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LINEAR ESTIMATION OF TWO-POINT BOUNDARY VALUE PROCESSES

by

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ABSTRACT

We consider estimation of an nxl vector non-causal stochastic process governed by the linear dynamics

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

with two-point boundary condition

$$v = V^0 x(0) + V^T x(T)$$

where u is an mxl white input noise with covariance parameter $Q(t)$ and v is an nxl zero mean Gaussion boundary value with nonsingular covariance matrix Π_v, uncorrelated with u on $[0,T]$. Estimation of processes satisfying models of this type was first suggested by Krener [1].

1. INTRODUCTION

The now classical solutions to linear filtering and smoothing problems for one-dimensional (1-D), nonstationary, causal processes are discussed in the review paper by Kailath [2]. Notable in the derivations of these solutions is the use of the Markovian nature of the models for these 1-D processes.   However, in as much as stochastic processes in higher dimensions (random fields) are typically noncausal and consequently are not Markovian in the usual sense, their estimators cannot be derived through a direct extension of these 1-D derivations. Thus, linear estimation problems for noncausal processes require new approaches, and a natural first step is to solve the estimation problem for 1-D noncausal processes.

With this as motivation, Krener [3] introduced a class of linear noncausal 1-D dynamic models. In his study of these models, Krener has developed results on controllability, observability and minimality and has solved a deterministic linear control problem. In addition, he has posed the fixed-interval linear smoothing problem for these systems and has derived intergral equations for both the weighting pattern and the error covariance for the optimal smoother. Working directly with these equations he has had some success in obtaining a dynamic realization of the smoother for a special "stationary-cyclic" class of these processes [4].

In this paper we solve the estimation problem for Krener's 1-D noncausal models through a generalization of the complementary models approach to linear estimation which was first introduced by Weinert and Desai [5].   In particular, we present a procedure for finding the complementary model which is appropriate for the 1-D

noncausal estimation problem and following the methodology introduced in [6], we augment the model of the process to be estimated with the complementary model and invert the combined system to obtain a differential realization of the smoother.

2. LINEAR STOCHASTIC TWO-POINT BOUNDARY VALUE PROCESS (TPBVP)

The General Solution

The 1-D noncausal process studied here is governed by an nth order linear stochastic differential equation together with a specified two-point boundary condition. Accordingly, the process will be referred to as a linear stochastic two-point boundary value process or TPBVP. In describing the model we employ the white noise formalism for representing linear stochastic differential equations. Let u(t) be a mxl white noise process with covariance parameter $Q(t)$. Let v be a nxl random vector, independent of u(t), with covariance matrix Π_v. The nxl boundary value process x(t) is governed on the interval $[0,T]$ by

$$\dot{x}(t) = A(t)x(t) = B(t)u(t) \qquad (1a)$$

with boundary condition

$$v = V^0 x(0) + V^T x(T) \quad . \qquad (1b)$$

It will be assumed that A and B are continuous on $[0,T]$ and that all random variables are zero-mean.

It is instructive to derive one form of the general solution for (1) as the approach we take in this derivation will be used later. The form of the solution which we obtain differs from the usual Green's function solution (e.g. see [1]). Let $\Phi(t,s)$ be the transition matrix associated with A(t). If x(0) were known, then x(t) could be represented in the variation-of-constants form

$$x(t) = \Phi(t,0)x(0) + x^0(t) \qquad (2a)$$

where $x^0(t)$ is the solution of (1a) with $x^0(0) = 0$:

$$x^0(t) = \int_0^t \Phi(t,s)b(s)u(s)ds \quad . \qquad (2b)$$

Substituting from (2a) at t = T into the boundary condition (1b), we can write

$$v = V^T x^0(t) = [V^0 + V^T \Phi(T,0)]x(0) \quad . \qquad (3a)$$

For a well-posed problem, there will be a unique x(0) for a given v and u on $[0,T]$. Thus

