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# INTEGRATION WITH RESPECT TO OPERATOR-VALUED MEASURES WITH APPLICATIONS TO QUANTUM ESTIMATION THEORY

by

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## 1. Introduction

The problem of quantum measurement has received a great deal of attention in recent years, both in the quantum physics literature and in the context of optical communications. An account of these ideas may be found in Davies [1976] and Holevo [1973]. The development of a theory of quantum estimation requires a theory of integration with respect to operator-valued measures. Indeed, Holevo [1973] in his investigations on the Statistical Decision Theory for Quantum Systems develops such a theory which, however, is more akin to Riemann Integration. The objective of this paper is to develop a theory which is analogous to Lebesque integration and which is natural in the context of quantum physics problems and show how this can be applied to quantum estimation problems. The theory that we present has little overlap with the theory of integration with respect to vector measures nor the integration theory developed by Thomas [1970].

We now explain how this theory is different from some of the known theories of integration with respect to operator-valued measures. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let X,Y be Banach spaces with normed duals $X^*,Y^*$. $C_o(S,X)$ denotes the Banach space of continuous X-valued functions $f: S \to X$ which vanish at infinity (for every $\varepsilon > 0$, there is a compact set $K \subset S$ such that $|f(s)| < \varepsilon$ for all $s \in S\setminus K$), with the supremum norm $|f|_\infty = \sup_{s\in S} |f(s)|$ . It is possible to identify every bounded linear map l: $C_o(S,X) \to Y$ with a representing measure m such that

$$Lf = \int_S m(ds)f(s) \qquad (1.1)$$

for every $f \in C_o(S,X)$. Here m is a finitely additive map $m: B \to L(X,Y^{**})$[(1)] with finite semivariation which satisfies:

1. for every $z \in Y^*$, $m_z: \mathcal{B} \to X^*$ is a regular $X^*$-valued Borel measure, where $m_z$ is defined by

$$m_z(E)x = \langle z, m(E)x \rangle \qquad E \in \mathcal{B},\ x \in X; \qquad (1.2)$$

2. the map $z \mapsto m_z$ is continuous for the w* topologies on $z \in Y^*$ and $m_z \in C_o(S,X)^*$.

The latter condition assures that the integral (1) has values in Y even though the measure has values in $L(X,Y^{**})$ rather than $L(X,Y)$ (we identify Y as a subspace of $Y^{**}$). Under the above representation of maps $L \in \mathbf{L}(C_o(S,X),Y)$, the maps for which $L_x: C_o(S) \to Y: g(\cdot) \mapsto L(g(\cdot)x)$ is weakly compact for every $x \in X$ are precisely the maps whose representing measures have values in $L(X,Y)$, not just in $L(X,Y^{**})$. In particular, if Y is reflexive or if Y is weakly complete or more generally if Y has no subspace isomorphic to $c_o$, then every map in $L(C_o(S,X),Y)$ is weakly compact and hence every $L \in L(C_o(S,X),Y)$ has a representing measure with values in $L(X,Y)$.

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(1) $L(X,Y)$ denotes the Banach space of bounded linear operators from X to Y.

We now develop some notation and terminology which will be needed. Let H be a complex Hilbert space. The real linear space of compact self-adjoint operators $\mathcal{K}_s(H)$ with the operator norm is a Banach space whose dual is isometrically isomorphic to the real Banach space $\mathcal{T}_s(H)$ of self-adjoint trace-class operators with the trace norm, i.e. $\mathcal{K}_s(H)^* = \mathcal{T}_s(H)$ under the duality

$$\langle A,B\rangle = \mathrm{tr}(AB) \leq |A|_{tr} \cdot |B| \qquad A \in \mathcal{T}_s(H),\ B \in \mathcal{K}_s(H).$$

Here $|B| = \sup\{|B\phi|: \phi \in H,\ |\phi| \leq 1\} = \sup\{\mathrm{tr}AB: A \in \mathcal{T}_s(H),\ |A|_{tr} \leq 1\}$ and $|A|_{tr}$ is the trace norm $\sum_i |\lambda_i| < +\infty$ where $A \in \mathcal{T}_s(H)$ and $\{\lambda_i\}$ are the eigenvalues of A repeated according to multiplicity. The dual of $\mathcal{T}_s(H)$ with the trace norm is isometrically isomorphic to the space of all linear bounded self-adjoint operators, i.e. $\mathcal{T}_s(H)^* = \mathcal{L}_s(H)$ under the duality

$$\langle A,B\rangle = \mathrm{tr}(AB) \qquad A \in \mathcal{T}_s(H),\ B \in \mathcal{L}_s(H).$$

Moreover the orderings are compatible in the following sense. If $\mathcal{K}_s(H)_+$, $\mathcal{T}_s(H)_+$, and $\mathcal{L}_s(H)_+$ denote the closed convex cones of nonnegative definite operators in $\mathcal{K}_s(H)$, $\mathcal{T}_s(H)$, and $\mathcal{L}_s(H)$ respectively, then

$$[\mathcal{K}_s(H)_+]^+ = \mathcal{T}_s(H)_+ \quad \text{and} \quad [\mathcal{T}_s(H)_+]^+ = \mathcal{L}_s(H)_+$$

where the associated dual spaces are to be understood in the sense defined above.

In the context of quantum mechanical measures with values in $L_s(H)$, one can identify every continuous linear map L: $C_o(S) \to L_s(H)$ (here X=R, $Y=L_s(H)$) with a representing measure with values in $L_s(H)$ rather than in $L_s(H)^{**}$, using fairly elementary arguments. Since $Y=L_s(H)$ is neither reflexive nor devoid of subspaces isomorphic to $C_o$, one might think at first sight this is incorrect. However, whereas in the usual approach it is assumed that the real-valued set function $zm(\cdot)x$ is countably additive for $x \in X$ and every $z \in Y^*$, we require that it be countably additive only for $x \in X$ and $z \in Z=T_s(H)$, where $Z=T_s(H)$ is a predual of $Y=L_s(H)$, and hence can represent all linear bounded maps L: $C_o(S,X) \to Y$ by measures with values in L(X,Y). In other words, by assuming that the measures m: $B \to L_s(H)$ are countably additive in the weak* topology rather than the weak topology (these are equivalent only when m has bounded variation), it is possible to represent every bounded linear map L: $C_o(S) \to L_s(H)$ and not just the weakly compact maps. This approach is generally applicable whenever Y is a dual space, and in fact yields the usual results by imbedding Y in Y**; moreover it clearly shows the relationships between various boundedness conditions on the representing measures and the corresponding spaces of linear maps. But first we must define what is meant by integration with respect to operator-valued measures. We shall always take the underlying field of scalars to be the reals, although the results extend immediately to the complex case.

## 2. Additive Set Functions

Throughout this section we assume that $\mathcal{B}$ is the $\sigma$-algebra of Borel sets of a locally compact Hausdorff space S, and X,Y are Banach spaces. Let $m: \mathcal{B} \to L(X,Y)$ be an additive set function, i.e. $m(E_1 \cup E_2) = m(E_1)+m(E_2)$ whenever $E_1, E_2$ are disjoint sets in B. The <u>semivariation</u> of m is the map $\bar{m}: \mathcal{B} \to \bar{R}_+$ defined by

$$\bar{m}(E) = \sup \left| \sum_{i=1}^{n} m(E_1)x_i \right|,$$

where the supremum is taken over all finite collections of disjoint sets $E_1,\ldots,E_n$ belonging to $\mathcal{B} \cap E$ and $x_1,\ldots,x_n$ belonging to $X_1$. By $\mathcal{B} \cap E$ we mean the sub-$\sigma$-algebra $\{E' \in \mathcal{B}: E' \subset E\} = \{E' \cap E: E \in \mathcal{B}\}$ and by $X_1$ we denote the closed unit ball in X. The <u>variation</u> of m is the map $|m|: \mathcal{B} \to \bar{R}_+$ defined by

$$|m|(E) = \sup \sum_{i=1}^{n} |m(E_i)|$$

where again the supremum is taken over all finite collections of disjoint sets in $\mathcal{B} \cap E$. The <u>scalar semivariation</u> of m is the map $\bar{\bar{m}}: \mathcal{B} \to \bar{R}_+$ defined by

$$\bar{\bar{m}}(E) = \sup \left| \sum_{i=1}^{n} a_i m(E_i) \right|$$

where the supremum is taken over all finite collections of disjoint sets $E_1,\ldots,E_n$ belonging to $\mathcal{B} \cap E$ and $a_1,\ldots,a_n \in R$ with $|a_i| \leq 1$. It should be noted that the notion of semivariation depends on the spaces X and Y;

in fact, if $m: \mathcal{B} \to L(X,Y)$ is taken to have values in $L(R,L(X,Y))$, $L(X,Y)$, $L(X,Y)^{**} = L(L(X,Y),R)$ respectively then

$$\bar{\bar{m}} = \bar{m}_{L(R,L(X,U))} \leq \bar{m} = \bar{m}_{L(X,Y)} \leq |m| = \bar{m}_{L(L(X,U)^*,R)}. \qquad (2.1)$$

When necessary, we shall subscript the semivariation accordingly. By $fa(\mathcal{B},W)$ we denote the space of all finitely additive maps $m: \mathcal{B} \to W$ where $W$ is a vector space.

<u>Proposition 2.1.</u> If $m \in fa(\mathcal{B},X^*)$ then $\bar{m} = |m|$. More generally, if $m \in fa(\mathcal{B},L(X,Y))$ then for every $z \in Y^*$ the finitely additive map $zm: \mathcal{B} \to X^*$ satisfies $\overline{zm} = |zm|$.

<u>Proof</u>. It is sufficient to consider the case $Y = R$, i.e. $m \in fa(\mathcal{B},X^*)$. Clearly $\bar{m} \leq |m|$. Let $E \in \mathcal{B}$ and let $E_1,\ldots,E_n$ be disjoint sets in $\mathcal{B} \cap E$. Then $\sum_i |m(E_i)| = \sup_{x_i \in X_1} \Sigma m(E_i)x_i = \sup_{x_i \in X_1} |\Sigma m(E_i)x_i| \leq m(E)$. Taking the supremum over all disjoint $E_i \in \mathcal{B} \cap E$ yields $|m|(E) \leq \bar{m}(E)$. □

We shall need some basic facts about variation and semivariation. Let $X,Y$ be normed spaces. A subset $Z$ of $Y^*$ is a <u>norming subset</u> of $Y^*$ if $\sup\{zy: z \in Z, |z| \leq 1\} = |y|$ for every $y \in Y$.

<u>Proposition 2.2.</u> Let $X,Y$ be normed spaces, $m \in fa(\mathcal{B}, L(X,Y))$. If $Z$ is a norming subset of $Y^*$, then

$$\bar{m}(E) = \sup_{z\in Z, |z|\leq 1} |zm|(E) \qquad , E \in \mathcal{S}$$

$$\bar{\bar{m}}(E) = \sup_{z\in Z, |z|\leq 1} \sup_{x\in X, |x|\leq 1} |zm(\cdot)x|(E) \quad , E \in \mathcal{S}$$

Moreover $|y^*m(\cdot)x|(E) \leq |x|\cdot|y^*m|(E) \leq |x|\cdot|y^*|\cdot|m|(E)$ for every $x \in X$, $y^* \in Y^*$, $E \in \mathcal{S}$.

Proof. Let $\{E_1,\ldots,E_n\}$ be disjoint sets in $\mathcal{S} \cap E$ and $x_1,\ldots,x_n \in X_1$. Then

$$|\sum_{i=1}^{n} m(E_i)x_i| = \sup_{z\in Z_1} \langle z, \sum_{i=1}^{n} m(E_i)x_i\rangle = \sum_{i=1}^{n} zm(E_i)x_i .$$

Taking the supremum over $\{E_i\}$ and $\{x_i\}$ yields $\bar{m}(E) = |zm|(E)$. Similarly,

$$\sup_{|a_i|\leq 1} |\sum_{i=1}^{n} a_i m(E_i)| = \sup_{|a_i|\leq 1} \sup_{x\in X_1} \sup_{z\in Z_1} \langle z, \sum_{i=1}^{n} a_i m(E_i)x\rangle$$

$$= \sup_{\substack{x\in X_1 \\ z\in Z_1}} \sum_{i=1}^{n} |zm(E_i)x|$$

and taking the supremum over finite disjoint collections $\{E_i\} \subset \mathcal{S} \cap E$ yields $\bar{\bar{m}}(E) = \sup_{|x|\leq 1} \sup_{|z|\leq 1} |zm(\cdot)x|(E)$.

It is straightforward to check the final statement of the theorem. □

Proposition 2.3. Let $m \in fa(\mathcal{D}, L(X,Y))$. Then $\bar{\bar{m}}$, $\bar{m}$, and $|m|$ are monotone and finitely subadditive; $|m|$ is finitely additive.

Proof. It is immediate that $\bar{m}$, $\bar{\bar{m}}$, $|m|$ are monotone. Suppose $E_1, E_2 \in \mathcal{D}$ and $E_1 \cap E_2 = \emptyset$, and let $F_1,\ldots,F_n$ be a finite collection of disjoint sets in $\mathcal{D} \cap (E_1 \cup E_2)$. Then if $|x_i| \leq 1$, $i = 1,\ldots,n$, we have

$$\left|\sum_{i=1}^{n} m(F_i)x_i\right| = \left|\sum_{i=1}^{n} (m(F_i \cap E_1)+m(F_i \cap E_2))x_i\right|$$

$$\leq \left|\sum_i m(F_i \cap E_1)x_i\right| + \left|\sum_i m(F_i \cap E_2)x_i\right|$$

$$\leq \bar{m}(E_1)+\bar{m}(E_2).$$

Taking the supremum over all disjoint $F_1,\ldots,F_n \in \mathcal{D} \cap (E_1 \cup E_2)$ yields $\bar{m}(E_1 \cup E_2) \leq \bar{m}(E_1)+\bar{m}_2(E_2)$. Using (2.1) we immediately have $\bar{\bar{m}}$, $|m|$ finitely subadditive. Since $|m|$ is always superadditive by its definition, $|m|$ is finitely additive. □

## 3. Integration with Respect to Additive Set Functions

We now define integration with respect to additive set functions $m: \mathcal{D} \to L(X,Y)$. Let $\mathcal{D} \otimes X$ denote the vector space of all X-valued measurable simple functions on S, that is all functions of the form $f(s) = \sum_{i=1}^{n} 1_{E_i}(s)x_i$ where $\{E_1,\ldots,E_n\}$ is a finite disjoint measurable partition of S, i.e. $E_i \in \mathcal{D}$ $\forall i$, $E_i \cap E_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{n} E_i = S$. Then the integral $\int_S m(ds)f(s)$ is defined unambiguously (by finite additivity) as

$$\int_S m(ds)f(s) = \sum_{i=1}^{n} m(E_i)x_i. \tag{3.1}$$

We make $\mathcal{B}\otimes X$ into a normed space under the uniform norm, defined for bounded maps $f: S \to X$ by

$$|f|_\infty = \sup_{s\in S}|f(s)|.$$

Suppose now that m has finite semivariation, i.e. $\overline{m}(s) < +\infty$. From the definitions it is clear that

$$\left|\int_S m(ds)f(s)\right| \leq \overline{m}(S)\cdot|f|_\infty , \tag{3.2}$$

so that $f \mapsto \int_S m(ds)f(s)$ is a bounded linear functional on $(\mathcal{B}\otimes X, |\cdot|_\infty)$; in fact, $\overline{m}(S) = \sup\{|\int m(ds)f(s)|: |f|_\infty \leq 1, f\in\mathcal{B}\otimes X\}$ is the bound. Thus, if $\overline{m}(S) < +\infty$ it is possible to extend the definition of the integral to the completion $M(S,X)$ of $\mathcal{B}\otimes X$ in the $|\cdot|_\infty$ norm. $M(S,X)$ is called the space of totally $\mathcal{B}$-measurable X-valued functions on S; every such function is the uniform limit of $\mathcal{B}$-measurable simple functions. For $f \in M(S,X)$ define

$$\int_S m(ds)f(s) = \lim_{n\to\infty} \int_S m(ds)f_n(s) \tag{3.3}$$

where $f_n \in \mathcal{B} \otimes X$ is an arbitrary sequence of simple functions which converge uniformly to f. The integral is well-defined since if $\{f_n\}$ is a Cauchy sequence in $\mathcal{B} \otimes X$ then $\{\int_S m(ds)f_n(s)\}$ is Cauchy in Y by (3.2) and hence converges. Moreover if two sequences $\{f_n\}$, $\{g_n\}$ in $\mathcal{B} \otimes X$ satisfy $|g_n - f|_\infty \to 0$ and $|f_n - f|_\infty \to 0$ then $|\int m(ds)f_n(s) - \int m(ds)g_n(s)| \leq \bar{m}(S)|f_n - g_n|_\infty \to 0$ so $\lim_{n\to\infty} \int_S m(ds)f_n(s) = \lim_{n\to\infty} \int_S m(ds)g_n(s)$.

