \wedge \ldots (x_{k-1} x_k) \in g_k\}$$

Then I_k is closed in $L^k \times C^{k+1}$. Now suppose that $(\bar{g}, \bar{x}) \in I_k$ is generic, then we have that $dim(\bar{g}, \bar{x}) = dim(x_1, \ldots, x_k/\bar{g}x_0) + dim(\bar{g}x_0)$. As $\bar{g}x_0$is generic in $pr(I_k) = L^k \times \mathcal{M}$ and each g_i is a finite relation, we must have that $dim(\bar{g}\bar{x}) = kdim(L) + 1$ and so $dimI_k = kdim(L) + 1$.

By pre smoothness of L, we know if I_k^c is an irreducible component, then

$$dimI_k^c \geq k(dim(I \times L^{k-1} \times \mathcal{M}^{k-1})) - (k-1)dim(L^k \times \mathcal{M}^{k+1})$$

$$= k(dim(L) + 1 + (k-1)(dim(L) + 1)) - (k-1)(kdim(L) + k + 1)$$

$$= kdim(L) + 1$$

Consider also the family $N^k \subset L^k$ where N is the subfamily of L consisting of curves passing through ab with ab generic in $\mathcal{M}^2$. Again N^k is closed in L^k and definable over ab. We set $J \subset L^k \times L^k \times \mathcal{M}^{k+1} \times \mathcal{M}^{k+1}$ to be

$$J = \{(g_k, \dots g_1, g'_k, \dots g'_1, x_0, \dots x_k, y_0, \dots y_k) : (g_k, \dots, g_1, x_0, \dots, x_k) \in I_k \wedge (g'_k, \dots g'_1, y_0 \dots y_k) \in I_k \wedge x_0 = y_0 \wedge x_k = y_k\}$$

In short, we have that $J(\bar{g}_1, \bar{g}_2, \bar{x}\bar{y})$ iff $I_k(\bar{g}_1, \bar{x}) \wedge I_k(\bar{g}_2, \bar{y}) \wedge \bar{y} \in \mathcal{H}_{\bar{g}_2}$, where $\mathcal{H}_{\bar{g}_2}$ is the hyperplane associated to $\bar{g}_2$ using the projection $pr : \mathcal{M}^{k+1} \to \mathcal{M}^2$ onto the first and last coordinates. Observe that $\bar{g}_2 \subset \mathcal{H}_{g_2}$.

Now suppose that $(\bar{g}_1\bar{g}_2\bar{x}\bar{y}) \in J$ with $(\bar{g}_1\bar{g}_2)$ generic in $L^k \times L^k$. Then we have $\bar{x}\bar{y} \in J(\bar{g}_1\bar{g}_2)$ iff $(x_0x_k) \in \bar{g}_1 \cap \bar{g}_2$. Now using the facts that independent generic curves intersect at finitely many points and trajectories are determined up to finite possibilities by the initial coordinate, we have that

$$dim(\bar{x}\bar{y}/\bar{g}_1\bar{g}_2) = dim(x_1 \dots x_{k-1}/x_0x_k\bar{g}_1\bar{g}_2) + dim(y_1 \dots y_{k-1}/y_0y_k\bar{g}_1\bar{g}_2) + dim(x_0x_k/\bar{g}_1\bar{g}_2) = 0$$

Hence $dim(J) \leq dim(L^k \times L^k)$, as if some component had dimension bigger than this, the above result implies it could not project onto $L^k \times L^k$ which would give that the dimension of a generic fibre should be at least 2 which is ridiculous.

Moreover, by pre smoothness the dimension of each component $dim(J^c) \geq dim(L^k \times L^k)$

We have the following lemmma,

Lemma 61. *If $\bar{g}_1, \bar{g}_2 \in N^k$ are independent generics then $J(\bar{g}_1\bar{g}_2)$ is finite.*

Proof. It is sufficient to show that $\bar{g}_1 \cap \bar{g}_2$ is finite.

First, carry out the following reduction on $K \subset N^k \times \mathcal{M}^2$ given by $K(\bar{g}, ab)$ iff $ab \in \bar{g}$ to make the generic fibre irreducible. Let $\bar{g} \in N^k$ be generic and $c = Cb(K(\bar{g})^c)$. Then we have $c \in acl(\bar{g})$. Let $a'b'$ be generic in $K(\bar{g})^c$ over ab and

$$L = locus(\bar{g}c, a'b'/acl(ab)) \subset locus(\bar{g}c)/acl(ab) \times \mathcal{M}^2$$

Then we have first a generically finite map

$$pr : locus(\bar{g}c)/acl(ab) \to N^k$$

and second

$$L(\bar{g}c) = K(\bar{g})^c$$

so the generic fibre of L is an irreducible curve.

