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A LAYER-STRIPPING SOLUTION OF THE INVERSE PROBLEM FOR A ONE-DIMENSIONAL ELASTIC MEDIUM

by

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ABSTRACT

A fast algorithm for recovering profiles of density and Lamé parameters as functions of depth for the inverse seismic problem in an elastic medium is obtained. The medium is probed with planar impulsive P and SV waves at oblique incidence, and the medium velocity components are measured at the surface. The interconversion of P and SV waves defines reflection coefficients from which the medium parameter profiles are obtained recursively. The algorithm works on a layer-stripping principle, and is specified in both differential and recursive forms. A physical interpretation of this procedure is given in terms of a lattice filter, where the first reflections of the downgoing waves in each layer yield the various reflection coefficients for that layer. A computer run of the algorithm on the synthetic impulsive plane wave responses of a twenty-layer medium shows that the algorithm works satisfactorily.

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INTRODUCTION

A layered elastic medium with depth-dependent material parameters is probed with impulsive plane waves at oblique incidence. The medium is assumed to support the propagation of seismic waves, so that there is continuing conversion between P waves and SV waves as the medium, with its depth-varying parameters, is penetrated. The goal is to recover profiles of the density $\rho(z)$ and Lamé parameters $\lambda(z)$ and $\mu(z)$ as functions of depth.

Previous work on this problem has yielded methods of solution that are computationally arduous to implement. For example, Coen (1983) solved this problem by employing solutions to the acoustic problem for the separate cases of P and SV impulsive plane waves at normal incidence, which are decoupled for a layered medium, and of SH waves at oblique incidence. This allowed the recovery of the parameter profiles by solving Marchenko integral equations, but sidesteps the issue of P-SV mode conversions. Blagoveschenskii (1967) exhibited several integral equations whose solutions yielded the parameter profiles as functions of travel times, and by combining the Gelfand-Levitan inverse scattering method with the solution of a Volterra equation, Carroll and Santosa (1982) were able to recover the parameter profiles as functions of depth. Baker (1982) solved the related problem of reconstructing radially varying parameters by using spherical harmonics and Marchenko integral equations. However, none of these methods can be considered to be attractive from a practical, computational perspective.

Clarke (1983) and Shiva and Mendel (1983) have recently given algorithms that utilize the layer-stripping principle employed by the algorithm given in this paper. However, their algorithms are much more complicated than the present algorithm, since Clarke's algorithm requires the iterative solution of an equation at each step, while Shiva and Mendel's algorithm requires the solution of a cubic equation and a maximum-likelihood estimation at each step. The present algorithm, in contrast, is quite straightforward, since the assumption of a continuous medium allows differential updates of the medium parameters.

In this paper we present a fast algorithm that recursively generates the parameter profiles $\rho(z)$, $\lambda(z)$ and $\mu(z)$ as functions of depth. There are no integral equations to solve; the computation is direct and only involves simple operations. The algorithm works on a layer-stripping principle that involves upgoing and downgoing P and SV waves and bears some similarity to the Schur algorithm and the downward continuation method which were used to solve the inverse acoustic problem in Yagle and Levy (1983) and Bube and Burridge (1983), and other inverse scattering problems in Dewilde, Fokkema and Widya (1981), Bruckstein, Levy and Kailath (1983) and Yagle and Levy (1984). The method of characteristics which was used by Symes (1981), Santosa and Schwetlick (1982) and Sondhi and Resnick (1983) to reconstruct the impedance of an acoustic medium relies also on a similar layer-stripping technique.

The paper is organized as follows. The problem is set up in detail in the next section. The algorithm is derived mathematically and exhibited in the following two sections. The next section discusses what the algorithm is doing and how it works with attention given to physical interpretations of the quantities appearing in the algorithm. A final section presents the results of a computer run of the algorithm, in which the algorithm reconstructs a twenty-layer medium from its impulsive plane wave responses.

PROBLEM FORMULATION

We consider a layered elastic medium with depth-dependent density $\rho(z)$ and Lamé parameters $\lambda(z)$ and $\mu(z)$. The following experiment is performed. An impulsive planar P wave is obliquely incident on the medium. The horizontal and vertical velocities are measured at the surface z=0 as functions of time. The experiment is then repeated with impulsive planar SV waves. The angles of incidence of the P and SV plane waves with respect to the vertical are chosen (i.e. the point source data is stacked) so that the horizontal ray parameter p is the same for both experiments. Of course, the algorithm may be run concurrently with many different values of p from a single point source experiment, the updated medium parameters at each depth from each run averaged, and the averaged values then used in the algorithms. This reduces the effect of noise in the data.

There is a choice for the boundary conditions at the surface (z=0) of the medium. In the presentation of the algorithm a free surface is assumed, so that the surface tractions τ_{zx} and τ_{zz} are both zero. Due to the vast difference in material parameters between the ground and the air, this assumption is quite reasonable. However, the medium may also be considered as being probed from an overlying homogeneous half-space in which case the upgoing P and SV waves at the surface are constructed from the velocities at the surface. Note that the assumption of planar source waves is often reasonable in practice, especially in the far field case (Aki and Richards, 1980, p.123). Also note that the desired responses for P and SV excitations could be obtained by an appropriate superposition of the responses to a P-wave source and to a mixed, P- and SV-wave source.

Since we are assuming an elastic medium, there will be continuous conversion between P-type and SV-type seismic waves as the inhomogeneous medium is penetrated. This makes the problem far more complex than the acoustic problem, for which a fast algorithm solution has already been found (Yagle and Levy, 1983). There may also be propagation of SH waves, if the impulsive SV-wave source is not completely polarized. Since any SH waves will be completely decoupled from the P and SV waves, these waves will not be considered further in this paper. It is the complexity of wave propagation in the elastic medium that allows the recovery of all three medium parameter profiles as functions of depth instead of travel time.

We now define the following quantities:

$$\alpha(z) = ((\lambda(z) + 2\mu(z))/\rho(z))^{1/2} = \text{local P-wave velocity} \tag{1a}$$

$$\beta(z) = (\mu(z)/\rho(z))^{1/2} = \text{local S-wave velocity} \tag{1b}$$

$$p = \text{horizontal ray parameter} \tag{1c}$$

$$\sin\theta_p(z) = \alpha(z)p = \text{sine of local angle between P-wave ray and vertical} \tag{1d}$$

$$\sin\theta_s(z) = \beta(z)p = \text{sine of local angle between S-wave ray and vertical} \tag{1e}$$

$$\alpha'(z) = \alpha(z)/\cos\theta_p(z) = \text{local vertical P-wave velocity} \tag{1f}$$

$$\beta'(z) = \beta(z)/\cos\theta_s(z) = \text{local vertical S-wave velocity.} \tag{1g}$$

We also define the vector

$$\underset{\sim}{g}(t,x,z) = \begin{bmatrix} u_x(t,x,z) \\ u_z(t,x,z) \\ \tau_{zx}(t,x,z) \\ \tau_{zz}(t,x,z) \end{bmatrix} \tag{2}$$

where u_x and u_z are the horizontal and vertical components of the displacement and where τ_{zx} and τ_{zz} are the horizontal and vertical tractions on an element perpendicular to the z axis.

An impulsive plane wave $\underset{\sim}{b}_0\delta(t-px-qz)$ is used to probe the elastic medium. Here $\delta(\cdot)$ denotes the Dirac delta function, and q is the vertical ray parameter just below the surface (for a free surface), or in the homogeneous half-space above the medium. The Fourier transform of this plane wave is $(\underset{\sim}{b}_0 \exp{-j\omega qz})\exp{-j\omega px}$. Since the horizontal ray parameter p is independent of depth, we may write the Fourier transform of the vector (2) for z>0 (inside the medium) as

$$\underset{\sim}{\hat{g}}(\omega,x,z) = \underset{\sim}{\hat{f}}(\omega,z)\exp{-j\omega px} \quad . \tag{3}$$

From Aki and Richards (1980, p.269), the propagation of seismic waves in an inhomogeneous, layered, continuous elastic medium is described by

$$\partial\underset{\sim}{\hat{f}}/\partial z = A(z)\underset{\sim}{\hat{f}}(\omega,z) \tag{4}$$

where

$$A(z) = \begin{bmatrix} 0 & -j\omega p & 1/\mu & 0 \\ -j\omega p\lambda/(\lambda+2\mu) & 0 & 0 & 1/(\lambda+2\mu) \\ 4\omega^2 p^2 \mu(\lambda+\mu)/(\lambda+2\mu)-\rho\omega^2 & 0 & 0 & -j\omega p\lambda/(\lambda+2\mu) \\ 0 & -\rho\omega^2 & -j\omega p & 0 \end{bmatrix} \quad (5)$$

In the next section we diagonalize equation (4), defining upgoing and downgoing P and SV waves. Appropriate weightings of the eigenvectors of A(z) will be necessary to put equation (4) into a form suitable for a fast algorithm.