Similarly, it is clear that (3.2) remains true for every $f \in M(S,X)$. More generally it is straightforward to verify that

$$\bar{m}(E) = \sup\{\int_S m(ds)f(s) : f \in M(S,X),\ |f|_\infty \leq 1,\ \mathrm{supp} f \subset E\}. \qquad (3.4)$$

<u>Proposition 3.1.</u> $C_o(S,X) \subset M(S,X)$.

Proof. Every $g(\cdot) \in C_o(S)$ is the uniform limit of simple real-valued Borel-measurable functions, hence every function of the form $f(s) = \sum_{i=1}^{n} g_i(s)x_i = \sum_{i=1}^{n} g_i \otimes x_i$ belongs to $M(S,X)$, for $g_i \in C_o(S)$ and $x_i \in X$. These functions may be identified with $C_o(S) \otimes X$, which is dense in $C_o(S,X)$ for the supremum norm (cf. Treves [1967], p. 448). Hence $C_o(S,X) = \mathrm{cl}\, C_o(S) \otimes X \subset M(S,X)$.

To summarize, if $m \in fa(\mathcal{B}, L(X,Y))$ has finite semivariation $\bar{m}(S) < +\infty$ then $\int_S m(ds)f(s)$ is well-defined for

$f \in M(S,X) \supset C_o(S,X)$, and in fact $f \mapsto \int_S m(ds)f(s)$ is a bounded linear map from $C_o(S,X)$ or $M(S,X)$ into Y.

Now let Z be a Banach space and L a bounded linear map from Y to Z. If $m: \mathcal{B} \to \mathbf{L}(X,Y)$ is finitely additive and has finite semivariation then $Lm: \mathcal{B} \to \mathbf{L}(X,Z)$ is also finitely additive and has finite semivariation $\overline{Lm}(S) \leq |L| \cdot \overline{m}(S)$. For every simple function $f \in \mathcal{B} \otimes X$ it is easy to check that $L\int_S m(ds)f(s) = \int_S Lm(ds)f(s)$. By taking limits of uniformly convergent simple functions we have proved

Proposition 3.2. Let $m \in fa(\mathcal{B}, \mathbf{L}(X,Y))$ and $\overline{m}(S) < +\infty$. Then $Lm \in fa(\mathcal{B}, \mathbf{L}(X,Z))$ for every bounded linear $L: Y \to Z$, with $\overline{Lm}(S) < +\infty$ and

$$L\int_S m(ds)f(s) = \int_S Lm(ds)f(s). \tag{3.5}$$

Since we will be considering measure representations of bounded linear operators on $C_o(S,X)$, we shall require some notions of countable additivity and regularity. Recall that a set function $m: \mathcal{B} \to W$ with values in a locally convex Hausdorff space W is countably additive iff

$m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$ for every countable disjoint sequence $\{E_i\}$ in $\mathcal{B}$. By the Pettis Theorem (cf. Dunford-Schwartz [1966]) countable additivity is equivalent to weak countable additivity, i.e. $m: \mathcal{B} \to W$ is countable additive iff it is countably additive for the weak topology on W, that is iff $w^*m: \mathcal{B} \to R$ is countably additive for every $w^* \in W^*$. If W is a Banach space, we denote by $ca(\mathcal{B}, W)$ the space of all countably additive maps $m: \mathcal{B} \to W$; $fabv(\mathcal{B}, W)$ and $cabv(\mathcal{B}, W)$ denote the spaces of finitely additive and countably additive maps $m: \mathcal{B} \to W$ which have bounded variation $|m|(S) < +\infty$.

If W is a Banach space, a measure $m \in fa(\mathcal{B}, W)$ is <u>regular</u> iff for every $\varepsilon > 0$ and every Borel set E there is a compact set $K \subset E$ and an open set $G \supset E$ such that $|m(F)| < \varepsilon$ whenever $F \in \mathcal{B} \cap (G \setminus K)$. The following theorem shows among other things that regularity actually implies countable additivity when m has bounded variation $|m|(S) < +\infty$ (this latter condition is crucial). By $rcabv(\mathcal{B}, W)$ we denote the space of all countably additive regular Borel measures $m: \mathcal{B} \to W$ which have bounded variation.

Let X,Z be Banach spaces. We shall be mainly concerned with a special class of $L(X,Z^*)$-valued measures which we now define. Let $\mathcal{M}(\mathcal{B}, L(X,Z^*))$ be the space of all $m \in fa(\mathcal{B}, L(X,Z^*))$ such that $\langle z, m(\cdot)x \rangle \in rcabv(\mathcal{B})$ for every $x \in X$, $z \in Z$. Note that such measures $m \in \mathcal{M}(\mathcal{B}, L(X,Z^*))$ need not be countably additive for the weak operator (equivalently, the strong operator) topology on $L(X,Z^*)$, since $z^{**}m(\cdot)x$ need not belong to $ca(\mathcal{B})$ for every $x \in X$, $z^{**} \in Z^{**}$.

The following theorem is very important in relating various countable additivity and regularity conditions.

Theorem 3.1. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let X,Y be normed spaces, $Z_1$ a norming subset of $Y^*$, $m \in fa(\mathcal{B}, L(X,Y))$. If $zm(\cdot)x: \mathcal{B} \to R$ is countably additive for every $z \in Z_1$, $x \in X$ then $|m|(\cdot)$ is countably additive $\mathcal{B} \to \overline{R}_+$. If $zm(\cdot)x: \mathcal{B} \to R$ is regular for every $z \in Z_1$, $x \in X$, and if $|m|(S) < +\infty$, then $|m|(\cdot) \in rcabv(\mathcal{B}, R_+)$. If $|m|(S) < +\infty$, then $m(\cdot)$ is countably additive iff $|m|$ is and $m(\cdot)$ is regular iff $|m|$ is.

Proof. Suppose $zm(\cdot)x \in ca(\mathcal{B},R)$ for every $z \in Z_1$, $x \in X$. Let $\{A_i\}$ be a disjoint sequence in $\mathcal{B}$. Let $\{B_1,\ldots,B_n\}$ be a finite collection of disjoint Borel subsets of $\bigcup_{i=1}^{\infty} A_i$. Then

$$\sum_{j=1}^{n} |m(B_j)| = \sum_{j=1}^{n} |m(\bigcup_{i=1}^{\infty} A_i \cap B_j)| = \sum_{j=1}^{n} \sup_{\substack{x_j \in X_1 \\ z_j \in Z_1}} |z_j m(\bigcup_{i=1}^{\infty} A_i \cap B_j) x_j| .$$

Since each $z_j m(\cdot) x_j$ is countably additive, we may continue with

$$= \sum_{j=1}^{n} \sup_{\substack{x_j \in X_1 \\ z_j \in Z_1}} \left| \sum_{i=1}^{\infty} a_j m(A_i \cap B_j) x_j \right| \leq \sum_{j=1}^{n} \sup_{\substack{x_j \in X_1 \\ z_j \in Z_1}} \sum_{i=1}^{\infty} |z_j m(A_i \cap B_j) x_j|$$

$$\leq \sum_{j=1}^{n} \sum_{i=1}^{\infty} |m(A_i \cap B_j)| = \sum_{i=1}^{\infty} \sum_{j=1}^{n} |m(A_i \cap B_j)| \leq \sum_{i=1}^{\infty} |m|(A_i).$$

Hence, taking the supremum over all disjoint $\{B_j\} \subset \bigcup_{i=1}^{\infty} A_i$, we have $|m|(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} |m|(A_i)$. Since $|m|$ is always countably superadditive, $|m|$ is countably additive.

Now assume $zm(\cdot)x$ is regular for every $z \in Z_1$, $x \in X$, and $|m|(S) < +\infty$. Obviously each $zm(\cdot)x$ has bounded variation since $|m|(S)<+\infty$, hence $zm(.)x \in ca(\mathcal{B})$ by (Dunford-Schwartz [1966], III.5.13) and $zm(.)x \in rcabv(\mathcal{B})$. We wish to show that $|m|$ is regular; we already know $|m| \in cabv(\mathcal{B})$. Let $E \in \mathcal{B}$, $\varepsilon > 0$. By definition of $|m|(E)$ there is a finite disjoint Borel partition $\{E_1,\ldots,E_n\}$ of $E$ such that $|m|(E) < \sum_{i=1}^{n} |m(E_i)| + \varepsilon/2$. Hence there are

$z_1,\ldots,z_n \in Z_1$ and $x_1,\ldots,x_n \in X$, $|x_i| \leq 1$, such that

$$|m|(E) < \sum_{i=1}^{n} z_i m(E_i) x_i + \varepsilon/2.$$

Now each $z_i m(\cdot) x_i$ is regular, so there are compact $K_i \subset E_i$

for which $|z_i m(E_i \setminus K_i)x_i| < \varepsilon/2n$, $i = 1,\ldots,n$. Hence

$$\begin{aligned}
|m|(E \setminus K) &= |m|(E) - |m|(K) \\
&< \sum_{i=1}^{n} z_i m(E_i)x_i + \frac{\varepsilon}{2} - \sum_{i=1}^{n} z_i m(E_i \cap K_i)x_i \\
&= \sum_{i=1}^{n} z_i m(E_i \setminus K_i)x_i + \varepsilon/2 \\
&< \varepsilon,
\end{aligned}$$

and we have shown that $|m|$ is inner regular. Since $|m|(s) < +\infty$, it is straightforward to show that $|m|$ is outer regular. For if $E \in \mathcal{B}$, $\varepsilon > 0$ then there is a compact $K \subset S \setminus E$ for which $|m|(S \setminus E) < |m|(K) + \varepsilon$ and so for the open set $G = S \setminus K \supset E$ we have

$$|m|(G \setminus E) = |m|(S \setminus E) - |m|(K) < \varepsilon.$$

Finally, let us prove the last statement of the theorem. We assume $m \in fa(\mathcal{B}, L(X,Y))$ and $|m|(S) < +\infty$. First suppose $m(\cdot)$ is countably additive. Then for every disjoint sequence $\{A_i\}$ in $\mathcal{B}$,

$|m(\bigcup_{i=1}^{\infty} A_i) - \sum_{i=1}^{n} m(A_i)| \to 0$, so certainly

$y^*m(\bigcup_{i=1}^{\infty} A_i)x - \sum_{i=1}^{n} y^*m(A_i)x_i \to 0$ for every $y^* \in Y^*$, $x \in X$

and by what we just proved $|m|$ is countably additive. Conversely, if $|m|$ is countably additive then for every disjoint sequence $\{A_i\}$ we have $|m(\bigcup_{i=1}^{\infty} A_i) - \sum_{i=1}^{n} m(A_i)| =$

$|m(\bigcup_{i=1}^{\infty} A_i)| \leq |m|(\bigcup_{i=1}^{\infty} A_i) = |m|(\bigcup_{i=1}^{\infty} A_i) - \sum_{i=1}^{n} |m|(A_i) \to 0.$

Similarly, if m is regular then every $y^*m(\cdot)x$ is regular and by what we proved already $|m|$ is regular. Conversely, if $|m|$ is regular it is easy to show that m is regular. □

Theorem 3.2. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let X,Z be Banach spaces. There is an isometric isomorphism $L \leftrightarrow m$ between the bounded linear maps $L: C_o(S) \to L(X,Z^*)$ and the finitely additive measures $m: \mathcal{B} \to L(X,Z^*)$ for which $zm(\cdot)x \in rcabv(\mathcal{B})$ for every $x \in X$, $z \in Z$. The correspondence $L \leftrightarrow m$ is given by

$$Lg = \int_S g(s)m(ds), \quad g \in C_o(S) \tag{3.6}$$

where $|L| = \bar{\bar{m}}(S)$; moreover, $zL(g)x = \int_S g(s)\, zm(ds)x$ and $|zL(\cdot)x| = |zm(\cdot)x|(S)$ for $x \in X$, $z \in Z$.

Remarks. The measure $m \in fa(\mathcal{B}, L(X,Z^*))$ need have neither finite semivariation $\bar{m}(s)$ nor bounded variation $|m|(S)$. It is also clear that $L(g)x = \int_S g(s)m(ds)x$ and $zL(g) = \int_S g(s)zm(ds)$, by Proposition 3.2.

Proof. Suppose $L \in \mathbb{L}(C_o(S), \mathbb{L}(X,Z^*))$ is given. Then for every $x \in X$, $z \in Z$ the map $g \mapsto zL(g)x$ is a bounded linear functional on $C_o(S)$, so there is a unique real-valued regular Borel measure $m_{x,z}\colon \mathcal{B} \to R$ such that

$$zL(g)x = \int_S f(s) m_{x,z}(ds). \tag{3.7}$$

For each Borel set $E \in \mathcal{B}$, define the map $m(E)\colon X \to Z^*$ by $\langle z, m(E)x\rangle = m_{xz}(E)$. It is easy to see that $m(E)\colon X \to Z^*$ is linear from (11); moreover it is continuous since

$$|m(E)| \leq \bar{\bar{m}}(S) = \sup_{\substack{|x|\leq 1 \\ |z|\leq 1}} |zm(\cdot)x|(S) = \sup_{\substack{|x|\leq 1 \\ |z|\leq 1}} |m_{xz}|(S) =$$

$$\sup_{\substack{|x|\leq 1 \\ |z|\leq 1}} |zL(\cdot)x| = |L|.$$

Thus $m(E) \in \mathbb{L}(X,Z^*)$ for $E \in \mathcal{B}$ and $m \in fa(\mathcal{B}, \mathbb{L}(X,Z^*))$ has finite scalar semivariation $\bar{\bar{m}}(S) = |L|$. Since $\bar{\bar{m}} = \bar{m}_{L(R,L(X,Z^*))}$ is finite, the integral in (3.6) is well-defined for $g \in C_o(S) \subset M(S,R)$ and is a continuous linear map $g \mapsto \int_S m(ds)g(s)$. Now (3.7) and Proposition 3.2 imply that

$$zL(g)x = \int_S zm(ds)xg(s) = \langle z, \int_S m(ds)g(s)\cdot x\rangle$$

for every $x \in X$, $z \in Z$. Thus (3.6) follows.