Third, we have a generically finite map

$$\bar{pr} : L \to K$$

extending the map pr, generically injective on fibres and with dense image

It follows that on some $U \subset locus(\bar{g}c)/acl(ab)$, U defines a $dim(c/ab) \geq 1$ family of curves. Now suppose that $dim(\bar{g}_1 \cap \bar{g}_2) \geq 1$. Let $pr(\bar{g}_1 c) = \bar{g}_1$ and $pr(\bar{g}_2 d) = \bar{g}_2$. It follows as

$$\bar{pr}(L(\bar{g}_1 c)) = \bar{pr}(L(\bar{g}_2 d)) = K(\bar{g})^c$$

that $L(\bar{g}_1 c) = L(\bar{g}_2 d)$.

Now let E be the equivalence relation on $U \times U$ given by

$$E(\bar{g}_1 c, \bar{g}_2 d) \text{ iff } L(\bar{g}_1 c) = L(\bar{g}_2 d)$$

As $\bar{g}_1 c$ and $\bar{g}_2 d$ are independent generics in U, it follows that the equivalence class of $\bar{g}_1 c$ is open in U. If $\bar{g}d$ is also generic, choose an automorphism between $\bar{g}_1 c$ and $\bar{g}d$. This takes the equivalence class of $\bar{g}_1 c$ to $\bar{g}d$ which must therefore also be open in U so the equivalence classes intersect. Hence, for all $\bar{g}d$ generic in U, $L(\bar{g}d) = K(\bar{g})^c$. This is a contradiction as if $dim(c/ab) = k \geq 1$, choose $a'b'$ in $K(\bar{g})^c$ generic over abc, then if $dim(a'b'/abc) = dim(a'b'/ab) = 1$, we must have that $c \in acl(ab)$ which is not the case, hence we have that ab and $a'b'$ are independent generics in $\mathcal{M}^2$. Choose another $a''b''$ generic over ab with $a''b'' \notin K(\bar{g})^c$ Then, again by automorphism we can find a generic element $\bar{g}d$ passing through $ab, a''b''$, which is a contradiction.

□

It follows that J is a generically finite cover of $N^k \times N^k$, though not necessarily irreducible. Let

$$U = \{(\bar{g}_1 \bar{g}_2) \in N^k \times N^k : dim J(\bar{g}_1 \bar{g}_2) = 0\}$$

Then U is open in $N^k \times N^k$. and $pr : J \rightarrow U$ is a finite reducible cover. Moreover, we may assume that U is presmooth as removing a finite set of curves from N, gives that $N^k \times N^k$ is presmooth.

Definition 22. *$\bar{g}_1 \bar{g}_2$ is ramified at $aba \ldots a$ if taking the union J' of components of J passing through $\bar{g}_1 \bar{g}_2 ab \ldots a, ab \ldots a$ with finite fibre over $\bar{g}_1 \bar{g}_2$, gives that $mult_{\bar{g}_1 \bar{g}_2}^{aba \ldots a} \geq 2$ in the sense of reducible covers.*

For ease of notation denote the concatenated tuple $ab \ldots a, ab \ldots a$ by $aba \ldots a^{(2)}$

Definition 23. *We will say* $\bar{g}_1 \mathcal{T}_{aba\ldots a} \bar{g}_2$ *iff* $\bar{g}_1\bar{g}_2$ *is ramified at* $aba\ldots a^{(2)}$ *or* $dim J(\bar{g}_1\bar{g}_2) \geq 1$ *with the infinite component passing through* $aba\ldots a^{(2)}$.

By properties of multiplicities outlined above, the first part of the tangency relation is definable on $(N^k \times N^k)$ with parameters ab. In order to see this, take all possible unions J_i of irreducible components of J, and take the union of the jacobians $j_2^{ab\ldots a}$ for each J_i. As adding components to a finite cover can only increase the size of multiplicity, we get the result. The second part follows by definability of dimension and presmoothness of each component of J, namely if we find an infinite component of a fibre containing $aba\ldots a^{(2)}$, then we may assume that the infinite component passes through $ab\ldots a^{(2)}$.

The following definition of tangency $\mathcal{T}'$ is given in [29]

Definition 24. $\bar{g}_1 \mathcal{T}' \bar{g}_2$ *iff* $\forall a' \in \mathcal{V}_a (\forall \bar{g}_1' \in (\mathcal{V}_{\bar{g}_1} \cap N^k) \exists \bar{g}_2' \in (\mathcal{V}_{\bar{g}_2} \cap N^k) \bar{g}_1'(a') = \bar{g}_2'(a'))$

Given this, we will show that

Theorem 62. *The new definition of tangency* $\mathcal{T}$ *is equivalent to the old definition* $\mathcal{T}'$.

Proof. Case 1. Left to right.

Suppose that we have $\bar{g}_1 \mathcal{T}'_{aba\ldots a} \bar{g}_2$ in the old sense. That is given $a' \in \mathcal{V}_a$, and $\bar{g}_1' \in \mathcal{V}_{\bar{g}_1} \cap N^k$, we can find $\bar{g}_2' \in \mathcal{V}_{\bar{g}_2} \cap N^k$ such that $\bar{g}_1'(a') = \bar{g}_2'(a')$.