TRANSFORMATION OF THE PROPAGATION EQUATION

It is well-known (e.g. Claerbout (1968)) that changing variables in equation (4) from $\hat{\underset{\sim}{f}}(\omega,z)$ to $R(z)\hat{\underset{\sim}{f}}(\omega,z)$, where R(z) is the matrix of row eigenvectors of A(z), diagonalizes equation (4) into upgoing and downgoing waves. In the present context it will be necessary to weight the row eigenvectors of A(z) in order to obtain a recursive algorithm. Thus we define

$$\hat{\underset{\sim}{w}}(\omega,z) = X(z)R(z)\hat{\underset{\sim}{f}}(\omega,z) \quad (6)$$

where X is a diagonal matrix whose elements weight the row eigenvectors of A(z). We may then write

$$\hat{\underset{\sim}{f}}(\omega,z) = R^{-1}X^{-1}\hat{\underset{\sim}{w}}(\omega,z) = CX^{-1}\hat{\underset{\sim}{w}}(\omega,z) \quad (7)$$

where $C(z) = R(z)^{-1}$ is the matrix of column eigenvectors of A(z).

Taking the partial derivative of equation (7) with respect to z and premultiplying by XR yields

$$\partial \hat{\underset{\sim}{w}}/\partial z = [\Lambda - (X(R\partial C/\partial z)X^{-1} + X(\partial/\partial z(X^{-1})))]\hat{\underset{\sim}{w}} \tag{8}$$

where

$$\Lambda = RAC = \mathrm{diag}[-j\omega/\alpha', -j\omega/\beta', j\omega/\alpha', j\omega/\beta']. \tag{9}$$

We now choose the elements of the diagonal matrix X so that the (diagonal) term $X\partial/\partial z(X^{-1}) = -(\partial/\partial z)\log|X|$ zeroes the diagonal elements of $X(R\partial C/\partial z)X^{-1}$. This is straightforward, and the result is

$$X = \mathrm{diag}[(\alpha\rho\cos\theta_p)^{1/2}, (\beta\rho\cos\theta_s)^{1/2}, (\alpha\rho\cos\theta_p)^{1/2}, (\beta\rho\cos\theta_s)^{1/2}]. \tag{10}$$

We recognize the components of X as the square roots of the P-wave and SV-wave impedances. Hence weighting the components of $R\hat{\underset{\sim}{f}}$ by these quantities normalizes the energy fluxes moving upwards and downwards.

Inserting equation (10) in equation (8) results in

$$\partial \hat{\underset{\sim}{w}}/\partial z = \begin{bmatrix} -j\omega/\alpha' & -t_c & -r_p & -r_c \\ t_c & -j\omega/\beta' & -r_c & -r_s \\ -r_p & -r_c & j\omega/\alpha' & -t_c \\ -r_c & -r_s & t_c & j\omega/\beta' \end{bmatrix} \hat{\underset{\sim}{w}} \tag{11}$$

where

$$r_p(z) = (1/2-2\beta^2p^2)(\partial/\partial z)\log\rho(z) - 4\beta^2p^2(\partial/\partial z)\log\beta(z)$$
$$+ 1/(2-2\alpha^2p^2)(\partial/\partial z)\log\alpha(z) \tag{12a}$$

$$r_c(z) = -(p/2)(\alpha'\beta')^{1/2}((1-2\beta^2p^2 + 2\beta^2/\alpha'\beta')(\partial/\partial z)\log \rho(z)$$
$$-(4\beta^2p^2 - 4\beta^2/\alpha'\beta')(\partial/\partial z) \log \beta(z)) \quad (12b)$$

$$r_s(z) = -(1/2 - 2\beta^2p^2)(\partial/\partial z) \log \rho(z)$$
$$-(1/(2-2\beta^2p^2) - 4\beta^2p^2)(\partial/\partial z) \log \beta(z) \quad (12c)$$

$$t_c(z) = (p/2)(\alpha'\beta')^{1/2}((1-2\beta^2p^2 - 2\beta^2/\alpha'\beta')(\partial/\partial z) \log \rho(z)$$
$$-(4\beta^2p^2 + 4\beta^2/\alpha'\beta')(\partial/\partial z) \log \beta(z)) \quad (12d)$$

and the quantities in equations (12) have the following interpretations:

$r_p(z)$ = reflection coefficient for a reflected P wave generated by a P wave;

$r_c(z)$ = reflection coefficient for a reflected wave generated by a wave of the opposite type;

$r_s(s)$ = reflection coefficient for a reflected SV wave generated by an SV wave;

$t_c(z)$ = transmission coefficient for a transmitted wave generated by a wave of the opposite type.

We have used here notations similar to those of Chapman (1974) and Kennett and Illingworth (1981). The physical meaning of the reflection coefficients is illustrated in Figure 1, which describes an infinitesimal section of a lattice filter structure which implements the elastic wave equation (11). Note that the elementary delay elements $D_p \triangleq \exp - j\omega\Delta/\alpha'(z)$ and $D_s \triangleq \exp -j\omega\Delta/\beta'(z)$ appearing in Figure 1 vary with depth. The lattice structure of Figure 1 can be viewed as a generalization of lattice filters used in speech processing (Markel and Gray, 1978) and linear estimation theory (Makhoul, 1977).

In the next section we use the transformed equation (11) to obtain a fast inversion algorithm.

INVERSION ALGORITHM

Recall that the first experiment consisted of probing the medium with a planar impulsive P wave. Since the first component of $\hat{\underset{\sim}{w}}(\omega,z)$ corresponds to a downgoing P wave, we may write its inverse Fourier transform $\underset{\sim}{w}(t,z)$ as

$$\underset{\sim}{w}(t,z) = \begin{bmatrix} b_p\delta(t-\tau_p(z)) \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} w_1(t,z) \\ w_2(t,z) \\ w_3(t,z) \\ w_4(t,z) \end{bmatrix} u(t - \tau_p(z)) \tag{13}$$

where

$$\tau_p(z) = \int_0^z d\ell/\alpha'(\ell) \tag{14}$$

denotes the vertical travel time for P waves, and

$$u(t) = \begin{cases} 1 & \text{for } t\geq 0 \\ 0 & \text{for } t<0 \end{cases} \tag{15}$$

is the unit step function. The second term in (13) reflects the causality of the excitation: There can be no wave at depth z until the excitation has had time to reach depth z.

Taking the inverse Fourier transform of equation (11), inserting the expression (13), and equating coefficients of $\delta(t-\tau_p)$ yields

$$r_p(z) = 2w_3(\tau_p(z),z)/(\alpha'(z)b_p) \tag{16a}$$

$$r_c(z) = w_4(\tau_p(z),z)(1/\alpha'(z) + 1/\beta'(z))/b_p. \tag{16b}$$

Now, for the second experiment, the excitation is a downgoing, impulsive SV wave. Since the second component of $\underset{\sim}{w}(t,z)$ corresponds to such a wave, we have for this experiment

$$\underset{\sim}{w}(t,z) = \begin{bmatrix} 0 \\ b_s\delta(t-\tau_s(z)) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} w_1(t,z) \\ w_2(t,z) \\ w_3(t,z) \\ w_4(t,z) \end{bmatrix} \tag{17a}$$

where the waves $w_i(t,z)$ have the form

$$w_i(t,z) = n_i(t,z)u(t-\tau_p(z)) + q_i(t,z)u(t-\tau_s(z)) \tag{17b}$$

and where the vertical travel time for SV waves has been defined as

$$\tau_s(z) = \int_0^z d\ell/\beta'(\ell) \quad . \tag{18}$$

Note that the form of equation (17) differs from that of equation (13). This is because in the SV experiment the impulsive excitation (an SV wave) does not coincide with the wavefront (a P wave). In the P experiment both of these were P waves and hence coincided.