Conversely suppose $m \in fa(\mathcal{B}, \mathcal{L}(X,Z^*))$ satisfies $zm(\cdot)x \in rcabv(\mathcal{B})$ for every $x \in X$, $z \in Z$. First we must show that $m$ has finite scalar semivariation $\bar{\bar{m}}(S) < +\infty$. Now $\sup_{E\in\mathcal{B}} |zm(E)x| \leq |zm(\cdot)x|(S) < +\infty$ for every $x \in X$, $z \in Z$. Hence successive applications of the uniform boundedness theorem yields $\sup_{E\in\mathcal{B}} |m(E)x| < +\infty$ for every $x \in X$ and $\sup_{E\in\mathcal{B}} |m(E)| < +\infty$, i.e. $m$ is bounded. But then by Proposition 2.2.

$$\bar{\bar{m}}(S) = \sup_{\substack{|x|\leq 1\\ |z|\leq 1}} |zm(\cdot)x|(S) = \sup_{\substack{|x|\leq 1\\ |z|\leq 1}} \sup_{E_i \text{ disjoint}} \sum_{i=1}^{n} |zm(E_i)x|$$

$$= \sup_{\substack{|x|\leq 1\\ |z|\leq 1}} \sup_{E_i \text{ disj}} \Sigma^+ zm(E_i)x - \Sigma^- zm(E_i)x$$

$$= \sup_{\substack{|x|\leq 1\\ |z|\leq 1}} \sup_{E_i \text{disj}} zm(\cup^+ E_i)x - zm(\cup^- E_i)x$$

$$\leq \sup_{\substack{|x|\leq 1\\ |z|\leq 1}} 2 \sup_{E\in\mathcal{B}} |zm(E)x| = 2 \sup_{E\in\mathcal{B}} |m(E)| < +\infty,$$

where $\Sigma^+$ and $\cup^+$ ($\Sigma^-$ and $\cup^-$) are taken over those $i$ for which $zm(E_i)x \geq 0$ ($zm(E_i)x < 0$). Thus $\bar{\bar{m}}(s)$ is finite so (3.6) defines a bounded linear map $L: C_o(S) \to \mathcal{L}(X,Z^*)$. □

We now investigate a more restrictive class of bounded linear maps. For $L \in \mathbf{L}(C_o(S), \mathbf{L}(X,Z^*))$ define the (not necessarily finite) norm

$$||L|| = \sup \left| \sum_{i=1}^{n} L(g_i)x_i \right|$$

where the supremum is over all finite collections $g_1,\ldots,g_n \in C_o(S)_1$ and $x_1,\ldots,x_n \in X_1$ such that the $g_i$ have disjoint support.

<u>Theorem 3.3.</u> Let $S$ be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let $X,Z$ be Banach spaces. There is an isometric isomorphism $L_1 \leftrightarrow m \leftrightarrow L_2$ between the linear maps $L_1: C_o(S) \to \mathbf{L}(X,Z^*)$ with $||L_1|| < +\infty$; the measures $m \in fa(\mathcal{B}, \mathbf{L}(X,Z^*))$ with finite semivariation $\bar{m}(S) < +\infty$ for which $zm(\cdot)x \in rcabv(\mathcal{B})$ for every $z \in Z$, $x \in X$; and the bounded linear maps $L_2: C_o(S,X) \to Z^*$ The correspondence $L_1 \leftrightarrow m \leftrightarrow L_2$ is given by

$$L_1 g = \int_S m(ds)g(s) \ , \quad g \in C_o(S) \tag{3.8}$$

$$L_2 f = \int_S m(ds)f(s) \ , \quad f \in C_o(S,X) \tag{3.9}$$

$$L_2(g(\cdot)x) = (L_1 g)x \ , \ g \in C_o(S), \ x \in X. \tag{3.10}$$

Moreover under this correspondence $||L_1|| = \bar{m}(S) = |L_2|$;

and $zL_2 \in C_o(S,X)^*$ is given by $zL_2 f = \int_S zm(ds)f(s)$ where $zm \in rcabv(\mathcal{B}, X^*)$ for every $z \in Z$.

Proof. From Theorem 3.2 we already have an isomorphism $L_1 \leftrightarrow m$; we must show that $||L_1|| = \bar{m}(S)$ under this correspondence. We first show that $||L_1|| \leq \bar{m}(S)$. Suppose $g_1, \ldots, g_n \in C_o(s)$ have disjoint support with $|g_i|_\infty \leq 1$; $x_1, \ldots, x_n \in X$ with $|x_i| \leq 1$; and $z \in Z$ with $|z| \leq 1$. Then

$$\langle z, \sum_{i=1}^{n} L_1(g_i)x_i \rangle = \sum_{i=1}^{n} \int_S zm(ds)x_i \cdot g_i(s)$$

$$\leq \sum_{i=1}^{n} |zm(\cdot)x_i|(\text{supp}g_i)$$

$$\leq \sum_{i=1}^{n} |zm|(\text{supp}g_i)$$

where the last step follows from Proposition 2.2 $|x_i| \leq 1$. Since $|zm|$ is subadditive by Proposition 2.3, we have

$$\langle z, \sum_{i=1}^{n} L_1(g_i)x_i \rangle \leq |zm|(\bigcup_{i=1}^{n} \text{supp}g_i) \leq |zm|(S).$$

Taking the supremum over $|z| \leq 1$, we have, again by Proposition 2.2.

$$\left| \sum_{i=1}^{n} L_1(g_i)x_i \right| \leq \sup_{|z| \leq 1} |zm|(S) = \bar{m}(S).$$

Since this is true for all such collections $\{g_i\}$ and $\{x_i\}$, $||L|| \leq \overline{m}(S)$. We now show $\overline{m}(S) \leq ||L||$. Let $\varepsilon > 0$ be arbitrary, and suppose $E_1,\ldots,E_n \in \mathcal{D}$ are disjoint, $|z| \leq 1$, $|x_i| \leq 1$, $i = 1,\ldots,n$. By regularity of $zm(\cdot)x_i$, there is a compact $K_i \subset E_i$ such that $|zm(\cdot)x_i|(E_i) < \frac{\varepsilon}{n} + |zm(\cdot)x_i|(K_i)$, $i = 1,\ldots,n$. Since the $K_i$ are disjoint, there are disjoint open sets $G_i \supset K_i$. By Urysohn's Lemma there are continuous functions $g_i$ with compact support such that $1_{K_i} \leq g_i \leq 1_{G_i}$. Then

$$\sum_{i=1}^{n} zm(E_i)x_i = \sum_{i=1}^{n} zL(g_i)x_i + \sum_{i=1}^{n} \int (1_{E_i}-g_i)(s)\,zm(ds)x_i$$

$$\leq \sum_{i=1}^{n} zL(g_i)x_i + \sum_{i=1}^{n} \int (1_{E_i}-1_{K_i})(s)\,zm(ds)x_i$$

$$\leq \sum_{i=1}^{n} zL(g_i)x_i + \sum_{i=1}^{n} |zm(\cdot)x|(E_i \setminus K_i) \leq \sum_{i=1}^{n} zL(g_i)x_i+\varepsilon$$

$$\leq \left|\sum_{i=1}^{n} L(g_i)x_i\right| + \varepsilon$$

$$\leq ||L|| + \varepsilon.$$

Taking the supremum over $|z| \leq 1$, finite disjoint collections $\{E_i\}$, $|x_i| \leq 1$ we get $\overline{m}(S) \leq ||L|| + \varepsilon$. Since $\varepsilon > 0$

was arbitrary $\bar{m}(S) \leq ||L||$ and so $\bar{m}(S) = ||L||$.

It remains to show how the maps $L_2 \in \mathcal{L}(C_o(S,X),Z^*)$ are related to $L_1$ and m. Now given $L_1$ or equivalently m, it is immediate from the definition of the integral (3.3) that (3.9) defines an $L_2 \in \mathcal{L}(C_o(S,X),Z^*)$ with $|L_2| = \bar{m}(S) < +\infty$. Conversely, suppose $L_2 \in \mathcal{L}(C_o(S,X),Z^*)$ is given. Then (3.10) defines a bounded linear map $L_1: C_o(S) \to \mathcal{L}(X,Z^*)$, with $|L_1| \leq |L_2|$; moreover it is easy to see that $||L_1|| \leq |L_2|$. Of course, $L_1$ uniquely determines a measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}(X,Z^*))$ with $\bar{m}(S) = ||L_1|| \leq |L_2|$ such that (3.8) holds. Now suppose $f(\cdot) = \sum_{i=1}^{n} g_i(\cdot)x_i \in C_o(S)\otimes X$; then

$$\int m(ds)f(s) = \sum_{i=1}^{n} L_1(g_i)x_i = \sum_{i=1}^{n} L_2(g_i(\cdot)x_i) = L_2(f).$$

Hence (3.10) holds for $f(\cdot) \in C_o(S)\otimes X$, and since $C_o(S)\otimes X$ is dense in $C_o(S,X)$ we have

$$|L_2| = \sup_{\substack{f\in C_o(S)\otimes X \\ |f|_\infty\leq 1}} |L_2 f| = \sup_{\substack{f\in C_o(S)\otimes X \\ |f|_\infty\leq 1}} \left|\int m(ds)f(s)\right|$$

$$\leq \sup_{\substack{f\in M(S,X) \\ |f|_\infty\leq 1}} \left|\int m(ds)f(s)\right| = \bar{m}(S).$$

Thus $\bar{m}(S) = |L_2|$.

Finally, it is immediate from Proposition 3.2 that $zL_2f = \int_S zm(ds)f(s)$ for $f \in C_o(S,X)$, $z \in Z$. We show that $zm \in rcabv(\mathcal{B},X^*)$ for $z \in Z$. Since $|zm|(S) \leq |z|\cdot\tilde{m}(S)$ by Proposition 2.2, $zm$ has bounded variation. Since for each $x \in X$, $zm(\cdot)x \in rcabv(\mathcal{B})$ we may apply Theorem (with $Y = R$) to get $|zm| \in rcabv(\mathcal{B})$ and $zm \in rcabv(\mathcal{B},X^*)$. □

The following interesting corollary is immediate from $||L_1|| = |L_2|$ in Theorem 3.3.

Corollary. Let $L_2\colon C_o(S,X) \to Y$ be linear and bounded, where $X,Y$ are Banach spaces and $S$ is a locally compact Hausdorff space. Then

$$|L_2| = \sup|L_2(\sum_{i=1}^{n} g_i(\cdot)x_i)|,$$

where the supremum is over all finite collections $\{g_1,\ldots,g_n\} \subset C_o(S)$ and all $\{x_1,\ldots,x_n\} \in X$ such that $\{\mathrm{supp}\,g_i\}$ are disjoint and $|g_i|_\infty \leq 1$, $|x_i| \leq 1$.

Proof. Take $Z = Y^*$ and imbed $Y$ in $Z^* = Y^{**}$. Then $L_2 \in \mathbb{L}(C_o(S,X),Z^*)$ and the result follows from $||L_1|| = |L_2|$ in Theorem 3.3.□

We now consider a subspace of linear operators $L_2 \in \mathbb{L}(C_o(S,X),Y)$ with even stronger continuity properties, namely those which correspond to bounded linear functionals on $C_o(S,X\otimes_\pi Z)$; equivalently, we shall see that these maps correspond to representing measures $m \in \mathcal{M}(\mathcal{B}, \mathbb{L}(X,Z^*))$ which have finite total variation $|m|(S) < +\infty$, so that $m \in rcabv(\mathcal{B}, \mathbb{L}(X,Z^*))$. For $L_2 \in \mathbb{L}(C_o(S,X),Y)$ we define the (not necessarily finite) norm

$$|||L_2||| = \sup_{\{f_i\}} \sum_{i=1}^{n} |L_2(f_i)|$$

where the supremum is over all finite collections $\{f_1,\ldots,f_n\}$ of functions in $C_o(S,X)$ having disjoint support and $|f_i|_\infty \leq 1$. In applying the definition to $L_1 \in \mathbb{L}(C_o(S), \mathbb{L}(X,Z^*)) = \mathbb{L}(C_o(S,R),Y)$ with $Y = \mathbb{L}(X,Z^*)$ we get

$$|||L_1||| = \sup_{\{g_i\}} \sum_{i=1}^{n} |L_1(g_i)|$$

where the supremum is over all finite collections $\{g_1,\ldots,g_n\}$ of functions in $C_o(S)$ having disjoint support and $|g_i|_\infty \leq 1$.

Before proceeding, we should make a few remarks about tensor product spaces. By $X \otimes Z$ we denote a tensor product space of $X$ and $Z$, which is the vector space of all finite linear combinations $\sum_{i=1}^{n} a_i x_i \otimes z_i$ where

$a_i \in R$, $x_i \in X$, $z_i \in Z$ (of course, $a_i$, $x_i$, $z_i$ are not uniquely determined). There is a natural duality between $X \otimes Z$ and $\mathbb{L}(X,Z^*)$ given by

$$\langle \sum_{i=1}^{n} a_i x_i \otimes z_i, L\rangle = \sum_{i=1}^{n} a_i \langle z_i, Lx_i\rangle.$$

Moreover the norm of $L \in \mathbb{L}(X,Z^*)$ as a linear functional on $X \otimes Z$ is precisely its usual operator norm $|L| = \sup_{\substack{|x|\le 1\\ |z|\le 1}} \langle z,Lx\rangle$ when $X \otimes Z$ is made into a normed space $X \otimes_\pi Z$ under the <u>tensor product norm</u> $\pi$ defined by

$$\pi(u) = \inf\{\sum_{i=1}^{n} |x_i|\cdot|z_i| : u = \sum_{i=1}^{n} x_i \otimes z_i\},\ u \in X \otimes Z.$$

It is easy to see that $\pi(x \otimes z) = |x|\cdot|z|$ for $x \in X$, $z \in Z$ (the canonical injection $X \times Z \to X \otimes Z$ is continuous) and in fact $\pi$ is the strongest norm on $X \otimes Z$ with this property. By $X \hat{\otimes}_\pi Z$ we denote the completion of $X \otimes_\pi Z$ for the $\pi$ norm. Every $L \in \mathbb{L}(X,Z^*)$ extends to a unique bounded linear functional on $X \hat{\otimes}_\pi Z$ with the same norm. $X \hat{\otimes}_\pi Z$ may be identified more concretely as infinite sums $\sum_{i=1}^{\infty} a_i x_i \otimes z_i$ where $x_i \to 0$ in $X$, $z_i \to 0$ in $Z$, and $\sum_{i=1}^{\infty} |a_i| < \infty$ (Schaeffer [1971], III.6.4) and we identify $(X \otimes_\pi Z)^*$ with $L(X,Z^*)$ by

$$\langle \sum_{i=1}^{\infty} a_i x_i \otimes z_i, L\rangle = \sum_{i=1}^{\infty} a_i \langle z_i, Lx_i\rangle .$$

The following theorem provides an integral representation of $C_o(S, X \hat{\otimes}_\pi Z)^*$.

Theorem 3.4. Let $S$ be a Hausdorff locally compact space with Borel sets $\mathcal{B}$. Let $X, Z$ be Banach spaces. There is an isometric isomorphism $L_1 \leftrightarrow m \leftrightarrow L_2 \leftrightarrow L_3$ between the linear maps $L_1: C_o(S) \to \mathbb{L}(X,Z^*)$ with $|||L_1||| < +\infty$; the finitely additive measures $m: \mathcal{B} \to \mathbb{L}(X,Z^*)$ with finite variation $|m|(S) < +\infty$ for which $zm(\cdot)x \in rcabv(\mathcal{B})$ for every $z \in Z$, $x \in X$; the linear maps $L_2: C_o(S,X) \to Z^*$ with $|||L_2||| < +\infty$; and the bounded linear functionals $L_3: C_o(S, X \hat{\otimes}_\pi Z) \to R$. The correspondence $L_1 \leftrightarrow m \leftrightarrow L_2 \leftrightarrow L_3$ is given by

$$L_1 g = \int_S m(ds g(s) \quad , \quad g \in C_o(S) \tag{3.11}$$

$$L_2 f = \int_S m(ds) f(s) \quad , \quad f \in C_o(S,X) \tag{3.12}$$

$$L_3 u = \int_S \langle u(s), m(ds)\rangle \quad , \quad u \in C_o(S, X \hat{\otimes}_\pi Z) \tag{3.13}$$

$$\langle z, (L_1 g)x\rangle = \langle z, L_2(g(\cdot)x)\rangle = L_3(g(\cdot)x \otimes z), \qquad g \in C_o(S),\ x \in X,\ z \in Z. \tag{3.14}$$

Under this correspondence $|||L_1||| = |m|(s) = |||L_2||| = |L_3|$, and $m \in rcabv(\mathcal{B}, L(X,Z^*))$.

<u>Proof</u>. From Theorem 3.3 we already have an isomorphism $L_1 \leftrightarrow m \leftrightarrow L_2$; we must show that the norms are carried over under this correspondence. As in Theorem 3.2, we assume that $L_1 \leftrightarrow m \leftrightarrow L_2$ with $||L_1|| = \overline{m}(s) = |L_2| < +\infty$.

