BOUNDARY VALUES AND RESTRICTIONS
OF
GENERALIZED FUNCTIONS WITH APPLICATIONS

By

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ABSTRACT

Let $\mathcal{N}$ be an imbedded submanifold of a smooth manifold $\mathcal{M}$. In this thesis, we study the problem of restricting generalized functions on $\mathcal{M}$ to $\mathcal{N}$. More generally, if $\mathcal{O}$ is open in $\mathcal{M}$ and $\mathcal{N}$ is contained in the closure of $\mathcal{O}$, we study the problem of defining boundary values on $\mathcal{N}$, respectively, a restriction to $\mathcal{N}$, for generalized functions defined on $\mathcal{O}$ where $\mathcal{N}$ is contained in the boundary of $\mathcal{O}$, respectively, the interior of $\mathcal{O}$. The generalized functions that we work with are continuous with respect to L^1-type seminorms which remain bounded when applied to sequences of densities converging weakly to a Dirac-type measure on $\mathcal{N}$.

A new sufficient condition for restrictability is given in terms of a refined version of the wave front set of a generalized function which we define and study. In terms of this refinement, we also derive sufficient conditions in order that the product of generalized functions is well defined.

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CONTENTS

CHAPTER 0
INTRODUCTION

Let $\mathcal{M}$ be a smooth manifold, $\mathcal{N}$ an imbedded submanifold of $\mathcal{M}$, and $\mathcal{O}$ an open set in $\mathcal{M}$. In this paper, we are interested in defining boundary values on $\mathcal{N}$, or a restriction to $\mathcal{N}$, for generalized functions defined on $\mathcal{O}$ depending on whether $\mathcal{N}$ is contained in the boundary or the interior of $\mathcal{O}$. The definition we give stems from the observation that if $f(x, t)$ is a smooth function on $\mathbf{R}^{n+1}$ and ν is a compactly supported density on $\mathbf{R}^n \times \{0\}$, we can define the value of the restriction of $f(x, t)$ on ν, $\int f(x, 0)\nu$, as the limit we get by integrating $f(x, t)$ against any sequence of the form $\{\nu \otimes \eta_n\}$ where $\{\eta_n\}$ converges weakly to the Dirac density at $t = 0$.

Hence, in Chapter I, we begin by defining a space of generalized functions on $\mathcal{O}$, $\mathcal{R}_{\mathcal{N},\mathcal{O}}$, that are continuous with respect to L^1-type seminorms that remain bounded on such sequences. Then, since we wish to work on a manifold and the form of such sequences is not coordinate invariant, we define permissible sequences $\{\mu_n\}$ which are just a coordinate invariant version of the above sequence $\{\nu \otimes \eta_n\}$. Then, if T is a generalized function, we say that T has a restriction to $\mathcal{N}$ if $\langle T, \mu_n \rangle$ converges as $n \to \infty$ and the limit depends only on ν. We then show that this is a local property and if a limit exists, it is given by a generalized function on $\mathcal{N}$, denoted $T_{\mathcal{N},\mathcal{O}}$.

We also define convergence in $\mathcal{R}_{\mathcal{N},\mathcal{O}}$ which is strong enough to insure that if T_n converges to T in $\mathcal{R}_{\mathcal{N},\mathcal{O}}$, then $(T_n)_{\mathcal{N},\mathcal{O}}$ converges to $T_{\mathcal{N},\mathcal{O}}$ weakly. In fact, the collection of restrictable generalized functions is seen to be sequentially closed in $\mathcal{R}_{\mathcal{N},\mathcal{O}}$. That is, if T_n converges to T in $\mathcal{R}_{\mathcal{N},\mathcal{O}}$ and $(T_n)_{\mathcal{N},\mathcal{O}}$ exists for all n, then $T_{\mathcal{N},\mathcal{O}}$ also exists and equals the weak limit of the $(T_n)_{\mathcal{N},\mathcal{O}}$. This property is used throughout the paper.

Since in practice it is difficult to test T against all such sequences $\{\mu_n\}$, it is natural to ask whether it is sufficient to work within a single coordinate system and there test T against only permissible sequences of the form $\{\nu \otimes \eta_n\}$.

In the first two sections of Chapter II, we show this to be the case for both bounday values and restrictions when $\mathcal{N}$ has codimension one. In fact, by a Tauberian-type argument, we show that with an auxiliary condition on the Fourier transform of η_0, it is sufficient to consider only permissible sequences of the form $\{\nu \otimes n\eta_0(nt)\,dt\}$. To prove this, we construct a sequence of smooth functions converging to T in $\mathcal{R}_{\mathcal{N},\mathcal{O}}$.

In these sections, we also study other properties of the spaces $\mathcal{R}_{\mathcal{N},\mathcal{O}}$. For example, if $\mathcal{N}$ is the boundary of $\mathcal{O}$, we show that if $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$, then T can be extended to a generalized function on $\mathfrak{M}$ in a natural way.

In section 3, we identify distributions on $\mathbf{R}^n$ with generalized functions in the natural way and show that if D is a distribution, φ a compactly supported smooth function with integral one, then $D * \varphi_t$ can be considered as a generalized function on $\mathbf{R}_+^{n+1}$ whose boundary value on $\mathbf{R}^n$ is given by D. Here, $\varphi_t = t^{-n}\varphi(\cdot/t)$.

In the two sections of Chapter III, we essentially follow the pattern of the first two sections of Chapter II only in the case where the codimension of $\mathfrak{N}$ is greater than one.

In Chapter IV, we are interested in studying the question of existence. To do this, we first refine the notion of the wave front set of a distribution of Hörmander [1] in section 1. Roughly speaking, we split the wave front set into orders of decay and then show that this splitting in local and coordinate invariant. These cones of variable decay are called k-wave front sets. We then study properties of these sets and their projections to the manifold, called the singular k-supports. For example, we show that if k is a positive integer, then the singular $(-k)$-support of a distribution D is contained in the complement of the set of points where D is locally in C^k.

In section 2, we extend these definitions to generalized functions in the natural way, and show that if $\mathfrak{N}$ has codimension l, and the $(-l)$-wave front set of T does not intersect the normal bundle of $\mathfrak{N}$, then T has a restriction to $\mathfrak{N}$. In some cases, we also derive a relationship between the k-wave front sets of T and $T_{\mathfrak{N},\mathcal{O}}$.

In section 3, we define the product of generalized functions as the restriction of the tensor product to the diagonal of the product manifold. We then derive sufficient conditions, in terms of k-wave front sets, in order that a product is well defined. Lastly, we discuss a couple of properties of products.

CHAPTER I
DEFINITIONS AND BASIC PROPERTIES

Let $\mathfrak{M}$ be a second countable C^∞ manifold of dimension n. Recall that a density μ on $\mathfrak{M}$ is a signed measure on $\mathfrak{M}$ which in every coordinate system $U = \{(x)\}$, μ can be expressed as $\mu = \varphi(x)\,dx$ where $\varphi \in C^\infty(\mathfrak{M})$. Note that if $U = \{(x)\}$, $V = \{(y)\}$ are two coordinate neighborhoods on $\mathfrak{M}$, say $\mu = \varphi(x)\,dx$ on U, $\mu = \psi(y)\,dy$ on V, then on $U \cap V$, if $y(x)$: $U \cap V \to U \cap V$ is the change of coordinates diffeomorphism, we have

$$\left|\frac{\partial y}{\partial x}\right| \psi(y(x)) = \varphi(x) \tag{1.0}$$

where $|\partial y/\partial x|$ is the absolute value of $\det(\partial y/\partial x)$. Hence if $\chi \in C_0^\infty(U \cap V)$, we have a coordinate invariant definition of

$$\int_{U \cap V} \chi\mu.$$

Also note that by (1.0), $\mu \neq 0$ at x_0 has an intrinsic definition. That is, if $x_0 \in U$, where $U = \{(y)\}$ is a coordinate neighborhood, then $\mu = \psi(y)\,dy$ on U and we say $\mu \neq 0$ at x_0 if $\psi(x_0) \neq 0$. Then by (1.0), $\varphi(x_0) \neq 0$ in any other coordinate neighborhood of x_0.

Now, let $\mathcal{O}$ be open in $\mathfrak{M}$. We denote by $\mathcal{B}(\mathcal{O})$ the collection of compactly supported densities on $\mathfrak{M}$ supported in $\mathcal{O}$ topologized as follows. Let $\{\mu_n\} \subset \mathcal{B}(\mathcal{O})$. We say that $\mu_n \to 0$ if:

i) $\operatorname{supp} \mu_n \subset K \subset \mathcal{O} \quad \forall n \quad$ where K is compact,

ii) If U is any coordinate patch in $\mathcal{O}$ with coordinate functions $(x_1, \ldots, x_n)$, and $\mu_n = \varphi_n(x)\,dx$ in U, then $|(\partial/\partial x)^\alpha \varphi_n| \to 0$ as $n \to \infty$ uniformly on every compact subset of U for any multi-index α.

Here we use the standard multi-index notation. That is, if $x = (x_1, \ldots, x_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers;

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Similarly, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$; $\alpha! = \alpha_1! \cdots \alpha_n!$; $|\alpha| = \sum_i \alpha_i$.

Note that ii) is coordinate invariant as above.

We denote by $\mathcal{B}'(\mathcal{O})$ the collection of continuous linear functionals on $\mathcal{B}(\mathcal{O})$. By continuous we mean the following. If T is a linear functional on $\mathcal{B}(\mathcal{O})$, we say that T is continuous if $\langle T, \mu_n\rangle \to 0$ for any collection $\{\mu_n\} \subset \mathcal{B}(\mathcal{O})$ such that $\mu_n \to 0$. If $T \in \mathcal{B}'(\mathcal{O})$, we call T a generalized function. If $\{T_n\} \subset \mathcal{B}'(\mathcal{O})$, we say that $T_n \to 0$ if $\langle T_n, \mu\rangle \to 0$ for every $\mu \in \mathcal{B}(\mathcal{O})$.

We denote by $\mathcal{O}^c$ the closure of $\mathcal{O}$.

Let $\mathfrak{N}$ be an imbedded submanifold of $\mathfrak{M}$ of codimension l. Let $\mathcal{O}$ be open in $\mathfrak{M}$ such that $\mathfrak{N} \subset \mathcal{O}^c$. Let $T \in \mathcal{B}'(\mathcal{O})$.

Definition 1.1. *We define* $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ *if given any* $x_0 \in \mathfrak{N}$ *there is an integer* m, *a constant* $c \in \mathbf{R}^+$, *and a coordinate patch* U_{x_0} *in* $\mathfrak{M}$, *with coordinate functions* $(x_1, \ldots, x_k, y_1, \ldots, y_l)$, *where* $U_{x_0} \cap \mathfrak{N} = \{(x_1, \ldots, y_l) \mid y_i = 0 \quad \forall i\}$, *so that*

$$|\langle T, \mu\rangle| \leqslant c \sum_{\substack{|\alpha|+|\beta|\leqslant m \\ |\beta|=|\beta'|}} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} y^{\beta} \left(\frac{\partial}{\partial y}\right)^{\beta'} \varphi \right\|_1 \tag{1.1}$$

for all $\mu \in \mathfrak{B}(U_{x_0} \cap \mathcal{O})$, *where on* U_{x_0}, $\mu = \varphi(x,y)\,dx\,dy$.

Here we make the convention that $(\partial/\partial x)^{\alpha} f = f$ if $\alpha = (0, \ldots, 0)$.

Clearly, if T has compact support, we may take c and m as independent of x_0. Also, if $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ and $\zeta \in C^{\infty}(\mathfrak{M})$, then $\zeta T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$.

Proposition 1.1. *Let* $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$, $x_0 \in \mathfrak{N}$, V_{x_0} *any coordinate patch in* $\mathfrak{M}$ *containing* x_0 *with coordinate functions* $(\bar{x}_1, \ldots, \bar{x}_k, \bar{y}_1, \ldots, \bar{y}_l)$ *where* $\mathfrak{N} \cap V_{x_0} = \{(\bar{x}_1, \ldots, \bar{y}_l) \mid \bar{y}_i = 0 \quad \forall i\}$. *Then there is a neighborhood of* x_0, $W_{x_0} \subset V_{x_0}$, *and an integer* $\bar{m}$ *so that*

$$|\langle T, \mu\rangle| \leqslant c' \sum_{\substack{|\alpha|+|\beta|\leqslant \bar{m} \\ |\beta|=|\beta'|}} \left\| \left(\frac{\partial}{\partial \bar{x}}\right)^{\alpha} \bar{y}^{\beta} \left(\frac{\partial}{\partial \bar{y}}\right)^{\beta'} \varphi \right\|_1 \tag{1.2}$$

for all $\mu \in \mathfrak{B}(\mathcal{O} \cap W_{x_0})$, $\mu = \varphi(\bar{x},\bar{y})\,d\bar{x}\,d\bar{y}$.