We may assume that $J(\bar{g}_1\bar{g}_2)$ is finite. Otherwise there exists a component of the fibre with dimension ≥ 1, and not passing through $ab\ldots a^{(2)}$. By presmoothness, this belongs to a component J' of J not passing through $\bar{g}_1\bar{g}_2 ab\ldots a^{(2)}$. Removing this component doesn't effect the following calculation as J' cannot contain any elements in an infintesimal neighborhood of $\bar{g}_1\bar{g}_2ab\ldots a^{(2)}$.

Let $a' \in \mathcal{V}_a$ be generic over a, so $a' \neq a$ and $\bar{g}_1'\bar{g}_2' \in \mathcal{V}_{\bar{g}_1} \times \mathcal{V}_{\bar{g}_2} \cap N^k \times N^k$ as above. Now $\bar{g}_1'\bar{g}_2' \in U$ by properties of infintesimals and $J(\bar{g}_1'\bar{g}_2'ab\ldots a^2)$ holds. Moreover, if $(a'b'\ldots a''), (a'b''\ldots a''$ are the trajectories associated to $\bar{g}_1', \bar{g}_2'$, then clearly $I_k(g_i', a'\ldots a')$ and moreover $(a'b'\ldots a'') \in \mathcal{H}_{\bar{g}_2'}$. Hence we have that $J(\bar{g}_1'\bar{g}_2', a'b'\ldots a'', a'b''\ldots a'')$ holds as well. Now by definition of multiplicities for reducible covers, we have that $mult_{\bar{g}_1\bar{g}_2}^{aba\ldots a} \geq 2$ as required.

Case 2. Right to left

Suppose that $mult_{\bar{g}_1\bar{g}_2}^{aba\ldots a} \geq 2$ and let j_2 be the Jacobian witnessing this, so $j_2 \subset J$. Now let $j_{2,a..a} = \{\bar{g}_1\bar{g}_2 : (\bar{g}_1, \bar{g}_2, a, \ldots, a^{(2)}) \in j_2\}$. Let N be the union of components of $J(\bar{g}_1\bar{g}_2\bar{x}_1\bar{x}_2) \wedge J(\bar{g}_1\bar{g}_2\bar{x}_3\bar{x}_4)$ witnessing tangency of $\bar{g}_1\bar{g}_2$ at $(aba\ldots a)^{(2)}$. Then by specialisation, we have

$$N(\bar{g}_1\bar{g}_2ab\ldots a^{(2)}, ab\ldots a^{(2)})$$

and moreover

$$N(\bar{g}_1\bar{g}_2ab\ldots a^{(2)}, ab\ldots a^{(2)}) \text{ iff } mult_{\bar{g}_1\bar{g}_2}^{ab\ldots a} \geq 2, \text{ that is } j_2^{a\ldots a}(\bar{g}_1\bar{g}_2)$$

Now fix $\bar{g}_1', ab\ldots a, a'b'\ldots a''$ such that

$$I_k(\bar{g}_1', a\ldots a) \wedge I_k(\bar{g}_1', a'\ldots a'')$$

with $\bar{g}_1' \in \mathcal{V}_{\bar{g}_1} \cap N^k$ generic and $a'a''$ generic over a

We have,

$dim(\bar{g}_2' \in N^k : J(\bar{g}_1'\bar{g}_2'ab\ldots a^{(2)}) \wedge \exists y_1 \ldots y_{k-1} J(\bar{g}_1'\bar{g}_2'a'b'\ldots a'', a'y_1\ldots y_{k-1}a'') = dimN^k - 1$

as we may assume that $a'a''$ is independent from aa and clearly $dim(a'a''/\bar{g}_2'aa) = 1$ as $\bar{g}_2'$ is a curve. Hence, by forking symmetry, and the fact that $\bar{g}_2' \in N^k$ gives $dim(\bar{g}_2'/aa'a'') \leq k-1$. Conversely if $\bar{g}_2$ is a generic element of N^k passing through $a'a''$, we have that $dim(\bar{g}_2a'a''/aa) = dim(\bar{g}_2/a'a''aa) + 2 = dim(a'a''\bar{g}_2/aa) = 1 + dimN^k$. Hence,

$dim(\bar{g}_2' \in N^k : \exists y_1 \ldots y_{k-1} N(\bar{g}_1'\bar{g}_2'ab\ldots a^{(2)}, a'b'\ldots a'', a'y_1\ldots y_{k-1}a'')) = dimN^k - 1$

However, $dim(\bar{g}_2' \in N^k : N(\bar{g}_1\bar{g}_2'ab\ldots a^{(2)}, ab\ldots a^{(2)})\} = dim(\bar{g}_2' \in N^k : j_2^{ab\ldots a}(\bar{g}_1\bar{g}_2')) \leq dimN^k - 1$ by properties of multiplicities, and hence