We first show $|||L_1||| \leq |||L_2|||$. Now if $\{g_1,\ldots,g_n\} \subset C_o(S)_1$ have disjoint support and $|x_i| \leq 1$, then $g_i(\cdot)x_i \in C_o(S,X)$ have disjoint support with $|g_i(\cdot)x_i|_\infty \leq 1$, so

$$\sum_{i=1}^{n} |L_1(g_i)x_i| = \sum_{i=1}^{n} |L_2(g_i(\cdot)x_i)| \leq |||L_2|||.$$

Taking the supremum over $|x_i| \leq 1$ yields

$\sum_{i=1}^{n} |L_1(g_i)| \leq |||L_2|||$, and hence $|||L_1||| \leq |||L_2|||$.

Next we show $|||L_2||| \leq |m|(s)$. Let $f_1,\ldots,f_n \in C_o(S,X)$ have disjoint support and $z_1,\ldots,z_n \in Z$ with $|z_i| \leq 1$. Then

$$\sum_{i=1}^{n} z_i L_2(f_i) = \sum_{i=1}^{n} \int_S z_i m(ds) f_i(s) \leq \sum_{i=1}^{n} |z_i m|(\mathrm{supp} f_i)$$

where the last inequality follows from (3.4) applied to $z_i m \in fa(\mathcal{B}, X^*)$. By Propositions 2.2 and 2.3 we now have

$$\sum_{i=1}^{n} z_i L_2(f_i) \le \sum_{i=1}^{n} |m|(\text{supp}f_i) = |m|(\bigcup_{i=1}^{n} \text{supp}f_i) \le |m|(S).$$

Taking the supremum over $|z_i| \le 1$ yields $\sum_{i=1}^{n} |L_2 f_i| \le |m|(S)$, and over $\{f_i\}$ yields $|||L_2||| \le |m|(S)$.

Now we show $|m|(S) \le |||L_1|||$. Let $\varepsilon > 0$ be arbitrary, and suppose $E_1,\ldots,E_n \in \mathcal{B}$ are disjoint and $|x_i| \le 1$, $|z_i| \le 1$, $i = 1,\ldots,n$. By regularity of $z_i m(\cdot)x_i$, there is a compact $K_i \subset E_i$ such that $|z_i m(\cdot)x_i|(E_i) < \frac{\varepsilon}{n} + |z_i m(\cdot)x_i|(K_i)$, $i = 1,\ldots,n$. Since the $K_i$ are disjoint, there are disjoint open sets $G_i \supset K_i$. Urysohn's Lemma then guarantees the existence of continuous functions $g_i$ with compact support wuch that $1_{K_i} \le g_i \le 1_{G_i}$. We have

$$\begin{aligned}
\sum_{i=1}^{n} z_i m(E_i)x_i &= \sum_{i=1}^{n} z_i L_1(g_i)x_i + \sum_{i=1}^{n} \int (1_{E_i} - g_i)(s) z_i m(ds) x_i \\
&\le \sum_{i=1}^{n} z_i L_1(g_i)x_i + \sum_{i=1}^{n} \int (1_{E_i} - 1_{K_i})(s) z_i m(ds) x_i \\
&\le \sum_{i=1}^{n} z_i L_1(g_i)x_i + \sum_{i=1}^{n} |z_1 m(\cdot)x_i|(E_i \setminus K_i) \\
&< \sum_{i=1}^{n} |L_1 g_i| + \varepsilon \le |||L_1||| + \varepsilon
\end{aligned}$$

Taking the supremum over $|x_i| \le 1$ and $|z_i| \le 1$ yields

$\sum_{i=1}^{n} |m(E_i)| \leq |||L_1||| + \varepsilon$, and the supremum over all disjoint $\{E_1,\ldots,E_n\}$ yields $|m|(S) \leq |||L_1||| + \varepsilon$. Since $\varepsilon$ was arbitrary, $|m|(S) \leq |||L_1|||$. We also note that if $|m|(S) < +\infty$, then $m \in \text{rcabv}(\mathcal{B}, \mathcal{L}(X,Z^*))$ by Theorem 3.1.

It remains to show how the maps $L_3 \in C_o(S, X \otimes_\pi Z)^*$ are related to $L_1$, m, and $L_2$. Suppose $L_3 \in C_o(S, X \otimes_\pi Z)^*$ is given. Define $L_1\colon C_o(S) \to \mathcal{L}(X,Z^*)$ by

$\langle z, L_1(g)x\rangle = L_3(g(\cdot)x \otimes z)$, $g \in C_o(S)$, $x \in X$, $z \in Z$. If $g_1,\ldots,g_n \in C_o(S)$ have disjoint support with $|g_i|_\infty \leq 1$, and if $|x_i| \leq 1$, $|z_i| \leq 1$ then $|\sum_{i=1}^{n} g_i(\cdot)x_i \otimes z_i|_\infty \leq 1$ and so

$$\sum_{i=1}^{n} z_i L_1(g_i)x_i = L_3\left(\sum_{i=1}^{n} g_i(\cdot)x_i \otimes z_i\right) \leq |L_3|.$$

Hence $\sum_{i=1}^{n} |L_1 g_i| \leq |L_3|$ and $|||L_1||| \leq |L_3|$. Conversely, let m correspond to $L_1$; since $|m|(S) = |||L_1||| \leq |L_3| < +\infty$ we know that $m \in \text{rcabv}(\mathcal{B}, \mathcal{L}(X,Z^*)) = \text{rcabv}(\mathcal{B}, (X \otimes_\pi Z)^*)$. Let us define $W = X \hat{\otimes}_\pi Z$. By Theorem 3.2 there is an isometric isomorphism between maps $L_3 \in C_o(S,W)^* = \mathcal{L}(C_o(S,W),R)$ and measures $m \in \text{rcabv}(\mathcal{B}, \mathcal{L}(W,R)) = \text{rcabv}(\mathcal{B}, W^*) = \text{rcabv}(\mathcal{B}, \mathcal{L}(X,Z^*))$; under this correspondence

$L_3 u = \int_S \langle u(s), m(ds) \rangle$ and $|L_3| = |m|(s)$. Thus (3.13) holds and the theorem is proved. □

Thus, to summarize, we have shown that there is a continuous canonical injection

$$C_o(S, X \otimes_\pi Z)^* \to \mathbb{L}(C_o(S,X), Z^*) \to \mathbb{L}(C_o(S), \mathbb{L}(X,Z^*));$$

each of these spaces corresponds to operator-valued measures $m \in \mathcal{M}(\mathcal{B}, \mathbb{L}(X,Z^*))$ which have finite variation $|m|(s)$, finite semivariation $\tilde{m}(s)$, and finite scalar semivariation $\tilde{\tilde{m}}(s)$, respectively. By posing the theory in terms of measures with values in an $\mathbb{L}(X,Z^*)$ space rather than an $\mathbb{L}(X,Y)$ space, we have developed a natural and complete representation of linear operators on $C_o(S,X)$ spaces. Moreover in the case that $Y$ is a dual space (without necessarily being reflexive), it is possible to represent all bounded linear operators $L \in \mathbb{L}(C_o(S,X),Y)$ by operator-valued measures $m \in \mathcal{M}(\mathcal{B}, \mathbb{L}(X,Y))$ with values in $\mathbb{L}(X,Y)$ rather than in $\mathbb{L}(X,Y^{**})$; this is important for the quantum applications we have in mind, where we would like to represent $\mathbb{L}(C_o(S), \mathbb{L}_s(\mathcal{H}))$ operators by $\mathbb{L}_s(\mathcal{H})$-valued operator measures rather than $\mathbb{L}_s(\mathcal{H})^{**}$-valued measures. We now give two examples to show how the usual representation theorems follow as corollaries by considering $Y$ as a subspace of $Y^{**}$.

<u>Corollary</u> (Dunford-Schwartz [1967], III.19.5). Let S be a locally compact Hausdorff space and X,Y Banach spaces. There is an isometric isomorphism between bounded linear maps L: $C_o(S,X) \to Y$ and finitely additive maps $m: \mathcal{B} \to \mathbf{L}(X,Y^{**})$ with finite semivariation $\overline{m}(s) < +\infty$ for which

1) $y^*m(\cdot) \in rcabv(\mathcal{B},X^*)$ for every $y^* \in Y^*$

2) $y^* \mapsto y^*m$ is continuous for the weak * topologies on $Y^*$, $rcabv(\mathcal{B},X^*) \cong C_o(S,X)^*$. This correspondence $L \leftrightarrow m$ is given by $Lf = \int m(ds)f(s)$ for $f \in C_o(S,X)$, and $|L| = \overline{m}(S)$.

<u>Proof</u>. Set $Z = Y^*$ and consider $Y$ as a norm-closed subspace of $Z^*$. An element $y^{**}$ of $Y^{**}$ belongs to $Y$ iff the linear functional $y^* \mapsto y^{**}(y^*)$ is continuous for the w* topology on $Y^*$. Hence the maps $L \in \mathbf{L}(C_o(S,X),Y^{**})$ which correspond to maps $L \in \mathbf{L}(C_o(S,X),Y)$ are precisely the maps for which $z \mapsto \langle z,Lf\rangle$ are continuous in the w*-topology on $Z = Y^*$ for every $f \in C_o(S,X)$, or equivalently those maps $L$ for which $z \mapsto L^*z$ is continuous for the w* topologies on $Z = Y^*$ and $C_o(S,X)^*$. The results then follow directly from Theorem 3.3, where we note that when $L \leftrightarrow m$,

$$\langle f,L^*z\rangle = \langle z,Lf\rangle = \int_S zm(ds)f(s). \quad \square$$

<u>Corollary</u> (Dobrakov [1971], 2.2). A bounded linear map

$L: C_o(S,X) \to Y$ can be uniquely represented as

$$Lf = \int_S m(ds)f(s) \quad , \quad f \in C_o(S,X)$$

where $m \in fa(\mathcal{B}, \mathbf{L}(X,Y))$ has finite semivariation $\overline{m}(s) < +\infty$ and satisfies $y^*m(\cdot)x \in rcabv(\mathcal{B})$ for every $x \in X$, $y^* \in Y$, if and only if for every $x \in X$ the bounded linear operator $L_x: C_o(S) \to Y: g(\cdot) \mapsto L(g(\ )x)$ is weakly compact. In that case $|L| = \overline{m}(s)$ and $L^*y^*$ is given by $(L^*y^*)f = \int_S y^*m(ds)f(s)$ where $y^*m \in rcabv(\mathcal{B},X^*)$ for every $y^* \in Y^*$.

Remark. Suppose $Y = Z^*$ is a dual space. Then by Theorem 2 every $L \in \mathbf{L}(C_o(S,X),Y)$ has a representing measure $m \in \mathcal{M}(\mathcal{B}, \mathbf{L}(X,Y))$. What this Corollary says is that the representing measure $m$ actually satisfies $y^*m(\cdot)x \in rcabv(\mathcal{B})$ for every $y^* \in Y^*$ (and not just for every $y^*$ belonging to the canonical image of $Z$ in $Z^{**} = Y^*$), if and only if $L_x$ is weakly compact $C_o(S) \to Y$ for every $x \in X$; i.e. in this case we have (in our notation) $m \in \mathcal{M}(\mathcal{B}, \mathbf{L}(X,\tilde{Y}^{**}))$ where $Y$ is injected into its bidual $Y^{**}$.

Proof. Again, let $Z = Y^*$ and define $J: Y \to Y^{**}$ to be the canonical injection of $Y$ into $Y^{**} = Z^*$. The bounded linear operator $L_x: C_o(S) \to Y$ is weakly compact iff

$L_x^{**}: C_o(S)^{**} \to Y^{**}$ has image $L_x^{**}C_o(X)^{**}$ which is a subset of JY (Dunford-Schwartz [1966], VI.4.2). First, suppose $L_x$ is weakly compact, so that $L_x^{**}: C_o(S)^{**} \to JY$ for every x. Now the map $\lambda \mapsto \lambda(E)$ is an element of $C_o(S)^{**}$ (where we have identified $\lambda \in \text{rcabv}(\mathcal{B}) \cong C_o(S)^*$) for $E \in \mathcal{B}$, and

$$L_x^{**}(\lambda \mapsto \lambda(E)) = (z \mapsto \langle z, m(E)x\rangle) \in Y^{**}$$

where $m \in \mathcal{M}(\mathcal{B}, \mathbf{L}(X,Z^*))$ is the representing measure of $JL: C_o(S,X) \to Y^{**}$. Since $L_x$ is weakly compact, $z \mapsto \langle z, m(E)x\rangle$ must actually belong to $JY \subset Y^{**}$, that is $z \mapsto \langle z, m(E)x\rangle$ is w* continuous and $m(E)x \in JY$. Hence m has values in $\mathbf{L}(X,JY)$ rather than just $\mathbf{L}(X,Y^{**})$.

Conversely if $m \in \mathcal{M}(\mathcal{B}, \mathbf{L}(X,JY))$ represents an operator $L \in \mathbf{L}(C_o(S,X),Y)$ by

$$JLf = \int m(ds)f(s),$$

then the map $L_x^*: Y^* \to C_o(S)^* \cong \text{rcabv}(\mathcal{B}): z \mapsto \langle z, m(\cdot)x\rangle$ is continuous for the weak topology on $Z = Y^*$ and the weak * topology on $C_o(S)^* \cong \text{rcabv}(\mathcal{B})$ since $m(E)x \in JY$ for every $E \in \mathcal{B}$, $x \in X$. Hence by (Dunford-Schwartz [1966], VI.4.7), $L_x$ is weakly compact. $\square$

## 4. Integration of real-valued functions with respect to operator-valued measures

In quantum mechanical measurement theory, it is nearly always the case that physical quantities have values in a locally compact Hausdorff space S, e.g. a subset of $R^n$. The integration theory may be extended to more general measurable spaces; but since for duality purposes we wish to interpret operator-valued measures on S as continuous linear maps, we shall always assume that the parameter space S is a locally compact space with the induced σ-algebra of Borel sets, and that the operator-valued measure is regular. In particular, if S is second countable then S is countable at infinity (the one-point compactification $S \cup \{\infty\}$ has a countable neighborhood basis at ∞) and every complex Borel measure on S is regular; also S is a complete separable metric space, so that the Baire sets and Borel sets coincide.

Let H be a complex Hilbert space. A (self-adjoint) <u>operator-valued regular Borel measure</u> on S is a map $m: \mathcal{B} \to \mathcal{L}_s(H)$ such that $\langle m(\cdot)\phi|\psi\rangle$ is a regular Borel measure on S for every $\phi, \psi \in H$. In particular, since for a vector-valued measure countable additivity is equivalent to weak countable additivity [DS, IV.10.1],

$m(\cdot)\phi$ is a (norm-) countably additive H-valued measure for every $\phi \in H$; hence whenever $\{E_n\}$ is a countable collection of disjoint subsets in $\mathcal{B}$ then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n),$$

where the sum is convergent in the strong operator topology. We denote by $\mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ the real linear space of all operator-valued regular Borel measures on S. We define <u>scalar semivariation</u> of $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ to be the norm

$$\bar{\bar{m}}(S) = \sup_{|\phi| \leq 1} |\langle m(\cdot)\phi|\phi\rangle|(s) \qquad (4.1)$$

where $|\langle m(\cdot)\phi|\phi\rangle|$ denotes the total variation measure of the real-valued Borel measure $E \mapsto \langle m(E)\phi|\phi\rangle$. The scalar semivariation is always finite, as proved in Theorem 3.2 by the uniform boundedness theorem (see previous sections for alternative definitions of $\bar{\bar{m}}(s)$; note that when $m(\cdot)$ is self-adjoint valued the identity $\bar{\bar{m}}(s) = \sup_{|\phi| \leq 1} \sup_{|\psi| \leq 1} |\langle m(\cdot)\phi|\psi\rangle|(s)$ reduces to (4.1)).

