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ON LOGARITHMIC TRANSFORMATIONS IN DISCRETE-TIME STOCHASTIC CONTROL

W.J. Runggaldier*
Seminario Matematico
Universita di Padova
Padova, Italy

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Introduction:

Logarithmic transformations for stochastic control and filtering in continuous time have been discussed e.g. in [1-5]. After an outline, given in the next Section 1, of some of the results contained in the above references, we extend them to the case of discrete time. In Section 2 we first present a general "duality result" as a rather straightforward extension of [6]. Using this general result, we then discuss possible ways of applying it to various particular situations, thereby obtaining results that in some sense parallel those available in continuous time. Comments on further possible uses of the results obtained here are in the concluding Section 3.

1. Summary of results available in continuous time

1.1 Introduction: Positive solutions to backwards equations and stochastic control problems

We follow [1]. Given a linear operator $L + V(x)$, where L is the generater of some Markov process x_t and $V(x)$ is a "potential", consider the backwards equation

$$(1.1)\quad \frac{\partial \phi(s,x)}{\partial s} + L\phi(s,x) + V(x)\phi(s,x) = 0, \quad s \leq T, \quad \phi(T,x) = \bar{\phi}(x)$$

Given a positive solution $\phi(s,x)$ of (1.1) we seek a probabilistic interpretation for

$$(1.2)\quad S(s,x): = -\lg \phi(s,x)$$

as optimal cost-to-go for a certain stochastic control problem. Relation (1.2) changes (1.1) into

$$(1.3)\quad \frac{\partial S}{\partial s} + H(S) - V(x) = 0, \quad S(T,x) = -\lg \bar{\phi}(x)$$

where

$$(1.4)\quad H(S): = -e^{S}L(e^{-S})$$

is a concave finction of S which we express as

(1.5) $$H(S)(x) = \inf_u \, [L^u S(x) + k(x,u)]$$

where L^u will be the generator of another (controlled) Markov process ξ_t. With (1.5) equation (1.3) becomes

(1.6) $$\frac{\partial S}{\partial s} + \inf_u [L^u S + k(x,u) - V(x)] = 0$$

which is the dynamic programming equation of the following stochastic control problem:

(CP) $$\begin{cases} \text{Find a feedback } u \text{ minimizing} \\ \quad J(s,x;u) = E_{s,x}\left\{ \int_s^T [k(\xi_t,u_t) - V(\xi_t)]\,dt + \psi(\xi_T) \right\} \\ \quad \text{where } \xi_t \text{ has generator } L^u \ (\xi_s = x) \text{ and } \psi(\cdot) = -\lg \phi(\cdot). \end{cases}$$

The probabilistic interpretation that we were looking for is then

(1.7) $$-\lg \phi(s,x) = S(s,x) = \min_u J(s,x;u)$$

In order to use these results to actually construct a control problem that corresponds (is dual) to (1.1) we need to solve the following

Problem: Given L, find L^u and $k(x,u)$ (and possibly an optimal control u^*).

In the next subsection 1.2 we shall present a general result to solve this problem. In many cases of interest however, an L^u and $k(x,u)$ are immediately suggested by the form of equation (1.3). This is the case e.g. in the following example where we shall be purely formal

Example 1.1 Let L be the generator of the following diffusion in R.

(1.8) $$dx_t = b(x_t)\,dt + dw_t$$

so that (1.3) becomes

(1.9) $$S_s + \tfrac{1}{2} S_{xx} - \tfrac{1}{2}(S_x)^2 + bS_x - V = 0$$

If we now consider the equation

(1.10) $$S_s + \min_u \left[\tfrac{1}{2} S_{xx} + (b+u)S_x + \tfrac{1}{2} u^2 - V \right] = 0$$

we immediately obtain the miniminizing

(1.11) $u^{*} = -S_x$

for which (1.10) becomes (1.9).

Equation (1.10) is the dynamic programming equation for

(1.12) $$\begin{cases} dx_t = [b(x_t) + u(x_t)]\, dt + dw_t \\ \min E\left\{ \int_0^T \tfrac{1}{2}u_t^2 dt - \lg \bar{\Phi}(x_T) \right\} \end{cases}$$

Analogous results are obtained in [1] also for L corresponding to a jump process by using

$$e^{r} = \max_{u>0} [ur + u - u \lg u]$$

1.2 A general result of Sheu

Consider (1.1) where for simplicity $V(x)=0$ and where L satisfies a positive maximum principle and let

(1.13) $\phi(T,x) = \bar{\phi}(x) = \exp(-\bar{\Phi}(x))$

It is proved in [5] that for S(s,x) as defined in (1.2) we then have

(1.14) $$\frac{\partial S}{\partial s} - e^{S}L(e^{-S}) = $$
$$= \frac{\partial S}{\partial s} + \inf_{g(\cdot)>0} [L^{g}S + k^{g}] = 0;\ S(T,x) + \bar{\Phi}(x)$$

where (g(·)>0 is such that the expressions below make sense)

(1.15) $$\begin{cases} L^{g}f: = \frac{1}{g}[L(fg) - fLg] \\ k^{g}: = L^{g}(\lg g) - \frac{Lg}{g} \end{cases}$$