$dim(\bar{g}_2' \in N^k : \exists y_1 \ldots y_{k-1} N(\bar{g}_1\bar{g}_2'ab\ldots a^{(3)}, ay_1\ldots y_{k-1}a)\} \leq dimN^k - 1,$

as removing a finite subset of points from $\mathcal{M}$ we may assume that there is no point (cd) distinct from (ab) such that every element of N passes through (cd)

So $\bar{g}_1, ab\ldots a, aa$ is regular for the cover

$\exists y_1 \ldots \exists y_{k-1} N(ab\ldots a^{(2)}) \rightarrow N^k \times \mathcal{M}^{k+1} \times \mathcal{M}^2$

It follows if we choose $\bar{g}_1', a'b'\ldots a'', a'a''$specialising to $\bar{g}_1, ab\ldots a, aa$ that we can find $\bar{g}_2' \in \mathcal{V}_{\bar{g}_2}$ such that $\exists y_1 \ldots y_{k-1} N(\bar{g}_1'\bar{g}_2'a\ldots a^{(2)}, a'b'\ldots a'', a'y_1\ldots y_{k-1}a'')$ and hence that

$N(\bar{g}_1'\bar{g}_2'a\ldots a^{(2)}, ab'\ldots a'', a'b''\ldots a''),$

where we may assume that $a'b''\dots a'' \in \mathcal{V}_{ab\dots a}$ as N is finite over $y_1 \dots y_k$

Now suppose we choose $a' \in \mathcal{V}_a$ and $\bar{g}_1' \in \mathcal{V}_{\bar{g}_1} \cap N^k$ inducing $a'b'\dots a'' \in \mathcal{V}_{ab\dots a}$. Then by the above we may find $\bar{g}_2' \in \mathcal{V}_{\bar{g}_2} \cap N^k$ with the property that $N(\bar{g}_1'\bar{g}_2'a\dots a^{(2)}, a'b'\dots a', a'b''\dots a')$ holds, that is $\bar{g}_2' \in N^k \cap \mathcal{V}_{\bar{g}_2}$ and $a'b''\dots a'' \in \mathcal{V}_{ab\dots a}$ is a trajectory induced by $\bar{g}_2'$ such that $\bar{g}_2'(a') = \bar{g}_1'(a')$. This is precisely the old definition.

The case of infinite intersections can be handled more easily. Note that the set Inf of $\bar{g}_1\bar{g}_2$ satisfying this property defines an equivalence relation on $N^k \times N^k$. Taking the quotient by this equivalence relation, and choosing a smooth set of representatives for our curves we can reduce to the case where distinct parameters in N^k/Inf define curves with finite intersection near $ab\dots a$. Now the case where $\bar{g}_1 = \bar{g}_2$ is trivial.

□

This shows easily that $\mathcal{T}$ defines an equivalence relation on $\mathcal{N}^k \times \mathcal{N}^k$

After defining tangency, one proceeds by finding a definable group inside $\mathcal{M}^{eq}$, using a similar technique to the 1-based case. We have the incidence relation $R \subset N^2 \times \mathcal{M}^3$ given by

$$R(g_1g_2xyz) \text{ iff } (xy) \in g_1, (yz) \in g_2^{-1}$$

In the stable case, the tangency relarion $\mathcal{T}$ defines an equivalence relation on N^2 and we take $G(\mathcal{M})$ to be the quotient $N^2/\mathcal{T}$. Then we can define composition on $G(M)^3$ by

$$m(\bar{a}, \bar{b}, \bar{c}) \text{ iff } \exists g_1g_2g_3 \in N^3(g_1g_2\mathcal{T}\bar{a} \wedge g_2g_3\mathcal{T}\bar{b} \wedge g_1g_3\mathcal{T}\bar{c})$$

If g_2 and $\bar{b}$ are fixed and independent, then in the stable case one can always find g_3 such that $g_2g_3\mathcal{T}\bar{b}$ from which it follows that m is defined for independent generic realisations of $G(\mathcal{M})$. Moreover, tangency preserves composition from which it follows that m is single valued. Finally, one checks associativity on independent generic realisations of $G(\mathcal{M})$. By the Hrushowski-Weil Theorem, the generic type ρ of $G(\mathcal{M})$ generates a 1-dimensional group in $\mathcal{M}^{eq}$ which by general facts on minimal groups must be abelian.

It is worth speculating on what could happen in the simple case. First, if we have a notion of specialisation, then it is unlikely we can show anything as strong as Theorem 58, rather we might have to weaken the conclusion to assert only the existence of $a' \in \mathcal{V}_a$ generic over a. In this case, the definition of tangency $g_1\mathcal{T}'g_2$ as given in [29] can be weakened to a purely existential statement requiring that the witnesses $(a'g_1'g_2')$ for tangency are generic over (ag_1g_2). Unfortunately, the new relation $\mathcal{T}'$ is no longer an equivalence relation. One can take the transitive closure of completions $\mathcal{T}'_i$ of such a relation and show it is definable using methods as in [14], unfortunately without rather strong assumptions this seems to lead to multi-valuedness of m. Alternatively, one can hope for sufficiently generic behaviour to show that the completions in fact coincide.