Proceeding as above, we obtain ($q_i(z)$ is defined in equation (17b))

$$r_c(z) = q_3(\tau_s(z),z)\ \ (1/\alpha'(z) + 1/\beta'(z))/b_s \tag{19a}$$

$$r_s(z) = 2q_4(\tau_s(z),z)\ /\ (\beta'(z)b_s) \quad . \tag{19b}$$

The importance of equations (16) and (19) is that they permit computation of the reflection coefficients at any depth provided the waves $\underset{\sim}{w}(t,z)$ are known at that depth.

Next, equations (12a-c) are written as a matrix equation:

$$\begin{bmatrix} r_p(z) \\ r_c(z) \\ r_s(z) \end{bmatrix} = M(z) \begin{bmatrix} (\partial/\partial z) \quad \log \rho(z) \\ (\partial/\partial z) \quad \log \beta(z) \\ (\partial/\partial z) \quad \log \alpha(z) \end{bmatrix} \tag{20a}$$

where

$$M(z) = \begin{bmatrix} 1/2 - 2\beta^2p^2 & -4\beta^2p^2 & 1/(2-2\alpha^2p^2) \\ -\ell(1-2\beta^2p^2 + 2\beta^2/\alpha'\beta') & \ell(4\beta^2p^2 - 4\beta^2/\alpha'\beta') & 0 \\ -(1/2 - 2\beta^2p^2) & -(1/(2-2\beta^2p^2)-4\beta^2p^2) & 0 \end{bmatrix} \tag{20b}$$

with $\ell(z) \triangleq (p/2)(\alpha'\beta')^{1/2}$. Inverting this equation gives

$$\begin{bmatrix} (\partial/\partial z) \quad \log \rho(z) \\ (\partial/\partial z) \quad \log \beta(z) \\ (\partial/\partial z) \quad \log \alpha(z) \end{bmatrix} = N(z)/m(z) \begin{bmatrix} r_c(z) \\ r_s(z) \\ r_p(z) \end{bmatrix} \tag{21a}$$

where

$$N(z) = \begin{bmatrix} -(1/(2-2\beta^2p^2) - 4\beta^2p^2) & -\ell(4\beta^2p^2 - 4\beta^2/\alpha'\beta') & 0 \\ 1/2 - 2\beta^2p^2 & -\ell(1-2\beta^2p^2 + 2\beta^2/\alpha'\beta') & 0 \\ -\dfrac{(1-\alpha^2p^2)(4\beta^2p^2-1)}{2(1-\beta^2p^2)} & -4\ell(1-\alpha^2p^2)(\beta^2p^2 + \beta^2/\alpha'\beta') & 2m(1-\alpha^2p^2) \end{bmatrix} \tag{21b}$$

and

$$\begin{aligned} m(z) &\triangleq (\det M(z))(2-2\alpha^2p^2) \\ &= \ell(1/2 - 3\beta^2p^2 - \beta^2/\alpha'\beta' + 2\beta^4p^4 + 2\beta^4p^2/\alpha'\beta')/(1-\beta^2p^2) \; . \end{aligned} \tag{22}$$

Equations (21) function as update equations for $\rho(z)$, $\beta(z)$, and $\alpha(z)$. Note that $\rho(z)$ and $\beta(z)$ are updated solely from $r_c(z)$ and $r_s(z)$, and then $r_p(z)$ is used to update $\alpha(z)$. From these three parameters, any other parameter of interest (e.g. $\lambda(z)$ and $\mu(z)$) may be quickly found. Chapman (1974, p. 67) gives equations similar to equations (20); however, Chapman's equations involve too many quantities ($\mu, \rho, \beta, \alpha'$, and β') and require the unobservable transmission coefficient t_c. Thus, they are unsuitable as update equations.

We have now specified all of the equations of the algorithm, in differential form. The algorithm consists of equation (11), twice (one for the experiment involving excitation by P waves; one for excitation by SV waves) for updating the up- and down-going waves; equations (16) and (19) for computing the reflection coefficients; equations (21) and (22) for updating the material parameters $\rho(z)$, $\beta(z)$, and $\alpha(z)$; and equation (12d) for computing the transmission coefficient $t_c(z)$ required to complete the matrix in equation (11). We then immediately have, for each z,

$$\mu(z) = \beta^2(z)\rho(z) \tag{23a}$$

$$\lambda(z) = (\alpha^2(z) - 2\beta^2(z))\rho(z). \tag{23b}$$

Next, the algorithm is discretized in order to clarify the recursions and specify in what order quantities should be computed.

Discretization

The depth coordinate z is discretized by $z=n\Delta$, where n is a positive integer and Δ is the discretization length. The time coordinate t is similarly discretized by $t = m\Delta t$, where Δt is the discretization time.

Initialization

It is assumed that all material parameters (λ, μ, ρ, and hence α, β, α', and β') are known at the earth's surface. Since we are assuming a free surface, the waves at the surface are determined by measuring the velocity components over time, for both the P and SV experiments.

Recursion

We start off with knowledge of $\alpha(z)$, $\beta(z)$, $\rho(z)$, $\alpha'(z)$, $\beta'(z)$ as well as that of all up- and down-going waves at depth z, from the previous iteration. Let $\underset{\sim}{w}^p(t,z)$ represent the waves in the P-wave source experiment, and $\underset{\sim}{w}^s(t,z)$ represent the waves in the S-wave source experiment. For convenience, we identify the dimensionless quantities

$$B(z) = \beta^2(z)p^2 = \sin^2\theta_s(z) \tag{24a}$$

$$G(z) = \beta^2(z)/\alpha'(z)\beta'(z) = (1/2)\sin 2\theta_s \cot \theta_p \quad . \tag{24b}$$

Then, taking the inverse Fourier transform of equation (11) and employing a simple Euler-Cauchy approximation to the various derivatives in the differential form of the algorithm yields the following recursive algorithm:

1) Computation of the reflection coefficients. From equations (16) and (19),

$$r_p(z) = 2w_3^p(\tau_p(z),\ z)/(\tilde{b}_p\Delta) \tag{25a}$$

$$r_c(z) = 2w_4^p(\tau_p(z),\ z)/(\tilde{b}_p\Delta) \tag{25b}$$

$$r_s(z) = 2(w_4^s(\tau_s(z)^+,z) - w_4^s(\tau_s(z)^-,z))/(\tilde{b}_s\Delta). \tag{25c}$$

where $\tilde{b}_p = b_p\alpha'(z)/\Delta$ and $\tilde{b}_s = b_s\beta'(z)/\Delta$ are the strengths of the discretized continuous impulses.

Upon going from continuous time to discrete time, the continuous-time impulse $b_i\delta(t)$ becomes a discrete-time impulse of hight b_i/D_i, where D_i is the differential delay time at depth z for wave type i (see Fig. 1). Since the impulse has been spread out over the time interval D_i, its height must be b_i/D_i in order to maintain its area b_i. For a P→P reflection $D_p = \Delta/\alpha'(z)$. For a P→S reflection the two-way delay is $D_p + D_s$, hence the one-way delay is half of this, or $(\Delta/2)(1/\alpha'(z) + 1/\beta'(z))$. Equations (16) and (19) are thus modified to eq. (25).

2) Computation of auxiliary quantities. From equations (22) and (24),

$$B(z) = \beta^2(z)p^2 \tag{26}$$

$$G(z) = \beta^2(z)/\alpha'(z)\beta'(z) \tag{27}$$

$$m(z) = (1/2 - 3B - G + 2B^2 + 2BG)\ell/(1-B) \tag{28}$$

$$t_c(z) = -\ell(1/2 - 3B + G + 2B^2 - 2BG)r_c(z)/((1-B)m(z)) + 2Br_s(z)/m(z) \tag{29}$$

where $\ell(z)$ is defined as above.