In addition

(1.16) $\arg\min [L^{g}S(s,x) + k^{g}(s,x)] = \phi(s,x) = \exp(-S(s,x))$

Example 1.2 Let us apply the previous result to the L of Example 1.1. Letting g be C^2 with respect to the x-variable, we have

(1.17) $L^{g}S = \frac{1}{2} S_{xx} + [b + (\lg g)_x]\, S_x$

Putting

(1.18) $u = (\lg g)_x$

we obtain from (1.17)

$$(1.19) \qquad L^u S = \frac{1}{2} S_{xx} + [b + u] S_x$$

On the other hand

$$(1.20) \qquad k^g = \frac{1}{2}((\lg g)_x)^2 = \frac{1}{2} u^2$$

and we reobtain the control problem (1.12) for which g^* (optimal g) = ϕ implies (see (1.11))

$$(1.21) \qquad u^* = -S_x$$

Analogous results can be obtained for jump-processes, discrete-state processes, etc.

1.3 Positive solutions to forward equations and stochastic control problems

Before applying the previous results to obtain a duality relation between stochastic filtering and control, let us analyse the situation when, instead of the backwards equation (1.1), we start from a forwards equation

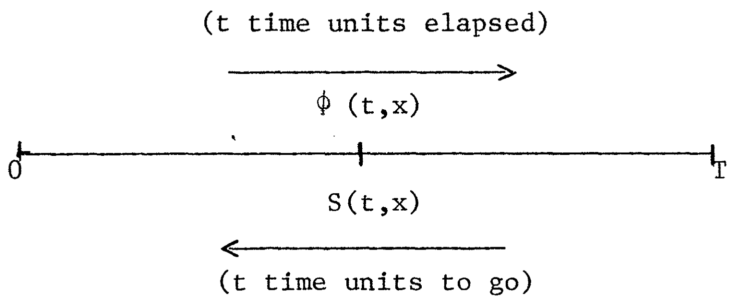
$$(1.22) \qquad \frac{\partial \phi}{\partial t} = L^* \phi \qquad ; \phi\,(0,x) = \tilde{\phi}(x)$$

For simplicity, let us assume that L^* corresponds to the diffusion (1.8) so that, if $\tilde{\phi}(x)$ is the initial density of the process x_t, $\phi(t,x)$ represents its density at time t. In this case (1.22) becomes

$$(1.23) \qquad \frac{\partial \phi}{\partial t} = \frac{1}{2} \phi_{xx} - (b\phi)_x =$$

$$= \frac{1}{2} \phi_{xx} - b\phi_x - b_x\phi$$

which, after a time inversion, becomes a backwards equation with potential of the form (1.1) to which we can apply our previous results obtaining a corresponding control problem. This then leads to a "duality relation," via (1.7), between positive solutions ϕ of (1.22) and optimal cost-to-go functions S of the corresponding control problem; the effect of the time inversion is shown in the following picture



where the controlled process runs backwards in time. It also follows that, if we can solve (1.22) to obtain a positive solution, we can solve the corresponding control problems.

1.4 Applications to filtering

An application of the previous results to filtering is discussed in [2,3]. Consider the nonlinear filtering model

$$\begin{cases} dx_t = b(x_t)dt + \sigma(x_t)\, dw_t \\ dy_t = h(x_t)dt + dv_t \end{cases}$$

where x_t is the signal process, y_t the observations and $\{w_t\}$ and $\{v_t\}$ are independent standard Wiener processes. Letting $q_t(x) = q(x,t)$ denote an unnormalized conditional density of x_t, given $y^t := \{y_s, s \leq t\}$, it is well known that it satisfies the Zakai equation

$$(1.24) \qquad dq_t = L^* q_t dt + h_t q_t dy_t$$

which is a linear stochastic PDE where L^* is the forward operator corresponding to $\{x_t\}$. The substitution

$$(1.25) \qquad q(x,t) = \exp\{y_t\, h(x_t)\}\, p(x,t)$$

transforms (1.24) into the following deterministic linear PDE

$$(1.26) \qquad p_t = \tfrac{1}{2}\mathrm{tr}\, a(x) p_{xx} + g^y(x,t) p_x + V^y(x,t) p$$

where $a(x) = \sigma(x)'\sigma^1(x)$ and $g^y(x,t), V^y(x,t)$ are parametized by y^t. Again, via a time inversion, (1.26) becomes a backwards equation with potential of the form (1.1) to which we can apply our previous results to obtain a corresponding control problem where the controlled process runs backwards in time.

The resulting "duality" between filtering and control problems is exploited in [2] to obtain upper and lower estimates on $p(x,t)$ and $S(x,t) = -\lg p(x,t)$ as $|x| \to \infty$. In [4], by essentially the same methods, but using the Stratonovich form of (1.24) without going through the substitution (1.25), a duality relation is established between filtering problems for diffusion processes and stochastic control problems, where the latter are of the type of a tracking problem.