well-posedness requires that the nxn matrix

$$F = V^0 + V^T \Phi(T,0) \qquad (3b)$$

be nonsingular. With F invertible, we can solve for x(0) as

$$x(0) = F^{-1}[v - V^T x^0(T)] \quad . \qquad (3c)$$

Substituting x(0) into (2a) gives the general solution for (1a,b) as

$$x(t) = \Phi(t,0)F^{-1}[v - V^T x^0(t)] + x^0(t) \quad . \qquad (4)$$

The noncausal nature of the TPBVP x(t) is clearly displayed if we correlate the value of x at t = 0 with future values of the input u:

$$E\{x(0)u'(t)\} = -F^{-1}V^T \Phi(T,t)B(t)Q(t) \quad t \varepsilon [0,T] \quad . \qquad (5)$$

Thus, the nth order model in (1) is not Markovian, and consequently Kalman filtering and associated smoothing technqiues are not directly applicable.

Below, as an alternative to (4), we present a second form for the general solution of (1) which leads to a numerically stable implementation. Consider the equivalent process obtained by transforming x as

$$\begin{bmatrix} x_f(t) \\ x_b(t) \end{bmatrix} = T(t)x(t) \qquad (6a)$$

where the transformation matrix T(t) is chosen so that 1) the dynamics of the system model in (1) become decoupled[1]:

$$\begin{bmatrix} \dot{x}_f \\ \dot{x}_b \end{bmatrix} = \begin{bmatrix} A_f & 0 \\ 0 & A_b \end{bmatrix}\begin{bmatrix} x_f \\ x_b \end{bmatrix} + \begin{bmatrix} B_f \\ B_b \end{bmatrix} u \qquad (6b)$$

and 2) A_f is exponentially stable in the forward direction and A_b is exponentially stable in the backward direction. For time-invariant systems this is always possible by assigning those modes associated with eigenvalues greater than or equal to zero to A_f and those less than zero to A_b. For time-varying dynamics, it may be difficult to determine the dynamics and boundary conditions for a transformation T(t) which transforms the system dynamics into this form. However, for the systems of interest to us later in this paper we can overcome this difficulty by invoking results obtained previously for smoothing solutions for causal processes. The boundary condition for the transformed process will be written as

$$v = \begin{bmatrix} V_f^0 & \vdots & V_b^0 \end{bmatrix}\begin{bmatrix} x_f(0) \\ x_b(0) \end{bmatrix} + \begin{bmatrix} V_f^T & \vdots & V_b^T \end{bmatrix}\begin{bmatrix} x_f(T) \\ x_b(T) \end{bmatrix} . \qquad (6c)$$

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[1] When there is no risk of confusion we will often omit explicit reference to the independent variable, i.e. A(t) → A.

The reason for our choice of subscripts f and b, denoting forward and backward respectively, will become apparent blow.

If $x_f(0)$ and $x_b(T)$ were known, then we could solve for $x_f(t)$ and $x_b(t)$ as

$$x_f(t) = \Phi_f(t,0)x_f(0) + x_f^0(t) \qquad (7a)$$

and

$$x_b(t) = \Phi_b(t,T)x_b(T) + x_b^0(t) \quad \cdot \qquad (7b)$$

where $x_f^0(t)$ is governed by (6b) with $x_f(0) = 0$ and $x_b^0(t)$ is governed by (7b) with $x_b^0(T) = 0$. Following a derivation similar to that used to obtain the general solution in (4), it can be shown that

$$\begin{bmatrix} x_f(t) \\ x_b(t) \end{bmatrix} = \begin{bmatrix} \Phi_f(t,0) & \vdots & 0 \\ - & - - & \vdots & - - \\ 0 & & \vdots & \Phi_b(t,T) \end{bmatrix} -F_{fb}^{-1}[v - V_f^T x_f^0(T)$$

$$- V_b^0 x_b^0(0)] + \begin{bmatrix} x_f^0(t) \\ x_b^0(t) \end{bmatrix} \qquad (8)$$

where

$$F_{fb} = [V_f^0 + V_f^T \Phi_f(T,0) \quad \vdots \quad V_b^T + V_b^0 \Phi_b(0,T)]. \qquad (9)$$

The TPBVP x is recovered from (8) by inverting (6a):

$$x(t) = T^{-1}(t)\begin{bmatrix} x_f(t) \\ x_b(t) \end{bmatrix} \quad . \qquad (10)$$

As we will see, the <u>two-filter</u> form of the general solution in (8) is the foundation for the implementation of the estimator that we develop later in this paper. The term two-filter is used to signify that the numerical solution of (8) requires the integration of a forward process x_f^0 and a backward process x_b^0.