Proof. Let U_{x_0} be as in Definition 1.1. Choose $W_{x_0} \subset U_{x_0} \cap V_{x_0}$. Then W_{x_0} has coordinate functions $(x_1, \ldots, y_l)$ and $(\bar{x}_1, \ldots, \bar{y}_l)$. Let $\mu \in \mathfrak{B}(\mathcal{O} \cap W_{x_0})$. Then $\mu = \varphi(\bar{x},\bar{y})\,d\bar{x}\,d\bar{y}$ and $\mu = \psi(x,y)\,dx\,dy$ with $\psi(x,y) = \varphi(\bar{x}(x,y), \bar{y}(x,y))\,|\partial(\bar{x},\bar{y})/\partial(x,y)|$ where $|\partial(\bar{x},\bar{y})/\partial(x,y)|$ is the absolute value of the Jacobian determinant of the C^{∞} diffeomorphism $(x,y) \to (\bar{x},\bar{y})$. Note that $\bar{x}(x,0) = \bar{x}$ and $\bar{y}(x,0) = 0$. Expanding the ith coordinate $\bar{y}_i(x,y)$ of $\bar{y}(x,y)$ by Taylor's formula, we have that

$$\bar{y}_i(x,y) = \sum_j y_j f_{ij}(x,y) \tag{1.3}$$

where $f_{ij}(x,y) = \int_0^1 (\partial \bar{y}_i/\partial y_j)(x, ty)\,dt$ are C^{∞} functions for all i, j.

Now the Jacobian matrix of the diffeomorphism $(x,y) \to (\bar{x},\bar{y})$ at $y = 0$ is given by

$$\begin{bmatrix} \left.\dfrac{\partial \bar{x}}{\partial x}\right|_{y=0} & \left.\dfrac{\partial \bar{x}}{\partial y}\right|_{y=0} \\ 0 & f_{ij}(x,0) \end{bmatrix}.$$

Hence, since this is a nonsingular matrix, we must have that $\det|f_{ij}(x,0)| \neq 0$. Hence $\det|f_{ij}(x,y)| \neq 0$ for (x,y) in a sufficiently small neighborhood of $U_{x_0} \cap V_{x_0} \cap \mathfrak{N}$. We modify W_{x_0} to be this neighborhood. Without loss of generality, choose $W_{x_0}^c$ to be compact, $W_{x_0}^c \subset U_{x_0} \cap V_{x_0}$.

Now, since $\mu \in \mathfrak{B}(W_{x_0} \cap \mathcal{O}) \subset \mathfrak{B}(U_{x_0} \cap \mathcal{O})$, we have by (1.1) that there is an m, c so that

$$|\langle T, \mu\rangle| \leqslant c \sum_{\substack{|\alpha|+|\beta|\leqslant m \\ |\beta|=|\beta'|}} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} y^{\beta} \left(\frac{\partial}{\partial y}\right)^{\beta} \psi(x,y) \right\|_1. \tag{1.4}$$

Now, by (1.3)

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial \bar{x}_j}{\partial x_i}\frac{\partial}{\partial \bar{x}_j} + \sum_{jk} \frac{\partial f_{jk}}{\partial x_i} y_k \frac{\partial}{\partial \bar{y}_j}$$

$$= \sum_j \frac{\partial \bar{x}_j}{\partial x_i}\frac{\partial}{\partial \bar{x}_j} + \sum_{jk} h_{jk}^{(i)}(x,y)\bar{y}_j \frac{\partial}{\partial \bar{y}_k}$$

by inverting $\bar{y} = A(x,y)y$ on W_{x_0}. Similarly,

$$\frac{\partial}{\partial y_i} = \sum_j \frac{\partial \bar{x}_j}{\partial y_i}\frac{\partial}{\partial \bar{x}_j} + \sum_j k_j^{i\omega}(x,y)\frac{\partial}{\partial \bar{y}_j}.$$

Hence, we have that

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} = \sum a_{\beta\gamma\gamma'}(x,y)\left(\frac{\partial}{\partial \bar{x}}\right)^{\beta}(\bar{y})^{\gamma}\left(\frac{\partial}{\partial \bar{y}}\right)^{\gamma'}$$

where the sum is over $|\beta| + |\gamma| \leqslant |\alpha|$, $|\gamma| = |\gamma'|$. Similarly,

$$\left(\frac{\partial}{\partial y}\right)^{\alpha} = \sum_{|\beta|+|\gamma|\leqslant|\alpha|} c_{\beta\gamma}(x,y)\left(\frac{\partial}{\partial \bar{x}}\right)^{\beta}\left(\frac{\partial}{\partial \bar{y}}\right)^{\gamma}.$$

And

$$(y)^{\alpha} = \sum_{|\beta|=|\alpha|} b_{\beta}(x,y)\bar{y}^{\beta},$$

where we note that all $a_{\beta\gamma\gamma'}$, $c_{\beta\gamma}$ and b_{β} are C^{∞} functions. Hence,

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} y^{\beta}\left(\frac{\partial}{\partial y}\right)^{\beta'} = \sum_{|\gamma|+|\delta|\leqslant m} d_{\gamma\delta\delta'}(x,y)\left(\frac{\partial}{\partial \bar{x}}\right)^{\gamma}(\bar{y})^{\delta}\left(\frac{\partial}{\partial \bar{y}}\right)^{\delta'},$$

where in the sum, we have $|\delta'| \leqslant |\delta|$ in general and $d_{\gamma\delta\delta'}$ are C^{∞} functions for all γ, δ, δ'.

From the above expression it follows that the right hand side of (1.4) is majorized by the right hand side of (1.2) with $\bar{m} = m$, and an appropriate constant c'. ■

DEFINITION 1.2. *Let* $\{\mu_n\} \subset \mathfrak{B}(\mathcal{O})$. *We say that* $\{\mu_n\}$ *is a permissible sequence on* $\mathcal{O}$ *if:*

i) supp $\mu_n \subset K$ *for all n, K a fixed compact set,*

ii) If U is any neighborhood of $\mathfrak{N}$, *then* supp $\mu_n \subset U \cap \mathcal{O}$ *for n large,*

iii) Given $x_0 \in \mathfrak{N}$, *there is a coordinate neighborhood* $U_{x_0} = \{(x,y)\}$ *where* $U \cap N = \{(x,y) \mid y = 0\}$, *so that for every* $\psi \in C_0^{\infty}(U_{x_0})$,

$$\sum_{\substack{|\alpha|+|\beta|\leqslant m\\ |\beta|=|\beta'|}} \left\|\left(\frac{\partial}{\partial x}\right)^{\alpha} y^{\beta}\left(\frac{\partial}{\partial y}\right)^{\beta'} \psi\varphi_n\right\|_1 \leqslant c_m \qquad \forall m$$

where c_m *is independent of n, and* $\mu_n = \varphi_n(x,y)\,dx\,dy$ *on U,*

iv) There exists a ν, $\nu \in \mathfrak{B}(\mathfrak{N})$ *such that*

$$\lim_n \langle F, \mu_n \rangle = \langle F|_{\mathfrak{N}}, \nu \rangle \qquad \textit{for all } F \in C^{\infty}(\mathfrak{M})$$

where $F|_{\mathfrak{N}}$ *is the restriction of F to* $\mathfrak{N}$.

(If $\{\mu_n\}$ is a permissible sequence, we will often say that μ_n converges to ν, where we mean in the sence of iv).

Clearly, if $\{\mu_n\}$ is a permissible sequence, $\zeta \in C^\infty(\mathfrak{M})$, then $\{\zeta\mu_n\}$ is a permissible sequence.

PROPOSITION 1.2. *Let $\{\mu_n\}$ be a permissible sequence, $x_0 \in \mathfrak{N}$, V_{x_0} any coordinate patch where $\mathfrak{N} \cap V_{x_0} = \{(\bar{x},\bar{y}) \mid \bar{y} = 0\}$. Then there exists $W_{x_0} \subset V_{x_0}$ so that for all $\psi \in C_0^\infty(W_{x_0})$.*

$$\sum_{\substack{|\alpha|+|\beta|\leqslant m\\ |\beta|=|\beta'|}} \left\| \left(\frac{\partial}{\partial \bar{x}}\right)^\alpha (\bar{y})^\beta \left(\frac{\partial}{\partial \bar{y}}\right)^{\beta'} \psi\varphi_n \right\|_1 \leqslant c_m \qquad \forall m \tag{1.5}$$

where c_m is independent of n, and $\mu_n = \varphi_n(\bar{x},\bar{y})\, d\bar{x}\, d\bar{y}$ on W_{x_0}.

PROOF. Identical to the proof of Proposition 1.1. ■

DEFINITION 1.3. *We say that $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$ has boundary values on $\mathfrak{N}$ [a restriction to $\mathfrak{N}$] if:*

i) *Given $x_0 \in \mathfrak{N}$, there is a neighborhood V_{x_0} of x_0 in $\mathfrak{N}$, $\nu \in \mathfrak{B}(V_{x_0})$, such that $\nu \neq 0$ at x_0, and a permissible sequence on $\mathcal{O}$ converging to ν,*

ii) $\lim_n \langle T, \mu_n\rangle$ *exists for all permissible sequences* $\{\mu_n\}$,

iii) $\lim_n \langle T, \mu_n\rangle = 0$ *if* $\nu = \lim_n \mu_n = 0$,

iv) $\mathfrak{N} \subset \partial\mathcal{O}$ $[\mathfrak{N} \subset \mathcal{O}]$.

Here, $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$.

Note that by (1.1) and (1.5), we have that for any $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$, any permissible sequence $\{\mu_n\}$,

$$|\langle T, \mu_n\rangle| \leqslant c \qquad \text{for all } n. \tag{1.6}$$

However, this does not imply, in general, the existence of a limit.

The following example shows that i) in Definition 1.3 is not superfluous.

EXAMPLE 1. Let $\mathfrak{M} = \mathbf{R}^2$, $\mathfrak{N} = \{(0,y)\}$. Let a_n be a sequence so that $a_0 = 1$, $a_n \to 0$ and $(a_{n-1} - a_n)/a_n < a_n^\alpha$ for $\alpha > 0$. Let $I_n = \{(x,y) \mid x \in (a_n, a_{n-1})\}$. Let $\mathcal{O} = \bigcup_n I_n$. Let $\{\mu_n\}$ be a sequence of densities satisfying i), ii) and iii) of Definition 1.2. Say supp $\varphi_n \subset I_n$. Then on I_m, if $\varphi^{(k)}(s,y) = (\partial/\partial s)^k \varphi(s,y)$,

$$|\varphi_n(x,y)| = \left| \int_{a_n}^x \frac{(s-a_n)^k}{k!} \varphi_n^{(k+1)}(s,y)\, ds \right| \leqslant |a_{n-1} - a_n|^k \left| \int_{a_n}^x \frac{\varphi_n^{(k+1)}(s,y)}{k!} ds \right|$$

$$\leqslant \frac{|a_{n-1} - a_n|^k}{|a_n|^{k+1}} \left| \int_{a_n}^x \frac{s^{k+1}\varphi_n^{(k+1)}(s,y)}{k!} ds \right|.$$

So on I_n,

$$|\varphi_n(x,y)| \leqslant \frac{a_n^{\alpha k}}{a_n} \int_{a_n}^{a_{n-1}} \left| \frac{s^{k+1}\varphi_n^{(k+1)}(s,y)}{k!} \right| ds.$$

Hence, $\|\varphi_n\|_1 \leqslant c_k a_n^{\alpha(k+1)}$ where c_k is independent of n by iii) of definition 1.2. So, as $n \to \infty$, $\|\varphi_n\|_1 \leqslant c_k a_n^{\alpha(k+1)} \to 0$. Hence, if $F \in C^\infty(\mathbf{R}^2)$,

$$\int F(x,y)\varphi_n(x,y)\, dx\, dy \to 0.$$

Thus, if $\nu \in \mathfrak{B}(\mathfrak{N})$, $\nu \not\equiv 0$, there is no permissible sequence on $\mathcal{O}$ converging to ν.

Before continuing, note that if $\{\mu_n\}$ is a permissible sequence converging to ν, then $\{\zeta\mu_n\}$ is a permissible sequence converging to $\zeta|_{\mathfrak{N}}\nu$ for any $\zeta \in C^\infty(\mathfrak{M})$.

Proposition 1.3. *Let $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ satisfy the conditions of Definition 1.3. Then there is a $T_{\mathfrak{N},\mathcal{O}} \in \mathcal{B}'(\mathfrak{N})$ so that $\lim_n\langle T,\mu_n\rangle = \langle T_{\mathfrak{N},\mathcal{O}},\nu\rangle$ for all $\nu \in \mathcal{B}(\mathfrak{N})$.*

Proof. First note, that by iii) of Definition 1.3, if $\{\mu_n\}$, $\{\bar{\mu}_n\}$ are permissible sequences converging to ν, then $\lim_n\langle T,\mu_n\rangle = \lim_n\langle T,\bar{\mu}_n\rangle$. Hence, we can define $\langle T_{\mathfrak{N},\mathcal{O}},\nu\rangle$ for any ν so that there is a permissible sequence $\{\mu_n\}$ converging to ν, and the value of $\langle T_{\mathfrak{N},\mathcal{O}},\nu\rangle$ is independent of the choice of $\{\mu_n\}$.