2. The discrete-time case

2.1 Introduction

In the previous Section we started our analysis from the backwards equation (1.1) as a natural equation to obtain a (backwards) dynamic programming equation. As discussed in subsection 1.3, via a time inversion, a forwards equation can, in continuous time, rather easily be reduced to a backwards equation of the form (1.1) that contains a potential term. The duality relation between positive solutions to forward equations and stochastic control problems had a natural extension to duality relations between filtering and control. Our purpose now is to obtain analogous results for the discrete-time case. Contrary to the continuous time case however, in discrete time the change from a forwards to a backwards representation is not any more as straightforward. On the other hand, one of our goals is to establish a duality relation between filtering and control, so let us start directly from a forward equation.

Given a Markov-transition kernel $p(x/y)$, corresponding to a discrete-time but continuous state-space process, consider the forward relation

$$(2.1) \qquad q_n(x_n) = \int p(x_n/x_{n-1})\, q_{n-1}(x_{n-1})\, dx_{n-1}\,,$$

$$q_o(x_o) \text{ given}$$

For a positive solution $q_n(x_n)$ of (2.1) we seek a probabilistic interpretation of

$$(2.2) \qquad S_n(x_n) := \lg q_n(x_n)$$

as optimal cost-to-go (n periods remaining) of a corresponding discrete time stochastic control problem. Notice that if $q_o(x_o)$ is the initial density of the Markov process $\{u_n\}$, then $q_n(x_n)$ represents its density in period n; there may however be other positive solutions to (2.1). Relation (2.2) changes (2.1) into

$$(2.3) \qquad S_n(x_n) = -\lg \int p(x_n/x_{n-1}) \exp(-S_{n-1}(x_{n-1}))dx_{n-1}$$

which we would like to express as "backwards dynamic programming" equation of a corresponding control problem. The fact of not having a time derivative as in (1.3) causes some difficulties with the "log" in (2.3), even the relation

$$(2.4) \quad \exp(-S(y)) = \exp_{u>0} [- S(y) + u - u \lg u]$$

leads to nowhere.

S.J. Sheu in an unpublished work [6] deals with an expression of the type

$$(2.5) \qquad \lg(\Sigma_j p_j e^{-S_j}) \quad ; p_j \geq 0, \Sigma_j p_j = 1$$

that corresponds to a discrete state-space problem and, noticing that it is convex, expresses it as the conjugate of its conjugate. Carrying over Sheu's result to the right-hand side in (2.3), namely to our continuous state-space problem, we obtain the general result of the next subsection.

2.2 A general result

Let p(x/y) be the Markov kernel in (2.1). For u(y)>0 define

$$(2.6) \qquad p^u(y/x) := \frac{p(x/y)u(y)}{\int p(x/y)u(y)dy}$$

and let $q_n(x_n)$ and $S_n(x_n)$ be as in (2.1), (2.2). Notice that (2.6) makes sense also if $\int p(x/y)dx \neq 1$ provided $\int p(x/y)u(y)dy < \infty$.

Theorem: We have

$$(2.7) \qquad -\lg \int p(x_n|x_{n-1}) \exp(-S_{n-1}(x_{n-1}))dx_{n-1} =$$

$$= \inf_{u(\cdot)>0} \left\{ \int p^u(x_{n-1}/x_n) S_{n-1}(x_{n-1})dx_{n-1} + \right.$$
$$\left. + \left[\int p^u(x_{n-1}/x_n) \lg u(x_{n-1})dx_{n-1} - \lg \int p(x_n/x_{n-1})u(x_{n-1})dx_{n-1} \right] \right\},$$

$$S_o(x_o) = -\lg q_o(x_o)$$

Furthermore, the minimizing u(•) is

(2.8) $\quad u^*(x_{n-1}) = q_{n-1}(x_{n-1})$

which implies

(2.9) $\quad p^{u^*}(x_{n-1}/x_n) = p(x_{n-1}/x_n)$

where $p(x_{n-1}/x_n)$ is the "backwards kernel" of the Markov process $\{x_n\}$ with the given "forward kernel" $p(x_n/x_{n-1})$.

Proof: (Adapted from [6]). Relations (2.7) and (2.8) follow immediately noticing that

$$\int p^u(x_{n-1}/x_n)\, S_{n-1}(x_{n-1})dx_{n-1} + \int p^u(x_{n-1}/x_n)\lg u(x_{n-1})dx_{n-1} -$$
$$- \lg\int p(x_n/x_{n-1})u(x_{n-1})dx_{n-1} + \lg\int p(x_n/x_{n-1})\exp(-S_{n-1}(x_{n-1}))dx_{n-1} =$$
$$= -\int p^u(x_{n-1}/x_n)\lg \frac{\exp(-S_{n-1}(x_{n-1}))}{u(x_{n-1})}\, dx_{n-1} +$$
$$+ \lg\int p^u(x_{n-1}/x_n) \frac{\exp(-S_{n-1}(x_{n-1}))}{u(x_{n-1})}\, dx_{n-1} \geq 0$$