3) Update of material parameters. From equations (21),

$$\rho(z+\Delta) = \rho(z) - \rho(z)((1/(2-2B) - 4B)r_c(z) + 4\ell\ (B-G)r_s(z))\Delta/m(z) \tag{30}$$

$$\beta(z+\Delta) = \beta(z) - \beta(z)((2B-1/2)r_c(z) + \ell(1-2B+2G)r_s(z))\Delta/m(z) \tag{31}$$

$$\alpha(z+\Delta) = \alpha(z) + \alpha(z)(1-\alpha^2(z)p^2)(2r_p(z) - ((2B-1/2)/(1-B)m(z))r_c(z) - \ell(4(B+G)/m(z))r_s(z))\Delta \tag{32}$$

$$\alpha'(z+\Delta) = \alpha(z+\Delta)/(1-\alpha^2(z+\Delta)p^2)^{1/2} \tag{33}$$

$$\beta'(z+\Delta) = \beta(z+\Delta)/(1-\beta^2(z+\Delta)p^2)^{1/2}\ . \tag{34}$$

4) Wave update. From the inverse Fourier transforms of equation (11),

$$w_1(t+\Delta/\alpha'(z),\ z+\Delta) = w_1(t,z) - (t_c(z)w_2(t,z) + r_p(z)w_3(t,z) + r_c(z)w_4(t,z))\Delta \tag{35a}$$

$$w_2(t+\Delta/\beta'(z),\ z+\Delta) = w_2(t,z) - (-t_c(z)w_1(t,z) + r_c(z)w_3(t,z) + r_s(z)w_4(t,z))\Delta \tag{35b}$$

$$w_3(t-\Delta/\alpha'(z),\ z+\Delta) = w_3(t,z) - (r_p(z)w_1(t,z) + r_c(z)w_2(t,z) + t_c(z)w_4(t,z))\Delta \tag{35c}$$

$$w_4(t-\Delta/\beta'(z),\ z+\Delta) = w_4(t,z) - (r_c(z)w_1(t,z) + r_s(z)w_2(t,z) - t_c(z)w_3(t,z))\Delta \tag{35d}$$

and these same recursions are used for both $\underset{\sim}{w}^p(t,z)$ and $\underset{\sim}{w}^s(t,z)$.

At this point, we have obtained $\rho(z+\Delta)$, $\alpha(z+\Delta)$, $\beta(z+\Delta)$, $\alpha'(z+\Delta)$ $\beta'(z+\Delta)$, and all eight waves at depth $z+\Delta$. Hence the recursion is complete. Each step in the recursion can be implemented as one stage or section of a ladder-type filter, which can be regarded as a more complex version of the lattice filter commonly encountered in spectral estimation theory. A typical section of this ladder filter is illustrated in Figure 1. The downgoing P and SV waves at depth z enter the filter section at the upper left, interact with each other, are reflected (due to the inhomogeneity of the medium), and exit at the upper right, now at depth $z+\Delta$. Upgoing P and SV waves undergo a similar experience in the lower half of the filter. Note how this filter illustrates the physical meaning of the reflection coefficients $r_p(z)$, $r_c(z)$ and $r_s(z)$, and of the transmission coefficient $t_c(z)$.

The recursions of the waves in z and t, given by equations (35), are slightly complicated, so the recursion patterns are illustrated in Figures 2a and 2b. We start off knowing the waves at depth z for all time, and wish to find the waves at depth $z+\Delta$. Although the simultaneous time and depth updates may make it seem as though information at early times is being lost, recall that by causality there can be no wave at depth z until the initial excitation has had time to reach depth z. Thus there is no information to lose at the early times.

The algorithm that we have described above for reconstructing $\rho(z)$, $\lambda(z)$ and $\mu(z)$ works even if some turning points exist for the P and SV waves propagating through the elastic medium. However, in this case ρ, λ and μ can only be reconstructed up to the depth z_p where the ray path for the P wave becomes horizontal. Note that along rays associated with the P and SV waves

$$\sin\theta_p(z)/\alpha(z) = \sin\theta_s(z)/\beta(z) = p = \text{constant} \tag{36}$$

so that unless $\alpha(z) < 1/p$ for all z (in which case we have also $\beta(z) < 1/p$), the angle $\theta_p(z)$ will become imaginary at some depth z_p. Physically, this situation results in evanescent waves where the waves decay exponentially with depth. This causes no problem in the reconstruction algorithm until $z=z_p$, at which point $\alpha'(z) \to \infty$. Then, the waves $\underset{\sim}{w}^p(z,t)$ and $\underset{\sim}{w}^s(z,t)$ cannot be propagated further, and the material parameters are reconstructed only up to depth z_p.

PHYSICAL INTERPRETATION OF THE ALGORITHM

The basic principle behind the algorithm is the concept of layer-stripping, which is discussed in Bruckstein, Levy, and Kailath (1983).

At each depth the downgoing and upgoing waves are being scattered (i.e., reflected and transmitted) due to the varying material parameters. By including an impulse in the initial excitation, the reflection coefficients may be measured, since the impulse is an easily recognizable tag. The reflection coefficients are then used to update the material parameters <u>at that depth</u>. When another infinitesimal layer is identified it is "stripped away", and the next layer is examined in the same manner.

The layer-stripping algorithm is closely related to the dynamic deconvolution algorithm, which is commonly associated with inverse acoustic problems (Robinson, 1982; Yagle and Levy, 1983; Bube and Burridge, 1983). Both algorithms employ up- and down-going waves being scattered by an inhomogeneous medium with the material parameters determined from reflection coefficients. However, layer-stripping algorithms may take a wide variety of forms, while dynamic deconvolution is generally associated with a specific form -- the Schur algorithm.

The waves (elements of $\underset{\sim}{w}$) used in the algorithm are, in the Fourier transform domain

$$\hat{T}_p / Z_p^{1/2} \mp j\omega Z_p^{1/2} \hat{U}_p \tag{37a}$$

$$\hat{T}_s / Z_s^{1/2} \mp j\omega Z_s^{1/2} \hat{U}_s \tag{37b}$$

where the upper sign is used for upgoing waves and the lower sign for downgoing waves, the impedances Z_p and Z_s are defined by

$$Z_p(z) = \alpha(z)\rho(z) \cos \theta_p(z) \tag{38a}$$

$$Z_s(z) = \beta(z)\rho(z) \cos \theta_s(z) \quad , \tag{38b}$$

and $\hat{T}_p$, $\hat{T}_s$, $\hat{U}_p$ and $\hat{U}_s$ are the stresses and displacements along the ray path for P waves, and perpendicular to the ray path for SV waves.

This shows that the up- and down-going waves have been normalized at each depth, so that the energy in a wave is the square of its amplitude, regardless of the medium around it. This preserves the equality of incoming and outgoing energy fluxes in any region. Note that in a homogeneous medium the waves (37) become simply the energy-normalized velocities, which were the state variables used by Shiva and Mendel (1983) for a discrete (each layer is homogeneous) layered elastic medium.

It should also be noted that if the medium is discretized, i.e. modelled as a welded stack of thin, homogeneous layers with material parameters varying only between different layers, then $R\partial C/\partial z$ may be interpreted as a scattering matrix for the layer at depth z. To see this, replace $(\partial/\partial z) \log \rho(z) = (\partial/\partial z)\rho(z)/\rho(z)$ by the discrete approximation $\Delta\rho(z)/\rho(z)$, and do the same for $\beta(z)$ and $\alpha(z)$. Then equations (12) become the reflection and transmission coefficients at an interface (Aki and Richards, 1980, p. 153). Thus discretization of the algorithm is equivalent to a physical discretization of the medium.

RESULTS OF A COMPUTER RUN OF THE ALGORITHM

The algorithm was tested by running it on the synthesized impulse response of a twenty-layer medium. The variation of medium parameters from one layer to another was made small (around 2%), in order to simulate a continuous layered medium. This is important, since the differential updates assume a continuously varying medium; the algorithm cannot handle sharp changes in medium properties unless the step size Δ is made smaller in such regions. The medium velocities and step size Δ were scaled down by a factor of 1000, so $\Delta = 0.1\text{m}$ instead of 100m.