Let $\{\nu_k\} \subset \mathcal{B}(\mathfrak{N})$, $\nu_k \to 0$. We will show that $\langle T_{\mathfrak{N},\mathcal{O}},\nu_k\rangle$ is defined and tends to 0 as $k \to \infty$. Let $K \subset \mathfrak{N}$ be compact so that $\operatorname{supp} \nu_k \subset K\ \forall k$. Let $x_0 \in K$; ν, V_{x_0} as in i) of Definition 1.3. Let U_{x_0} be as in Definition 1.1. Choose $W_{x_0} \subset \mathfrak{N}$ so that $W_{x_0} \subset V_{x_0}$, $\nu \neq 0$ on W_{x_0}. Let ω_n be the permissible sequence on $\mathcal{O}$ converging to ν. Clearly, if $\psi \in C_0^\infty(W_{x_0})$,

$$\psi\nu_k = \varphi_k\nu \qquad \text{where } \varphi_k \in C_0^\infty(W_{x_0})$$

and $\varphi_k \to 0$. Let $\bar{\varphi}_k$ be in $C_0^\infty(U_{x_0})$ so that $\bar{\varphi}_k|Vx_0 = \varphi_k$. Then if ω_n is permissible and converges to ν, $\bar{\varphi}_k\omega_n$ is a permissible sequence and converges to $\varphi_k\nu = \psi\nu_k$. Clearly, we can choose $\bar{\varphi}_k \in C_0^\infty(U_{x_0})$ so that $\bar{\varphi}_k \to 0$. Hence, $\langle T_{\mathfrak{N},\mathcal{O}},\psi\nu_k\rangle$ is defined $\forall_k$, and by definition:

$$\langle T_{\mathfrak{N},\mathcal{O}},\psi\nu_k\rangle = \lim_n\langle T,\bar{\varphi}_k\omega_n\rangle.$$

Now, since $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$, we have for all k, that

$$|\langle T,\bar{\varphi}_k\omega_n\rangle| \leqslant c \sum_{\substack{|\alpha|+|\beta|\leqslant m\\ |\beta|=|\beta'|}} \left\|\left(\frac{\partial}{\partial x}\right)^\alpha y^\beta\left(\frac{\partial}{\partial y}\right)^{\beta'}\bar{\varphi}_k\theta_n\right\|_1,$$

where c is independent of n and $\omega_n = \theta_n(x,y)\,dx\,dy$. Hence,

$$|\langle T_{\mathfrak{N},\mathcal{O}},\psi\nu_k\rangle| = \lim_n|\langle T,\bar{\varphi}_k\omega_n\rangle| \leqslant c\sup_n \sum_{\substack{|\alpha|+|\beta|\leqslant m\\ |\beta|=|\beta'|}} \left\|\left(\frac{\partial}{\partial x}\right)^\alpha y^\beta\left(\frac{\partial}{\partial y}\right)^{\beta'}\bar{\varphi}_k\theta_n\right\|_1.$$

But $\bar{\varphi}_k \to 0$ in $C_0^\infty(U_{x_0})$ clearly implies that

$$\lim_{k\to\infty}\left[\sup_n \sum_{\substack{|\alpha+\beta|\leqslant m\\ |\beta|=|\beta'|}} \left\|\left(\frac{\partial}{\partial x}\right)^\alpha y^\beta\left(\frac{\partial}{\partial y}\right)^{\beta'}\bar{\varphi}_k\theta_n\right\|_1\right] = 0.$$

Hence, $\lim_k\langle T_{\mathfrak{N},\mathcal{O}},\psi\nu_k\rangle = 0$.

Now if we choose $\{\psi_i\}$, $i = 1, \ldots, N$ as a partition of unity over K, we have that

$$\langle T_{\mathfrak{N},\mathcal{O}},\nu_k\rangle = \sum_{i=1}^N \langle T_{\mathfrak{N},\mathcal{O}},\psi_i\nu_k\rangle \to 0 \text{ as } k \to \infty.$$

So, $T_{\mathfrak{N},\mathcal{O}} \in \mathcal{B}'(\mathfrak{N})$ by noting that the above construction shows that $T_{\mathfrak{N},\mathcal{O}}$ is defined on all $\nu \in \mathcal{B}(\mathfrak{N})$. ■

Definition 1.4. *Let $\mathfrak{M}$, $\mathfrak{N}$, $\mathcal{O}$ be as above. Let $\{T_n\} \subset \mathcal{R}_{\mathfrak{N},\mathcal{O}}$. We will say that $T_n \to 0$ in $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$ if given $x_0 \in \mathfrak{N}$, there is a coordinate patch $U_{x_0} = \{(x,y)\}$ in $\mathfrak{M}$ where $U_{x_0} \cap \mathfrak{N} = \{(x,y)|y = 0\}$, an integer m, and a sequence $c_n \to 0$ so that*

$$|\langle T_n,\mu\rangle| \leqslant c_n \sum_{\substack{|\alpha|+|\beta|\leqslant m\\ |\beta|=|\beta'|}} \left\|\left(\frac{\partial}{\partial x}\right)^\alpha y^\beta\left(\frac{\partial}{\partial y}\right)^{\beta'}\varphi\right\|_1 \tag{1.7}$$

for any $\mu \in \mathcal{B}(U_{x_0} \cap \mathcal{O})$, $\mu = \varphi(x,y)\,dx\,dy$ *on* U_{x_0}. *If* $\{T_n, T\} \subset \mathcal{R}_{\mathfrak{N},\mathcal{O}}$, *we say that* $T_n \to T$ *in* $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$ *if* $T_n - T \to 0$.

If $T_n \to 0$ in $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$ and $\zeta \in C^\infty(\mathfrak{M})$, clearly $\zeta T_n \to 0$ in $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$.

PROPOSITION 1.4. *Referring to Definition 1.4, if* V_{x_0} *is any other coordinate patch containing* x_0, *say* $V_{x_0} = \{(\bar{x},\bar{y})\}$ *where* $V_{x_0} \cap \mathfrak{N} = \{(\bar{x},\bar{y}) \mid \bar{y} = 0\}$, *then there is a neighborhood* $W_{x_0} \subset V_{x_0}$, *an integer* $\bar{m}$, *and a sequence* $\bar{c}_n \to 0$ *so that*

$$|\langle T_n, \mu\rangle| \leqslant \bar{c}_n \sum_{\substack{|\alpha|+|\beta|\leqslant \bar{m}\\ |\beta|=|\beta'|}} \left\| \left(\frac{\partial}{\partial \bar{x}}\right)^{\alpha} (\bar{y})^{\beta} \left(\frac{\partial}{\partial \bar{y}}\right)^{\beta'} \varphi \right\|_1$$

for any $\mu \in \mathcal{B}(W_{x_0} \cap \mathcal{O})$, $\mu = \varphi(\bar{x},\bar{y})\,d\bar{x}\,d\bar{y}$ *on* W_{x_0}.

PROOF. Similar to the proof of Proposition 1.1 and is omitted. ■

PROPOSITION 1.5. *Let* $T_n \to T$ *in* $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$ *and assume* $T_{\mathfrak{N},\mathcal{O}}$ *and* $(T_n)_{\mathfrak{N},\mathcal{O}}$ *exist. Then* $(T_n)_{\mathfrak{N},\mathcal{O}} \to T_{\mathfrak{N},\mathcal{O}}$ *in* $\mathcal{B}'(\mathfrak{N})$.

PROOF. Let $x_0 \in \mathfrak{N}$, U_{x_0} as in Definition 1.4. Let $U_{x_0} \cap \mathfrak{N} = V_{x_0}$. Let $\nu \in \mathcal{B}(V_{x_0})$, $\{\mu_m\} \subset \mathcal{B}(U_{x_0} \cap \mathcal{O})$ a permissible sequence converging to ν. Note that $\{\mu_m\}$ exists as in the proof of Proposition 1.3. Then

$$\langle (T_n)_{\mathfrak{N},\mathcal{O}} - T_{\mathfrak{N},\mathcal{O}}, \nu\rangle = \lim_m \langle T_n - T, \mu_m\rangle.$$

Now by (1.7), we have since $\{\mu_m\}$ is permissible, $|\langle T_n - T, \mu_m\rangle| \leqslant c \cdot c_n$ where c is independent of μ_m, and $c_n \to 0$ as $n \to \infty$. Hence, $\langle (T_n)_{\mathfrak{N},\mathcal{O}} - T_{\mathfrak{N},\mathcal{O}}, \nu\rangle \to 0$ for all $\nu \in \mathcal{B}(V_{x_0})$.

Now if $\nu \in \mathcal{B}(\mathfrak{N})$, we can write $\nu = \sum_i \psi_i \nu$ where the $\{\psi_i\}$ is a finite partition of unity over supp ν that is subordinate to V_{x_i} as above. Then clearly, as $n \to \infty$,

$$\langle (T_n)_{\mathfrak{N},\mathcal{O}} - T_{\mathfrak{N},\mathcal{O}}, \nu\rangle = \sum_{i=1}^{M} \langle (T_n)_{\mathfrak{N},\mathcal{O}} - T_{\mathfrak{N},\mathcal{O}}, \psi_i \nu\rangle \to 0. \quad ■$$

The following generalization of Proposition 1.5 will be useful later.

PROPOSITION 1.6. *Let* $T_n \to T$ *in* $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$. *Assume that* $(T_n)_{\mathfrak{N},\mathcal{O}}$ *exists for all n. Then* $T_{\mathfrak{N},\mathcal{O}}$ *exists.*

PROOF. Let $x_0 \in \mathfrak{N}$; U_{x_0}, V_{x_0} as in Proposition 1.5. Let $\nu \in \mathcal{B}(V_{x_0})$, $\{\mu_j\} \subset \mathcal{B}(U_{x_0} \cap \mathcal{O})$ a permissible sequence converging to ν. Then

$$|\langle T, \mu_i - \mu_j\rangle| \leqslant |\langle T_n, \mu_i - \mu_j\rangle| + |\langle T - T_n, \mu_i - \mu_j\rangle|,$$

$$|\langle T_n, \mu_i - \mu_j\rangle| \leqslant |\langle T, \mu_i - \mu_j\rangle| + |\langle T - T_n, \mu_i - \mu_j\rangle|.$$

Hence, since $|\langle T - T_n, \mu_i - \mu_j\rangle| \leqslant 2c_n$ for all i, j; we have that for each n,

$$|\langle T_n, \mu_i - \mu_j\rangle| - 2c_n \leqslant |\langle T, \mu_i - \mu_j\rangle| \leqslant |\langle T_n, \mu_i - \mu_j\rangle| + 2c_n. \tag{1.8}$$

By assumption, $\lim_{ij} \langle T_n, \mu_i - \mu_j\rangle = 0$ for all n. So using this in (1.8), we obtain that

$$\lim_{ij} |\langle T, \mu_i - \mu_j\rangle| \leqslant 2c_n \qquad \text{for all } n.$$

Hence, $\lim_i \langle T, \mu_i\rangle$ exists.

Now if $\nu = 0$, we can proceed as above to obtain

$$|\langle T_n, \mu_i\rangle| - c_n \leqslant |\langle T, \mu_i\rangle| \leqslant |\langle T_n, \mu_i\rangle| + c_n. \tag{1.9}$$

Hence, $\lim_i \langle T_n, \mu_i\rangle = 0$ implies that $\lim_i \langle T, \mu_i\rangle = 0$. So, $T_{\mathfrak{N},\mathcal{O}}$ exists for $\nu \in \mathcal{B}(V_{x_0})$. In general, if $\nu \in \mathcal{B}(\mathfrak{N})$, we proceed as in the proof of Proposition 1.5, using a partition of unity argument.

It will be convenient later to have each $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ supported in a coordinate patch. To this end, let $\{U_i\}$ be a locally finite covering of $\mathfrak{N}$, U_i open in $\mathfrak{M}$. Let $\{\psi_i\}$ be a partition of unity subordinate to this cover.

PROPOSITION 1.7.

a) $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ *if and only if* $\psi_i T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ *for all i.*

b) $T_n \to 0$ *in* $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$ *if and only if* $\psi_i T_n \to 0$ *in* $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$ *for all i.*

c) $T_{\mathfrak{N},\mathcal{O}}$ *exists if and only if* $(\psi_i T)_{\mathfrak{N},\mathcal{O}}$ *exists for all i. In this case, we have*

$$T_{\mathfrak{N},\mathcal{O}} = \sum_i (\psi_i T)_{\mathfrak{N},\mathcal{O}}. \tag{1.10}$$

PROOF. Note that a) and b) are clear by Definition 1.1 and Definition 1.4. As for c), we need only check that (1.10) is independent of $\{U_i\}$ and $\{\psi_i\}$. For this, let $\{V_j\}$ be another locally finite covering of $\mathfrak{N}$; $\{\varphi_j\}$ a partition of unity subordinate to $\{V_j\}$. Let $\{\mu_n\}$ be a permissible sequence converging to ν. Then

$$\langle \varphi_j T, \mu_n\rangle = \sum_i \langle \psi_i T, \varphi_j \mu_n\rangle.$$

Hence, $\lim_n \langle \varphi_j T, \mu_n\rangle$ exists for all j, and

$$\lim_n \langle \varphi_j T, \mu_n\rangle = \sum_i \langle (\varphi_i T)_{\mathfrak{N},\mathcal{O}}, \varphi_j|_{\mathfrak{N}}\nu\rangle. \tag{1.11}$$

From (1.11), it is clear that $\lim_n \langle \varphi_j T, \mu_n\rangle = 0$ if $\nu = 0$. Hence, $(\varphi_j T)_{\mathfrak{N},\mathcal{O}}$ exists for all j. Further, from (1.11), we see that

$$\sum_j \langle (\varphi_j T)_{\mathfrak{N},\mathcal{O}}, \nu\rangle = \sum_i \langle (\psi_i T)_{\mathfrak{N},\mathcal{O}}, \nu\rangle$$

since $\sum_j \varphi_j|_{\mathfrak{N}}\nu = \nu$. Hence, (1.10) is independent of $\{U_i\}$, $\{\psi_i\}$ and the proposition is proved. ■

Thus, without loss of generality, we will study in the following sections, the existence of boundary values or restrictions of generalized functions supported in coordinate patches.