with equality for

$$u(x_{n-1}) = \exp(-S_{-1}(x_{n-1})) = q_{n-1}(x_{n-1}). \boxtimes$$

Letting

(2.10) $$k^u(x) := \int p^u(y/x)\lg u(y)dy - \lg\int p(x/y)u(y)dy$$

we can rewrite (2.7) as

(2.11) $$S_n(x_n) = \inf_{u(\cdot)>0} \{k^u(x_n) + \int p^u(x_{n-1}/x_n)S_{n-1}(x_{n-1})dx_{n-1}\},$$

$$S_o(x_o) = =\lg q_o(x_o)$$

which is now indeed a dynamic programming recursion for a discrete-time stochastic control problem: The controlled process $\{x_n\}$ is a Markov process that runs backwards in time with kernel $p^u(x_{n-1}/x_n)$, which is parametized by the control $u(\cdot)$, and the running cost is $k^u(x_n)$, again parametized by $u(\cdot)$. Notice that, via the normalization in (2.6), we automatically accomplish the time inversion that was needed in continuous time to pass from the forward relation (1.22) to a backwards relation of the form (1.1).

The kernel $p^u(x_{n-1}/x_n)$ in (2.11) gives a very general, but implicit representation of the controlled process in the control problem that is dual to (2.1).

It would now be desirable that, given an explicit model for the process $\{x_n\}$ that corresponds to a given (forward) Markov kernel $p(x_n/x_{n-1})$, we would be able to construct an explicit control model that generates the kernel $p^u(x_{n-1}/x_n)$, thereby obtaining also an explicit expression for the running cost $k^u(x)$ in (2.10)

The key is the product $p(x/y)u(y)$ in the definition (2.6). According to the Theorem, the "inf" in (2.7) has to be taken over all positive functions $u(\cdot)$ so that, for a given $p(x/y)$, the product $p(x/y)u(y)$ has to be computed for all $u(\cdot)>0$ making it impossible to obtain an explicit expresssion. On the other hand, that same Theorem gives us $u^*(x_{n-1}) = q_{n-1}(x_{n-1})$ so that, without loss of generality, we can restrict the "inf" in (2.7) to any subclass of positive $u(\cdot)$, provided this class contains $u^*(\cdot)$. This implies that for a given kernel $p(x/y)$, we may obtain many dual control problems: First, for various $q_o(\cdot)>0$ we may obtain various $q_n(\cdot)>0$. Second, given a solution $q_n(\cdot)>0$ to (2.1), and therefore a $u^*(\cdot)$, we may choose various subclasses of positive $u(\cdot)$ that contain $u^*(\cdot)$.

This also implies that, if we can explicitly solve (2.1) to obtain at least one positive solution, we can also explicitly solve all corresponding control problems.

In the next subsection we exemplify this situation for a specific and very simple kernel $p(x/y)$.

2.3 Example of construction of (solvable) control problems corresponding to positive solutions of a given forward equation

Consider the model

(2.12) $\quad x_n = x_{n-1} + w_n \quad , (x_o = w_o)$

where $\{w_n\}$ is i.i.d. $\sim N(0,1)$, so that the corresponding kernel is

(2.13) $$p(x_n/x_{n-1}) = \frac{1}{\sqrt{2\Pi}} \exp\{-\tfrac{1}{2}(x_n - x_{n-1})^2\}$$

One positive solution to (2.1) for the kernel in (2.13) is

(2.14) $$q_n(x_n) = \frac{1}{\sqrt{2\Pi(n+1)}} \exp\{-\frac{1}{2(n+1)} x_n^2\}$$

to which corresponds

(2.15) $$S_n(x_n) = \lg \sqrt{2\Pi(n+1)} + \frac{x_n^2}{2(n+1)}$$

We want to construct control problems for which (2.15) is the optimal cost-to-go function. From the previous Theorem we know that

(2.16) $$u^*(x_{n-1}) = q_{n-1}(x_{n-1}) = \frac{1}{\sqrt{2\Pi n}} \exp\{-\frac{1}{2n} x_{n-1}^2\}$$

By perturbing $u^*(x_{n-1})$ to obtain various classes of positive functions $u(x_{n-1})$ that contain $u^*(x_{n-1})$, we may obtain as many corresponding explicit and solvable control problems.

Since the "inf" in (2.7) is taken over functions $u(\cdot)$ of x_{n-1} for any given x_n, we may more generally consider the "inf" in (2.7) over functions of the form $u(x_{n-1};x_n)$.