The response of the medium to impulsive plane P and SV waves was generated in the frequency domain using the reflectivity method (Aki and Richards, 1980, p. 393). A FORTRAN program given by Kind (1976) was used to compute the plane wave transfer functions R_{pp}, R_{ps}, and R_{ss} at 512 frequency points (integer multiples of 0.78 Hz). Each of these was divided by $j2\pi f$ and a discrete inverse Fourier transform taken. This synthesized sample step responses; taking differences and dividing by the discretization time $\Delta t = 0.005s$ yielded the discretized impulse responses. It should be noted that careful attention must be paid to signs in going from potential reflection responses to velocity reflection responses; see (Aki and Richards, 1980, p. 191).

The impulse responses, scaled by $1/\Delta t$ for convenience, are plotted in Figure 3. Although the responses were computed for t = 0 up to t = 2.565 to avoid aliasing problems, the responses beyond t = 1.3s were essentially zero and are not shown. Note that the peaks corresponding to strong primary reflections are smeared out. This is due in part to the use of a DFT, which in this case is tantamount to bandpass-filtering the data with a filter with pass band 0.78 Hz - 400 Hz. Since the strengths of the primary reflections are especially important to the algorithm, this smearing might be expected to hamper its performance. However, this evidently did not happen.

The impulsive plane wave responses were then used to initialize the upgoing P and SV waves, and the algorithm was run on a VAX-11/782 computer. Results are shown in Figure 4. It can be seen that the agreement between the actual and algorithm-generated medium parameter profiles is quite good, with less than 5% error everywhere.

It should be noted that the algorithm was not tested under perfect conditions. Bandlimiting of the frequency response resulted in the time response being smeared over two or three samples, and the medium itself was discrete, so that some error may be expected in the update equations. Nevertheless, the algorithm performed quite well. Small amounts of additive noise in the data may be handled by running the algorithm concurrently at several different values of the stacking parameter p, averaging the updated medium parameters at each depth, and using these averaged updates in the algorithms. Sharper variations in the parameter profiles could be handled by temporarily reducing the step size Δ if the reflection coefficients get too large; this would also reveal the sharp variation in more detail. These possibilities are subjects of current research.

CONCLUSION

A fast algorithm for recovery of material parameter profiles as functions of depth for the case of seismic wave propagation through a continuous elastic medium has been given. The algorithm has been specified both in differential form and in a simple discretized version that details its recursive nature. The algorithm works on a layer-stripping principle, and appears to be much faster and easier computationally than previous solutions to the inverse seismic problem for an elastic medium. A physical interpretation of the algorithm was also discussed in terms of a lattice filter concept showing how the first reflection of various wave types at each depth yields the medium reflection coefficients at that depth, from which the medium parameters at that depth can be differentially updated to the next depth. Results of a computer run of the algorithm on the impulsive plane wave response of a twenty-layer medium show that the algorithm works satisfactorily.

More work needs to be done in investigating the speed, stability, and performance of this algorithm on real-life data. The effects of noise and modelling errors on this algorithm should also be the subject of further research.

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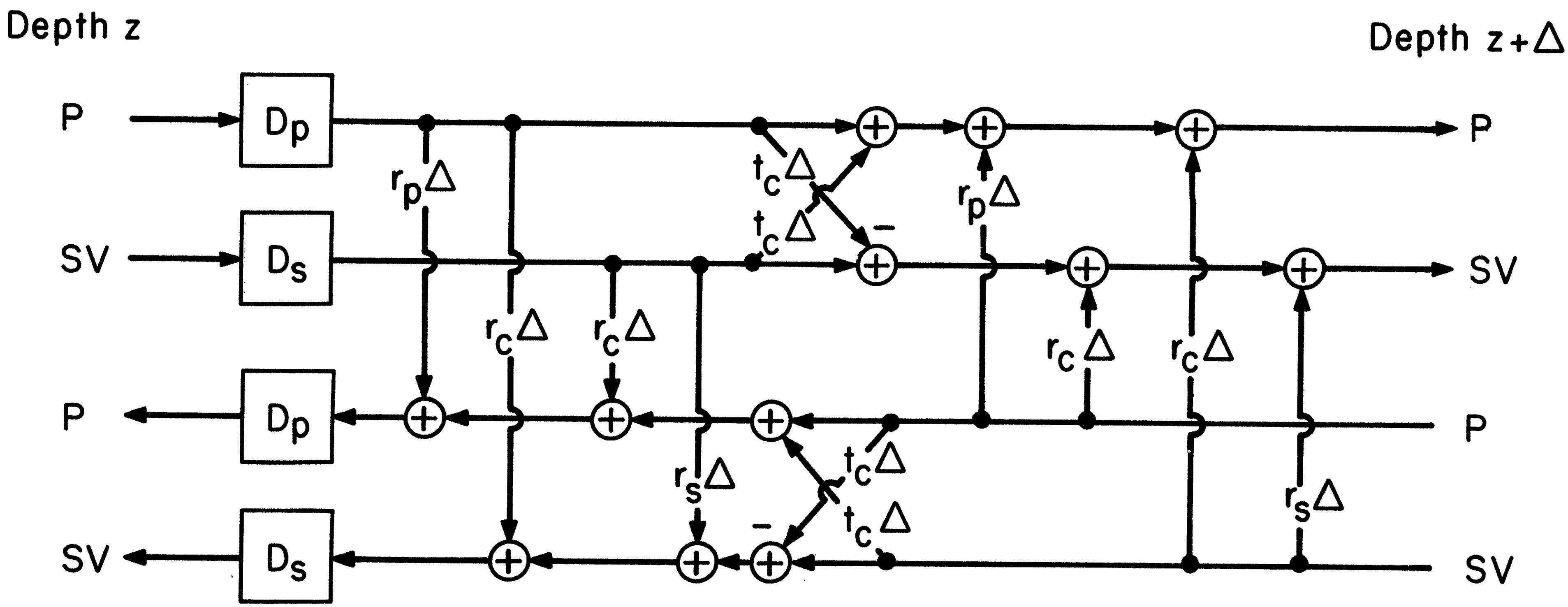

FIG. 1

An infinitesimal section of the ladder filter which implements the elastic wave equation.

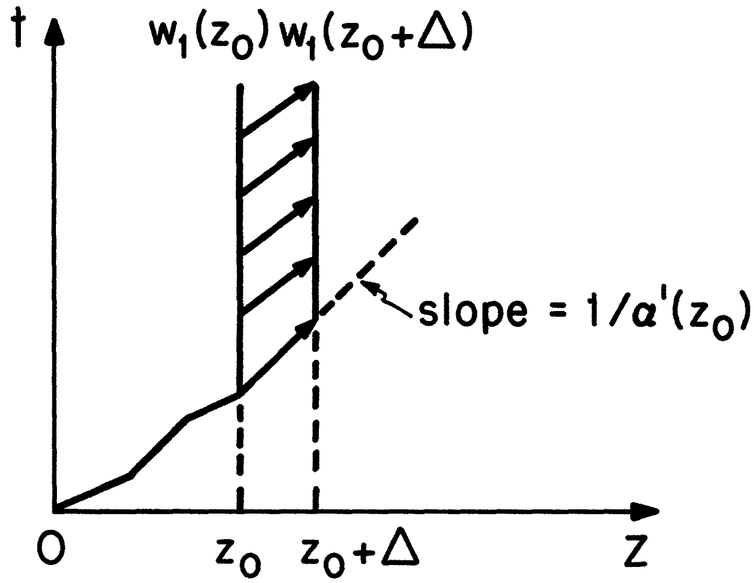

FIG. 2a

Recursion pattern for updating the downgoing waves.