Let $T \in \mathcal{B}'(\mathcal{O})$, μ a nowhere vanishing density on $\mathfrak{M}$. Then the product $T\mu$ is a distribution $D \in \mathcal{D}'(\mathcal{O})$ defined by:

$$\langle D, \varphi\rangle = \langle T, \varphi\mu\rangle \qquad \text{for all } \varphi \in C_0^\infty(\mathcal{O}).$$

It is easy to show that if $\psi \in C^\infty(\mathfrak{M})$, then

$$(\psi T)\mu = \psi(T\mu) = (\psi\mu)T.$$

Also, since μ is nonvanishing, we clearly have that the mapping $T \to T\mu$ is bijective and bicontinuous from $\mathcal{B}'(\mathcal{O})$ to $\mathcal{D}'(\mathcal{O})$. Hence, if $D \in \mathcal{D}'(\mathcal{O})$, $D/\mu \in \mathcal{B}'(\mathcal{O})$ is well defined. That is, if $\mu_1, \in \mathcal{B}(\mathcal{O})$, we have that $\mu_1 = \varphi\mu$ where $\varphi \in C_0^\infty(\mathcal{O})$. Then $D/\mu = T$ is defined by,

$$\langle T, \mu_1 \rangle = \langle D, \varphi \rangle.$$

Hence, given $D \in \mathcal{D}'(\mathcal{O})$, and μ, we can say $D \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ if $D/\mu \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$. This makes sense, since if μ' is another nowhere vanishing density on $\mathfrak{M}$, then $\mu = f\mu'$ where $f \in C^\infty(\mathfrak{M})$, $f(x) \neq 0$ for all $x \in \mathfrak{M}$. Hence $D/\mu' = f(D/\mu)$ and so $D/\mu' \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ if and only if $D/\mu \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$.

Also, D/μ' has a restriction to, or boundary values on, $\mathfrak{N}$ if and only if D/μ has. In this case, we have that $(D/\mu')_{\mathfrak{N},\mathcal{O}} = f|_{\mathfrak{N}}(D/\mu)_{\mathfrak{N},\mathcal{O}}$.

Now if ν is a nowhere vanishing density on $\mathfrak{N}$, and $(D/\mu)_{\mathfrak{N},\mathcal{O}}$ exists, we can define $D_{\mathfrak{N},\mathcal{O}} = D_{\mathfrak{N},\mathcal{O}}(\mu, \nu)$ by

$$D_{\mathfrak{N},\mathcal{O}}(\mu, \nu) = [(D/\mu)_{\mathfrak{N},\mathcal{O}}]\nu. \tag{1.12}$$

As noted above, the existence of $D_{\mathfrak{N},\mathcal{O}}$ is independent of μ and ν, but the value of $D_{\mathfrak{N},\mathcal{O}}$ is not. In fact, if μ' and ν' are other choices, $\mu = f\mu'$, $\nu' = g\nu$ where f and g are non-vanishing smooth functions on $\mathfrak{M}$ and $\mathfrak{N}$ respectively, then

$$D_{\mathfrak{N},\mathcal{O}}(\mu', \nu') = [f|_{\mathfrak{N}}g] D_{\mathfrak{N},\mathcal{O}}(\mu, \nu). \tag{1.13}$$

For most purposes, it is more natural not to choose μ and ν independently of each other, but as in $\mathbf{R}^n$, to be related by a Riemannian structure. To this end, let $\langle\, , \rangle_x$ be a Riemannian metric on $\mathfrak{M}$. That is, for each x, $\langle\, , \rangle_x$ is a positive definite, symmetric, bilinear form on $T_x(\mathfrak{M})$ so that if α, β: $\mathfrak{M} \to T^*(\mathfrak{M})$ are smooth sections, then $\langle \alpha(x), \beta(x) \rangle_x$ is a smooth function on $\mathfrak{M}$. This Riemannian structure induces, in a natural way, a nowhere vanishing density μ on $\mathfrak{M}$ defined as follows. If U is a coordinate neighborhood of $\mathfrak{M}$ with coordinate functions $(x_1, \ldots, x_n)$, we define μ on $T(U)$ by

$$\mu\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) = \left|\det \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_x \right|^{\frac{1}{2}}. \tag{1.14}$$

Clearly, μ is nonvanishing on $T(U)$ and hence is smooth. If V is another coordinate neighborhood with coordinate functions $(y_1, \ldots, y_n)$, $V \cap U \neq \varnothing$, then a calculation shows that on $V \cap U$,

$$\mu\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) = \left|\det \frac{\partial y}{\partial x}\right| \mu\left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right)$$

where by $\partial y/\partial x$ we mean the Jacobian matrix. Hence, μ is indeed a density on $\mathfrak{M}$. Now since $\mathfrak{N}$ is an imbedded submanifold of $\mathfrak{M}$, we can identify $T_x(\mathfrak{N})$ with a subspace of $T_x(\mathfrak{M})$ for all $x \in \mathfrak{N}$. Hence, the Riemannian metric on $\mathfrak{M}$ gives rise to a Riemannian metric on $\mathfrak{N}$, and as above, this gives rise to a nowhere vanishing density ν on $\mathfrak{N}$.

Hence, to work with distributions $D \in \mathcal{D}'(\mathcal{O})$, we could assume that $\mathfrak{M}$ is given a Riemannian structure and define $D \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ or $D_{\mathfrak{N},\mathcal{O}}$ as above with respect to the 'natural' densities μ and ν. Note that if $f \in C^\infty(\mathfrak{M})$, dit f nequ 0 ditend on $\mathfrak{M}$ and $\mu' = f\mu$, $\nu' = f|_{\mathfrak{N}}\nu$, then by (1.13)

$$D_{\mathfrak{N},\mathcal{O}}(\mu, \nu) = D_{\mathfrak{N},\mathcal{O}}(\mu', \nu'). \tag{1.15}$$

This observation is a convenience locally, since given any Riemannian metric on $\mathfrak{M}$, $U = \{(x_1, \ldots, x_n)\}$ a coordinate neighborhood, we can choose f above so that on U, $\mu' = f\mu = 1\,dx$. Then, if $\mu_1 \in \mathcal{B}(\mathcal{O})$, $\mu_1 = \varphi(x)\,dx$ on U, we have that $\langle D/\mu', \mu_1 \rangle = \langle D, \varphi \rangle$ as in $\mathbf{R}^n$. Hence, locally we can suppress the identification $D \leftrightarrow T$, and work as in $\mathbf{R}^n$.

Although we will work exclusively with generalized functions, we note that the above construction gives a natural way of applying the results of this paper to distributions on a Riemannian manifold.

CHAPTER II
CODIM $\mathfrak{N} = 1$

1. Boundary Values of Generalized Functions

In this section, we make the following assumptions:

(2.1.0)
i) $\mathfrak{N} \subset \partial\mathcal{O}$
ii) For each $x_0 \in \mathfrak{N}$, there is some coordinate neighborhood U of x_0 in $\mathfrak{M}$ such that $U \cap \mathfrak{N} = \{(x,t) | t = 0\} = V$ and $V \times (0, \delta) \subset \mathcal{O}$ for some $\delta > 0$.

By Proposition (1.7), we can assume without loss of generality, that supp $T \subset U$, a coordinate patch. Combining this with (2.1.0), we use the following model:

(2.1.0)′
i) $\mathfrak{M} = \mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$
ii) $\mathfrak{N} \subset \mathbf{R}^n$ is open
iii) $\mathfrak{N} \times (0, \infty) \subset \mathcal{O}$
iv) $\mathfrak{N} \subset \partial\mathcal{O}$
v) $T \in \mathfrak{B}'(\mathcal{O})$, supp T is relatively compact.

In this setting, there always exists permissible sequences for any $\nu \in \mathfrak{B}(\mathfrak{N})$. For example, if $\eta \in C_0^\infty(\mathbf{R}^+)$, $\int \eta \, dt = 1$, then $\omega_n = \nu \otimes n\eta(nt)\, dt$ converges to ν, clearly, and for all n, supp $\omega_n \subset \mathcal{O}$. Also, a simple calculation shows that ω_n satisfies iii) of Definition 1.2. This type of sequence is easier to work with in most cases than the general one. Our interest here is to justify the exclusive use of such sequences in applications. That is, we are interested in the following question:

Let $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$. If $\lim_n \langle T, \nu \otimes n\eta(nt)\, dt \rangle$ exists for all $\nu \in \mathfrak{B}(\mathfrak{N})$, some $\eta \in C_0^\infty(\mathbf{R}^+)$, $\int \eta \, dt = 1$, does T have boundary values on $\mathfrak{N}$ in the sense of Definition 1.3?

The following example shows that without an auxiliary condition on η, the answer is in general in the negative.

Example 1. Let $\eta_0 \in C_0^\infty(\mathbf{R}^+)$, $\int \eta_0 \, dt = 1$. Assume that $\mathfrak{F}(\eta_0(e^t))(z_0) = 0$ for $\operatorname{Im} z_0 = -1$, $\operatorname{Re} z_0 \neq 0$, where $\mathfrak{F}(\eta_0(e^t))(z_0)$ is defined by:

$$\mathfrak{F}(\eta_0(e^t))(z_0) = \int \eta_0(e^t) e^{itz_0}\, dt.$$

Then, $\int \eta_0(e^{t-s}) e^{itz_0}\, dt = 0$ for all $s \in \mathbf{R}$. Letting $u = e^t$, we have that

$$\int \eta_0\left(\frac{u}{e^s}\right) e^{iz_0 \log u} \frac{du}{u} = 0 \qquad \text{for all } s.$$

Hence,

$$\lim_{\epsilon \to 0} \int \eta_0\left(\frac{u}{\epsilon}\right) \frac{1}{\epsilon} g(u)\, du = 0$$

where $g(u) = (e^{iz_0 \log u}/u)$ is defined for $u > 0$. Let $T = g(u) \otimes \psi(x)$ where $\psi \in C_0^\infty(\mathbf{R}^n)$. Then since $\operatorname{Im} z_0 = -1$, we have that $|(e^{iz_0 \log u}/u)| = 1$ and hence, if $\mu = \varphi(u,x)\,du\,dx \in \mathcal{B}(\mathcal{O})$, where $\mathcal{O}$ is equal to $\mathbf{R}^+ \times \mathbf{R}^n$, then

$$|\langle T, \mu\rangle| \leqslant c\|\varphi\|_1.$$

That is, $T \in \mathcal{R}_{\mathbf{R}^n, \mathbf{R}^+ \times \mathbf{R}^n}$. Also, for all $\nu \in \mathcal{B}(\mathbf{R}^n)$,

$$\lim_{\epsilon \to 0} \langle T, \eta_0\left(\frac{t}{\epsilon}\right)\frac{1}{\epsilon}dt \otimes \nu\rangle = 0.$$

However, if $\eta \in C_0^\infty(\mathbf{R}^+)$, say $\operatorname{supp} \eta \subset [1,2]$, then:

$$\langle T, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\,dt \otimes \nu\rangle = e^{ia \log \epsilon}\int_1^2 \eta(t) e^{ia \log t''}\,dt \int \psi\nu,$$

where $a = \operatorname{Re} z_0 \neq 0$. Clearly then, the limit as $\epsilon \to 0$ does not exist in general.

Hence, we must at least demand that $\mathcal{F}(\eta(e^t))(z) \neq 0$ for $\operatorname{Im} z = -1$, $\operatorname{Re} z \neq 0$. That is, $\mathcal{F}(\eta(e^t)e^t)(s) \neq 0$ for $s \in \mathbf{R}$. In this case, we have the following.

THEOREM 2.1.1. *Let* $\eta_0 \in C_0^\infty(\mathbf{R}^+)$, $\int \eta_0\,dt = 1$, $\mathcal{F}(\eta_0(e^t)e^t)(s) \neq 0$ *for* $s \in \mathbf{R}$. *Let* $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ *where* $\mathfrak{N}$, $\mathcal{O}$, T *satisfy (2.1.0)′. Assume that*

$$\lim_{n \to \infty} \langle T, \nu \otimes n\eta_0(nt)\,dt\rangle \tag{2.1.1}$$

exists for all $\nu \in \mathcal{B}(\mathfrak{N})$. *Then* T *has boundary values on* $\mathfrak{N}$ *in the sense of Definition 1.3. That is,*

i) $\lim\langle T, \mu_n\rangle$ *exists for all permissible sequences* $\{\mu_n\}$,

ii) $\lim\langle T, \mu_n\rangle = 0$ *if* $\nu = 0$.

The proof of Theorem 2.1.1 will be obtained from a series of results.