A <u>positive</u> operator-valued regular Borel measure is a measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ which satisfies

$$m(E) \geq 0 \qquad \forall E \in \mathcal{B},$$

where by $m(E) \geq 0$ we mean $m(E)$ belongs to the positive cone $\mathcal{L}_s(H)_+$ of all nonnegative-definite operators. A probability operator measure (POM) is a positive operator-valued measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ which satisfies

$$m(S) = I.$$

If $m$ is a POM then every $\langle m(\cdot)\phi|\phi\rangle$ is a probability measure on $S$ and $\bar{\bar{m}}(S) = 1$. In particular, a resolution of the identity is an $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ which satisfies $m(S) = I$ and $m(E)m(F) = 0$ whenever $E \cap F = \emptyset$; it is then true that $m(\cdot)$ is projection-valued and satisfies

$$m(E \cap F) = m(E)m(F), \qquad E, F \in \mathcal{B}.^{+}$$

We now consider integration of real-valued functions with respect to operator-valued measures. Basically, we identify the regular Borel operator-valued measures

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+Proof. First, $m(\cdot)$ is projection valued since by finite additivity

$$m(E) = m(E)m(S) = m(E)[m(E)+m(S\setminus E)] = m(E)^2+m(E)m(S\setminus E),$$

and the last term is 0 since $E \cap (S\setminus E) = \emptyset$. Moreover we have by finite additivity

$$m(E)m(F) = [m(E\cap F)+m(E\setminus F)]\cdot[m(E\cap F)+m(F\setminus E)]$$

$$= m(E\cap F)^2+m(E\cap F)m(F\setminus E)+m(E\setminus F)m(E\cap F)+m(E\setminus F)m(F\setminus E),$$

where the last three terms are 0 since they have pairwise disjoint sets.

$m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ with the bounded linear operators $L: C_o(S) \to \mathcal{L}_s(H)$, using the integration theory of Section 3 to get a generalization of the Riesz Representation Theorem.

<u>Theorem 4.1</u>. Let $S$ be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let $H$ be a Hilbert space. There is an isometric isomorphism $m \leftrightarrow L$ between the operator-valued regular Borel measures $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ and the bounded linear maps $L \in \mathbb{L}(C_o(S), \mathcal{L}_s(H))$. The correspondnece $m \leftrightarrow L$ is given by

$$L(g) = \int_S g(s)m(ds), \quad g \in C_o(S) \tag{4.2}$$

where the integral is well-defined for $g(\cdot) \in M(S)$ (bounded and totally measurable maps $g: S \to R$) and is convergent for the supremum norm on $M(S)$. If $m \leftrightarrow L$, then $\bar{\bar{m}}(S) = |L|$ and $\langle L(g)\phi|\psi\rangle = \int_S g(s)\langle m(\cdot)\phi|\psi\rangle(ds)$ for every $\phi, \psi \in H$. Moreover $L$ is positive (maps $C_o(S)_+$ into $\mathcal{L}_S(H)_+$) iff $m$ is a positive measure; $L$ is positive and $L(1) = I$ iff $m$ is a POM; and $L$ is an algebra homomorphism with $L(1) = I$ iff $m$ is a resolution of the identity, in which case $L$ is actually an isometric algebra homomorphism of $C_o(S)$ onto a norm-closed subalgebra of $\mathcal{L}_s(H)$.

Proof. The correspondence $L \leftrightarrow m$ is immediate from Theorem 3.2. If $m$ is a positive measure, then $\langle m(E)\phi|\phi\rangle \geq 0$ for every $E \in \mathcal{B}$ and $\phi \in H$, so $\langle L(g)\phi|\phi\rangle = \int_S g(s)\langle m(\cdot)\phi|\phi\rangle(ds) \geq 0$ whenever $g \geq 0$, $\phi \in H$ and $L$ is positive. Conversely, if $L$ is positive then $\langle m(\cdot)\phi|\phi\rangle$ is a positive real-valued measure for every $\phi \in H$, so $m(\cdot)$ is positive. Similarly, $L$ is positive and $L(1) = I$ iff $m$ is a POM. It only remains to verify the final statement of the theorem.

Suppose $m(\cdot)$ is a resolution of the identity. If $g_1(s) = \sum_{j=1}^{n} a_j i_{E_j}(s)$ and $g_2(s) = \sum_{j=1}^{m} b_j 1_{F_j}(s)$ are simple functions, where $\{E_1,\ldots,E_n\}$ and $\{F_1,\ldots,F_n\}$ are each finite disjoint subcollections of $\mathcal{B}$, then

$$\int g_1(s)m(ds)\cdot\int g_2(s)m(ds) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_j b_k m(E_j)m(F_k)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{m} a_j b_k m(E_1 \cap F_k)$$

$$= \int g_1(s)g_2(s)m(ds).$$

Hence $g \mapsto \int g(s)m(ds)$ is an algebra homomorphism from the algebra of simple functions on $S$ into $\mathcal{L}_s(H)$. Moreover we show that the homomorphism is isometric on simple functions. Clearly

$$|\int g(s)m(ds)| \leq \bar{\bar{m}}(s)|g|_\infty = |g|_\infty .$$

Conversely, for $g = \sum_{j=1}^{n} a_j 1_{E_j}$ we may choose $\phi_j$ to be in the range of the projection $m(E_j)$, with $|\phi_j| = 1$, to get

$$|\int g(s)m(ds)| \geq \max_{j=1,\ldots,n} \langle \int g(s)m(ds)\cdot\phi_j|\phi_j\rangle$$

$$= \max_{j=1,\ldots,n} |a_j|\langle m(E_i)\phi_j|\phi_j\rangle$$

$$= \max_{j=1,\ldots,n} |a_j| = |g|_\infty .$$

Thus $g \mapsto \int g(s)m(ds)$ is isometric on simple functions. Since simple functions are uniformly dense in $M(S)$, it follows by taking limits of simple functions that $\int g_1(s)m(ds)\cdot\int g_2(s)m(ds) = \int g_1(s)g_2(s)m(ds)$ and $|\int g_1(s)m(ds)| = |g_1|_\infty$ for every $g_1, g_2 \in M(S)$. Of course, the same is then true for $g_1, g_2 \in C_o(S) \subset M(S)$. Since $C_o(S)$ is complete, it follows that $L$ is an isometric isomorphism of $C_o(S)$ onto a closed subalgebra of $\mathcal{L}_s(H)$.

Now assume that $L$ is an algebra homomorphism and $L(1) = I$. Clearly $m(S) = L(1) = I$. Since $L(g^2) = L(g)^2 \geq 0$ for every $g \in C_o(S)$, $L$ and hence $m$ are positive. Let

$$M_1 = \{g \in M(S): \int g(s)m(ds)\cdot\int h(s)m(ds) = \int g(s)h(s)m(ds)$$
$$\text{for every } h \in C_o(S)\}.$$

Then $M_1$ contains $C_o(S)$. Now if $g_n \in M(S)$ is a uniformly bounded sequence which converges pointwise to $g_o$ then $\int g_n(s)m(ds)$ converges in the weak operator topology to $\int g_o(s)m(ds)$ by the dominated convergence theorem applied to each of the regular Borel measures $\langle m(\cdot)\phi|\psi\rangle$, $\phi,\psi \in H$ (the integrals actually converge for the norm topology on $\mathcal{L}_s(H)$ whenever $|g_n - g_o|_\infty \to 0$). Hence $M_1$ is closed under pointwise convergence of uniformly bounded sequences, and so equals all of M(S) by regularity. Similarly, let

$$M_2 = \{h \in M(S): \int g(s)m(ds)\cdot\int h(s)m(ds) = \int g(s)h(s)m(ds)$$
$$\text{for every } g \in M(S)\}.$$

Then $M_2$ contains $C_o(S)$ and must therefore equal all of M(S). It is now immediate that whenever E,F are disjoint sets in $\mathcal{B}$ then

$$m(E)m(F) = \int 1_E dm\cdot\int 1_F dm = \int 1_{E\cap F}(s)m(ds) = 0.$$

Thus m is a resolution of the identity. □

Remark. Since every real-linear map from a real-linear subspace of a complex space into another real-linear subspace of a complex space corresponds to a unique "Hermitian" complex-linear map on the complex linear spaces, we could just as easily identify the (self-adjoint) operator-valued regular measures $\mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ with the complex-linear maps $L: C_o(S,\mathbb{C}) \to \mathcal{L}(H)$ which satisfy

$$L(g) = L(\overline{g})^*, \quad g \in C_o(S,\mathbb{C}).$$

## 5. Integration of $\mathcal{T}_s(H)$-valued functions

We now consider $\mathcal{L}(H)$ as a subspace of the "operations" $\mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$, that is, bounded linear maps from $\mathcal{T}(H)$ into $\mathcal{T}(H)$. This is possible because if $A \in \mathcal{T}(H)$ and $B \in \mathcal{L}(H)$ then $AB$ and $BA$ belong to $\mathcal{T}(H)$ and

$$\begin{aligned} |AB|_{tr} &\leq |A|_{tr}|B| \\ |BA|_{tr} &\leq |A|_{tr}|B| \\ tr(AB) &= tr(BA). \end{aligned} \tag{5.1}$$

Then every $B \in \mathcal{L}(H)$ defines a bounded linear function $L_B\colon \mathcal{T}(H) \to \mathcal{T}(H)$ by

$$L_B(A) = AB, \quad A \in \mathcal{T}(H)$$

with $|B| = |L_B|$. In particular, $A \mapsto trAB$ defines a continuous (complex-) linear functional on $A \in \mathcal{T}(H)$, and in fact every linear functional in $\mathcal{T}(H)^*$ is of this form for some $B \in \mathcal{L}(H)$. We note that if $A$ and $B$ are selfadjoint then $trAB$ is real (although it is <u>not</u> necessarily true that AB is self-adjoint unless AB = BA). Thus, it is possible to identify the space $\mathcal{T}_s(H)^*$ of real-linear continuous functionals on $\mathcal{T}_s(H)$ with $\mathcal{L}_s(H)$, again under the pairing $\langle A,B\rangle = \mathrm{tr}AB$, $A \in \mathcal{T}_s(H)$, $B \in \mathcal{L}_s(H)$. For our purposes we shall be especially interested in this latter duality between the spaces $\mathcal{T}_s(H)$ and $\mathcal{L}_s(H)$, which we shall use later to formulate a dual problem for the quantum estimation situation. However, we will also need to consider $\mathcal{L}_s(H)$ as a subspace of $\mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$ so that we may integrate $\mathcal{T}_s(H)$-valued functions on S with respect to $\mathcal{L}_s(H)$-valued operator measures to get an element of $\mathcal{T}(H)$.

Suppose $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ is an operator-valued regular Borel measure, and $f: S \to \mathcal{T}_s(H)$ is a simple function with finite range of the form

$$f(s) = \sum_{j=1}^{n} 1_{E_j}(s)\rho_j$$

where $\rho_j \in \mathcal{T}_s(H)$ and $E_j$ are disjoint sets in $\mathcal{B}$, that is $f \in \mathcal{B} \otimes \mathcal{T}_s(H)$. Then we may unambiguously (by finite additivity of m) define the integral

$$\int_S f(s)m(ds) = \sum_{j=1}^{n} m(E_j)\rho_j.$$

The question, of course, is to what class of functions can we properly extend the definition of the integral? Now <u>if</u> $m$ has finite total variation $|m|(s)$, then the map $f \mapsto \int_S f(s)m(ds)$ is continuous for the supremum norm $|f|_\infty = \sup_s |f(s)|_{tr}$ on $\mathcal{B} \otimes \mathcal{T}_s(H)$, so that by continuity the integral map extends to a continuous linear map from the closure $M(S, \mathcal{T}_s(H))$ of $\mathcal{B} \otimes \mathcal{T}_s(H)$ with the $|\cdot|_\infty$ norm into $\mathcal{T}(H)$. In particular, the integral $\int_S f(s)m(ds)$ is well-defined (as the limit of the integrals of uniformly convergent simple functions) for every bounded and continuous function $f: S \to \mathcal{T}_s(H)$. Unfortunately, it is <u>not</u> the case that an arbitrary POM m has finite total variation. Since we wish to consider general quantum measurement processes as represented by POM's $m$ (in particular, resolutions of the identity), we can only assume that $m$ has finite scalar semivariation $\bar{\bar{m}}(S) < +\infty$. Hence we must put stronger restrictions on the class of functions which we integrate.

We may consider every $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ as an element of $\mathcal{M}(\mathcal{B}, \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$ in the obvious way: for $E \in \mathcal{B}$, $\rho \in \mathcal{T}(H)$ we put

$$m(E)(\rho) = \rho m(E).$$

Moreover, the scalar semivariation of $m$ as an element of $\mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ is the same as the scalar semivariation of $m$ as an element of $\mathcal{M}(\mathcal{B}, \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H)))$, since the norm of $B \in \mathcal{L}_s(H)$ is the same as the norm of $B$ as the map $\rho \mapsto \rho B$ in $\mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$. By the representation Theorem 3.2 we may uniquely identify $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)) \subset \mathcal{M}(\mathcal{B}, \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H)))$ with a linear operator $L \in \mathcal{L}(C_o(S), \mathcal{L}_s(H)) \subset \mathcal{L}(C_o(S), \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H)))$. Now it is well-known that for Banach spaces $X,Y,Z$ we may identify (Treves [1967], III.43.12)

$$\mathcal{L}(X \hat{\otimes}_\pi Y, Z) \cong \beta(X, Y\cdot, Z) \cong \mathcal{L}(X, \mathcal{L}(Y,Z))$$

where $X \hat{\otimes}_\pi Y$ denotes the completion of the tensor product space $X \otimes Y$ for the projective tensor product norm

$$|f|_\pi = \inf\{\sum_{j=1}^{n} |x_j|\cdot|y_j| : f = \sum_{j=1}^{n} x_j \otimes y_j\},\ f \in X \otimes Y;$$

$\beta(X,Y:Z)$ denotes the space of continuous bilinear forms $B: X \times Y \to Z$ with norm

$$|B|_{\beta(X,Y;Z)} = \sup_{|x|\leq 1}\ \sup_{|y|\leq 1} |B(x,y)|;$$

and $\mathcal{L}(X, \mathcal{L}(Y,Z))$ of course denotes the space of continuous linear maps $L_2: X \to \mathcal{L}(X,Z)$ with norm

$$|L_2|_{(X,\mathcal{L}(Y,Z))} = \sup_{|x|\leq 1} |L_2 x|_{\mathcal{L}(Y,Z)}.$$

The identification $L_1 \leftrightarrow B \leftrightarrow L_2$ is given by

$$L_1(x \otimes y) = B(x,y) = L_2(x)y.$$

In our case we take $X = M(S)$, $Y = Z = \mathcal{T}(H)$ to identify

$$\mathcal{L}(M(S) \hat{\otimes}_\pi \mathcal{T}(H), \mathcal{T}(H)) \cong \mathcal{L}(M(S), \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))). \quad (5.2)$$

Since the map $g \mapsto \int g(s)m(ds)$ is continuous from $M(S)$ into $\mathcal{L}_s(H) \subset \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$ for every $m \in \mathcal{M}(\mathcal{S}, \mathcal{L}_s(H))$, we see that we may identify $m$ with a continuous linear map $f \mapsto \int f dm$ for $f \in M(S) \hat{\otimes}_\pi \mathcal{T}(H)$. Clearly if $f \in M(S) \otimes \mathcal{T}(H)$, that is if

$$f(s) = \sum_{j=1}^{n} g_j(s)\rho_j$$

for $g_j \in M(S)$ and $\rho_j \in \mathcal{T}(H)$, then

$$\int_S f(s)m(ds) = \sum_{j=1}^{n} \rho_j \int g_j(s)m(ds).$$

Moreover the map $f \mapsto \int_S f(s)m(ds)$ is continuous and linear for the $\|\cdot\|_\pi$-norm on $M(S) \otimes \mathcal{T}(H)$, so we may extend the definition of the integral to elements of the completion $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ by setting

$$\int f m(ds) = \lim_{n\to\infty} \int f_n(s)m(ds)$$

where $f_n \in M(S) \otimes \mathcal{T}(H)$ and $f_n \to f$ in the $\|\cdot\|_\pi$-norm. In the section which follows we prove that the completions $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ and $C_o(S) \hat{\otimes}_\pi \mathcal{T}(H)$ may be identified with subspaces of $M(S, \mathcal{T}(H))$ and $C_o(S, \mathcal{T}(H))$ respectively, i.e. we can treat elements $f$ of $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ as totally measurable functions $f: S \to \mathcal{T}(H)$. We shall show that under suitable conditions the maps $f: S \to \mathcal{T}(H)$ we are interested in for quantum estimation problems do belong to $C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$, and hence are integrable against arbitrary operator-valued measures $m \in \mathcal{M}(\mathcal{B}, \mathcal{T}_s(H))$.