2.3.1 Dual problem I

Consider the class of controls $u(x_{n-1};x_n)$ of the form

(2.17) $$u(x_{n-1};x_n) = \frac{1}{\sqrt{2\Pi n}} \exp\left\{-\frac{1+v_n(x_n)}{2n} x_{n-1}^2\right\}$$

which contains the optimal u^* for $v(\cdot)=0$. We have

(2.18) $$p(x_n/x_{n-1})u(x_{n-1};x_n) =$$

$$= \frac{1}{2\Pi\sqrt{n}} \exp\left\{-\frac{1+n+v_n}{2n}\left(x_{n-1} - \frac{nx_n}{1+n+v_n}\right)^2\right\} \cdot$$

$$\cdot \exp\left\{-\frac{1+v_n}{2(1+n+v_n)} x_n^2\right\}$$

from which, by (2.6), we conclude that a model for $p^u(x_{n-1}/x_n)$ is

$$(2.19)\qquad x_{n-1} = \frac{nx_n}{1+n+v_n} + \sqrt{\frac{n}{1+n+v_n}}\, w_{n-1}$$

where $\{w_n\}$ are i.i.d $\sim N(0,1)$. Also, the running cost (2.10) becomes

$$(2.20)\qquad k_n(x,v) = \lg\sqrt{\frac{1+n+v_n}{n}} + \frac{(1+v_n)^2(x^2_n-1)-(1+v_n)n}{2(1+n+v_n)^2}$$

and

$$(2.21)\qquad S_o(x_o) = \lg\sqrt{2\Pi} + (x_o^2/2)$$

Notice that the control problem thus obtained is nonlinear. Notice also that, for $v_n=0$ (optimal control), (2.19) becomes in fact the backwards model corresponding to (2.12) (generates the same $q_n(x_n)$).

2.3.2 Dual problem II

Consider this time the class of controls $u(x_{n-1};x_n)$ of the form

$$(2.22)\qquad u(x_{n-1};x_n) = \frac{1}{\sqrt{2\Pi n}} \exp\left\{-\frac{1}{2n}x^2_{n-1} + v_n(x_n)x_{n-1}\right\}$$

We have

$$(2.23)\qquad p(x_n/x_{n-1})u(x_{n-1};x_n) =$$

$$= \frac{1}{2\Pi\sqrt{n}} \exp\left\{-\frac{1}{2}\frac{n+1}{n}\left[x_{n-1} - \frac{n}{n+1}\left(x_n + v_n(x_n)\right)\right]^2\right\}.$$

$$\cdot \exp\left\{\frac{n}{2(n+1)}(v_n^2 + 2v_nx_n) - \frac{1}{2(n+1)}x_n^2\right\}$$

from which we conclude that a model for $p^u(x_{n-1}/x_n)$ is

$$(2.24)\qquad x_{n-1} = \frac{n}{n+1}x_n + \frac{n}{n+1}v_n(x_n) + \sqrt{\frac{n}{n+1}}\, w_{n-1}$$

with running cost

$$(2.25)\qquad k_n(x,v) = \lg\sqrt{\frac{n+1}{n}} - \frac{1}{2(n+1)} +$$

$$+ v_n^2\left[-\frac{1}{2}\left(\frac{n}{n+1}\right)^2 + \frac{n}{n+1}\right] - v_nx_n\frac{n}{(n+1)^2} + x_n^2\frac{1}{2(n+1)^2}$$

and $S_o(x_o)$ as in (2.21).

Notice that this time the control problem is linear-quadratic. Corresponding to the optimal control $v_n=0$, it coincides with the previous problem (2.19)-(2.20);

in fact, the optimal value for the two problems is the same. Notice furthermore that, for $n \to \infty$, the present problem becomes

$$(2.26)\quad \begin{cases} x_{n-1} = x_n + v_n + w_{n-1} \\ k_n(x,v) = \frac{1}{2} v^2 \\ S_o(x_o) = \lg \sqrt{2\Pi} + x_o^2/2 \end{cases}$$

2.3.3 Dual problem III

The previous dual problems I and II were based on the positive solution (2.14) to the equation (2.1) for the kernel (2.13). Another such positive solution is

$$(2.27)\quad q_n(x_n) = 1$$

(not a density) to which corresponds

$$(2.28)\quad S_n(x_n) = 0$$

Since $u^*(x_{n-1}) = q_{n-1}(x_{n-1}) = 1$, by considering the class of controls of the form

$$(2.29)\quad u(x_{n-1};x_n) = \exp\left\{v_n(x_n)x_{n-1}\right\}$$

which contains the optimal u^* for $v_n(\cdot)=0$, we obtain, quite intuitively, the following control problem

$$(2.30)\quad \begin{cases} x_{n-1} = x_n + v_n + w_{n-1} \\ k_n(x,v) = \frac{1}{2} v^2 \\ S_o(x_o) = 0 \end{cases}$$

2.4 Example of positive solutions to a forward equation corresponding to a solvable control problem

So far the Theorem of this Section 2 has been used to construct solvable control problems that are "dual" to a given forward equation. The Theorem can however also be used to go the opposite way. The purpose of this subsection is to exemplify such procedure in two simple cases.