LEMMA 2.1.2. *Let* $g, f \in \mathcal{S}(\mathbf{R})$, $\hat{f}(t) \neq 0$ *for* $t \in \mathbf{R}$. *Then given* $M \in \mathbf{Z}^+$, $\epsilon > 0$, *there is* $\xi_1, \ldots, \xi_N$; $s_1, \ldots, s_N$ *so that*

$$g(t) = \sum_{i=1}^N \xi_i f(t - s_i) + r(t)$$

where $\|(d/dt)^j r\|_1 < \epsilon$ *for* $j = 0, \ldots, M$. *Further,*

$$|\hat{g}(0) - \hat{f}(0)\sum_{i=1}^N \xi_i| < \epsilon.$$

PROOF. First note, that there is an $h \in \mathcal{S}(\mathbf{R})$ so that $\hat{h} \in C_0^\infty(\mathbf{R})$ and $\|(d/dt)^j(h-g)\|_1 < \epsilon/2$, $j \leqslant M$. To see this, let $\zeta \in C_0^\infty(\mathbf{R})$, $\zeta \equiv 1$ near 0. Clearly, as $\delta \to 0$ we have for all j that

$$\|[\zeta(\delta t)(-it)^j\hat{g}(t) - (-it)^j\hat{g}(t)]^\vee\|_1 \to 0$$

where $\check{k}$ is the inverse Fourier transform of k. Choose δ_0 so that

$$2\pi\|[\zeta(\delta_0 t)(-it)^j\hat{g}(t) - (-it)^j\hat{g}(t)]^\vee\|_1 < \frac{\epsilon}{2}$$

for $j = 0, \ldots, M$. Let $\hat{h}(t) = \zeta(\delta_0 t)\hat{g}(t)$. Clearly, $\hat{h}(t) \in C_0^\infty(\mathbf{R})$. Also, by the above inequality, we have that

$$\left\|\left(\frac{d}{dt}\right)^j(h-g)\right\|_1 < \frac{\epsilon}{2} \qquad \text{for } j \leqslant M.$$

Now $\hat{f}(t) \neq 0$ for all t implies that $\hat{h}(t)/\hat{f}(t) \in C_0^\infty(\mathbf{R})$. Hence, $\hat{h}/\hat{f} = \hat{k}$ with $k \in \mathbb{S}(\mathbf{R})$. So, $h^j(t) = f^j * k(t)$ where $h^j = (d/dt)^j h$ and $f^j * k(t) = \int f^j(s)k(t-s)\,ds$.

To complete the proof, we now show that if $f, k \in \mathbb{S}(\mathbf{R})$, then the Riemann sums of $f * k$ converge to $f * k$ in $L^1(\mathbf{R})$. To see this, first assume that $f, k \in C_0^\infty(\mathbf{R})$. Then by partitioning $\mathbf{R}$ into sufficiently small intervals, it is clear that for each x, and for all ϵ, there is $\{y_i\}_{i=1}^m$ so that

$$|f(x-y)k(y)\,dy - \sum_{j=1}^m f(x-y_j)k(y_j)m_j| < \epsilon$$

where m_j = measure of the jth interval. Now since $f * k \in C_0^\infty(\mathbf{R})$, we can have this estimate uniformly in x. Hence for all ϵ, there is $\{y_i\}_{i=1}^m$ so that

$$\|f * k(x) - \sum_1^m \xi_j f(x-y_j)\|_1 < \epsilon.$$

Note that $\xi_j = k(y_j)m_j$ and $\sum_{j=1}^m |\xi_j| \leqslant \|k\|_1$ for all m. Now, if $f \in \mathbb{S}(\mathbf{R})$, $k \in C_0^\infty(\mathbf{R})$, we choose $\{f_n\} \subset C_0^\infty(\mathbf{R})$ so that $\|f - f_n\|_1 < 1/n$. Then, by the above argument, we can choose $\{\xi_i\}_1^m$, $\{y_i\}_1^m$ so that

$$\|f_n * k(x) - \sum_{j=1}^m \xi_j f_n(x-y_j)\|_1 < \frac{1}{n}.$$

Then

$$\|f * k(x) - \sum_{j=1}^m \xi_j f(x-y_j)\|_1 \leqslant \|f * k(x) - f_n * k(x)\|_1 + \|f_n * k(x) - \sum_{j=1}^m \xi_j f_n(x-y_j)\|_1$$
$$+ \|\sum_{j=1}^m \xi_j[f_n(x-y_j) - f(x-y_j)]\|_1$$
$$< \|f - f_n\|_1 \|k\|_1 + \frac{1}{n} + \sum_{j=1}^m |\xi_j|\,\|f - f_n\|_1 < \frac{1}{n}(1 + 2\|k\|_1).$$

Finally, in the general case, we choose $k_n \in C_0^\infty(\mathbf{R})$, so that $\|k - k_n\|_1 < 1/n$. We can then choose $\{\xi_i\}$, $\{y_i\}$ so that

$$\|f * k_n(x) - \sum_{i=1}^m \xi_i f(x-y_i)\|_1 < \frac{1}{n}$$

by the above argument. Then,

$$\|f * k(x) - \sum_{i=1}^m \xi_i f(x-y_i)\|_1 \leqslant \|f * k(x) - f * k_n(x)\|_1 + \|f * k_n(x) - \sum_{i=1}^m \xi_i f(x-y_i)\|_1$$
$$\leqslant \frac{1}{n}(1 + \|k\|_1).$$

Hence, we can choose $\{\xi_i\}$, $\{s_i\}$ so that

$$\|h^j(t) - \sum_{i=1}^m \xi_i f^j(t-s_i)\|_1 < \frac{\epsilon}{2}, \qquad j \leqslant M,$$

since $h^j = k * f^j$. Combining this with the estimate $\|h^j - g^j\|_1 < \epsilon/2$ obtained above completes the proof of the first part of the lemma.

By construction, we have that

$$|\int g - \sum_{i=1}^{m} \xi_i \int f| = |\int g - \sum_{i=1}^{m} \int \xi_i f(x - y_i)\,dx| \leqslant \|g - \sum_{i=1}^{m} \xi_i f(x - y_i)\|_1 < \epsilon$$

Noting that $\int g = \hat{g}(0)$, $\int f = \hat{f}(0)$, completes the proof. ■

PROPOSITION 2.1.3. *Let $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$, η_0 satisfy the conditions of Theorem 2.1.1. Then for all $\eta \in C_0^\infty(\mathbf{R}^+)$, $\int \eta \neq 0$*

$$\lim_{n\to\infty} \langle T, \nu \otimes n\eta(nt)\,dt\rangle \qquad \textit{exists for all } \nu \in \mathfrak{B}(\mathfrak{N}). \tag{2.1.2}$$

Further, if $\int \eta = 1$, the value of (2.1.2) is the same as the value of (2.1.1) for all ν.

PROOF. Let $\eta \in C_0^\infty(\mathbf{R}^+)$. Clearly, $\eta_0(e^t)e^t$, $\eta(e^t)e^t$ are in $\mathcal{S}(\mathbf{R})$ and $[\eta_0(e^t)e^t]\hat{}\,(s) \neq 0$ for all $s \in \mathbf{R}$. So for $\delta > 0$, $M \in \mathbf{Z}^+$, we have by Lemma 2.1.2 that there exists $\xi_1, \ldots, \xi_M$; $s_1, \ldots, s_M$ such that

$$\eta(e^u)e^u = \sum_{i=1}^{M} \xi_i \eta_0(e^{u-s_i})e^{u-s_i} + r(u)$$

where $\|(d/du)^j r\|_1 < \delta$ for $j = 0, \ldots, M$. Letting $e^u = t/\epsilon$, we obtain

$$\frac{1}{\epsilon}\eta\Big(\frac{t}{\epsilon}\Big) = \sum_{i=1}^{M} \xi_i \eta_0\Big(\frac{t}{\epsilon e^{s_i}}\Big)\frac{1}{\epsilon e^{s_i}} + \frac{1}{\epsilon}\Big[\frac{r(\log(t/\epsilon))}{t/\epsilon}\Big].$$

Now by assumption, $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$. Hence, if $\mu = \varphi(x,t)\,dx\,dt$ has support in $\mathcal{O}$, we have

$$|\langle T, \mu\rangle| \leqslant c \sum_{|\alpha|+j\leqslant N} \Big\|\Big(\frac{\partial}{\partial x}\Big)^\alpha t^j \Big(\frac{\partial}{\partial t}\Big)^j \varphi\Big\|_1. \tag{2.1.3}$$

Choose M above so that $N \leqslant M$. Then using (2.1.3), if $\nu = \varphi(x)\,dx \in \mathfrak{B}(\mathfrak{N})$,

$$\Big|\Big\langle T, \Big(\eta\Big(\frac{t}{\epsilon}\Big)\frac{1}{\epsilon} - \sum_{i=1}^{M} \xi_i \eta_0\Big(\frac{t}{\epsilon e^{s_i}}\Big)\frac{1}{\epsilon e^{s_i}}\Big)dt \otimes \nu\Big\rangle\Big|$$

$$\leqslant c \sum_{|\alpha|+j\leqslant M} \Big\|\Big(\frac{\partial}{\partial x}\Big)^\alpha \varphi\Big\|_1 \Big\|t^j\Big(\frac{d}{dt}\Big)^j\Big[\frac{r(\log(t/\epsilon))}{t/\epsilon}\Big]\frac{1}{\epsilon}\Big\|_1 \leqslant C'\delta,$$

since by substituting $u = \log(t/\epsilon)$,

$$\Big\|t^j\Big(\frac{d}{dt}\Big)^j\Big[\frac{r(\log(t/\epsilon))}{t/\epsilon}\Big]\frac{1}{\epsilon}\Big\|_1 \leqslant c\sum_{l=0}^{j}\Big\|\Big(\Big(\frac{d}{du}\Big)^l\Big[\frac{r(u)}{e^u}\Big]\Big)e^u\Big\|_1 \leqslant c\sum_{l=0}^{j}\Big\|\Big(\frac{d}{du}\Big)^l r\Big\|_1 \leqslant c\delta.$$

Note that we have easily by induction,

$$t^j\Big(\frac{d}{dt}\Big)^j = \sum_{l=0}^{j} c_{jl}\Big(\frac{d}{du}\Big)^l.$$

So,

$$\Big[\overline{\lim_{\epsilon}} - \underline{\lim_{\epsilon}}\Big]\Big|\Big\langle T, \Big(\eta\Big(\frac{t}{\epsilon}\Big)\frac{1}{\epsilon} - \sum_{i=1}^{M} \xi_i \eta_0\Big(\frac{t}{\epsilon e^{s_i}}\Big)\frac{1}{\epsilon e^{s_i}}\Big)dt \otimes \nu\Big\rangle\Big|$$

$$= \Big[\overline{\lim_{\epsilon}} - \underline{\lim_{\epsilon}}\Big]\Big|\Big\langle T, \Big(\eta\Big(\frac{t}{\epsilon}\Big)\frac{1}{\epsilon} - \Big(\sum_{i=1}^{M} \xi_i\Big)\eta_0\Big(\frac{t}{\epsilon}\Big)\frac{1}{\epsilon}\Big)dt \otimes \nu\Big\rangle\Big| \leqslant c\delta.$$

If $\int \eta = 1$, then

$$\left|\int \eta(e^u)e^u - \left(\sum_{i=1}^{M} \xi_i\right)\int \eta_0(e^u)e^u\right| = \left|1 - \sum_{i=1}^{M} \xi_i\right| < \delta$$

by Lemma 2.1.2. Also, $|\langle T, \eta_0(t/\epsilon)(1/\epsilon)\,dt \otimes \nu\rangle| \leqslant c\ \forall\epsilon$. Hence,

$$\left[\overline{\lim_{\epsilon}} - \underline{\lim_{\epsilon}}\right]\left|\left\langle T, (\eta(\frac{t}{\epsilon})\frac{1}{\epsilon} - \eta_0(\frac{t}{\epsilon})\frac{1}{\epsilon})\,dt \otimes \nu\right\rangle\right|$$

$$\leqslant \left[\overline{\lim_{\epsilon}} - \underline{\lim_{\epsilon}}\right]\left|\left\langle T, \left(\eta(\frac{t}{\epsilon})\frac{1}{\epsilon} - \left(\sum_{i=1}^{M} \xi_i\right)\eta_0(\frac{t}{\epsilon})\frac{1}{\epsilon}\right)dt \otimes \nu\right\rangle\right|$$

$$+\left[\overline{\lim_{\epsilon}} - \underline{\lim_{\epsilon}}\right]\left|\left\langle T, \left(1 - \sum_{i=1}^{M} \xi_i\right)\eta_0(\frac{t}{\epsilon})\frac{1}{\epsilon}\,dt \otimes \nu\right\rangle\right| < c\delta.$$

Hence the limit in (2.1.2) exists and is equal to the limit in (2.1.1) for all ν.

If $\int \eta = c \neq 0$, then by considering $\bar{\eta} = \eta/c$, we can see from the above argument that the limit in (2.1.2) exists and equals c times the limit in (2.1.1). ■

In order to generalize the result of Proposition 2.1.3 to more general permissible sequences, we first construct a sequence of smooth functions converging to T as in (1.7). Note, however, that if $\varphi \in C_0^\infty(\mathbf{R}^{n+1})$, $\int \varphi = 1$, we need not have $\varphi_\epsilon * T \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathfrak{O}}$ where

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^{n+1}}\varphi\left(\frac{x}{\epsilon}, \frac{t}{\epsilon}\right).$$

For example, in $\mathbf{R}^{n+1}$, identifying distributions and generalized functions, the Dirac delta $\delta \in \mathfrak{R}_{\mathfrak{N},\mathfrak{O}}$ and $(\delta)_{\mathfrak{N},\mathfrak{O}} = 0$, where $\mathfrak{N} = \mathbf{R}^n$ and $\mathfrak{O} = \mathbf{R}^n \times (0, \infty)$. However, we cannot have that $\varphi_\epsilon * \delta = \varphi_\epsilon$ converges to δ in $\mathfrak{R}_{\mathfrak{N},\mathfrak{O}}$ or else we would conclude that $(\varphi_\epsilon)_{\mathfrak{N},\mathfrak{O}} \to 0$ weakly by Proposition 1.5. Clearly,

$$(\varphi_\epsilon)_{\mathfrak{N},\mathfrak{O}} = \frac{1}{\epsilon^{n+1}}(\varphi(\frac{x}{\epsilon}, 0))$$

cannot converge weakly. We will modify the above convolution though, so that only the values of T in $\{(x,t)|t > 0\}$ are smoothed. This is natural, since if $S, T \in \mathfrak{B}'(\mathfrak{O})$, $\operatorname{supp}(S - T) \subset \{(x,t)|t \leqslant 0\}$, then $S - T \in \mathfrak{R}_{\mathfrak{N},\mathfrak{O}}$ and $(S - T)_{\mathfrak{N},\mathfrak{O}} = 0$.