Covariance of the TPBVP x(t)

The covariance of x(t)

$$P_x(t) = E\{x(t)x'(t)\} \qquad (11)$$

can be derived from (4) as

$$P_x(t) = P_x^0(t) + \Phi(t,0)F^{-1} [\Pi_v + V^T P_x^0(T)V^{T'}]$$

$$F^{-1'}\Phi'(t,0) - \Phi(t,0)F^{-1}V^T P_x^0(t) - P_x^0(t)V^{T'}F^{-1'}\Phi'(t,0) \qquad (12a)$$

where $P_x^0(t)$ is the covariance of $x^0(t)$ satisfying

$$\dot{P}_x^0 = AP_x^0 + P_x^0 A' + BQB' \; ; \; P_x^0(0) = 0 \; . \qquad (12b)$$

An additional expression for P_x can be derived from the two-filter form of the general solution (equation (8)). However, this expression is somewhat complex, and we will wait until a later publication to present it in the context of the study of the estimation error covariance [7].

Green's Identity

In establishing the solution to our estimation problem we will make use of Green's Identity for the differential operator:

$$L:D(L) \rightarrow R(L) \; ; \; (Lx)(t) = \dot{x}(t) - A(t)x(t) \qquad (13)$$

where $D(L)$ is the space of once continuously differentiable $n \times 1$ vector functions on $[0,T]$ and $R(L)$ is the Hilbert space of square integrable $n \times 1$ vector functions on $[0,T]$. Let E be a $2n \times 2n$ matrix partitioned into $n \times n$ blocks with:

$$E = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \qquad (14a)$$

and define the $2n \times 1$ vector

$$x_b = \begin{bmatrix} x(0) \\ x(T) \end{bmatrix} \; . \qquad (14b)$$

The formal adjoint of the operator L is

$$(L^\dagger \lambda)(t) = -\dot{\lambda}(t) - A'(t)\lambda(t) \; . \qquad (14c)$$

Given these definitions, the Green's Identity for L on the interval $[0,T]$ is obtained directly by integration by parts, yielding

$$\langle Lx, \lambda \rangle_{L_2^n[0,T]} = \langle x, L^\dagger \lambda \rangle_{L_2^n[0,T]} + \langle x_b, E\lambda_b \rangle_{R^{2n}} \qquad (15)$$

where λ_b takes a form similar to x_b in (14b).

3. PROBLEM STATEMENT

The fixed-interval smoothing problem for the noncausal process $x(t)$ defined earlier is stated as follows. Let $r(t)$ be a $p \times 1$ white noise process uncorrelated with v and $u(t)$ and with continuous covariance parameter $R(t)$. Let $C(t)$ be a $p \times n$ matrix whose elements are continuous on $[0,T]$. The observations of $x(t)$ are given by the $p \times 1$ vector stochastic process:

$$y(t) = C(t)x(t) + r(t) \; . \qquad (16)$$

In addition, we assume that there may be available a boundary observation y_b defined as follows. Let r_b be a $q \times 1$ random vector uncorrelated

with $r(t)$, $u(t)$ and v with covariance matrix Π_b. Define a $q \times 2n$ matrix W partitioned into $q \times n$ blocks as

$$W = [W^0 \; \vdots \; W^T] \; . \qquad (17a)$$

The boundary observation is the $q \times 1$ random vector:

$$y_b = Wx_b + r_b \; . \qquad (17b)$$

Define an $n \times 2n$ matrix V as

$$V = [V^0 \; \vdots \; V^T] \qquad (18a)$$

so that the boundary condition in (1b) can be written as

$$v = Vx_b \qquad (18b)$$

We will assume that the rows of W and the rows of V are linearly independent [8].