In the stable case, the next step is to transfer the (multiplicative) group $G(\mathcal{M})$ back to $U \subset \mathcal{M}$ and define a notion of addition to give a ring structure on U. This is sufficient to define a non-nilpotent matrix group M of rank 2 inside U. It is then possible to find a mimimal subgroup M' on which M acts non trivially by conjugation. The field is then recovered inside M^{eq} as $End_M M'$, see [8] for the general idea behind this construction, and section 15 for some progress towards carrying out this step for simple theories.

Chapter 13

Zariski Structures and Simple Theories

We will make a few remarks with reference to the example of an algebraically closed field F with a generic predicate P, see also remarks in Section 13. As is shown in [7], the completions of $T_{ACF,P}$ are obtained by specifying P on $acl_{ACF}(\emptyset)$. Working in a saturated model (F, P), if $\bar{a}$ and $\bar{b}$ are tuples and $A \subset (F, P)$, then $tp(\bar{a}/A) = tp(\bar{b}/A)$ iff there is an isomorphism of $\mathcal{L}_{T_{ACF,P}}$ structures taking $acl_{ACF}(A(\bar{a}))$ to $acl_{ACF}(A(\bar{b}))$. Then clearly any n type $p(\bar{x})$ is determined by formulae of the form $\exists\bar{y}\phi(\bar{x},\bar{y})$ with $\phi(\bar{x},\bar{y})$ quantifier free in $\mathcal{L}_{T_{ACF,P}}$. Enumerating the n-types containing a given formula $\theta(\bar{x})$, it follows by compactness that

$$\theta(\bar{x}) \equiv \vee_{i=1}^{m} \exists\bar{y}\phi_i(\bar{x},\bar{y})$$

where $\phi_i(\bar{x},\bar{y})$ is of the form $C(\bar{x},\bar{y}) \wedge P^{+,-}(\bar{x},\bar{y})$ with C a constructible set in ACF and $P^{+,-}$ some assignment of P or $\neg P$ to the variables $\bar{x}\bar{y}$.

Now we work inside the reduct $P(F)$, which is simple of SU rank 1. We will take our closed sets $X \subset P(F)^n$ to be of the form $C(\bar{x}) \wedge P(\bar{x})$ where $C(\bar{x})$ is a Zariski closed set inside F^n or the complement of $\exists\bar{y}(C(\bar{x},\bar{y}) \wedge P(\bar{x},\bar{y}))$ where $C(\bar{x},\bar{y})$ is constructible and a finite cover of $\exists\bar{y}C(\bar{x},\bar{y})$ in F^n. The need for taking both forms of closed set will be apparent soon. Now we consider the Zariski structure axioms;

(L) is clear.

(P) This is problematic and the introduction of the second form of closed set is one possible solution. Let $C(\bar{x},y) \wedge P(\bar{x},y) \subset P(F)^{n+1}$ be a quantifier free closed set, the other case will be similar. Let $pr : P(F)^{n+1} \to P(F)^n$ be a projection. We have that $\{\bar{a} : C(\bar{a}) = F\}$ is Zariski closed and definable by a formula $D(\bar{x})$. By the axioms for the generic predicate, using the fact that the fibres over D are infinite, the quantifier free formula $D(\bar{x}) \wedge P(\bar{x})$ is equivalent to $pr(C(\bar{x},y) \wedge D(\bar{x}) \wedge P(\bar{x},y))$. Let $U(\bar{x})$ be the complement to $D(\bar{x})$ in $\exists y C(\bar{x},y)$, then $U(\bar{x}) \wedge C(\bar{x},y)$ is constructible and a finite cover of $U(\bar{x})$ in F^{n+1}. Then $pr(U(\bar{x}) \wedge C(\bar{x},y) \wedge P(\bar{x},y))$ is just $\exists y(U(\bar{x}) \wedge C(\bar{x},y) \wedge P(\bar{x},y))$ which is an open set of the second form. Then $pr(C(\bar{x},y) \wedge P(\bar{x},y))$ is constructible.