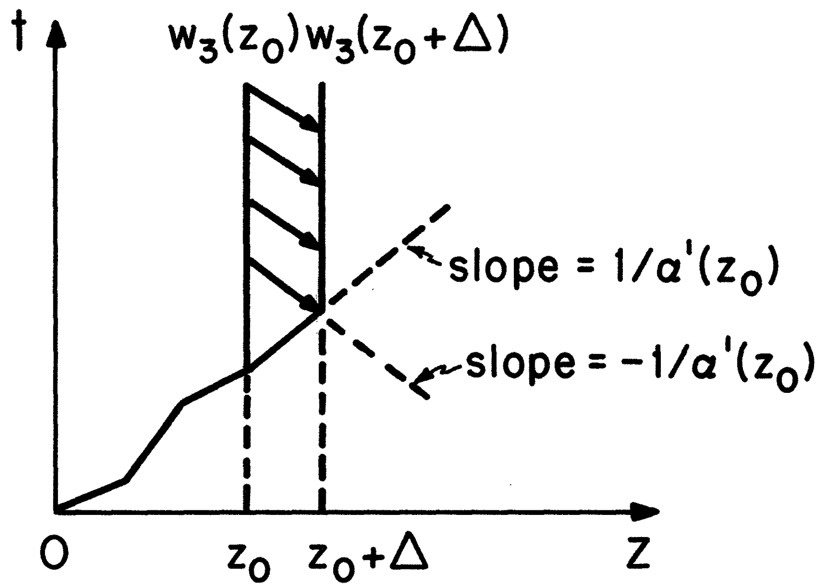

FIG. 2b

Recursion pattern for updating the upgoing waves.

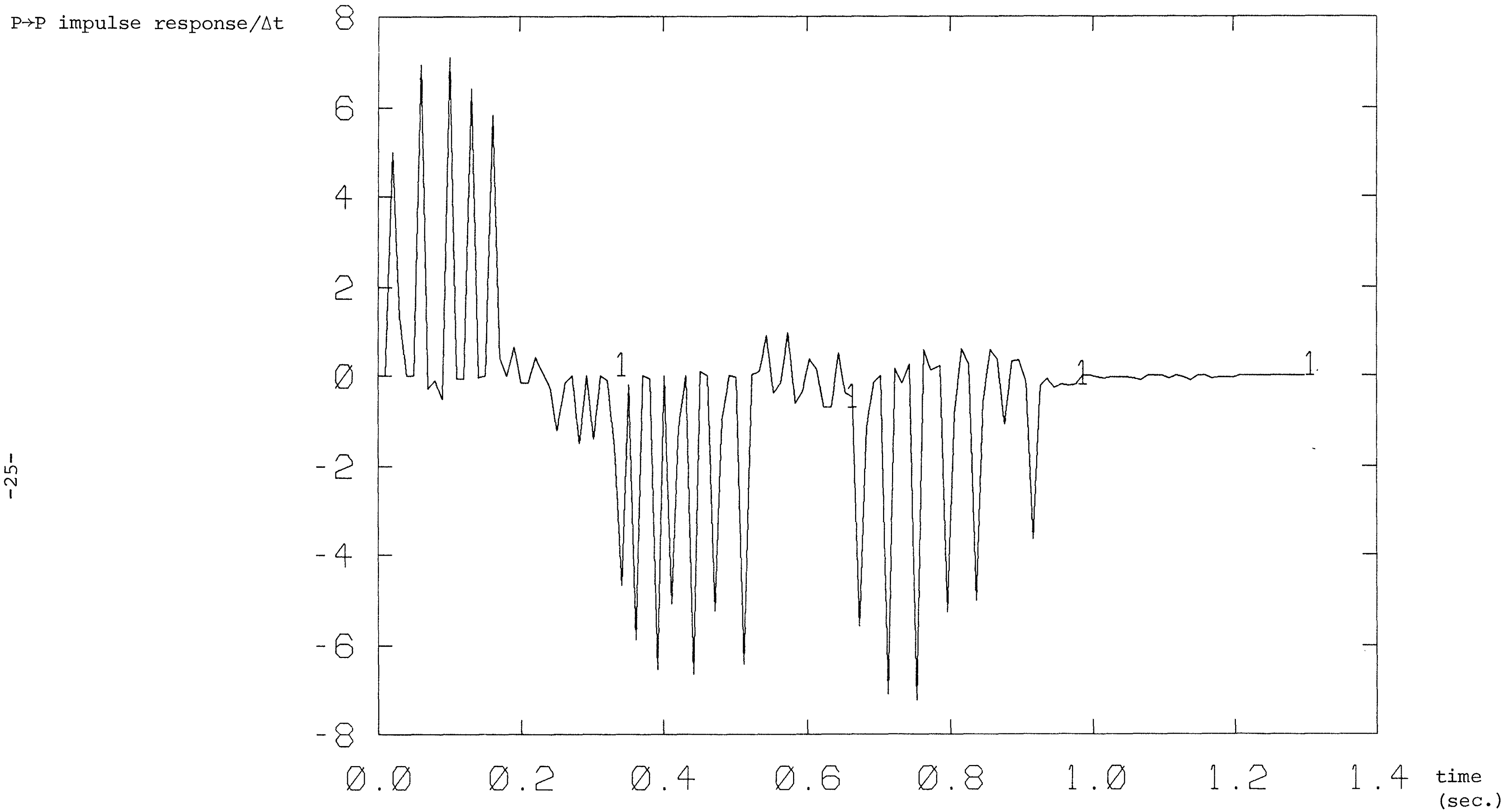

Fig. 3a. P→P impulse response, scaled by 1/Δt for convenience.

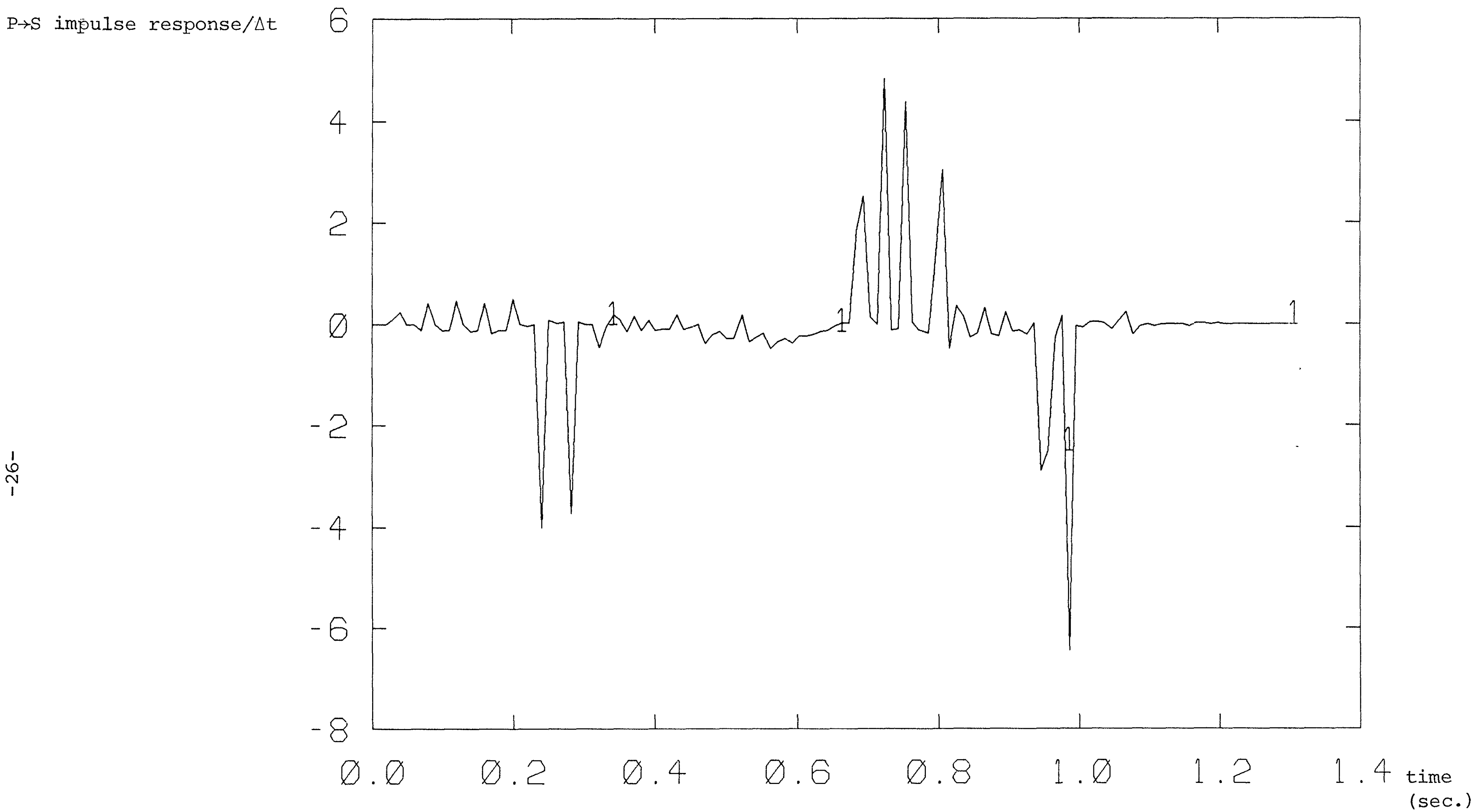

Fig. 3b. P→S impulse response, scaled by 1/Δt for convenience.