The fixed-interval smoothing problem is to find the linear minimum variance estimate of the noncausal TPBVP $x(t)$, $t \in [0,T]$, given the complete observation set Y:

$$Y = \{y_b, \; y(t) \; \vdots \; t \in [0,T]\} \; . \qquad (19)$$

4. SOLUTION TO THE ESTIMATION PROBLEM

Our approach to the solution of the estimation problem is based on an application of the method of complementary models introduced by Weinert and Desai [ ] in deriving the smoother for causal stochastic processes. A brief discussion of the basis for this approach follows. Define the underlying process ξ as

$$\xi = \begin{bmatrix} u \\ v \\ r \\ r_b \end{bmatrix} \qquad (20)$$

The complementary process $Z = \{z, z_b\}$ is orthogonal to the observations $Y \{y, y_b\}$ and when combined with the observations, they span the space spanned by ξ. In particular, we can express the observations Y and the process to be estimated x as linear mapping of the underlying process

$$Y = \begin{bmatrix} y \\ y_b \end{bmatrix} = [H \; \vdots \; I] \xi \qquad (21a)$$

$$\triangleq M_y \xi$$

and

$$x = M_x \xi \; . \qquad (21b)$$

The complementary process

$$Z = \begin{bmatrix} z \\ z_b \end{bmatrix} \tag{22a}$$

is a linear mapping of ξ

$$Z = M_z \xi \tag{22b}$$

which satisfies: (1) the orthogonality condition $Z \perp Y$, i.e.

$$E\left\{ \begin{bmatrix} y(t) \\ y_b \end{bmatrix} [z'(t), z_b] \right\} = 0 \tag{23a}$$

and (2) the complementation condition: The augmented mapping M defined by

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} M_y \\ M_z \end{bmatrix} \xi \triangleq M\xi \tag{23b}$$

is invertible so that ξ can be written as

$$\xi = N_y Y + N_z Z$$

$$\triangleq \xi_y + \xi_z \tag{23c}$$

where the partitions of M^{-1} are

$$M^{-1} = [N_y \vdots N_z] .$$

Given (23a) and (23c), the linear minimum variance estimate of ξ given Y is ξ_y, the projection of (23c) into Span (Y). Since x is a linear mapping of ξ, the linear minimum variance estimate of x is

$$\hat{x} = M_x \xi_y$$

$$= M_x M_y Y \tag{24a}$$

It is straight-forward to show that the estimation error is the minimum variance estimate given Z:

$$\tilde{x} = x - \hat{x}$$

$$= M_x \xi_z = M_x N_y z$$

$$= M_x N_y M_z \xi . \tag{24b}$$

Here we have expressed the error as a linear mapping of the underlying process ξ, and the estimation error variance is therefore a linear function of the variance of ξ.

For the above construction to be useful we need to know the mapping M_z in (22b) and the inverse of the augmented map M in (23b). In [8] it is proved that the orthogonality and complementation conditions are satisfied by M_z

given by

$$M_z = [-I \vdots H*] \Sigma_\xi^{-1} . \tag{25a}$$

where H* is the Hilbert adjoint operator of H in (21a) and Σ_ξ is the integral operator whose kernel is the correlation function of the underlying process ξ. From the definition of the underlying process for our problem

$$Z = \begin{bmatrix} z \\ z_b \end{bmatrix} = H* \begin{bmatrix} R^{-1}r \\ \Pi_b^{-1}r_b \end{bmatrix} - \begin{bmatrix} Q^{-1}u \\ \Pi_v^{-1}v \end{bmatrix} . \tag{25b}$$