Let $T \in \mathfrak{R}_{\mathfrak{N},\mathfrak{O}}$; $\mathfrak{N}$, $\mathfrak{O}$, T satisfying the conditions of Theorem 2.1.1. Let $\eta \in C_0^\infty(\mathbf{R})$, $\varphi \in C_0^\infty(\mathbf{R}^n)$, even functions; $0 \notin \operatorname{supp}\eta$, say $\operatorname{supp}\eta = [-2,-1] \cup [1,2]$. Assume also that $\int \eta = 1$, $\int \varphi = 1$. Define for $v > 0$:

$$T_m(y, v) = \left\langle T, \varphi(m[y - x])m^n\,dx \otimes \frac{m}{u}\eta\left(m \ln\frac{u}{v}\right)du\right\rangle. \tag{2.1.4}$$

Note that since T has compact support, $T_m(y,v)$ vanishes for large v uniformly in m.

PROPOSITION 2.1.4. *The function $T_m(y,v)$ is continuous in $v \geqslant 0$, C^∞ for $v > 0$ and $T_m \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathfrak{O}}$.*

To prove the proposition, we need the following lemma.

LEMMA 2.1.5. *Let $\eta \in C_0^\infty(\mathbf{R}^n)$, $\int \eta = 1$, η an even function. Let $f \in L^1_{k+1}(\mathbf{R}^n)$. Then*

$$\|\eta_\epsilon * f - f\|_{L^1_k} \leqslant c\epsilon\|f\|_{L^1_{k+1}}, \tag{2.1.5}$$

where c is independent of f.

PROOF. For $F(x) \in L_1^1(\mathbf{R}^n)$, we have that $F(0) = -\sum_{i=1}^n \int_0^\infty F_i(t\nu)\nu_i\,dt$, where $|\nu| = 1$ and $F_i = \partial F/\partial x_i$. Hence, letting $\omega = \int_{S^{n-1}} d\sigma$, we have that

$$F(0) = -\frac{1}{\omega}\int_{S^{n-1}} d\sigma \int_0^\infty \sum_{i=1}^n F_i(t\nu)\frac{\nu_i}{t^{n-1}}t^{n-1}\,dt = -\frac{1}{\omega}\sum_{i=1}^n \int F_i(x)\frac{x_i}{|x|^n}dx.$$

Substituting $F(y) = f(x-y)$ into this expression, we get

$$f(x) = \frac{1}{\omega}\sum_{i=1}^n \int f_i(x-y)\frac{y_i}{|y|^n}dy = \frac{1}{\omega}\int \nabla f(x-y)\cdot\frac{y}{|y|^n}dy \equiv \frac{1}{\omega}\left(\nabla f \mathbin{\dot{*}} \frac{y}{|y|^n}\right)(x).$$

Hence,

$$f - \eta_\epsilon * f = \frac{1}{\omega}\left[\left(\nabla f \mathbin{\dot{*}} \frac{x}{|x|^n}\right) - \left(\nabla f \mathbin{\dot{*}} \frac{x}{|x|^n}\right) * \eta_\epsilon\right] = \frac{1}{\omega}\nabla f \mathbin{\dot{*}} \left[\frac{x}{|x|^n} - \frac{x}{|x|^n} * \eta_\epsilon\right].$$

So

$$\|f - \eta_\epsilon * f\|_{L_k^1} \leqslant \frac{c}{\omega}\|f\|_{L_{k+1}^1}\left\|\frac{x}{|x|^n} - \frac{x}{|x|^n} * \eta_\epsilon\right\|_{L^1}. \tag{2.1.6}$$

To complete the proof, we need only show that

$$\left\|\frac{x}{|x|^n} - \frac{x}{|x|^n} * \eta_\epsilon\right\|_{L^1} \leqslant c\epsilon. \tag{2.1.7}$$

Looking at components,

$$\left\|\frac{x_i}{|x|^n} - \left(\frac{y_i}{|y|^n} * \eta_\epsilon\right)(x)\right\|_1 = \int_{|x|<2\epsilon} dx + \int_{|x|\geqslant 2\epsilon}\left|\frac{x_i}{|x|^n} - \left(\frac{y_i}{|y|^n} * \eta_\epsilon\right)(x)\right| dx = I + II.$$

Now

$$I \leqslant \int_{|x|<2\epsilon}\left|\frac{x_i}{|x|^n}\right| dx + \int_{|x|<2\epsilon}\left|\frac{y_i}{|y|^n} * \eta_\epsilon\right| dx \leqslant c\epsilon.$$

For II, we expand $(x_i - y_i)/|x-y|^n$ about $y = 0$ by Taylor's formula and obtain;

$$\frac{x_i - y_i}{|x-y|^n} = \frac{x_i}{|x|^n} + \sum_{i=1}^n g_i(x)y_i + III, \qquad \text{where } g_i(x) = \frac{\partial}{\partial y_i}\left[\frac{x_i - y_i}{|x-y|^n}\right]_{y=0}.$$

Hence, since $\int \eta_\epsilon(y)\,dy = 1$, and $\operatorname{supp}\eta \subset \{x \mid |x| < 1\}$ say, we have that

$$\begin{aligned}\frac{x_i}{|x|^n} - \left(\frac{y_i}{|y|^n} * \eta_\epsilon\right)(x) &= \int_{|y|\leqslant\epsilon}\left(\frac{x_i}{|x|^n} - \frac{x_i - y_i}{|x-y|^n}\right)\eta_\epsilon(y)\,dy \\ &= -\int_{|y|<\epsilon}\sum_{i=1}^n g_i(x)y_i\eta_\epsilon(y)\,dy - \int_{|y|\leqslant\epsilon}(III)\eta_\epsilon(y)\,dy \\ &= 0 + \int_{|y|\leqslant\epsilon}(III)\eta_\epsilon(y)\,dy\end{aligned}$$

since η is an even function. So

$$II \leqslant \int_{|x|\geqslant 2\epsilon}\int_{|y|<\epsilon}|(III)\eta_\epsilon(y)|\,dy\,dx \leqslant \int_{|x|>2\epsilon}dx\int_{|y|<\epsilon}\frac{|y|^2}{|x-\theta|^{n+1}}\eta_\epsilon(y)\,dy$$

where $0 \leqslant |\theta| \leqslant |y|$. Hence, since $|x| > 2\epsilon$, $|y| < \epsilon$ and $|x - \theta| \geqslant ||x| - |\theta||$ we have that

$$II \leqslant \epsilon^2 \int_{|x|>2\epsilon} \frac{dx}{||x| - \epsilon|^{n+1}} \leqslant c\epsilon.$$

Hence (2.1.7) is proved and so is the lemma. ■

PROOF OF PROPOSITION 2.1.4. First, $T_m \in C^\infty$ for $v > 0$ since

$$\operatorname{supp} \eta(m \ln \frac{u}{v}) \subset \{u | ve^{-\frac{2}{m}} \leqslant u \leqslant ve^{\frac{2}{m}}\}.$$

Also, $\int (m/u)\eta(m \ln(u/v))\, du = 1\ \forall v, m$. So letting $\eta_v(u)\, du = (m/u)\eta(m \ln(u/v))\, du$ in Proposition 2.1.3, we see that for all m, each y, $\lim_{v\to 0} T_m(y,v)$ exists. To show that $T_m(y,v)$ can be extended continuously to $v = 0$, we will show that

$$\left|\frac{\partial}{\partial y_i} T_m(y,v)\right| \leqslant c_m. \tag{2.1.8}$$

Then,

$$\begin{aligned} |T_m(y_1,v) - T_m(y_2,0)| &\leqslant |T_m(y_1,v) - T_m(y_2,v)| + |T_m(y_2,v) - T_m(y_2,0)| \\ &\leqslant c_m|y_1 - y_2| + |T_m(y_2,v) - T_m(y_2,0)| \to 0 \text{ as } (y_1,v) \to (y_2,0). \end{aligned}$$

Hence, T_m is continuous in $v \geqslant 0$.

For (2.1.8), note that the y_i difference quotients of $\varphi(m[y - x])\, dx$ converge in $\mathfrak{B}(\mathcal{O})$. Hence,

$$\frac{\partial}{\partial y_i} T_m(y,v) = \langle T, \frac{m}{u}\eta(m \ln \frac{u}{v})\, du \otimes \varphi_i(m[y - x])m^{n+1}\, dx\rangle \qquad \text{where } \varphi_i = \frac{\partial}{\partial y_i}\varphi.$$

Now, since $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$ and

$$\operatorname{supp}[\frac{m}{u}\eta(m \ln \frac{u}{v})\, du \otimes \varphi_i(m[y - x])m^{n+1}\, dx] \subset \mathcal{O} \qquad \text{for all } m,\ v > 0,$$

we have that

$$\begin{aligned} \left|\frac{\partial}{\partial y_i} T_m\right| &\leqslant c \sum_{|\alpha|+j \leqslant M} \left\|\left(\frac{\partial}{\partial x}\right)^\alpha \varphi_i(m[y - x])m^{n+1}\right\|_1 \left\|u^j\left(\frac{\partial}{\partial u}\right)^j \frac{m}{u}\eta\left(m \ln \frac{u}{v}\right)\right\|_1 \\ &\leqslant c \sum_{|\alpha|+j \leqslant M} \left\|\left(\frac{\partial}{\partial x}\right)^\alpha \varphi_i\right\| m^{|\alpha|+1} \left\|\left(\frac{d}{dt}\right)^j \eta\right\|_1 m^j \leqslant c_m, \end{aligned}$$

proving (2.1.8). To complete the proof of Proposition 2.1.4, we must show that there exists an M, $c_m \to 0$ so that

$$|\langle T - T_m, \mu\rangle| \leqslant c_m \sum_{|\alpha|+j \leqslant M} \left\|\left(\frac{\partial}{\partial y}\right)^\alpha v^j \left(\frac{\partial}{\partial v}\right)^j \varphi\right\|_1 \qquad \text{for all } \mu = \psi(y,v)\, dy\, dv \in \mathfrak{B}(\mathcal{O}). \tag{2.1.9}$$

Before continuing, observe that if $T \in \mathfrak{B}'(\mathcal{O})$, $\varphi(x,y) \in C_0^\infty(\mathcal{O} \times \mathbf{R}^{n+1})$, then

$$\int \langle T, \varphi(x,y)\, dx\rangle\, dy = \langle T, [\int \varphi(x,y)\, dy]\, dx\rangle,$$

since the Riemann sums of $[\int \varphi(x,y)\, dy]\, dx$ converge in $\mathfrak{B}(\mathcal{O})$. Let $\mu = \psi(y,v)\, dy\, dv$ be given. Then

$$\langle T_m(y,v),\mu\rangle = \int T_m(y,v)\psi(y,v)\,dy\,dv = \int \langle T, \frac{m}{u}\eta(m\ln\frac{u}{v})\varphi(m[y-x])m^n\,dx\,du\rangle\psi\,dy\,dv$$

$$= \langle T, (\int \frac{m}{u}\eta(m\ln\frac{u}{v})\varphi(m[y-x])m^n\psi(y,v)\,dy\,dv)\,dx\,du\rangle$$

by the above observation since

$$\frac{m}{u}\eta(m\ln\frac{u}{v})\varphi(m(y-x))m^n\psi(y,v)$$

has uniform compact support in (y,v,x,u)-space for all m. Let

$$f_m(x,u) = \int \frac{m}{u}\eta(m\ln\frac{u}{v})\varphi(m[y-x])m^n\psi(y,v)\,dy\,dv.$$

Then for sufficiently large m, $\operatorname{supp} f_m \subset \mathcal{O}$. Hence, since $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$, we have that;

$$|\langle T - T_m, \psi(x,u)\,dx\,du\rangle| = |\langle T, [f_m(x,u) - \psi(x,u)]\,dx\,du\rangle|$$

$$\leqslant c \sum_{|\alpha|+j\leqslant N} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} u^j \left(\frac{\partial}{\partial u}\right)^{j} [f_m - \psi] \right\|_1$$

$$= c \sum_{|\alpha|+j\leqslant N} \int \left| \left(\frac{\partial}{\partial x}\right)^{\alpha} u^j \left(\frac{\partial}{\partial u}\right)^{j} \right.$$

$$\left. \times [\int \frac{m}{u}\eta(m\ln\frac{u}{v})\varphi(m[y-x])\psi(y,v)m^n\,dy\,dv - \psi(x,u)] \right| dx\,du$$

$$= c \sum_{|\alpha|+j\leqslant N} \int \left| \left(\frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{\partial}{\partial t}\right)^{j} [\int \eta(m[t-s])\varphi(m[x-y])m^{n+1}\psi(y,e^s)e^s\,dy\,ds - \psi(x,e^t)e^t] \right| dx\,dt$$

$$= c \sum_{|\alpha|+j\leqslant N} \int |(\eta_m \otimes \varphi_m) * f_{\alpha j} - f_{\alpha j}|\,dx\,dt$$

where $V = e^s$, $u = e^t$, $f_{\alpha j} = \left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial t}\right)^{j}[\psi(x,e^t)e^t]$.