(DCC) This completely fails. Consider the family of curves $\{C_n(x,y) : n \geq 2\}$ inside F^2 given by $y = x^n$. Let $\phi_n(x, y_2 y_3 \ldots y_n)$ be the formula $C_2(xy_2) \wedge C_3(xy_3) \wedge \ldots \wedge C_n(xy_n)$. Then clearly I can find a tuple $xy_2 \ldots y_n$ satisfying ϕ_n with $xy_2 \ldots y_n$ distinct and disjoint from $acl_{ACF}(\emptyset)$. By the axioms for generic predicate, I can find such a tuple with the assignment P to x, $\neg P$ to $y_2 \ldots y_{n-1}$ and P to y_n. Now consider the closed sets of the second form given by $X_n(x) = \neg\exists y(C_n(xy) \wedge P(xy))$. Then by construction we have that the sequence $\{X_2 \cap \ldots \cap X_n : n < \omega\}$ forms an strictly decreasing chain of closed sets inside $P(F)$. Note that if we restrict ourselves to quantifier free definable sets then the DCC holds trivially, but we then have no analogue of P. If a closed set X is definable over $\bar{a}$ and we consider only closed sets defined over $acl_{ACF}(\bar{a})$, then clearly X can have at most countably many irreducible components. We also have that if $\bar{a}$ is a tuple in $P(F)^n$ and $A \subset P(F)^n$, then in general $locus(\bar{a}/A)$ will now only be type definable.

(DIM) If we allow for possibly type definable closed sets, which the above seems to require, then for X closed irreducible over $\bar{a}$, one can hope to define $dim(X)$ to be the length of the maximal chain of irreducible closed subsets of X over $acl_{ACF}(\bar{a})$, such a chain will consist of type definable sets! In the case of DCF, one generally needs to pass outside $acl_{DCF}(\bar{a})$ to count the dimension of a closed set defined over $\bar{a}$, it would then be useful to consider what happens in the example of DCF with a generic predicate.

(PS) Again, this seems likely to fail, but by analogy with what happens in the case of a vector space with generic predicate, one hopes to recover intersections by finding "parallel" curves.

In the case of (F, P), one expects that the right version of these axioms will be enough to prove the following version of Noether normalisation. Namely, X is an irreducible (type definable) closed set of dimension k iff there exists a generically finite map of X onto $P(F)^k$. That is a map $pr : X \to P(F)^k$ such that $\bar{pr(X)} = P(F)^k$ and $\{\bar{x} \in P(F)^k : \exists^{=\infty}\bar{y}(X(\bar{x}, \bar{y}))$ has dimension strictly less then k. This would be enough to equate dim with SU-rank.

Clearly, $P(F)$ has type definable independence which would be sufficient to recover an analogue of (DF).

Ultimately, one hopes to obtain (GF), the generic fibres lemma and show the notion of dim is additive.

Chapter 14

Interpreting a field in T_{SCF_p}

In this section, we show a partial result towards interpreting a field inside any simple non abelian group defined in T_{SCF_p} using only the group language. An analogous question holds for simple non abelian group definable in pseudofinite fields. In [29], Zilber is able to find a non-nilpotent group after defining tangency. Therefore, positive answers to these questions provide further support for carrying out the Zariski construction for simple theories.

Let $G(L)$ be a simple, non abelian group definable in a seperably closed field L of characteristic p, we aim to show that $G(L)$ interprets the field L in the group language. By results already proved in [26] we know that $G(L)$ may be considered as the L-rational points of an algebraic group G defined over the original field L. We consider a series of reductions of G to a semisimple linear algebraic group defined over the prime subfield F_p, and use some standard facts about the structure of such groups.

First, by Chevalley's theorem, we can find a maximal normal subgroup N of G, such that the following sequence;

$e \rightarrow N \rightarrow G \rightarrow G/N \rightarrow e$

is exact, N is linear algebraic, and G/N is an abelian variety. We claim first that

N is defined over L. Otherwise, working in a large algebraically closed field $K \supset L$, we can find $\sigma \in Gal(K/L)$ such that $N^\sigma \neq N$ and clearly $N^\sigma \subset G$. Consider the group $H =< N, N^\sigma >$ containing N. We will show that H is linear algebraic, which clearly gives the result.

First choose an affine embedding θ of N into $GL(n, K)$ for some n. Then extend θ to $N \cup N^\sigma$ by setting $\theta(x^\sigma) = \theta(x)^\sigma$ for $x \in N$, clearly this map is well defined. Moreover, if $w(x_1, \dots x_n)$ is a word of length n belonging to $N \cup N^\sigma$, then I claim that $\theta(w(x_1, \dots, x_n)) = w(\theta(x_1), \dots, \theta(x_n))$. This is seen by induction on n, so let $w(x_1, \dots, x_{n+1})$ be a word of length $n+1$, then without loss of generality we may assume that $w(x_1, \dots, x_{n+1}) \in N$ and some $x_i \in N$. Now using the fact that both N and N^σ are normal subgroups of G, we can replace $w(x_1, \dots, x_{n+1})$ by

$$x_i(x_i^{-1}w(x_1)x_i)\dots(x_i^{-1}w(x_{i-1})x_i)w(x_{i+1})w(x_{i+1}, \dots x_{n+1}).$$

This gives a word of the same length and equal to the original word, so we may assume that $x_1 \in N$. Then it follows that $w(x_2, \dots, x_{n+1})$ is also in N. Now using the fact that θ is a homomorphism on N, we obtain $\theta(w(x_1, \dots, x_{n+1}) = w(\theta(x_1))\theta(w(x_2, \dots, x_{n+1}) = w(\theta(x_1), \dots, \theta(x_n))$ by the induction hypothesis.