Given M_z, the final step in determining the estimator is to augment M_y and M_z as in (23b) and invert. This step can be simplified if we first determine an internal differential realization for Z in a form similar to that for Y. Then, following [5] the internal differential realizations for Y and Z can be augmented and inverted in a straightforward manner. From (25b) we see that an internal differential realization for Z implies an internal realization for H*. To obtain such a realization, start with the inner product identity for Hilbert adjoints

$$\left\langle H \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} \right\rangle_{R(H)} = \left\langle \begin{bmatrix} u \\ v \end{bmatrix}, H* \begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} \right\rangle_{D(H)} \tag{26}$$

where $\begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix}$ is an arbitrary element in the range of H, R(H), and $\begin{bmatrix} u \\ v \end{bmatrix}$ is an arbitrary element in the domain of H, D(H). Employing

$$H \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} Cx \\ Wx_b \end{bmatrix} \tag{27a}$$

and defining

$$\begin{bmatrix} \lambda \\ \psi_b \end{bmatrix} = \tilde{H}* \begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix} \tag{27b}$$

where

$$H = \tilde{H} \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} , \quad \left(\text{i.e. } H* = \begin{bmatrix} B* & 0 \\ 0 & I \end{bmatrix} \tilde{H}* \right) \tag{27c}$$

(26) can be rewritten as

$$\langle x, C*u_\lambda \rangle + \langle x_b, W*v_\lambda \rangle = \langle Lx, \lambda \rangle$$

$$+ \langle x_b, V*\psi_b \rangle . \tag{28a}$$

Applying Green's identity to replace $\langle Lx, \lambda \rangle$ in (28a) gives

$$\langle x, [C^*u_\lambda - \overset{*}{L}\lambda] \rangle = \langle x_b, [E\lambda_b + V^*\psi_b - W^*v_\lambda] \rangle$$

(28b)

Now if W_c is a matrix such that

$$\Gamma = \begin{bmatrix} -V \\ W_c \\ W \end{bmatrix} .$$

(29a)

is invertiable, we can define

$$\begin{bmatrix} W_\lambda \\ V_{\lambda c} \\ V_\lambda \end{bmatrix} = (\Gamma')^{-1} E .$$

(29b)

Then it is shown in [8] that the map

$$\Psi = \begin{bmatrix} \psi \\ \psi_b \end{bmatrix} = H^* \begin{bmatrix} u_\lambda \\ v_\lambda \end{bmatrix}$$

(30a)

has an internal differential realization with interval process λ satisfying

$$\overset{*}{L}\lambda = C'u_\lambda$$

(30b)

with boundary condition

$$\begin{bmatrix} v_\lambda \\ 0 \end{bmatrix} = \begin{bmatrix} v_\lambda \\ v_{\lambda c} \end{bmatrix} \lambda_b$$

(30c)

and output

$$\begin{bmatrix} \psi \\ \psi_b \end{bmatrix} = \begin{bmatrix} B'\lambda \\ W_\lambda \lambda_b \end{bmatrix}$$

(30d)

and that this choice of internal realization satisfies the inner product identity (28b) with both left and right hand sides identically zero.

Augmenting Z defined by (25b) and H* given by (30) with the internal differential realization for Y and inverting as in [6] and projecting the solution onto the span of Y (see (23)) we find [8] that an internal differential realization of the estimator is given by the $2n^{th}$ order system:

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{\lambda}} \end{bmatrix} = H \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} + \begin{bmatrix} 0 \\ C'R^{-1} \end{bmatrix} y$$

(31a)

with two-point boundary condition

$$\begin{bmatrix} \dot{W'\Pi_b^{-1}y_b} \end{bmatrix} = V_{x\lambda}^0 \begin{bmatrix} \hat{x}(0) \\ \hat{\lambda}(0) \end{bmatrix} + V_{x\lambda}^T \begin{bmatrix} \hat{x}(T) \\ \hat{\lambda}(T) \end{bmatrix}$$

(31b)

where

$$H = \begin{bmatrix} A & BQB' \\ C'R^{-1}C & -A' \end{bmatrix} ,$$

(31c)

$$V_{x\lambda}^0 = \begin{bmatrix} V^{0'}\Pi_v^{-1}V^0 + W^{0'}\Pi_b^{-1}W^0 & : & -I \\ V^{T'}\Pi_v^{-1}V^0 + W^{T'}\Pi_b^{-1}W^0 & : & 0 \end{bmatrix}$$