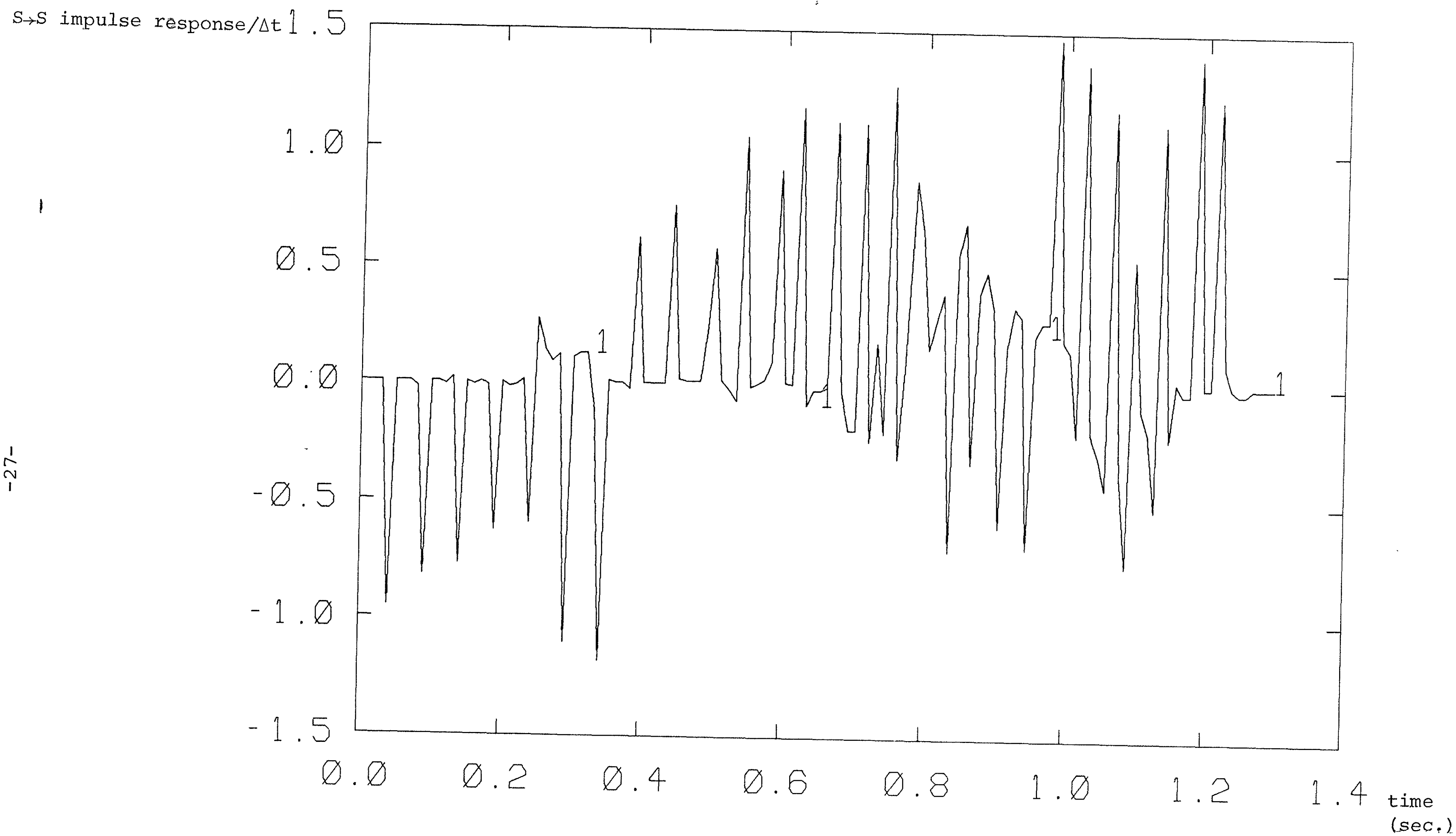

Fig. 3c. S→S impulse response, scaled by 1/Δt for convenience.

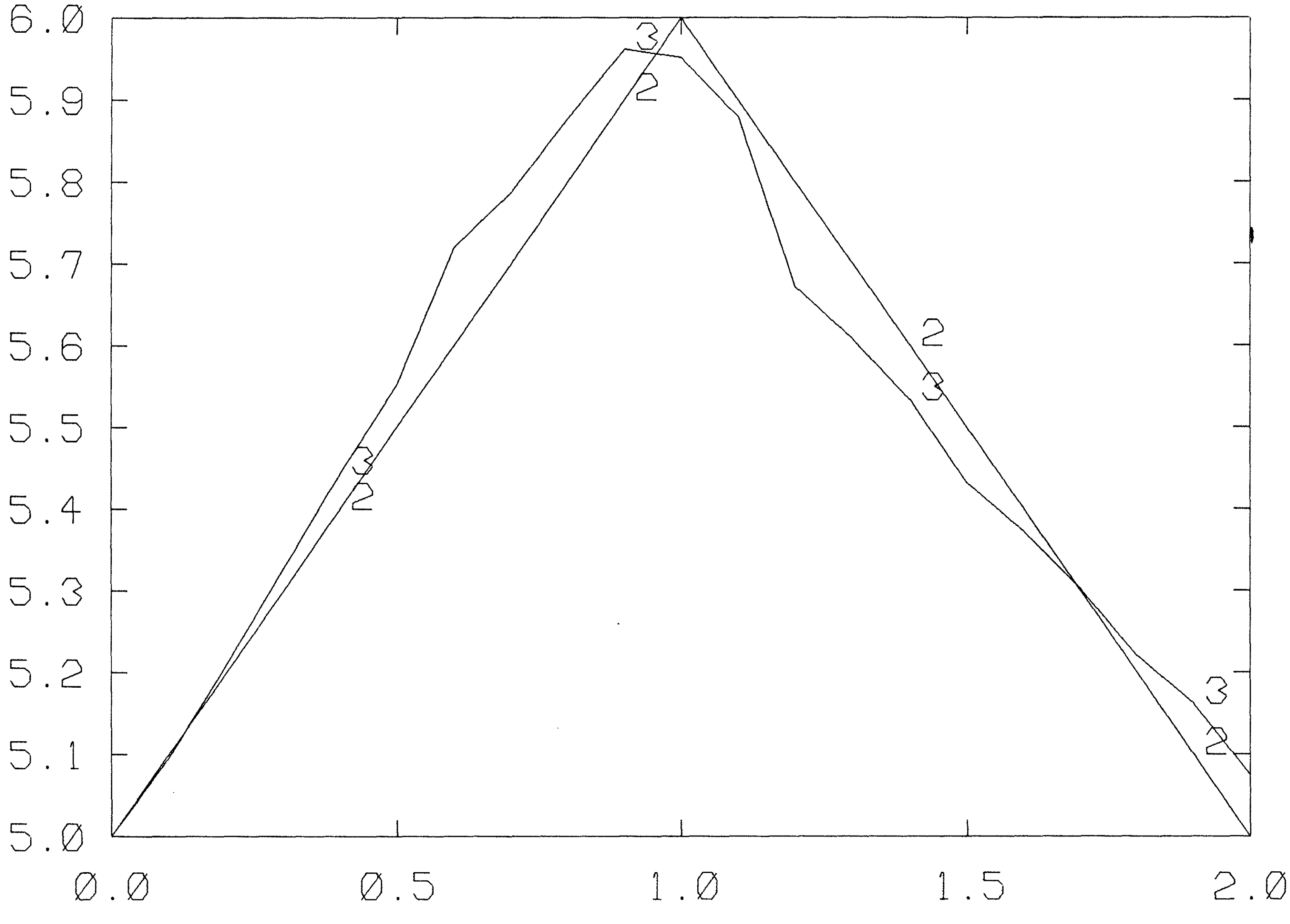

Fig. 4a Comparison between actual and computed P wave speed profiles (2 = actual, 3 = computed). Both profiles and depth have been scaled by 1000.