<u>Theorem 5.1</u>. Let $S$ be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let $H$ be a Hilbert space. There is an isometric isomorphism $L_1 \leftrightarrow m \leftrightarrow L_2$ between the bounded linear maps $L_1: C_o(S) \hat{\otimes}_\pi \mathcal{T}(H) \to \mathcal{T}(H)$, the operator-valued regular Borel measures $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H)))$, and the bounded linear maps $L_2: C_o(S) \to \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$. The correspondence $L_1 \leftrightarrow m \leftrightarrow L_2$ is given by the relations

$$L_1(f) = \int_S f(s)m(ds), \quad f \in C_o(S) \hat{\otimes}_\pi \mathcal{T}(H)$$

$$L_2(g)\rho = L_1(g(\cdot)\rho) = \rho \int g(s)m(ds), \; g \in C_o(S), \; \rho \in \mathcal{T}(H)$$

and under this correspondence $|L_1| = \bar{\bar{m}}(s) = |L_2|$. Moreover the integral $\int_S f(s)m(ds)$ is well-defined for every $f \in M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ and the map $f \mapsto \int_S f(s)m(ds)$ is bounded and linear from $M(S) \hat{\otimes}_{\pi} \mathcal{T}(H)$ into $\mathcal{T}(H)$.

Proof. From Theorem 6.1 of section 6 (see next section), we may identify $M(S) \hat{\otimes}_{\pi} \mathcal{T}(H)$, and hence $C_0(S) \hat{\otimes}_{\pi} \mathcal{T}(H)$, as a subspace of the totally measurable (that is, uniform limits of simple functions) functions $f: S \to \mathcal{T}(H)$. The results then follow from Theorem 3.2 and the isometric isomorphism

$$\mathcal{L}(C_0(S) \hat{\otimes}_{\pi} \mathcal{T}(H), \mathcal{T}(H)) \cong \mathcal{L}(C_0(S), \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H)))$$

as in (5.2). We note that by a $\mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$-valued regular Borel measure we mean a map $m: \mathcal{B} \to \mathcal{L}(\mathcal{T}(H), \mathcal{T}(H))$ for which $\mathrm{tr} Cm(\cdot)\rho$ is a complex regular Borel measure for every $\rho \in \mathcal{T}(H)$, $C \in \mathcal{K}(H)$, where in the application of Theorem 3.2 we have taken $X = \mathcal{T}(H)$, $Z = \mathcal{K}(H)$, $Z^* = \mathcal{T}(H)$. In particular this is satisfied for every $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$. □

Corollary 5.1. If $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ then the integral $\int_S f(s) m(ds)$ is well-defined for every $f \in M(S) \hat{\otimes}_{\pi} \mathcal{T}(H)$.

Remark. It should be emphasized that the $|\cdot|_{\pi}$ norm is strictly stronger than the supremum norm $|f|_{\infty} = \sup_s |f(s)|_{tr}$. Hence, if $f_n, f \in M(S) \otimes_{\pi} \mathcal{T}(H)$ satisfy $f_n(s) \to f(s)$ uniformly, it is not necessarily true that $|f_n - f|_{\pi} \to 0$ or that $\int_S f_n(s) m(ds) \to \int_S f(s) m(ds)$.

5.2. $M(S)\ \hat{\otimes}_\pi \mathcal{T}(H)$ is a subspace of $M(S, \mathcal{T}(H))$

6. A Result in Tensor Product Spaces.

The purpose of this section is to show that we may identify the tensor product space $M(S)\ \hat{\otimes}_\pi\ \mathcal{T}_s(H)$ with a subspace of the totally measurable functions $f: S \to \mathcal{T}_s(H)$ in a well-defined way. The reason why this is important is that the functions $f \in M(S)\ \hat{\otimes}_\pi\ \mathcal{T}_s(H)$ are those for which we may legitimately define an integral $\int_S f(s)m(ds)$ for arbitrary operator-valued measures $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$, since $f \mapsto \int_S f(s)m(ds)$ is a continuous linear map from $M(S)\ \hat{\otimes}_\pi\ \mathcal{T}(H)$ into $\mathcal{T}(H)$. In particular, it is obvious that $C_o(S) \otimes \mathcal{T}_s(H)$ may be identified with a subspace of continuous functions $f: S \to \mathcal{T}_s(H)$ in a well-defined way, just as it is obvious how to define the integral $\int_S f(s)m(ds)$ for finite linear combinations $f(s) = \sum_{j=1}^{n} g_j(s)\rho_j \in C_o(S) \otimes \mathcal{T}_s(H)$. What is not obvious is that the completion of $C_o(S)\ \hat{\otimes}\ \mathcal{T}_s(H)$ in the tensor product norm $\pi$ may be identified with a subspace of continuous functions $f: S \to \mathcal{T}_s(H)$.

Before proceeding, we review some basic facts about tensor product spaces. Let $X, Z$ be normed spaces. By $X \otimes Z$ we denote a tensor product space of $X$ and $Z$, which is the vector space of all linear finite combinations

$\sum_{j=1}^{n} a_j x_j \otimes z_j$ where $a_j \in R$, $x_j \in X$, $z_j \in Z$ (of course, $a_j, x_j, z_j$ are not uniquely determined). There is a natural duality between $X \otimes Z$ and $\mathcal{L}(X,Z^*)$ given by

$$<\sum_{j=1}^{n} a_j x_j \otimes z_j, L> = \sum_{j=1}^{n} a_j <z_j, Lx_j>.$$

Moreover the norm of $L \in \mathcal{L}(X,Z^*)$ as a linear functional on $X \otimes Z$ is precisely its usual operator norm $|L| = \sup_{|z|\leq 1} \sup_{|x|\leq 1} <z,Lx>$ when $X \otimes Z$ is made into a normed space $X \otimes_\pi Z$ under the <u>tensor</u> <u>product</u> <u>norm</u> $|\cdot|_\pi$ defined by

$$|f|_\pi = \inf\{\sum_{j=1}^{n} |x_j|\cdot|z_j| : f = \sum_{j=1}^{n} x_j \otimes z_j\}, \ f \in X \otimes Z.$$

It is easy to see that $|x \otimes z|_\pi = |x|\cdot|z|$ for $x \in X$, $z \in Z$ (the canonical injection $X \times Z \to X \otimes Z$ is continuous with norm 1) and in fact $|\cdot|_\pi$ is the strongest norm on $X \otimes Z$ with this property. By $X \hat{\otimes}_\pi Z$ we denote the completion of $X \otimes_\pi Z$ for the $|\cdot|_\pi$ norm. Every $L \in \mathcal{L}(X,Z^*)$ extends to a unique bounded linear functional on $X \hat{\otimes}_\pi Z$ with the same norm as its operator norm, so that we identify $(X \hat{\otimes}_\pi Z)^* \cong \mathcal{L}(X,Z^*)$. The space $X \hat{\otimes}_\pi Z$ may be identified more concretely as all infinite sums

$\sum_{j=1}^{\infty} a_j x_j \otimes z_j$ where $x_j \to 0$ in $X$, $z_j \to 0$ in $Z$, and $\sum_{j=1}^{\infty} |a_j| < +\infty$ (Schaeffer [1971], III.6.4), and the pairing between $X \hat{\otimes}_\pi Z$ and $\mathcal{L}(X,Z^*)$ by

$$<\sum_{j=1}^{\infty} a_j x_j \otimes z_j, L> = \sum_{i=1}^{\infty} a_j <z_i, Lx_i>.$$

A second important topology on $X \otimes Z$ is the $\varepsilon$-topology, with norm

$$|\sum_{i=1}^{n} a_i x_i \otimes z_i|_\varepsilon = \max_{|x^*|\leq 1} \max_{|z^*|\leq 1} |\sum_{i=1}^{n} a_i <x_i, x^*><z_i, z^*>|$$

It is easy to see that $|\cdot|_\varepsilon$ is a cross-norm, i.e. $|x \otimes z|_\varepsilon = |x|\cdot|z|$, and that $|\cdot|_\varepsilon \leq |\cdot|_\pi$, i.e. the $\pi$-topology is finer than the $\varepsilon$-topology. We denote by $X \otimes_\varepsilon Z$ the tensor product space $X \otimes Z$ with the $\varepsilon$-norm, and by $X \hat{\otimes}_\varepsilon Z$ the completion of $X \otimes Z$ in the $\varepsilon$-norm. Now the canonical injection of $X \otimes_\pi Z$ into $X \hat{\otimes}_\varepsilon Z$ is continuous (with norm 1 and dense image); this induces a canonical continuous map $X \hat{\otimes}_\pi Z \to X \hat{\otimes}_\varepsilon Z$. It is not known, in general, whether this map is one-to-one. In the case that $X,Z$ are Hilbert spaces we may identify $X \hat{\otimes}_\pi Z$ with the nuclear or trace-class maps $\mathcal{T}(X^*,Z)$ and $X \hat{\otimes}_\varepsilon Z$ with the compact operators $\mathcal{K}(X^*,Z)$, and it is well known that the canonical map

$X \hat{\otimes}_{\pi} Z \to X \hat{\otimes}_{\varepsilon} Z$ is one-to-one (cf Treves [1967], III.38.4). We are interested in the case that $X = C_o(S)$ and $Z = \mathcal{T}_s(H)$; we may then identify $C_o(S) \hat{\otimes}_{\varepsilon} \mathcal{T}_s(H)$ with $C_o(S, \mathcal{T}_s(H))$ (since the $|\cdot|_{\varepsilon}$ is precisely the $|\cdot|_{\infty}$ norm when $C_o(S) \otimes \mathcal{T}_s(H)$ is identified with a subspace of $C_o(S, \mathcal{T}_s(H))$, and $C_o(S) \otimes \mathcal{T}_s(H)$ is dense in $C_o(S, \mathcal{T}_s(H))$) and we would like to be able to consider $C_o(S) \hat{\otimes}_{\pi} \mathcal{T}_s(H)$ as a subspace of $C_o(S, \mathcal{T}_s(H))$. Similarly we want to consider $M(S) \hat{\otimes}_{\pi} \mathcal{T}(H)$ as a subspace of $M(S, \mathcal{T}(H))$.

Theorem 6.1. Let $X$ be a Banach space and $H$ a Hilbert space. Then the canonical mapping of $X \hat{\otimes}_{\pi} \mathcal{T}(H)$ into $X \hat{\otimes}_{\varepsilon} \mathcal{T}(H)$ is one-to-one.

Proof. It suffices to show that the adjoint of the mapping in question has weak * dense image in $(X \hat{\otimes}_{\pi} \mathcal{T}(H))^* \cong \mathcal{L}(X, \mathcal{L}(H))$, where we have identified $\mathcal{T}(H)^*$ with $\mathcal{L}(H)$. Note that the adjoint is one-to-one, since the image of the canonical mapping is clearly dense. What we must show is that the imbedding of $(X \hat{\otimes}_{\varepsilon} \mathcal{T}(H))^*$, the so-called integral mappings $X \to \mathcal{L}(H) \cong \mathcal{T}(H)^*$, into $\mathcal{L}(X, \mathcal{L}(H))$ has weak * dense image. Of course, the set of linear continuous maps $L_o: X \to \mathcal{L}(H)$ with finite dimensional image belongs to the integral mappings

$(X \hat{\otimes}_\varepsilon \mathcal{T}(H))^*$; we shall actually show that these finite-rank operators are weak* dense in $\mathcal{L}(X, \mathcal{L}(H))$. We therefore need to prove that for every $f \in (X \hat{\otimes}_\pi \mathcal{T}(H))$, $L \in \mathcal{L}(X, \mathcal{L}(H))$, $\varepsilon > 0$ there is an $L_o$ in $\mathcal{L}(X, \mathcal{L}(H))$ with finite rank such that $|\langle f, L-L_o \rangle| < \varepsilon$. Now $f$ has the representation

$$f = \sum_{j=1}^{\infty} a_j x_j \otimes z_j \tag{6.1}$$

with $\sum_{j=1}^{\infty} |a_j| < +\infty$, $x_j \to 0$ in X, and $z_j \to 0$ in $\mathcal{T}(H)$ (Schaeffer [1971], III.6.4), and

$$\langle f, L-L_o \rangle = \sum_{j=1}^{\infty} a_j \langle z_j, (L-L_o)x_j \rangle. \tag{6.2}$$

The lemma which follows proves the following fact: to every compact subset K of X and every 0-neighborhood V of $\mathcal{L}(H)$, there is a continuous linear map $L_o\colon X \to \mathcal{L}(H)$ with finite rank such that $(L-L_o)(K) \subset V$. Using the representation (6.1), we take $K = \{x_j\}_{j=1}^{\infty} \cup \{0\}$ and $V = \{y_1, y_2, \ldots\}^o \cdot \varepsilon / \sum_{j=1}^{\infty} |a_j|$. We then have $|\langle f, L-L_o \rangle| < \varepsilon$ as desired. □

The lemma required for the above proof, which we give below, basically amounts to showing that $Z^* = \mathcal{L}(H)$ satisfies the <u>approximation property</u>, that is for every

Banach space X the finite rank operators are dense in $\mathcal{L}(X,Z^*)$ for the topology of uniform convergence on compact subsets of X. It is not known whether every locally convex space satisfies the approximation property; this question (as in the present situation) is closely related to when the canonical mapping $X \hat{\otimes}_\pi Z \to X \hat{\otimes}_\varepsilon Z$ is one-to-one.

Lemma 6.1. Let X be a Banach space, H a Hilbert space. For every $L \in \mathcal{L}(X, \mathcal{L}(H))$, every compact subset K of X, and every 0-neighborhood V in $\mathcal{L}(H)$ there is a continuous linear map $L_o: X \to \mathcal{L}(H)$ with finite rank such that

$$(L-L_o)(K) \subset V.$$

Proof. Let $P_n$ be projections in H with $P_n \uparrow I$, where I is the identity operator on H (e.g. take any complete orthonormal basis $\{\phi_j, j \in J\}$ for H; let N be the family of all finite subsets of J, directed by set inclusion; and for $n \in N$ define $P_n$ to be the projection operator $P_n(\phi) = \sum_{j \in n} <\phi|\phi_j>\phi_j$ for $\phi \in H$). Suppose $L \in \mathcal{L}(X, \mathcal{L}(H))$. Then $P_n L \in \mathcal{L}(X, \mathcal{L}(H))$ has finite rank and converges pointwise to L, since $(P_n L)(x) = P_n(Lx) \to Lx$. Moreover $\{P_n L\}$ is uniformly bounded, since $|P_n L| \leq |P_n| \cdot |L| = |L|$. Thus, by the Banach-Steinhaus Theorem or by the

Arzela-Ascoli Theorem the convergence $P_n L \to L$ is uniform on compact sets. This means that for every 0-neighborhood V in $\mathcal{L}(H)$ and every compact subset K of X, it is true that for n sufficiently large

$$(L-P_n L)(K) \subset V. \quad \square$$

<u>Corollary 6.2</u>. Let S be a locally compact Hausdorff space, H a Hilbert space. The canonical mapping $C_o(S) \hat{\otimes}_\pi \mathcal{T}(H) \to C_o(S, \mathcal{T}(H))$ is one-to-one, and the canonical mapping $M(S) \hat{\otimes}_\pi \mathcal{T}(H) \to M(S, \mathcal{T}(H))$ is one-to-one.

<u>Proof</u>. This follows from the previous theorem and the fact that $C_o(S) \hat{\otimes}_\varepsilon Z$ may be identified with $C_o(S,Z)$ with the supremum norm, for Z a Banach space. Similarly $M(S) \hat{\otimes}_\varepsilon Z = M(S,Z)$ with the supremum norm. $\square$

<u>Remark</u>. In Theorem 3.4, we explicitly identified $(C_o(S) \hat{\otimes}_\pi \mathcal{T}(H))^* = \mathcal{L}(C_o(S), \mathcal{L}(H))$ and $(C_o(S) \hat{\otimes}_\varepsilon \mathcal{T}(H))^* = C_o(S, \mathcal{T}(H))^*$ with the measures $m \in \mathcal{M}(\mathcal{D}, \mathcal{L}(H))$ having finite semivariation and finite total variation, respectively.