Now let $H' =< \theta(N), \theta(N)^\sigma >$, we want to extend θ from H to H'. First, as N and N^σ are connected groups, it follows by Zilber's indecomposability theorem that elements of the group H may be written as words of bounded length n in elements from $N \cup N^\sigma$. We therefore extend θ by setting $\theta(w(x_1, \dots, x_n)) = w(\theta(x_1), \dots, \theta(x_n))$. We show that this is well defined. So suppose that $w(x_1, \dots, x_n) = w'(y_1, \dots, y_n)$, then concatenating w and w', we have a longer word $w''(x_1, \dots x_n, y_1, \dots, y_n) = e$. Now applying θ and using the above result gives $w''(\theta(x_1), \dots \theta(x_n), \theta(y_1), \dots \theta(y_n)) = e$, which gives the result

We have shown how to extend θ to an abstract isomorphism between H and H', it remains to show that is an algebraic isomorphism.

In order to see this, note that we can use θ to pull back the Zariski topology on H' to H, so it only remains to check that this topology is compatible with the original multiplication μ on H. So let $\Gamma' \subset H^3$ be the graph of μ', the pullback of multiplication on H' and let $(a, b, \mu'(a, b))$ be generic in Γ' over the field F of definition of μ'. We clearly have that $(a, b, \mu'(a, b))$ is in the graph Γ which is defined over $L \subset F$. Hence, it follows that $\Gamma' \subset \Gamma$. Moreover, as Γ and Γ' are irreducible, if strict inequality holds, we must have that $dim(\Gamma') < dim(\Gamma)$. Using the fact that the domains agree, taking a generic fibre this implies that μ is multivalued which is absurd...(this argument is not essential!)

The above shows that H is defined over L. We now consider the subgroup $H \cap G(L)$ of $G(L)$ and claim this is definable inside $G(L)$, no proof is given. As $G(L)$ was assumed simple, this forces $G(L) \cap H$ to be e or $G(L)$.

Now consider the canonical map,

$\pi : G \to G/H$

By elimination of imaginaries, it is straightforward to see this map is also defined over L. If $G(L) \cap H = e$, then π gives a definable embedding of $G(L)$ into the L-rational points $G/H(L)$ of an abelian variety. This implies that $G(L)$ is abelian which is not the case. Hence, we may assume that $G(L)$ consists of the L-rational points $H(L)$ of a linear algebraic group H defined over L.

We now assume that H is definable over L in the stronger algebraic sense, that is the ideal $I(H)$ of H is generated by polynomials with coefficients in L. Let N be the maximal normal solvable subgroup of H. Again we can observe that N is definable over L in the weaker sense, as if not we can find a $\sigma \in Gal(K/L)$ such that $N^\sigma \neq N$

and $N^\sigma \subset H$. We have the group $< N, N^\sigma >$ is also solvable, as;

$$< N, N^\sigma > /N \cong N/N \cap N^\sigma$$

The right hand side is clearly solvable as the quotient of a solvable group, hence as N is solvable, so is the left hand side.

We cannot deduce that N is strongly definable over L. However, we may apply the Frobenius map Fr^n to the coefficients of polynomials defining $I(N)$. This defines a bijective morphism (not an isomorphism!) between H and H^{Fr^n} which is strongly definable over L^{p^n} and whose maximal normal solvable subgroup N^{Fr^n} is strongly definable over L. This map also sets up an isomorphism between $H(L)$ and $H^{Fr^n}(L^{p^n})$

We now work with the L rational points of H^{Fr^n}. Again, assume that $H^{Fr^n}(L) \cap N^{Fr^n} = e$ and let

π be the canonical map

$$\pi : H^{Fr^n} \to H^{Fr^n}/N^{Fr^n}.$$

Let $f_1, \ldots, f_m$ be local uniformisers at e for H^{Fr^n}/N^{Fr^n}. As π is dominant, it follows that $\pi^* f_1, \ldots, \pi^* f_m$ are algebraically independent in the ring of functions $R(G^{Fr^n})$ of G^{Fr^n}. Then, as they clearly vanish on N, it follows that $Rad(< \pi^* f_1, \ldots, \pi^* f_m >) = I(N)$. We want to show more generally that the $\pi^* f_i$ generate the normal bundle J/J^2 for N^{Fr^n}. Suppose not, then taking a uniformisers g at $x \in N$, we can find integers $n \geq 2$ such that $g^n = \pi^* f_m$for some m. Then, as $\pi^* f_m$ is N invariant, it follows that so is g. Now as $f_1, \ldots f_m$ are local uniformisers, we can find h integral over $K[f_1, \ldots f_m]$ such that $f_1, \ldots, f_m, h$ generate the ring of functions $R(H^{Fr^n}/N^{Fr^n})$, hence $\pi^* f_1, \ldots, \pi^* f_m, \pi^* h$ generates the ring of N invariants. This gives us a relation of the form

$$g = p(\pi^* f_1, \ldots, \pi^* f_m, \pi^* h)$$

and hence

$$\pi^* f_m = g^n = p(\pi^* f_1, \ldots, \pi^* f_m, \pi^* h)^n$$

which gives an algebraic dependence between the f_i's contrary to hypothesis. It follows that the $\pi^* f_i$ generate the ideal sheaf J and in particular as N^{Fr^n} is affine, generate $I(N^{Fr^n})$. By hypothesis, we can take the f_i to have coefficients in L, and similarilily for h being integral over the f_i,