Now, using the assumption that η, φ are even functions, and applying Lemma 2.1.5 we get

$$|\langle T_m - T, \mu\rangle| \leqslant \frac{c}{m}\sum_{|\alpha|+j\leqslant N} \|f_{\alpha j}\|_{L_1^1} \leqslant \frac{c}{m}\sum_{|\alpha|+j\leqslant N+1} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial t}\right)^{j}\psi(x,e^t)e^t \right\|_1$$

$$= \frac{c}{m}\sum_{|\alpha|+j\leqslant N+1} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} u^j \left(\frac{\partial}{\partial u}\right)^{j}\psi(x,u) \right\|_1$$

proving (2.1.9) with $c_m = c/m$, $M = N + 1$. ■

Now let

$$\tilde{T}_m(x,t) = \begin{cases} T_m(x,t) & t \geqslant 0 \\ T_m(x,-t) & t < 0. \end{cases}$$

Since $\tilde{T}_m$ have uniform compact support, we can choose $S_m \in C_0^{\infty}(\mathbf{R}^{n+1})$ with uniform compact support so that

$$\sup|\tilde{T}_m - S_m| < \frac{1}{m}. \tag{2.1.10}$$

Clearly, $S_m \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}}$ since if $\mu = \varphi(x,t)\,dx\,dt$,

$$|\langle S_m - T, \mu\rangle| \leqslant |\langle S_m - T_m, \mu\rangle| + |\langle T_m - T, \mu\rangle| \leqslant \frac{1}{m}\|\varphi\|_1 + \frac{c}{m}\sum_{|\alpha|+j\leqslant M}\left\|\left(\frac{\partial}{\partial x}\right)^{\alpha} t^j \left(\frac{\partial}{\partial t}\right)^{j}\varphi\right\|_1.$$

We now restate and prove the main theorem of this section.

THEOREM 2.1.1. *Let $\eta_0 \in C_0^\infty(\mathbf{R}^+)$, $\int \eta_0 = 1$ and $\mathcal{F}(\eta_0(e^t)e^t)(s) \neq 0$ for $s \in \mathbf{R}$. Let T be a generalized function on $\mathcal{O}$, $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$ where $\mathfrak{N}$, $\mathcal{O}$, T satisfy (2.1.0)′. Assume that*

$$\lim_{n\to\infty} \langle T, \nu \otimes n\eta_0(nt)\,dt\rangle \tag{2.1.1}$$

exists for all $\nu \in \mathfrak{B}(\mathfrak{N})$. Then T has boundary values on $\mathfrak{N}$ in the sense of Definition 1.3.

PROOF. Let μ_n be a permissible sequence on $\mathcal{O}$ converging to ν. Let $\eta \in C_0^\infty(\mathbf{R}^+)$, $\int \eta = 1$. Then by Proposition 2.1.3, $\lim_{n\to\infty}\langle T, n\eta(nt)\,dt \otimes \nu\rangle$ exists. We will show that

$$\lim_n \langle T, \mu_n\rangle = \lim_n \langle T, n\eta(nt)\,dt \otimes \nu\rangle. \tag{2.1.11}$$

Let $\mu_n = \varphi_n(x,t)\,dx\,dt$, $\nu = \psi(x)\,dx$, S_m the sequence constructed in (2.1.10). Choose a subsequence, also denoted by S_m, so that $|\langle T - S_m, \mu_n\rangle| \leqslant 1/m$ for all n. This is possible since μ_n is a permissible sequence and $S_m \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}}$.

Since $S_m \in C_0^\infty(\mathbf{R}^{n+1})$, we have that for all n, there is a j_n so that

$$|\langle S_m, \mu_j - j\eta(jt)\,dt \otimes \nu\rangle| < \frac{1}{n} \qquad \text{for } j > j_n.$$

Combining, we have that for each m, there is a j_m so that

$$|\langle T, \mu_j\rangle - \langle S_m, j\eta(jt)\,dt \otimes \nu\rangle| < \frac{2}{m} \qquad \text{for } j > j_m.$$

Letting $j \to \infty$, we see that

$$\left|\lim_j \langle T, \mu_j\rangle - \int S_m(x,0)\psi(x)\,dx\right| < \frac{2}{m} \qquad \text{for all } m. \tag{2.1.12}$$

Now by construction, $S_m \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}}$. Hence,

$$\lim_{j\to\infty}\langle S_m, j\eta(jt)\,dt \otimes \nu\rangle = \int S_m(x,0)\psi(x)\,dx \to \lim_{j\to\infty}\langle T, j\eta(jt)dt \otimes \nu\rangle \qquad \text{as } m \to \infty.$$

Letting $m \to \infty$ in (2.1.12), we have that

$$\lim_{j\to\infty}\langle T, \mu_j\rangle = \lim_{j\to\infty}\langle T, j\eta(jt)\,dt \otimes \nu\rangle$$

proving (2.1.11) and hence the theorem. ■

Let $U = \{(y,s)\}$, $V = \{(x,t)\}$ be bounded open sets in $\mathbf{R}^{n+1}$. Let $\mathfrak{N}_1$ be the slice $\{s = 0\}$ of U, $\mathfrak{N}_2$ the slice $\{t = 0\}$ of V. Let $\chi: U \to V$ be a diffeomorphism such that $\chi(y,0) \to (x,0)$ and assume that χ extends diffeomorphically to a neighborhood of U^c. Let $\bar{\chi}$ be the induced diffeomorphism, $\bar{\chi}: \mathfrak{N}_1 \to \mathfrak{N}_2$. Let $\mathcal{O}_2 \subset V$ be open so that $\mathfrak{N}_2 \subset \partial\mathcal{O}_2$ and $\{(x,t)\,|\,0 < t < \delta, x \in \mathfrak{N}_2\} \subset \mathcal{O}_2$ for some δ. Let $\mathcal{O}_1 = \chi^{-1}(\mathcal{O}_2)$.

Now for $T \in \mathfrak{B}'(\mathcal{O}_2)$, $\mu = \varphi(y,s)\,dy\,ds \in \mathfrak{B}(\mathcal{O}_1)$, define

$$\left\langle \chi^* T, \mu\right\rangle = \left\langle T, \varphi(\chi^{-1}(x,s))\left|\frac{\partial\chi^{-1}}{\partial(x,t)}\right| dx\,dt\right\rangle. \tag{2.1.13}$$

Clearly, as in Proposition 1.1, if $T \in \mathcal{R}_{\mathfrak{N}_2,\mathcal{O}_2}$ then $\chi^* T \in \mathcal{R}_{\mathfrak{N}_1,\mathcal{O}_1}$.

PROPOSITION 2.1.6. *In the notation above, if $T_{\mathfrak{N}_2,\mathcal{O}_2}$ exists, $(\chi^* T)_{\mathfrak{N}_1,\mathcal{O}_1}$ also exists and we have that for all $\nu \in \mathcal{B}(\mathfrak{N}_1)$,*

$$\langle (\chi^* T)_{\mathfrak{N}_1,\mathcal{O}_1}, \nu\rangle = \langle \bar{\chi}^*(T_{\mathfrak{N}_2,\mathcal{O}_2}), \nu\rangle. \tag{2.1.14}$$

PROOF. If T is given by a smooth function, the result is trivial. In the general case, let S_m be the sequence constructed in (2.1.10). Then since χ extends to a neighborhood of U^c, an application of the proof of Proposition 1.1 yields that

$$\chi^* S_m \to \chi^* T \text{ in } \mathcal{R}_{\mathfrak{N}_1,\mathcal{O}_1}.$$

Therefore, by Proposition 1.6, $(\chi^* T)_{\mathfrak{N}_1,\mathcal{O}_1}$ exists and $(\chi^* S_m)_{\mathfrak{N}_1,\mathcal{O}_1} \to (\chi^* T)_{\mathfrak{N}_1,\mathcal{O}_1}$ weakly. By the above observation, $(\chi^* S_m)_{\mathfrak{N}_1,\mathcal{O}_1} = \bar{\chi}^*(S_m)_{\mathfrak{N}_2,\mathcal{O}_2}$ since $S_m \in C_0^\infty(\mathbf{R}^{n+1})$. Hence,

$$\bar{\chi}^*(S_m)_{\mathfrak{N}_2,\mathcal{O}_2} \to (\chi^* T)_{\mathfrak{N}_1,\mathcal{O}_1} \tag{2.1.15}$$

weakly. Now, since $S_m \to T$ in $\mathcal{R}_{\mathfrak{N}_2,\mathcal{O}_2}$ we have that $(S_m)_{\mathfrak{N}_2,\mathcal{O}_2} \to T_{\mathfrak{N}_2,\mathcal{O}_2}$ weakly. So clearly, we have

$$\bar{\chi}^*(S_m)_{\mathfrak{N}_2,\mathcal{O}_2} \to \bar{\chi}^*(T_{\mathfrak{N}_2,\mathcal{O}_2}) \tag{2.1.16}$$

weakly. Combining (2.1.15) with (2.1.16) we obtain (2.1.14). ■

We conclude this section with a theorem concerning the extension of generalized functions.

THEOREM 2.1.7. *Let $\mathcal{O} \subset \mathfrak{M}$ be open so that $\partial\mathcal{O}$ is an embedded submanifold of $\mathfrak{M}$. Then for every $T \in \mathcal{R}_{\partial\mathcal{O},\mathcal{O}}$ there exists a $\tilde{T} \in \mathcal{B}'(\mathfrak{M})$ so that*

$$T = \tilde{T}|_{\mathcal{B}(\mathcal{O})}. \tag{2.1.17}$$

In fact, we have that $\tilde{T} \in \mathcal{R}_{\partial\mathcal{O},\mathfrak{M}}$, and there is a natural injection,

$$\mathcal{R}_{\partial\mathcal{O},\mathcal{O}} \to \mathcal{R}_{\partial\mathcal{O},\mathfrak{M}}. \tag{2.1.18}$$

PROOF. Let $T \in \mathcal{R}_{\partial\mathcal{O},\mathcal{O}}$. Using a partition of unity argument as before, it is clear that it is enough to show the result locally. That is, it is sufficient to assume that T has relatively compact support in a coordinate neighborhood $U = \{(x,t)\}$ of $\mathfrak{M}$ where $\mathfrak{N} \equiv U \cap \partial\mathcal{O} = \{(x,0)\}$, $\{(x,t) \in U \mid t > 0\} \subset \mathcal{O}$ and there show that $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}\cap U}$ can be extended to an element $\tilde{T} \in \mathcal{B}'(U)$. In fact, we will show that $\tilde{T} \in \mathcal{R}_{\mathfrak{N},U}$. To define $\tilde{T}$, we first construct a partition of unity. For each $k \in \mathbf{Z}$, let $I_k \subset \mathbf{R}$ be defined by

$$I_k = [(k + \tfrac{1}{2})\ln 2, (k + \tfrac{3}{2})\ln 2].$$

Let χ_k be the characteristic function of I_k. Let

$$\zeta \in C_0^\infty(\mathbf{R}), \qquad \int \zeta = 1, \qquad \operatorname{supp} \zeta \subset \left[-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right].$$

Then $\zeta * \chi_k(t) = \zeta * \chi_0(t - k \ln 2)$ and $\operatorname{supp} \zeta * \chi_k \subset [k \ln 2, (k+2)\ln 2]$. Let

$$\eta(t) = \zeta * \chi_0(\ln t), \qquad t > 0. \tag{2.1.19}$$

Then if $t > 0$,

$$\sum_{k=-\infty}^{\infty} \eta(2^k t) = \sum_{k=-\infty}^{\infty} \zeta * \chi_0(\ln t + k \ln 2) = \zeta * \sum_{k=-\infty}^{\infty} \chi_k(\ln t) = \zeta * 1 = \int \zeta = 1.$$

Note also, that supp $\eta \subset [1,4]$. Now for $\mu = \varphi(x,t)\,dx\,dt \in \mathfrak{B}(U)$, we define $\tilde{T}$ by,

$$\langle \tilde{T}, \mu \rangle = \lim_{n\to\infty} \sum_{k=-\infty}^{n} \langle T, \eta(2^k t)\mu \rangle. \tag{2.1.20}$$

Note that for each n, the above sum is finite since μ has compact support. Also, since supp $\eta(2^k t)\mu \subset \mathcal{O}$, we see that $\langle T, \eta(2^k t)\mu\rangle$ is well defined.