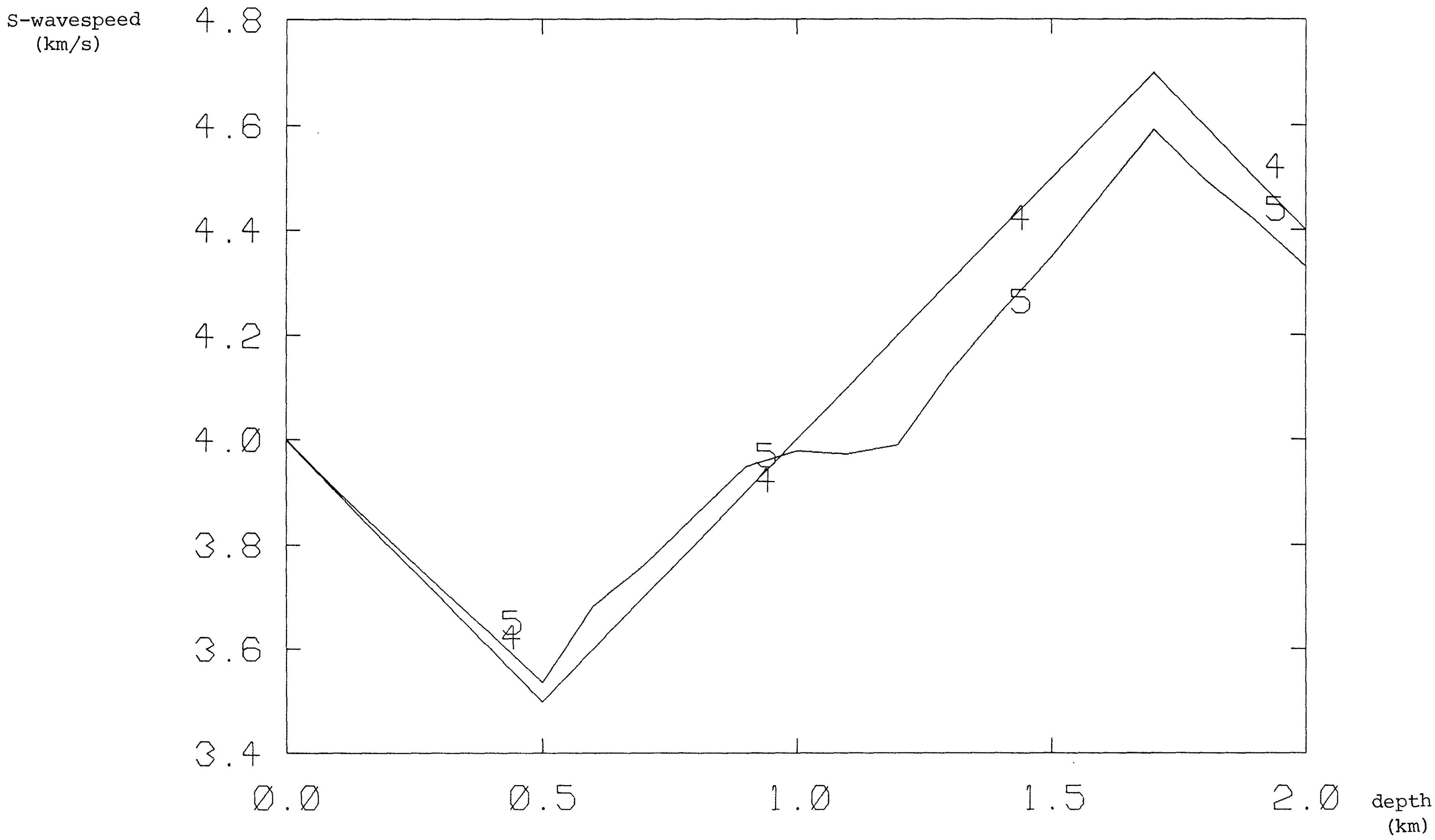

Fig. 4b Comparison between actual and computed SV wave speed profiles (4 = actual, 5 = computed). Both profiles and depth have been scaled by 1000.

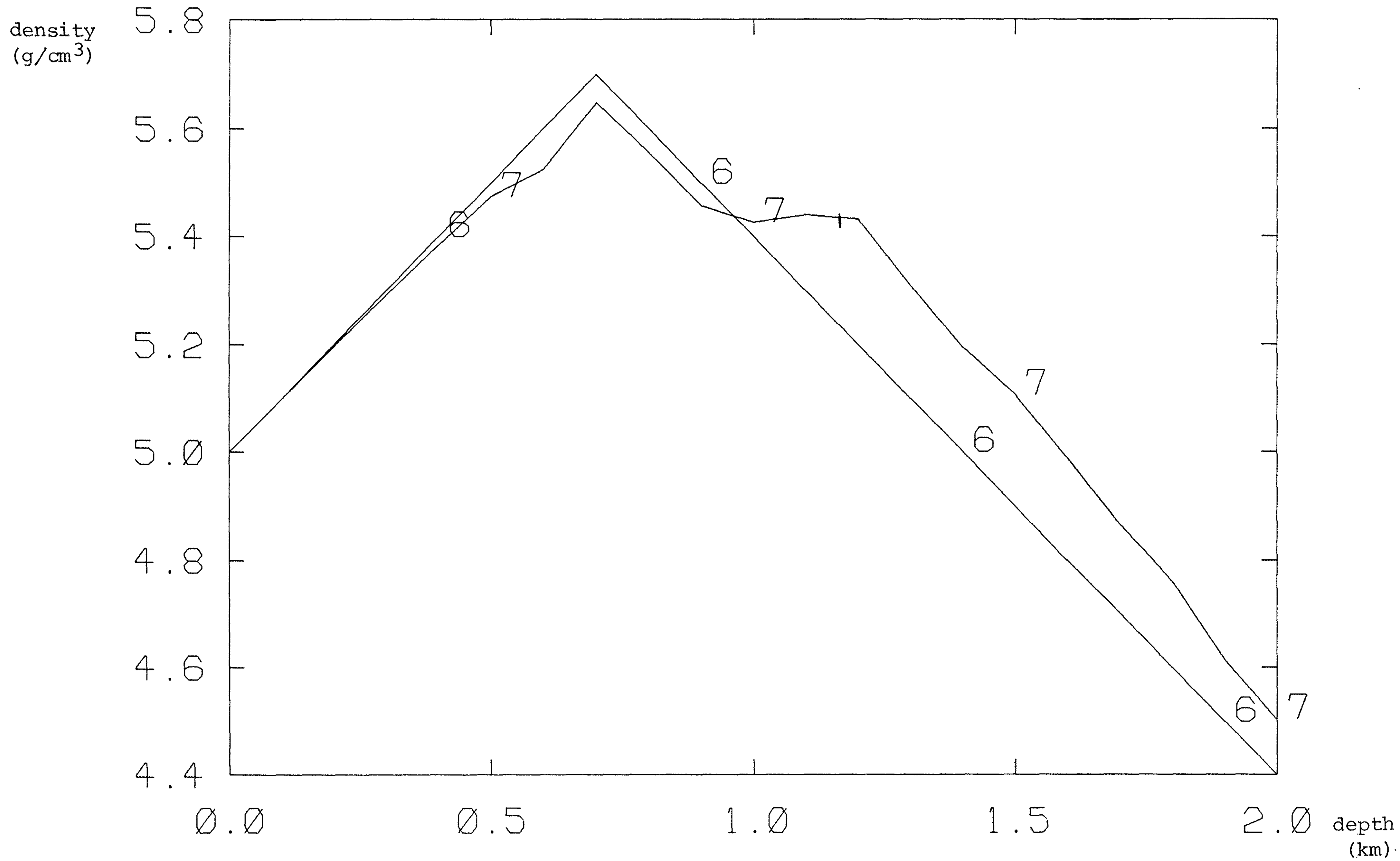

Fig. 4c Comparison between actual and computed density profiles (6 = actual, 7 = computed). The depth has been scaled by 1000.