Now as N^{Fr^n} is smooth, the map

$$d : J/J^2 \rightarrow \Omega_H \otimes \mathcal{O}_{N^{Fr^n}}$$

is injective, see Theorem 40, and in particular the differentials $d\pi^* f_i$ are non zero. Hence, we have that the map π defined by the polynomials $f_1, \ldots, f_m, h$ is seperable and in particular the polynomials $f_1, \ldots, f_m, h$ are seperable over L. It follows that π defines an isomorphism between the L rational points $G^{Fr^n}(L)$ and $G^{Fr^n}/H^{Fr^n}(L)$ using the fact that L is seperably closed.

We have now reduced to the case of considering L rational points for a semisimple algebraic group G defined over L. We first aim

to descend the field of definition of G to F_p^{alg}. So let the tuple $\bar{l}$ define G and let $\bar{a}$ be a generic point of G over $F_p^{alg} \cup \bar{l}$. Then consider

$$V = locus(\bar{a}\bar{l}/F_p^{alg})$$

As multiplication μ on G is defined over F_p, the statement $\phi(\bar{x}) \subset pr(V)$ given by μ defines a multiplication on the fibre $V(\bar{x})$ is algebraic, defined over F^p and holds for the generic point $\bar{l}$ of $pr(V)$ over F_p^{alg}. Hence, the fibres of V are linear algebraic groups almost everywhere. It can also be shown using Zilber's indecomposability theorem, and definabilty of the dimension of fibres that the statement $\phi'(\bar{x})$ given by "the fibre $V(\bar{x})$ is a semisimple algebraic group of dimension n" is also definable over F_p. Again, it follows this holds almost everywhere on $pr(V)$. Now using the fact that $F_p^{alg} \prec L^{alg}$, we can find a parameter $\bar{f} \in \phi'(F_p^{alg})$, and hence the fibre $V(\bar{f})$ defines a semisimple algebraic group.

It remains to show that $V(\bar{l})$ and $V(\bar{f})$ are biregularly isomorphic as algebraic groups over L^{alg}. However, this follows from the fact that $\phi(x')$ defines a continously varying family of semisimple algebraic groups of given dimension. Using the isomorphism theorem for such groups, there can only be finitely many isomorphism types for the fibres over L^{alg} and hence the isomorphism type is constant.

Now let θ be an isomorphism defined over L^{alg} between G and G' where the latter is defined over F_p^{alg}. Again, we alter θ by Frobenius to get rid of the inseperability in the coefficients defining θ;

$\theta: \; G \to G'$

$Fr^n : \downarrow \qquad \downarrow$

$\theta^{Fr^n} : G^{Fr^n} \to G'$

Then similarily to before Fr^n defines an isomorphism between $G(L)$ and $G^{Fr^n}(L^{p^n})$, and one easily shows that θ^{Fr^n} defines an isomorpism between $G^{Fr^n}(L)$ and $G'(L)$.

We now finally descend the field of definition of G' from F_p^{alg} to F_p. As the field of definition is a finite extension of F_p, its normal closure is finite and hence seperable over F_p, (the Frobenius is onto). Therefore we may assume the field of definition to be

Galois over F_p. Now the result is classical, see [33], we obtain a biregular isomorphism between G' and G'' defined over F_p^{alg}. This clearly preserves L-rational points.

We are now in the situaion of considering the L rational points of a semisimple group G'' defined over F_p. Using the theory of Borel subgroups and the Frobenius which fixes G'', it looks fairly straightforward to interpret the field L inside $G''(L)$

We still have to prove the following result though;

If $G(L)$ and $G(L^{p^n})$ are the sets of L and L^{p^n} rational points for an algebraic group defined over L^{p^n}, then if $G(L)$ interprets the field L in the group language, then so does $G(L^{p^n})$.

This seems very plausible given that the fields L and L^{p^n} are elementarily equivalent.

Alternatively, the proof goes through if the following, for which I know no counterexample, is true for linear algebraic groups.

Given a linear algebraic group defined over L a seperably closed field, $Rad(G)$, the maximal normal solvable group is defined over L in the sense of algebraic geometry (We know it must be defined over some purely inseperable extension $L^{p^{-n}}$.) (*)

Given $(*)$, we have tentatively,

Theorem 63. *Any simple, non-abelian group G defined in $T_{SCF,p}$ interprets a field using only the group language.*

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