# 7. Quantum Estimation Theory

## 7.1 Introduction

The integration theory developed in the previous sections is needed in studying the problem of Quantum Estimation Theory. We now develop estimation theory for quantum systems.

In the classical formulation of Bayesian estimation theory it is desired to estimate the unknown value of a random parameter $s \in S$ based on observation of a random variable whose probability distribution depends on the value s. The procedure for determining an estimated parameter value s, as a function of the experimental observation, represents a decision strategy; the problem is to find the optimal decision strategy.

In the quantum formulation of the estimation problem, each parameter $s \in S$ corresponds to a state $\rho(s)$ of the quantum system. The aim is to estimate the value of s by performing a measurement on the quantum system. However, the quantum situation precludes exhaustive measurements of the system. This contrasts with the classical situation, where it is possible in principle to measure all relevant variables determining the state of the system and to specify meaningful probability density functions for the resulting values. For the quantum estimation problem it is necessary to specify not only the best procedure for processing experimental data, but also what to measure in the first place. Hence the quantum decision problem is to determine an optimal measurement procedure, or, in mathematical terms, to determine the optimal probability operator measure corresponding to a measurement procedure.

We now formulate the quantum estimation problem. Let $H$ be a separable complex Hilbert space corresponding to the physical variables of the system under consideration. Let $S$ be a parameter space, with measurable sets $\mathcal{B}$. Each $s \in S$ specifies a state $\rho(s)$ of the quantum system, i.e. every $\rho(s)$ is a nonnegative-definite selfadjoint trace-class operator on $H$ with trace 1. A general decision strategy is determined by a measurement process $m(\cdot)$, where $m: \mathcal{B} \to \mathcal{L}_s(H)$ is a positive operator-valued measure (POM) on the measurable space $(S, \mathcal{B})$ -- $m(E) \in \mathcal{L}_s(H)_+$ is a positive selfadjoint bounded linear operator on $H$ for every $E \in \mathcal{B}$, $m(S) = I$, and $m(\cdot)$ is countably additive for the weak operator topology on $\mathcal{L}_s(H)$. The measurement process yields an estimate of the unknown parameter; for a given value $s$ of the parameter and a given measurable set $E \in \mathcal{B}$, the probability that the estimated value $\hat{s}$ lies in $E$ is given by

$$\Pr\{\hat{s} \in E \mid s\} = \mathrm{tr}[\rho(s) m(E)]. \tag{7.1}$$

Finally, we assume that there is a cost function $c(s,\hat{s})$ which specifies the relative cost of an estimate $\hat{s}$ when the true value of the parameter is s.

For a specified decision procedure corresponding to the POM $m(\cdot)$, the risk function is the conditional expected cost given the parameter value s, i.e.

$$R_m(s) = \mathrm{tr}[\rho(s)\int_S c(s,t)m(dt)]. \tag{7.2}$$

If now $\mu$ is a probability measure on $(S,\mathcal{B})$ which specifies a prior distribution for the parameter value s, the Bayes cost is the posterior expected cost

$$R_m = \int_S R_m(s)\mu(ds). \tag{7.3}$$

<u>The quantum estimation problem is to find a POM $m(\cdot)$ for which the Bayes expected cost $R_m$ is minimum</u>.

A formal interchange of the order of integration yields

$$R_m = \mathrm{tr}\int_S f(s)m(ds) \tag{7.4}$$

where $f(s) = \int_S c(t,s)\rho(t)\mu(dt)$. Thus, formally at least, the problem is to minimize the linear functional (7.4) over all POM's $m(\cdot)$ on $(S,\mathcal{B})$. We shall apply duality theory for optimization problems to prove existence of a solution and to determine necessary and sufficient conditions for a decision strategy to be optimal, much as in the detection problem with a finite number of hypotheses (a special case of the estimation problem where $S$ is a finite set). Of course we must first rigorously define what is meant by an integral of the form (7.4); note that both the integrand and the measure are operator-valued. We must then show the equivalence of (7.3) and (7.4); this entails proving a Fubini-type theorem for operator-valued measures. Finally, we must identify an appropriate dual space for POM's consistent with the linear functional (7.4) so that a dual problem can be formulated.

Before proceeding, we summarize the results in an informal way to be made precise later. Essentially, we shall see that there is always an optimal solution, and that necessary and sufficient conditions for a POM m to be optimal are

$$\int_S f(s)m(ds) \leq f(t) \quad \text{for every} \quad t \in S.$$

It then turns out that $\int_S f(s)m(ds)$ belongs to $\mathcal{T}_s(H)$ (that is, selfadjoint) and the minimum Bayes posterior expected cost is

$$R_m = \mathrm{tr}\int_S f(s)m(ds).$$

## 7.2 A Fubini theorem for the Bayes posterior expected cost

In the quantum estimation problem, a decision strategy corresponds to a probability operator measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ with posterior expected cost

$$R_m = \int_S tr[\rho(s) \int_S C(t,s) m(dt)] \mu(dt)$$

where for each $s$, $\rho(s)$ specifies a state of the quantum system, $C(t,s)$ is a cost function, and $\mu$ is a prior probability measure on $S$. We would like to show that the order of integration can be interchanged to yield

$$R_m = tr \int_S f(s) m(ds)$$

where

$$f(s) = \int_S C(t,s) \rho(t) \mu(dt)$$

is a map $f: S \to \mathcal{T}_s(H)$ that belongs to the space $M(S) \hat{\otimes}_\pi \mathcal{T}(H)$ of functions integrable against operator-valued measures.

Let $(S, \mathcal{B}, \mu)$ be a finite nonnegative measure space, $X$ a Banach space. A function $f: S \to X$ is <u>measurable</u> iff there is a sequence $\{f_n\}$ of simple measurable functions converging pointwise to f, i.e. $f_n(s) \to f(s)$ for every $s \in S$. A useful criterion for measurability is the following [Dunford-Schwartz (1966), III 6.9]: f is measurable if it is separably-valued and for every open subset V of X, $f^{-1}(V) \in \mathcal{B}$. In particular, every $f \in C_o(S,X)$ is measurable, when S is a locally compact Hausdorff space with Borel sets $\mathcal{B}$. A function f: S → X is integrable iff it is measurable and $\int_S |f(s)| \cdot \mu(ds) < +\infty$, in which case the integral $\int_S f(s)\mu(ds)$ is well-defined as Bochner's integral; we denote by $L_1(S, \mathcal{B}, \mu; X)$ the space of all integrable functions f: S → X, a normed space under the $L_1$ norm $|f|_1 = \int_S |f(s)|\mu(ds)$. The uniform norm $|\cdot|_\infty$ on f8nctions f: S → X is defined by $|f|_\infty = \sup_{s \in S} |f(s)|$; M(S,X) denotes the Banach space of all uniform limits of simple X-valued functions, with norm $|\cdot|_\infty$, i.e. M(S,X) is the closure of the simple X-valued functions with the uniform norm. We abbreviate M(S,R) to M(S).

Proposition 7.1. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$, μ a probability measure on S, and H a Hilbert space. Suppose $\rho: S \to \mathcal{T}_s(H)$ belongs to $M(S, \mathcal{T}_s(H))$, and C: S × S → R is a real-valued map satisfying

$$t \mapsto C(t,\cdot) \in L_1(S, \mathcal{B}, \mu; M(S)).$$

Then for every $s \in S$, f(s) is well-defined as an element of $\mathcal{T}_s(H)$ by the Bochner integral

$$f(s) = \int_S C(t,s)\rho(t)\mu(dt); \tag{7.5}$$

moreover $f \in M(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ and for every operator-valued measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$, we have

$$\int_S f(s)m(ds) = \int_S \rho(t)[\int_S C(t,s)m(ds)]\mu(dt) \tag{7.6}$$

Moreover if $t \mapsto C(t,\cdot)$ in fact belongs to $L_1(S,\mathcal{B},\mu;C_o(S))$ then $f \in C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$.

Proof. Since $t \mapsto C(t,\cdot) \in L_1(S,\mathcal{B},\mu;M(S))$, for each $n$ there is a simple function $C_n \in L_1(S,\mathcal{B},\mu;M(S))$ such that

$$\int_S |C(t,\cdot)-C_n(t,\cdot)|_\infty \mu(dt) < \frac{1}{n^{2n}}. \tag{7.7}$$

Each simple function $C_n$ is of the form

$$C_n(t,s) = \sum_{k=1}^{k_n} g_{nk}(s)1_{E_{nk}}(t)$$

where $E_{n,1},\ldots,E_{nk_n}$ are disjoint subsets of $\mathcal{B}$ and $g_{n1},\ldots,g_{nk_n}$ belong to $M(S)$ (in the case that $t \mapsto C(t,\cdot)$ $L_1(S,\mathcal{B},\mu;C_o(S))$ we take $g_{n1},\ldots,g_{nk_n}$ in $C_o(S)$). Since $\rho \in M(S,\mathcal{T}_s(H))$, for each $n$ there is a simple measurable function $\rho_n\colon S \to \mathcal{T}_s(H)$ such that

$$\sup_t |\rho(t)-\rho_n(t)| < \frac{1}{n^{2n}}. \tag{7.8}$$

We may assume, by replacing each set $E_{nk}$ with a disjoint subpartition corresponding to the finite number of values taken on by $\rho_n$, that each $\rho_n$ is in fact of the form

$$\rho_n(t) = \sum_{k=1}^{k_n} \rho_{nk} 1_{E_{nk}}(t).$$

Define $f_n\colon S \to \mathcal{T}_s(H)$ by

$$f_n(s) = \int_S C_n(t,s)\rho_n(t)\mu(dt)$$

$$= \sum_{k=1}^{k_n} g_{nk}(s)\rho_{nk}\mu(E_{nk}).$$

Of course, each $f_n$ belongs to $M(S) \otimes \mathcal{T}_s(H)$. We shall show that $\{f_n\}$ is a Cauchy sequence for the $|\cdot|_\pi$ norm on $M(S) \otimes \mathcal{T}_s(H)$, and that $f_n(s) \to f(s)$ for every $s \in S$; since the $|\cdot|_\pi$-limit of the sequence $f_n$ is a unique function by Theorem 6.1, we see that $f$ is the $|\cdot|_\pi$-limit of $\{f_n\}$ and hence $f$ belongs to the completion $M(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$.

We calculate an upper bound for $|f_{n+1}-f_n|_\pi$. Now

$$f_{n+1}(s)-f_n(s) =$$

$$\sum_{j=1}^{k_{n+1}} \sum_{k=1}^{k_n} \{g_{n+1,j}(s)[\rho_{n+1,j}-\rho_{n,k}]+[g_{n+1,j}(s)-g_{n,k}(s)]\rho_{n,k}\}\mu(E_{n+1,j}\cap E_{n,k})$$

and hence

$$|f_{n+1}-f_n|_\pi \leq \tag{7.9}$$

$$\sum_{j=1}^{k_{n+1}} \sum_{k=1}^{k_n} \{|g_{n+1,j}|_\infty \cdot |\rho_{n+1,j}-\rho_{n,k}|_{tr}+|g_{n+1,j}-g_{n,k}|_\infty \cdot |\rho_{nk}|_{tr}\}\mu(E_{n+1,j}\cap E_{n,k})$$

Suppose $E_{n+1,j}\cap E_{n,k} \neq \emptyset$, i.e. there exists a $t_o \in E_{n+1,j} \cap E_{n,k}$. Then from (7.8) we have

$$|\rho_{n+1,j}-\rho_{n,k}|_{tr} \leq |\rho_{n+1,j}-\rho(t_o)|_{tr} + |\rho_{n,k}-\rho(t_o)|_{tr}$$

$$\leq \frac{1}{(n+1)2^{n+1}} + \frac{1}{n^{2n}} < \frac{1}{n2^{n+1}} .$$

Thus, the first half of the summation in (7.6) is bounded above by

$$\frac{1}{n2^{n-1}} \sum_{j=1}^{k_{n+1}} \sum_{k=1}^{k_n} |g_{n+1,j}|_\infty \mu(E_{n+1,j}\cap E_{n,k}) = \frac{1}{n2^{n-1}} \int_S |C_{n+1}(t,\cdot)|_\infty \mu(dt)$$

$$= \frac{1}{n2^{n-1}}||C_{n+1}||_1$$

$$\leq \frac{1}{n2^{n-1}}(1+||C||_1)$$

where by $||C||_1$ we mean the norm of $t \mapsto C(t,\cdot)$ as a element of $L_1(S,\mathcal{B},\mu;M(S))$; and the last inequality follows from (7.7). Similarly the second half of the summation is bounded above by

$$(|\rho|_\infty+1)\cdot\sum_{j=1}^{k_{n+1}}\sum_{k=1}^{k_n}|g_{n+1,j}-g_{n,k}|_\infty\cdot\mu(E_{n+1,j}\cap E_{n,k})$$

$$=(|\rho|_\infty+1)\cdot||C_{n+1}-C_n||_1$$

$$<(|\rho|_\infty+1)\cdot\frac{1}{n2^{n-1}}$$

where again the last inequality follows since $||C_n-C||_1 < \frac{1}{n2^n}$ by (7.7). Let $a$ be a constant larger than $1+||C||_1$ and $1+|\rho|_\infty$; adding the last two inequalities from (7.9) we have

$$|f_{n+1}-f_n|_\pi < \frac{a}{n2^{n-2}} .$$

Hence for every $m > n \geq 1$ it follows that

$$|f_m-f_n|_\pi \leq \sum_{j=n}^{m-1}|f_{j+1}-f_j|_\pi < \sum_{j=n}^{\infty}\frac{a}{n2^{n-2}} < \frac{1}{n}\sum_{j=1}^{\infty}\frac{a}{2^{n-2}} = \frac{3a}{n} .$$

Thus $\{f_n\}$ is a Cauchy sequence for the $|\cdot|_\pi$ norm on $M(S)\otimes\mathcal{T}_s(H)$, and hence has a limit $f_o\in M(S)\otimes_\pi\mathcal{T}_s(H)$. Since it certainly follows that $f_n\to f_o$ pointwise (in fact in the uniform norm since $|\cdot|_\infty\leq|\cdot|_\pi$), and since it is straightforward to show that $f_n(s)\to f(s)$ for every $s\in S$, $f_o=f$. Moreover in the case that $t\mapsto C(t,\cdot)\in L_1(S,\mathcal{B},\mu;C_o(S))$, we have $f_n\in C_o(S)\otimes\mathcal{T}_s(H)$

and hence $f = |\cdot|_\pi\text{-}\lim f_n$ belongs to $C_o(S) \otimes_\pi \mathcal{T}_s(H)$.