(31d)

and

$$V_{x\lambda}^T = \begin{bmatrix} V^{0'}\Pi_v^{-1}V^T + W^{0'}\Pi^{-1}W^T & : & 0 \\ V^{T'}\Pi_v^{-1}V^T + W^{T'}\Pi_b^{-1}W^T & : & I \end{bmatrix}$$

(31e)

Hamiltonian Diagonalization

The solution of the $2n^{th}$ order boundary value process in (31) could be implemented by either of the two forms of the general solution derived earlier. However, by considering the "time"-invariant case we can anticipate that there may be numerical instabilities associated with the first of those methods. In the time-invariant case the eigenvalues of the $2n \times 2n$ Hamiltonian matrix H defined in (1a) are symmetric about the imaginary axis [9], i.e. there are n eigenvalues in each of the left and right half planes. Thus, for the time-invariant case, the right half plane eigenvalues will result in numerical instabilities for the unidirectional implementation suggested by (4). This stability problem can be avoided in general by transforming the smoother dynamics into the stable forward/backward form in (8). To achieve this second form, we need a transformation which diagonalizes the dynamics of H into two $n \times n$ blocks, one stable in the forward direction and the other backwards stable.

Since the dynamics of our smoother as represented by H are identical to those of the smoother for causal processes as originally derived by Bryson and Frazier [10], any transformation which results in a two-filter smoother for causal processes will also diagonalize our smoother. However, choosing a diagonalizing transformation for our problem requires special considerations not encountered in the causal case. First, because the two-point boundary condition provides incomplete information for both the initial and final values of the process, we will choose a transformation which corresponds to a two-filter solution for causal processes with both filters in information form. Second, it is important to choose the boundary conditions properly for the Riccati differential

equations which govern the time-varying elements of the diagonalizing transformation. Our choice of diagonalizing transformation and corresponding boundary conditions makes it possible to formulate a numerically stable two-filter form for our smoother which is remarkably similar to two-filter smoothers for causal processes.

Define the time-varying transformation $T(t)$ as the $2n \times 2n$ matrix partitioned in $n \times n$ blocks as

$$T(t) = \begin{bmatrix} \theta_f(t) & \vdots & -I \\ - & - & - & - & - \\ \theta_b(t) & \vdots & I \end{bmatrix} \quad . \tag{32a}$$

Let the transformed process be denoted by

$$q(t) = \begin{bmatrix} q_f(t) \\ q_b(t) \end{bmatrix} = T(t) \begin{bmatrix} \hat{x}(t) \\ \hat{\lambda}(t) \end{bmatrix} \quad . \tag{32b}$$

Also define

$$H_q = \dot{T}T^{-1} + THT^{-1} \tag{33a}$$

and

$$G_q = TG \tag{33b}$$

so that the dynamics of the transformed process can be written as

$$\begin{bmatrix} q_f \\ q_b \end{bmatrix} = H_q \begin{bmatrix} q_f \\ q_b \end{bmatrix} + G_q y \quad . \tag{33c}$$

If we use the following form for the inverse of T:

$$T^{-1}(t) = \begin{bmatrix} I & \vdots & I \\ - & - & - & - & - \\ -\theta_b(t) & \vdots & \theta_f(t) \end{bmatrix} \begin{bmatrix} P_s(t) & \vdots & 0 \\ - & - & - & - \\ 0 & \vdots & P_s(t) \end{bmatrix} \tag{34a}$$

where

$$P_s(t) = [\theta_f(t) + \theta_b(t)]^{-1} \quad , \tag{34b}$$

and if we choose the dynamics for θ_f and θ_b as

$$-\dot{\theta}_f = \theta_f A + A'\theta_f + \theta_f BQB'\theta_f - C'R^{-1}C \tag{34c}$$