Example 2.4.1 Consider the very simple control problem

$$(2.31)\quad \begin{cases} x_{n-1} = x_n + v_n + w_{n-1} \\ \min \frac{1}{2} E\{\Sigma_n v_n^2\} \\ S_o(x_o) = 0 \end{cases}$$

for which the optimal control is $v_n^*=0$ with optimal cost-to-go $S_n(x_n)=0$. The purpose is to construct a forward kernel $p(x_n/x_{n-1})$ so that $q_n(x_n)=\exp(-S_n(x_n))=1$ is a positive solution to the corresponding equation (2.1). From (2.31) we have

$$(2.32)\quad p^u(x_{n-1}/x_n) = \frac{1}{\sqrt{2\Pi}} \exp\left\{-\frac{1}{2}(x_{n-1}-x_n-v_n)^2\right\}$$

and

$$(2.33)\quad p^{u^*}(x_{n-1}/x_n) = \frac{1}{\sqrt{2\Pi}} \exp\left\{-\frac{1}{2}(x_{n-1}-x_n)^2\right\}$$

From (2.9) we know that $p^{u^*}(x_{n-1}/x_n)$ coincides with the backward kernel $p(x_{n-1}/x_n)$ of the Markov process $\{x_n\}$ whose forward kernel $p(x_n/x_{n-1})$ we are looking for: therefore, as expected

$$(2.34)\quad p(x_n/x_{n-1}) = \frac{p(x_{n-1}/x_n)q_n(x_n)}{\int p(x_{n-1}/x_n)q_n(x_n)dx_n} =$$

$$= \frac{1}{\sqrt{2\Pi}} \exp\left\{-\frac{1}{2}(x_n-x_{n-1})^2\right\}$$

This example is the counterpart of the previous subsection 2.3.3 for which $q_n(x_n)$ is not a density. Notice in fact that (2.34) makes sense even if $\int q(x)dx = \infty$ as long as $\int p(x/y)q(y)dy < \infty$ (this is analagous to the case of improper prior distributions in Bayesian Statistics). The next example considers a case where $q_n(x_n)$ is not a constant, but is integrable and positive and therefore a density up to a multiplicative factor.

<u>Example 2.4.2</u> The control problem is again the linear-quadratic problem (2.31) except that now we take

$$(2.35)\quad S_o(x_o) = \lg\sqrt{2\Pi} + \frac{1}{2}x_o^2$$

One easily finds the optimal control as

$$(2.36)\quad v_n^*(x_n) = -\frac{1}{n+1}x_n$$

with optimal cost-to-go

$$(2.37) \qquad S_n(x_n) = \frac{1}{2(n+1)} x_n^2 + \alpha_n$$

where

$$(2.38) \qquad \alpha_n = \alpha_{n-1} + \frac{1}{2n} \; ; \; \alpha_o = \lg \sqrt{2\Pi}$$

From (2.32), which remains the same also for the present Example, and (2.36) we have

$$(2.39) \qquad p^{u*}(x_{n-1}/x_n) = \frac{1}{\sqrt{2\Pi}} \exp\left\{-\frac{1}{2}\left(x_{n-1} - \frac{n}{n+1} x_n\right)^2\right\}$$

which, according to (2.9), is the backwards form $p(x_{n-1}/x_n)$ of the kernel $p(x_n/x_{n-1})$ we are looking for. Since

$$(2.40) \quad q_n(x_n) = \exp(-S_n(x_n)) = \exp(-\alpha_n) \exp\left\{-\frac{1}{2(n+1)} x_n^2\right\}$$

which is a density up to a multiplicative factor, we obtain the desired forward kernel as

$$(2.41) \qquad p(x_n/x_{n-1}) = \frac{p(x_{n-1}/x_n)q_n(x_n)}{\int p(x_{n-1}/x_n)q_n(x_n)dx_n} =$$

$$= \frac{\sqrt{n^2+n+1}}{\sqrt{2\Pi}(n+1)} \exp\left\{-\frac{1}{2}\frac{n^2+n+1}{(n+1)^2}\left(x_n - \frac{n^2+n}{n^2+n+1} x_{n-1}\right)^2\right\}$$

Notice that, for $n \to \infty$, (2.41) becomes

$$(2.42) \qquad p(x_n/x_{n-1}) = \frac{1}{\sqrt{2\Pi}} \exp\left\{-\frac{1}{2}(x_n - x_{n-1})^2\right\}$$

which is exactly the starting kernel (2.13) for the examples of the previous subsection 2.3. In fact, the present example can be considered as counterpart of subsection 2.3.2: Here we started with a control problem that coincides with the problem (2.26) that was obtained there in the limit for $n \to \infty$. It is therefore quite plausible that here we could obtain the starting kernel for the problems there only in the limit for $n \to \infty$. Notice however that, for $n \to \infty$, $q_n(x_n) \to 1/\sqrt{2\Pi}$, $S_n(x_n) \to \lg \sqrt{2\Pi}$, so that in the limit we essentially return to the situation of Example 2.4.1 (again in analogy to Bayesian Statistics where improper priors can be looked at as limiting situations.)

2.5 Applications to filtering

Analogously to the continuous-time case, the previous results can be applied to obtain a form of "duality" between filtering and control problems. Given a "signal" process $\{x_n\}$, that we assume Markov with transition kernel $p(x_n/x_{n-1})$, as well as an "observation" process $\{y_n\}$, that we assume probabilistically related to the signal $\{x_u\}$ via e.g.

$$(2.43) \qquad y_n = h\ (x_n) + v_n \quad ; \{v\ \} \text{ i.i.d} \sim N(0,1)$$

we can recursively compute the conditional ("filtering") density $q_n(x\ ) = q_n(x_n;y^n)$ of x_n given all the past and present observations $y^n:=(y_o,\ldots,y_n)$ by using the recursive Bayes formula (discrete-time Zakai-equation) namely

$$(2.44) \qquad q_n(x_n) = C(y^n) \exp\left\{-\frac{1}{2}(y_n-h\ (x_n))^2\right\} \int p(x_n/x_{n-1})q_{n-1}(x_{n-1})dx_{n-1}$$

where $C(y^n)$ is a normalizing factor.