It only remains to show that (7.6) holds. Essentially this follows from the approximations we have already made with simple functions. Now clearly

$$\int f_n(s)m(ds) = \sum_{k=1}^{k_n} \rho_{nk}\mu(E_{nk})\int_S g_{nk}(s)m(ds)$$

$$\int_S \rho_n(t)[\int C_n(t,s)m(ds)]\mu(dt), \tag{7.10}$$

so that (7.6) is satisfied for the simple approximations. We have already shown that $f_n \to f$ in $M(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$, so that $|\int f_n(m)m(ds) - \int f(s)m(ds)|_{tr} \leq |f_n - f|_\pi \cdot \bar{\bar{m}}(S) \to 0$ and the LHS of (7.10) converges to $\int f(s)m(ds)$. We need only show that the RHS of (7.10) converges to the RHS of (7.6) But applying the triangle inequality to (7.10) yields

$$|\int \rho_n(t)[\int C_n(t,s)m(ds)]\mu(dt) - \int \rho(t)[\int C(t,s)m(ds)]\mu(dt)|_{tr}$$

$$\leq |\rho_n(t)\int[C_n(t,s)-C(t,s)]m(ds)|_{tr}\mu(dt)$$

$$+ \int|(\rho_n(t)-\rho(t))\cdot\int C(t,s)m(ds)|_{tr}\mu(dt)$$

$$\leq |\rho_n|_\infty \cdot \int|C_n(t,\cdot)-C(t,\cdot)|_\infty \cdot \bar{\bar{m}}(S)\mu(dt)$$

$$+ |\rho_n - \rho|_\infty \cdot \int|C(t,\cdot)|_\infty \bar{\bar{m}}(S)\mu(dt)$$

$$\leq (|\rho|_\infty + 1)\cdot\bar{\bar{m}}(S)\cdot||C_n - C||_1 + |\rho_n - \rho|_\infty \bar{\bar{m}}(S)||C||_1$$

$$\leq (|\rho|_\infty + 1)\cdot\bar{\bar{m}}(S)\cdot\frac{1}{n2^n} + \frac{1}{n2^n}\bar{\bar{m}}(S)\cdot||C||_1 \to 0$$

where the last inequality follows from (7.7) and (7.8) and again $||C||_1 = \int_S |C(t,\cdot)|_\infty \mu(dt)$ denotes the norm of $C$ as an element of $L_1(S, \mathcal{B}, \mu; M(S))$. □

## 7.3 The quantum estimation problem and its dual

We are now prepared to precisely formulate the quantum estimation problem in the framework of duality theory of optimization and calculate the associated dual problem. Let $S$ be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let $H$ be a Hilbert space associated with the physical variables of the system under consideration. For each parameter value $s \in S$ let $\rho(s)$ be a state or density operator for the quantum system, i.e. every $\rho(s)$ is a nonnegative-definite selfadjoint trace-class operator on $H$ with trace 1; we assume $\rho \in M(S, \mathcal{T}_s(H))$. We assume that there is a cost function $C: S \times S \to R$, where $C(s,t)$ specifies the relative cost of an estimate t when the true parameter value is s. If the operator-valued measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ corresponds to a given measurement and decision strategy, then the posterior expected cost is

$$R_m = \mathrm{tr}\int_S \rho(t)\left[\int_S C(t,s)m(ds)\right]\mu(dt),$$

where $\mu$ is a prior probability measure on $(S,\mathcal{B})$. By Proposition 7 this is well-defined whenever the map $t \mapsto C(t,\cdot)$ belongs to $L_1(S,\mathcal{B},\mu;M(S))$, in which case we may interchange the order of integration to get

$$R_m = \mathrm{tr}\int_S f(s)m(ds) \qquad (7.11)$$

where $f \in M(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ is defined by

$$f(s) = \int_S \rho(t)C(t,s)\mu(ds).$$

The quantum estimation problem is to minimize (7.11) over all operator-valued measures $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ which are POM's, i.e. the constraints are that $m(E) \geq 0$ for every $E \in \mathcal{B}$ and $m(S) = I$.

We shall now assume that the reader is familiar with the duality theory of optimization in infinite-dimensional spaces as for example development in [Rockafellar (1973)]. To form the dual problem we take perturbations on the equality constraint $m(S) = I$. Define the convex function $F: \mathcal{M}(\mathcal{B}, \mathcal{L}_x(H)) \to \overline{R}$ by

$$F(m) = \delta_{\geq o}(m) + tr\int_S f(s)m(ds), \quad m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)),$$

where $\delta_{\geq o}$ denotes the indicator function for the positive operator-valued measures, i.e. $\delta_{\geq o}(m)$ is 0 if $m(\mathcal{B}) \subset \mathcal{L}_s(H)_+$ and $+\infty$ otherwise. Define the convex function $G: \mathcal{L}_s(H) \to \overline{R}$ by

$$G(x) = \delta_{\{o\}}(x), \quad x \in \mathcal{L}_s(H)$$

i.e. $G(x)$ is 0 if $x = 0$ and $G(x) = +\infty$ if $x \neq 0$. Then the quantum detection problem may be written

$$P_o = \inf\{F(m)+G(I-Lm) : m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))\}$$

where $L: \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)) \to \mathcal{L}_s(H)$ is the continuous linear operator

$$L(m) = m(S).$$

We consider a family of perturbed problems defined by

$$P(x) = \inf\{F(m)+G(x-Lm): m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))\}, \quad x \in \mathcal{L}_s(H).$$

Thus we are taking perturbations in the equality constraint, i.e. the problem $P(x)$ requires that every feasible m be nonnegative and satisfy $m(S) = x$; of course, $P_o = P(I)$. Since F and G are convex, $P(\cdot)$ is convex $\mathcal{L}_s(H) \to \overline{R}$.

In order to construct the dual problem corresponding to the family of perturbed problems $P(x)$, we must calculate the conjugate functions of F and G denoted as $F^*$ and $G^*$. We shall work in the norm topology of the constraint space $\mathcal{L}_s(H)$, so that the dual problem is posed in $\mathcal{L}_s(H)^*$. Clearly $G^* \equiv 0$. The adjoint of the operator L is given by

$$L^*: \mathcal{L}_s(H)^* \to \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))^*: y \mapsto (m \mapsto y \cdot m(S)).$$

To calculate $F^*(L^*y)$, we have the following lemma.

<u>Lemma 7.2</u>. Suppose $y \in \mathcal{L}_s(H)^*$ and $f \in M(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ satisfy

$$y \cdot m(S) \leq \operatorname{tr} \int_S f(s) m(ds) \tag{7.12}$$

for every positive operator-valued measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)_+)$. Then $y_{sg} \leq 0$ and $y_{ac} \leq f(s)$ for every $s \in S$, where $y = y_{ac}+y_{sg}$ is the unique decomposition of $y$ into

$y_{ac} \in \mathcal{T}_s(H)$ and $y_{sg} \in \mathcal{K}_s(H)^{\perp}$.

Proof. Fix any $s_o \in S$. Let $x$ be an arbitrary element of $\mathcal{L}_s(H)_+$, and define the positive operator-valued measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)_+)$ by

$$m(E) = \begin{cases} x & \text{if } s_o \in E \\ 0 & \text{if } s_o \notin E \end{cases}, \quad E \in \mathcal{B}.$$

Then $y \cdot m(S) = y(x) = tr(y_{ac}x) + y_{sg}(x)$, and $tr\int f(s)m(ds) = trf(s_o)x$. Thus, by (7.12) $tr[y_{ac} - f(s_o)]x + y_{sg}(x) \leq 0$; since $x \in \mathcal{L}_s(H)_+$ was arbitrary, it follows that $y_{ac} \leq f(s_o)$ (i.e. $f(s_o) - y_{ac} \in \mathcal{T}_s(H)_+$) and $y_{sg} \leq 0$ (i.e. $-y_{sg} \in [\mathcal{L}_s(H)_+]^+ \cap \mathcal{K}_s(H)^{\perp}$). □

With the aid of this lemma it is now easy to verify that

$$F^*(L^*y) = \begin{cases} 0 & \text{if } y_{ac} \leq f(s) \quad s \in S, \text{ and } y_{sg} \leq 0 \\ +\infty & \text{otherwise} \end{cases}$$

$$= \delta_{\leq f}(y_{ac}) + \delta_{\leq o}(y_{sg}).$$

It now follows that $P^*(y) = F^*(L^*y) + G^*(y)$ is $0$ if $y_{sg} \leq 0$ and $y_{ac} \leq f(s)$ for every $s \in S$, and $P^*(y) = +\infty$ otherwise. The dual problem $D_o = {}^*(P^*)(I) = \sup_y [y(I) - P^*(y)]$ is thus given by

$$D_o = *(P^*)(I)$$

$$= \sup\{\mathrm{tr}\, y_{ac} + y_{sg}(I) : y \in \mathcal{L}_s(H)^*,\ y_{sg} \le 0,\ y_{ac} \le f(s)\ \forall s \in S\}.$$

We show that $P(\cdot)$ is norm continuous at I, and hence there is no duality gap ($P_o = D_o$) and $D_o$ has solutions. Moreover we shall show that the optimal solutions for $D_o$ will always have 0 singular part, i.e., will be in $\mathcal{T}_s(H)$.

Proposition 7.3. The perturbation function P(.) is continuous at I, and hence $\partial P(I) \neq \emptyset$, where $\partial P$ denotes the subgradient of P. In particular, $P_o = D_o$ and the dual problem $D_o$ has optimal solutions. Moreover every solution $y \in \mathcal{L}_s(H)^*$ of the dual problem $D_o$ has 0 singular part, i.e. $\hat{y}_{sg} = 0$ and $\hat{y} = \hat{y}_{ac}$ belongs to the canonical image of $\mathcal{T}_s(H)$ in $\mathcal{T}_s(H)^{**}$.

Proof. We show that $P(\cdot)$ is bounded above on a unit ball centered at I. Suppose $x \in \mathcal{L}_s(H)$ and $|x| \le 1$. Then it is easily seen that $I+X \ge 0$. Let $s_o$ be an arbitrary element of S and define the positive operator-valued measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)_+)$ by

$$m(E) = \begin{cases} I+x & \text{if } s_o \in E \\ 0 & \text{if } s_o \notin E \end{cases}, \quad E \in \mathcal{B}.$$

Then m is feasible for $P(x)$ and has cost

$$\mathrm{tr}\int f(s)m(ds) = \mathrm{tr} f(s_o)(I+x) \leq 2|f(s_o)|_{tr}.$$

Thus $P(I+x) \leq 2|f(s_o)|_{tr}$ whenever $|x| \leq 1$, so $P(\cdot)$ is bounded above on a neighborhood of $I$ and so by convexity is continuous at I, and hence from standard results in convex analysis, it follows that $\partial P(x_o) \neq \emptyset$, hence $P_o = D_o$ and $D_o$ has solutions. Suppose now that $\hat{y} \in \mathcal{L}_s(H)^*$ is an optimal solution for $D_o$. If $\hat{y}_{sg} \neq 0$, then since $\hat{y}_{sg} \leq 0$ and $I \in \mathrm{int}\mathcal{L}_s(H)_+$ it follows that $\mathrm{tr}(\hat{y}_{ac}) + \hat{y}_{sg}(I) < \mathrm{tr}(\hat{y}_{ac})$. Hence the value of the dual objective function is strictly improved by setting $\hat{y}_{sg} = 0$, while the constraints remain satisfied, so that if $\hat{y}$ is optimal it must be true that $\hat{y}_{sg} = 0$. □

In order to show that the problem $P_o$ has solutions, we could define a family of dual perturbed problems $D(v)$ for $v \in C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ and show that $D(\cdot)$ is continuous. Or we could take the alternative method of showing that the set of feasible POM's $m$ is weak* compact and the cost function is weak*-lsc when $\mathcal{M}(\mathcal{B}, \mathcal{L}_s(H)) \cong \mathcal{L}(C_o(S), \mathcal{L}_s(H))$ is identified as the normed dual of the space $C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ under the pairing

$$\langle f, m \rangle = \mathrm{tr}\int f(s)m(ds).$$

Note that both methods require that $f$ belong to the predual $C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ of $\mathcal{M}(\mathcal{B}, \mathcal{T}_s(H))$; by Proposition 7.1 it suffices to assume that $t \mapsto C(t,\cdot)$ belongs to $L_1(S,\mathcal{B},\mu;C_o(S))$.

Proposition 7.4. The set of POM's is compact for the weak* $\equiv w(\mathcal{M}(\mathcal{B},\mathcal{L}_s(H)), C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H))$ topology. If $t \mapsto C(t,\cdot) \in L_1(S,\mathcal{B},\mu;C_o(S))$ then $P_o$ has optimal solutions $\hat{m}$.

Proof. Since $\mathcal{M}(\mathcal{B},\mathcal{L}_s(H))$ is the normed dual of $C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ it suffices to show that the set of POM's is bounded; in fact, we show that $\bar{\bar{m}}(S) = 1$ for every POM m. If $\phi \in H$ and $|\phi| = 1$, then $\langle\phi m(\cdot)|\phi\rangle$ is a regular Borel probability measure on S whenever m is a POM, so that the total variation of $\langle\phi m(\cdot)|\phi\rangle$ is precisely 1. Hence

$$\bar{\bar{m}}(S) = \sup_{\substack{\phi\in H \\ |\phi|\le 1}} |\langle\phi m(\cdot)|\phi\rangle|(S) = \sup_{\substack{\phi\in H \\ |\phi|=1}} |\langle\phi m(\cdot)|\phi\rangle|(S) = 1.$$

Thus the set of POM's is a weak*-closed subset of the unit ball in $\mathcal{M}(\mathcal{B},\mathcal{L}_s(H))$, hence weak*-compact. If now $t \mapsto C(t,\cdot)$ belongs to $L_1(S,\mathcal{B},\mu;C_o(S))$ then $f \in C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ by Proposition 7.1, so $m \mapsto \mathrm{tr}\int f(s)m(ds)$ is a weak*-continuous linear function and hence attains its infimum on the set of POM's. Thus $P_o$ has solutions. □

The following theorem summarizes the results we have obtained so far, as well as providing a necessary and sufficient characterization of the optimal solution.

Theorem 7.5. Let $H$ be a Hilbert space, $S$ a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let $\rho \in M(S, \mathcal{T}_s(H))$, $C: S \times S \to R$ a map satisfying $t \mapsto C(t,\cdot) \in L_1(S,\mathcal{B},\mu;C_o(S))$, and $\mu$ a probability measure on $(S,\mathcal{B})$. Then for every $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$,

$$\mathrm{tr}\int_S \rho(t)\left[\int_S C(t,s)m(ds)\right]\mu(dt) = \mathrm{tr}\int_S f(s)m(ds)$$

where $f \in C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$ is defined by

$$f(s) = \int_S \rho(t)C(t,s)\mu(ds).$$

Define the optimization problems

$$P_o = \inf\{\mathrm{tr}\int_S f(s)m(ds):\ m\in\mathcal{M}(\mathcal{B},\mathcal{L}_s(H)), m(S)=I, m(E)\geq 0 \text{ for every } E\in\mathcal{B}\}$$

$$D_o = \sup\{\mathrm{tr}y:\ y\in \mathcal{T}_s(H),\ y \leq f(s) \text{ for every } s \in S\}.$$

Then $P_o = D_o$, and both $P_o$ and $D_o$ have optimal solutions. Moreover the following statements are equivalent for $m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H))$, assuming $m(S) = I$ and $m(E) \geq 0$ for every $E \in \mathcal{B}$:

1) $m$ solves $P_o$

2) $\int_S f(s)m(ds) \leq f(t)$ for every $t \in S$

3) $\int_S m(ds)f(s) \leq f(t)$ for every $t \in S$.

Under any of the above conditions it follows that $y = \int_S f(s)m(ds) = \int_S m(ds)f(s)$ is selfadjoint and is the unique solution of $D_o$, with

$$P_o = D_o = \operatorname{tr} y.$$

<u>Proof</u>. We need only verify the equivalence of 1)-3); the rest follows from Propositions 7.3 and 7.4. Suppose $m$ solves $P_o$. Then there is a $y \in \mathcal{T}_s(H)$ which solves D, so that $y \leq f(t)$ for every $t$ and

$$\operatorname{tr}\int_S f(s)m(ds) = \operatorname{tr} y.$$

Equivalently $0 = \operatorname{tr}\int_S f(s)m(ds) - \operatorname{tr} y = \operatorname{tr}\int_S (f(s)-y)m(ds)$. Since $f(s)-y \geq 0$ for every $s \in S$ and $m \geq 0$ it follows that $0 = \int_S (f(s)-y)m(ds) = \int_S f(s)m(ds) - y$ and hence 2) holds. This last equality also shows that $y$ is unique.

Conversely, suppose 2) holds. Then $y = \int_S f(s)m(ds)$ is feasible for $D_o$, and moreover $\operatorname{tr}\int_S f(s)m(ds) = \operatorname{tr} y$. Since $P_o \geq D_o$, it follows that $m$ solves $P_o$ and $y$ solves $D_o$, so that 1) holds.

Thus 1) <=> 2) is proved. The proof of 1) <=> is identical, assuming that $\mathrm{tr}\int f(s)m(ds) = \mathrm{tr}\int m(ds)f(s)$ for every $f \in C_o(S) \hat{\otimes}_\pi \mathcal{T}_s(H)$. But the latter is true since $\mathrm{tr}AB = \mathrm{tr}BA$ for every $A \in \mathcal{T}_s(H)$, $B \in \mathcal{L}_s(H)$ and hence it is true for every $f \in C_o(S) \otimes \mathcal{T}_s(H)$. □

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