and

$$-\dot{\theta}_b = \theta_b A + A'\theta_b - \theta_b BQB'\theta_b + C'R^{-1}C \quad , \tag{34d}$$

then carrying out the calculation in (33a), it can be shown that H_q is diagonalized with diagonal blocks

$$H_f = -[A' + \theta_f BQB'] \tag{34e}$$

and

$$H_b = [-A' - \theta_b BQB'] \quad . \tag{34f}$$

Thus the dynamics of q_f and q_b are decoupled and are given by

$$\dot{q}_f = H_f q_f + C'R^{-1}y \tag{35a}$$

and

$$\dot{q}_b = H_b q_b - C'R^{-1}y \quad . \tag{35b}$$

If we assume for time-invariant dynamics that $\{A,B\}$ is stabilizable and that $\{A,C\}$ is detectable and for time-varying dynamics that $\{A,B\}$ is uniformly completely controllable and $\{A,C\}$ is uniformly completely reconstructable, then the invertibility of P_s in (34b) is guaranteed if both $\theta_f(0)$ and $\theta_b(T)$ are nonnegative definite [9]. Furthermore, these conditions guarantee that θ_f and θ_b and their derivatives are bounded and that H_f and H_b are forward and backward stable respectively.

Under the transformation (32a), the boundary condition (31b) becomes

$$\begin{bmatrix} W^{0'}\Pi_b^{-1}y_b \\ W^{T'}\Pi_b^{-1}y_b \end{bmatrix} = V_q^0 \begin{bmatrix} q_f(0) \\ q_b(0) \end{bmatrix} + V_q^T \begin{bmatrix} q_f(T) \\ q_b(T) \end{bmatrix} \tag{36a}$$

where

$$V_q^0 = V_{x\lambda}^0 T^{-1}(0) \tag{36b}$$

and

$$V_q^T = V_{x\lambda}^T T^{-1}(T) \quad . \tag{36c}$$

To simplify the expressions for the boundary value coefficient matrices in (36b) and (36c), choose the following nonnegative definite initial and final conditions for the Riccati equations (34c) and (34e):

$$\theta_f(0) = V^{0'}\Pi_v^{-1}V^0 + W^{0'}\Pi_b^{-1}W^0 \tag{37a}$$

and

$$\theta_b(T) = V^{T'}\Pi_v^{-1}V^T + W^{T'}\Pi_b^{-1}W^T \quad . \tag{37b}$$

Then defining θ_c as the following $n \times n$ matrix:

$$\theta_c = V^{T'}\Pi_v^{-1}V^0 + W^{T'}\Pi_b^{-1}W^0 \quad , \tag{38}$$

it can be shown that the boundary value coeffi-
cient matrices can be written as

$$V_q^0 = \begin{bmatrix} I & : & 0 \\ - - - & : & - - - \\ \theta_c P_s(0) & : & \theta_c P_s(0) \end{bmatrix}$$

$$\equiv [\quad V_f^0 \quad : \quad V_b^0 \quad] \quad , \quad (39a)$$

and

$$V_q^T = \begin{bmatrix} \theta_c' P_s(T) & : & \theta_c' P_s(T) \\ - - - & : & - - - \\ 0 & : & I \end{bmatrix}$$

$$\equiv [\quad V_f^T \quad : \quad V_b^T \quad] \quad . \quad (39b)$$

Since the dynamics of q_f and q_b are de-
coupled, the only coupling between the two enters
through the boundary condition. By our choice
of initial and final conditions for the Riccati
equations, we have been able to display this
coupling solely as a function of the matrix θ_c.