Letting $p_n(x_n)$ denote the "predictive" density of x_n, given y^{n-1}, we can write (2.44) alternatively as

$$(2.45a) \qquad q\ (x_n) = C(y^n) \exp\left\{-\frac{1}{2}(y_n-h\ (x\ ))^2\right\} p_n(x_n)$$

$$(2.45b) \quad p_n(x_n)=\int\left[p(x_n/x_{n-1})C(y^{n-1})\exp\left\{-\frac{1}{2}(y_{n-1}-h\ (x_{n-1}))^2\right\}\right]p_{n-1}(x_{n-1})dx_{n-1}$$

$$:= \int p^y(x_n/x_{n-1})\ p_{n-1}(x_{n-1})dx_{n-1}$$

Notice that, while (2.44) corresponds to the Zakai equation (1.24), the relation (2.45a) corresponds to the substitution (1.25) so that (2.45b) is the analog to (1.26). In fact, the transformation of the Zakai equation (1.24) into its ("robust") form (1.26) was performed to obtain, via a time inversion, a backward equation with potential of the form (1.1) from which we had started our analysis.

Analogously, the transformation (2.45a) allows us to transform (2.44) into (2.45b), which in form concides with (2.1) from which we had started our analysis in discrete-time.

In the next subsection 2.5.1 we parallel the analysis in [4] where, without using the substitution (1.25), one starts directly from the Stratonovich form of (1.24) to obtain a corresponding control problem in the form of a "tracking" problem.

In the following subsection 2.5.2 we then take advantage of the substitution (2.45a) leading us to (2.45b), which then allows us to proceed in the filtering case in complete analogy to what was done in subsections 2.3 and 2.4.

2.5.1 Example where the dual problem has a "tracking" interpretation

Using the transformation $q_n(x_n)=\exp(-S_n(x_n))$ and the theorem of Section 2.2 we obtain from (2.44)

$$(2.46)\qquad \begin{aligned} S_n(x_n) &= -\lg C(y^n) + \frac{1}{2}(y_n - h(x_n))^2 - \\ &\quad -\lg \int p(x_n/x_{n-1})q_{n-1}(x_{n-1})dx_{n-1} = \\ &= -\lg C(y^n) + \frac{1}{2}(y_n - h(x_n))^2 + \\ &\quad + \inf_{u(\cdot)>0}\left\{\int p^u(x_{n-1}/x_n)\, S_{n-1}(x_{n-1})dx_{n-1} + k(x_n,u)\right\} \end{aligned}$$

with

$$(2.47)\qquad u^*(x_{n-1}) = q_{n-1}(x_{n-1})$$

As in Section 2.3, in order to obtain a more explicit expression for $p^u(x_{n-1}/x_n)$ and $k(x_n,u)$, we have to consider a specific filtering model. For the sake of exemplifying the procedure let us therefore consider the following extremely simple model

$$(2.48)\qquad \begin{cases} x_n = x_{n-1} + w_n \\ y_n = x_n + v_n \end{cases}$$

where $\{w_n\}$ and $\{v_n\}$ are independent Gaussian white noise sequences. In this case the filtering density $q_n(x_n)$ is $N(m_n, \tau_n^2)$ where the mean m_n depends on the past observations y^n and τ_n^2 can be precomputed independently of y^n, more precisely

$$(2.49)\quad q_{n-1}(x_{n-1}) = \frac{1}{\sqrt{2\Pi}\ \tau_{n-1}} \exp\left\{-\frac{1}{2\tau^2_{n-1}}(x_{n-1}-m_{n-1})^2\right\} = u^*(x_{n-1})$$

Considering a class of controls of the form

$$(2.50)\quad u(x_{n-1};x_n,y^{n-1}) = \frac{1}{\sqrt{2\ \Pi}\ \tau_{n-1}} \exp\left\{-\frac{1}{2\tau^2_{n-1}}(x_{n-1}-m_{n-1})^2+v_n(x_n,m_{n-1})x_{n-1}\right\}$$

which contains the optimal u^* for $v_n(\cdot) = 0$ we have

$$(2.51)\quad p(x_n/x_{n-1})u(x_{n-1};x_n,\ y^{n-1}) =$$

$$= \frac{1}{2\Pi\ \tau_{n-1}} \exp\left\{-\frac{1}{2}\frac{1+\tau^2_{n-1}}{\tau^2_{n-1}}\left[x_{n-1} - \frac{\tau^2_{n-1}}{1+\tau^2_{n-1}}(x_n + v_n + \frac{m_{n-1}}{\tau^2_{n-1}})\right]^2\right\}.$$