The smoothed estimate of x is recovered by
inverting T(t) in (32b) so that we obtain

$$\hat{x}(t) = P_s(t)[q_f(t) + q_b(t)] \quad . \quad (40)$$

Following (8), an explicit expression for the
two-filter solution for q_f and q_b is formulated
as follows. Let q_f^0 and q_b^0 be governed by
(35a) and (35b) respectively with boundary
conditions: $q_f^0(0) = 0$ and $q_b^0(T) = 0$. Define
F_{fb} and Φ_{fb} as the 2nx2n matrices

$$F_{fb} = [V_f^0 + V_f^T \Phi_f(T,0) : V_b^T + V_b^0 \Phi_b(0,T)] \quad (41a)$$

$$= \begin{bmatrix} I + \theta_c' P_s(T)\Phi_f(T,0) & : & \theta_c' P_s(T) \\ - - - - - - - - - & : & - - - - - - - - \\ \theta_c P_s(0) & & I + \theta_c P_s(0)\Phi_b(0,T) \end{bmatrix}$$

and

$$\Phi_{fb}(t) = \begin{bmatrix} \Phi_f(t,0) & : & 0 \\ - - - & : & - - - \\ 0 & & \Phi_b(t,T) \end{bmatrix} \quad . \quad (41b)$$

Then the two-filter solution for q(t) is given
by

$$\begin{bmatrix} q_f(t) \\ q_b(t) \end{bmatrix} = \Phi_{fb}(t) F_{fb}^{-1} \{ \begin{bmatrix} w^{0\prime} \\ w^{T\prime} \end{bmatrix} \Pi_b^{-1} y_b$$

$$- \begin{bmatrix} \theta_c' P_s(T) q_f^0(T) \\ \theta_c P_s(0) q_b^0(0) \end{bmatrix} \} + \begin{bmatrix} q_f^0(t) \\ q_b^0(t) \end{bmatrix} \quad . \quad (41c)$$

The computational complexity of the non-
causal smoother implementation suggested by (41)
is nearly the same as that of the two-filter
smoothers for causal processes such as the
Mayne-Fraser form [11,12]. We note, however,
that before q_f and q_b can be evaluated for any
t ε [0,T], both q_f^0 and q_b^0 must be computed and
stored along with P_s and Φ_{fb} for the entire in-
terval [0,T]. Thus, the required storage exceeds
that of the smoother for causal processes. In-
deed, the Mayne-Fraser solution and ours differ
significantly in one aspect. That is, for our
smoother the contribution of the forward filter
to the smoothed estimate at some point t depends
not only on past observations, as does the Mayne-
Fraser solution, but also on future observations
through the term $\theta_c' P_s(T) q_f^0(T)$ in (14). A
similar statement applies for the backward
filter contribution.

5. SMOOTHING ERROR

From (24b) we see that the smoothing error
is the solution of the augmented and inverted
system projected onto the span of the complemen-
tary process Z. By expressing the complementary
process as a linear function of the underlying
process and employing the internal differential
realization of the inverted system it can be shown
[8], that the smoothing error has the same
Hamiltonian dynamics so the smoother (31a) and
that the boundary condition for the smoothing
error can be written in terms of $V_{x\lambda}^0$ and $V_{x\lambda}^T$ in
(31d) and (31e). As a consequence, the same
diagonalizing transformation (32a) can be used
to write the error as a linear combination of a
forward stable process and backward stable pro-
cess. From this representation it can be shown
[8] that the error covariance can be expressed
in terms of θ_f, θ_b and θ_c in (34c), (34d) and
(38) plus the solution of one additional matrix
Riccati equation.

6. CONCLUSIONS

An internal differential realization of the
fixed-interval smoother for an nth order non-
causal two-point boundary value stochastic pro-
cess (TPBVP) has been shown to have the same $2n^{th}$
order Hamiltonian dynamics as the fixed-interval
smoother for causal processes. The simplicity of
this two-filter form is achieved by employing an
information form for the diagonalizing transfor-
mation with carefully chosen boundary conditions

for the differential equations governing its elements. The significant difference between our two-filter implementation and that for causal processes is that in the noncausal case the smoothed estimate at a given point in the interval is a noncausal function of each of the forward and backward processes (see (40) and (41c)).

In closing we note that differential realizations for estimators of both discrete and continuous parameter multidimensional stochastic processes can be formulated as well by the method of complementary models. The solutions to these multidimensional estimators and problems associated with their implementation are addressed in [8].

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