$$.\exp\left\{-\frac{1}{2}\frac{1}{1+\tau^2_{n-1}}(x^2_n + m^2_{n-1}) + \frac{1}{2}\frac{\tau^2_{n-1}}{1+\tau^2_{n-1}}(v^2_n + 2v_nx_n) + \right.$$

$$\left. + \frac{1}{1+\tau^2_{n-1}}(x_n + v_n)m_{n-1}\right\}$$

so that a model for $p^u(x_{n-1}/x_n)$ is

$$(2.52)\quad x_{n-1} = \frac{\tau^2_{n-1}}{1+\tau^2_{n-1}}(x_n + v_n + \frac{m_{n-1}}{\tau^2_{n-1}}) + \sqrt{\frac{\tau^2_{n-1}}{1+\tau^2_{n-1}}}\,w_{n-1}$$

with $\{w_n\}$ Gaussian white noise.

The cost function $k(x_n,v_n)$ turns out to be a rather lenghty expression which is a quadratic in the pair (x_n,v_n) that, for large values of τ^2_n, can be approximated by $\frac{1}{2}v^2_n$.

We therefore conclude that a control model that can be considered dual to (2.48) is given by the linear state equation (2.52) and a running cost (to be minimized) that, besides a quadratic term in the state x_n and the control v_n, also contains the term $\frac{1}{2}(y_n-x_n)^2$ which leads to the tracking interpretation.

Notice that for large values of the variance we have $\tau^2_n/(1 + \tau^2_n) \simeq 1$, so that in this latter case the model just described corresponds in form to the one obtained in [4].

2.5.2 A general form of duality between filtering and control

In this subsection, instead of starting directly from (2.44), we take advantage of the substitution (2.45a) in order to start from (2.45b) which coincides in form with (2.1). This then allows us to proceed exactly as we did in Sections 2.3 and 2.4. Since the analogy is very close, we simply mention that, proceeding as in 2.3 where now $p(x_n/x_{n-1})$ is given by $p^y(x_n/x_{n-1})$ defined in (2.45b) and $9_n(x_n)$ is given by the predictive density $p_n(x_n)$, we can (see comments at the end of Section 2.2) obtain various control problems that are dual to a given filtering problem.

The analogy to Section 2.4 consists in constructing filtering models that are "dual" to a given solvable stochastic control problem: given in fact a solvable stochastic control problem (whose data depend on present and future values of certain "observations" y_n), by proceeding as in Section 2.4 we may be able to construct a kernel $p^y(x_n/x_{n-1})$ which is parametized by the same observations y_n that now however belong to the past. The possibility of ending up with the construction of an explicit (dual) filtering model is related to the possibility of factoring (modulo a constant of proportionality) the $p^y(x_n/x_{n-1})$ into the product, see (2.45b), of a kernel $p(x_n/x_{n-1})$ (the transition kernel of the original process x_n) and a conditional density $p(y_{n-1}/x_{n-1})$(density of the observation y_{n-1}, given the original x_{n-1}).

3. Conclusions

Using logarithmic transformations we have been able to establish also in discrete time a duality relation that was known in continuous time; more precisely: given a forward equation, in particular a filtering equation, of which an explicit analytic solution is known, we are able to construct various corresponding (dual) control problems. The duality automatically provides the solution to the control problem.

Although more involved, also the opposite is possible: given a stochastic control problem whose solution is known, we automatically obtain a solution to a corresponding (dual) forward equation, which under certain circumstances might be interpreted as solution to a filtering problem.

The procedure has been exemplified by using the simplest possible models. It is hoped that, by working out the procedure for more complicated filtering or control problems, for which an explicit analytic solution is known (see e.g. [7] for some control examples), one might be able to obtain as duals new explicitly solvable filtering or control problems. This is not only of interest in itself, but also because continuous-time filtering and control problems can be arbitrarily closely approximated by corresponding discrete-time problems (see e.g. [8], [9], [10]).

The basic logarithmic transformation between a positive solution $\phi(t,x)$ of a forward (filtering) equation and the optimal cost-to-go $S(t,x)$ of a corresponding control problem is given by (see (2.2))

$$\phi(t,x) = \exp(-S(t,x) \qquad (3.1)$$

This implies that $\phi(t,x)$ cannot be a (convex) combination of densities. In [8] a discrete-time filtering model is considered that admits an explicitly computable finite-dimensional filter given exactly by a combination of densities. It seems therefore that for such problems we cannot use our procedure to obtain corresponding solvable control problems. The particular model in [8] however considers step functions which essentially reduces the problem to one with a finite number of states, for which we can use the original formulation of Sheu in [6] (see (2.5)) and derive a completely analogous duality relation as was done here for continuous-state problems.

We finally mention that, analogously to Bayesian Statistics where the exponential families of distributions play an essential role in the existence of (finitely parametrized) conjugate families, one could expect that exponential families play an important role also in the existence of finite-dimensional filters in discrete-time (see e.g. [11], [12]). The duality relation (3.1) expressing a filtering density $q(t,x)$ as the exponential of $-S(t,x)$ could help in establishing similar results.

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