Let $S_n = \sum_{-\infty}^{n} \langle T, \eta(2^k t)\mu\rangle$. Then for $n \geqslant m$,

$$|S_n - S_m| \leqslant |\langle T, \sum_{k=m}^{n} \eta(2^k t)\mu\rangle| \leqslant c \sum_{|\alpha|+j \leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} t^j \left(\frac{\partial}{\partial t}\right)^{j} \left[\sum_{k=m}^{n} \eta(2^k t)\varphi\right] \right\|_1$$

$$\leqslant c \sum_{i+|\alpha|+j \leqslant M} \left\| \left[t^i \left(\frac{\partial}{\partial t}\right)^{i} \left[\sum_{k=m}^{n} \eta(2^k t)\right]\right] \left[\left(\frac{\partial}{\partial x}\right)^{\alpha} t^j \left(\frac{\partial}{\partial t}\right)^{j} \varphi \right] \right\|_1.$$

Now

$$\left| t^i \left(\frac{\partial}{\partial t}\right)^{i} \left(\sum_{k=m}^{n} \eta(2^k t)\right)\right| \leqslant \left| t^i \left(\frac{\partial}{\partial t}\right)^{i} \sum_{\substack{k=m\\ k \text{ even}}}^{n} \eta(2^k t)\right| + \left| t^i \left(\frac{\partial}{\partial t}\right)^{i} \sum_{\substack{k=m\\ k \text{ odd}}}^{n} \eta(2^k t)\right| \leqslant c_i \qquad \text{for all } t, n, m.$$

Hence,

$$|S_n - S_m| \leqslant c \sum_{\alpha,j} \int_{2^{-n}}^{2^{-m+2}} \int \left| \left(\frac{\partial}{\partial x}\right)^{\alpha} t^j \left(\frac{\partial}{\partial t}\right)^{j} \varphi \right| dx\,dt.$$

Clearly, then, $|S_n - S_m| \to 0$ as $n, m \to \infty$. Thus, $\lim_n S_n$ exists for all μ. To see that $\tilde{T} \in \mathfrak{R}_{\mathfrak{N},U}$, let

$$\chi_1^n(t) = \sum_{\substack{k=-\infty\\ k \text{ even}}}^{n} \eta(2^k t), \qquad \chi_2^n(t) = \sum_{\substack{k=-\infty\\ k \text{ odd}}}^{n} \eta(2^k t).$$

Then,

$$|\langle \tilde{T}, \mu\rangle| \leqslant \sup_n \left| \left\langle T, \sum_{-\infty}^{n} \eta(2^k t)\mu \right\rangle \right| \leqslant \sup_n \left\{ c \sum_{|\alpha|+j \leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} t^j \left(\frac{\partial}{\partial t}\right)^{j} [\Sigma\, \eta(2^k t)\varphi] \right\|_1 \right\}$$

$$\leqslant \sup_n \left\{ c \sum_{|\alpha|+j \leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} t^j \left(\frac{\partial}{\partial t}\right)^{j} \chi_1^n(t)\varphi \right\|_1 \right\} \tag{2.1.21}$$

$$+ \sup_n \left\{ c \sum_{|\alpha|+j \leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} t^j \left(\frac{\partial}{\partial t}\right)^{j} \chi_2^n(t)\varphi \right\|_1 \right\}.$$

Differentiating the products $\chi_1^n \varphi$, $\chi_2^n \varphi$, as before and noting that

$$\left| t^i \left(\frac{\partial}{\partial t}\right)^{i} \chi_1^n \right| \leqslant c_i \quad \text{for all } n, \qquad \left| t^i \left(\frac{\partial}{\partial t}\right)^{i} \chi_2^n \right| \leqslant d_i \quad \text{for all } n,$$

we obtain that (2.1.21) is dominated by

$$c \sum_{|\alpha|+j \leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} t^{j} \left(\frac{\partial}{\partial t}\right)^{j} \varphi \right\|_{1}. \tag{2.1.22}$$

Hence $\tilde{T} \in \mathfrak{R}_{\mathfrak{N},U}$.

To show that the mapping; $\mathfrak{R}_{\partial\mathcal{O},\mathcal{O}} \to \mathfrak{R}_{\partial\mathcal{O},\mathfrak{N}}$ taking $T \to \tilde{T}$ is natural, we now show that the value of $\tilde{T}$ as defined in (2.1.20) is coordinate invariant.

Let $U = \{(y,s)\}$, $V = \{(x,t)\}$ be coordinate neighborhoods on $\mathfrak{N}$ so that $U \cap V \cap \mathfrak{N} \neq \varnothing$ and $U \cap \mathfrak{N} = \{(y,s) | s = 0\}$, $V \cap \mathfrak{N} = \{(x,t) | t = 0\}$. Let Ψ: $U \to V$ be a diffeomorphism $(y,s) \to (x,t)$ so that Ψ: $(y,0) \to (x,0)$. Let $t = \Psi_t(y,s)$. Then since the Jacobian of Ψ at $s = 0$ is nonsingular, we have that $(\partial\Psi_t/\partial s)(y,0) \neq 0$ for all $y \in U \cap V$. Hence, given $K \subset U \cap V$, K compact, there exists an $\epsilon > 0$, c_1, c_2 so that if $0 < s < \epsilon$,

$$c_1 s \leqslant \Psi_t(y,s) \leqslant c_2 s \qquad \text{uniformly for } y \in K. \tag{2.1.23}$$

Clearly, we can assume $(\operatorname{supp} T)^c \subset U \cap V$ and is compact. Let ϵ, c_1, c_2 be given for $K = (\operatorname{supp} T)^c$. Let $\mu \in \mathfrak{B}(U \cap V)$, $\mu = \varphi(y,s)\,dy\,ds$ on U, $\mu = \bar{\varphi}(x,t)\,dx\,dt$ on V. Then on U,

$$\langle \tilde{T}, \mu \rangle \equiv \lim_{n\to\infty} \sum_{k=-\infty}^{n} \langle T, \eta(2^k s)\mu \rangle = \lim_{n\to\infty} a_n.$$

Also, on V,

$$\langle \tilde{T}, \mu \rangle \equiv \lim_{m\to\infty} \sum_{k=-\infty}^{m} \langle T, \eta(2^k t)\mu \rangle = \lim_{m\to\infty} b_m.$$

Hence,

$$a_n - b_m = \left\langle T, \left[\sum_{k=-\infty}^{n} \eta(2^k s) - \sum_{k=-\infty}^{m} \eta(2^k \Psi_t(y,s))\right] \varphi(y,s)\,dy\,ds \right\rangle. \tag{2.1.24}$$

Now $\operatorname{supp} \eta(2^k s) \subset \{s | 2^{-k} \leqslant s \leqslant 2^{-k+2}\}$ and $\operatorname{supp} \eta(2^k \Psi_t(y,s)) \subset \{s | d_1 2^{-k} \leqslant s \leqslant d_2 2^{-k+2}\}$ for all $y \in (\operatorname{supp} T)^c$ by (2.1.23). Hence, it is clear that

$$\operatorname{supp}\left[\sum_{k=-\infty}^{n} \eta(2^k s) - \sum_{k=-\infty}^{m} \eta(2^k \Psi_t(y,s))\right] \subset \{(y,s) | f(n,m) \leqslant s \leqslant g(n,m)\} \tag{2.1.25}$$

where f, g tend to 0 as n, $m \to \infty$. Hence, using (2.1.25) and the fact that $T \in \mathfrak{R}_{\partial\mathcal{O},\mathcal{O}}$, we can estimate $|a_n - b_m|$ from (2.1.24) as we estimated $|s_n - s_m|$ above, to obtain that $|a_n - b_m| \to 0$ as n, $m \to \infty$. Hence, $\tilde{T}$ as defined in (2.1.20) has value independent of the coordinate system chosen. ∎

2. Restrictions of Generalized Functions

Its now relatively simple to obtain a theorem for restrictions corresponding to Theorem 2.1.1. We assume $\mathcal{O} \subset \mathbf{R}^{n+1}$ is open, $\mathbf{R}^{n+1} = \{(x,t)\}$ and $\mathcal{O} \cap \{(x,0)\} = \mathfrak{N}$. Let $\mathcal{O}^+ = \{(x,t) \in \mathcal{O} | t > 0\}$, $\mathcal{O}^- = \{(x,t) \in \mathcal{O} | t < 0\}$. Let $T \in \mathfrak{B}'(\mathcal{O})$ and as before, we assume that supp T is compact. Note that $\mathcal{O}^+$ and $\mathcal{O}^-$ satisfy conditions iii) and iv) of (2.1.0)′.

We first relate convergence in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}}$ to convergence in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}^+}$ and $\mathfrak{R}_{\mathfrak{N},\mathcal{O}^-}$. Our goal being to show that the sequence S_m constructed in (2.1.10) actually converges in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}}$.

PROPOSITION 2.2.1. *Let T_n, $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ and assume that $T_n \to T$ in both $\mathcal{R}_{\mathfrak{N},\mathcal{O}^+}$ and $\mathcal{R}_{\mathfrak{N},\mathcal{O}^-}$. Then $T_n \to T$ in $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$.*

PROOF. Let $\mu \in \mathcal{B}(\mathcal{O})$, $\eta \in C_0^\infty(\mathbf{R})$ so that $\eta = 1$ for $|t| < 1$; $\eta = 0$ for $|t| > 2$. Then,

$$|\langle T - T_n, \mu\rangle| \leqslant |\langle T - T_n, (1 - \eta(\tfrac{t}{\epsilon}))\mu\rangle| + |\langle T - T_n, \eta(\tfrac{t}{\epsilon})\mu\rangle| = I + II.$$

By assumption, we clearly have that $T_n \to T$ in $\mathcal{R}_{\mathfrak{N},\mathcal{O}^+ \cup \mathcal{O}^-}$. Hence since $\operatorname{supp}(1 - \eta(t/\epsilon))\mu \subset \mathcal{O}^+ \cup \mathcal{O}^-$, we obtain that

$$\begin{aligned} I &\leqslant c_n \sum_{|\alpha|+j\leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha t^j \left(\frac{\partial}{\partial t}\right)^j [(1 - \eta(\tfrac{t}{\epsilon}))\varphi] \right\|_1 \\ &\leqslant c_n \sum_{|\alpha|+j\leqslant M} \left\| (1 - \eta(\tfrac{t}{\epsilon})) \left(\frac{\partial}{\partial x}\right)^\alpha t^j \left(\frac{\partial}{\partial t}\right)^j \varphi \right\|_1 \\ &\quad + c_n \sum_{\substack{|\alpha|+j+k\leqslant M \\ k>0}} \left\| \left[t^k \left(\frac{\partial}{\partial t}\right)^k (1 - \eta(\tfrac{t}{\epsilon}))\right] \left[\left(\frac{\partial}{\partial x}\right)^\alpha t^j \left(\frac{\partial}{\partial t}\right)^j \varphi \right] \right\|_1 = III + IV. \end{aligned}$$

Now

$$III \leqslant c_n \sum_{|\alpha|+j\leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha t^j \left(\frac{\partial}{\partial t}\right)^j \varphi \right\|_1,$$

clearly. Also,

$$IV \leqslant c \sum_{|\alpha|, j} \int_{|t|<\epsilon} \left[\int \left| \left(\frac{\partial}{\partial x}\right)^\alpha t^j \left(\frac{\partial}{\partial t}\right)^j \varphi \right| dx \right] dt$$

since

$$\left| t^k \left(\frac{\partial}{\partial t}\right)^k (1 - \eta(\tfrac{t}{\epsilon})) \right| \leqslant c_k \qquad \text{for all } \epsilon.$$

Hence, $IV \to 0$ with ϵ and we have that

$$I \leqslant c_n \sum_{|\alpha|+j\leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha t^j \left(\frac{\partial}{\partial t}\right)^j \varphi \right\|_1.$$

Now since T_n, T are in $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$, we have

$$\begin{aligned} II &\leqslant |\langle T, \eta(\tfrac{t}{\epsilon})\mu\rangle| + |\langle T_n, \eta(\tfrac{t}{\epsilon})\mu\rangle| \\ &\leqslant c_1 \sum_{|\alpha|+j\leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha t^j \left(\frac{\partial}{\partial t}\right)^j \eta(\tfrac{t}{\epsilon})\varphi \right\|_1 + c_2 \sum_{|\alpha|+j\leqslant N} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha t^j \left(\frac{\partial}{\partial t}\right)^j \eta(\tfrac{t}{\epsilon})\varphi \right\|_1. \end{aligned}$$

Calculating as with IV, we see that $II \to 0$ with ϵ. Hence, combining the above estimates for I and II completes the proof. ■

COROLLARY 2.2.2. *Let $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ such that $T_{\mathfrak{N},\mathcal{O}^+}$, $T_{\mathfrak{N},\mathcal{O}^-}$ exist and are equal. Then there exists a sequence $S_m \in C_0^\infty(\mathbf{R}^{n+1})$ such that $S_m \to T$ in $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$.*

PROOF. Let $T_m^{\pm}$ be the sequences of functions as in Proposition 2.1.4 so that

$$T_m^{\pm} \to T \text{ in } \mathcal{R}_{\mathfrak{N},\mathcal{O}^{\pm}}.$$

Recall that $T_m^+(x,t)[T^-(x,t)]$ was continuous in $t \geqslant 0[t \leqslant 0]$ and C^∞ in $t > 0[t < 0]$. Also, note by construction,

$$T_m^{\pm}(x,0) = \langle T_{\mathfrak{N},\mathcal{O}^{\pm}}, \varphi(m[x-y])m^n\, dy\rangle.$$

So by assumption, $T_m^+(x,0) = T_m^-(x,0)$. Therefore, let $S_m \in C_0^\infty(\mathbf{R}^{n+1})$ be chosen so that

$$|S_m - \tilde{T}_m| < \frac{1}{m}$$

where $\tilde{T}_m \in C_0^\infty(\mathbf{R}^{n+1})$ is defined by:

$$\tilde{T}_m = \begin{cases} T_m^+ & t \geqslant 0 \\ T_m^- & t \leqslant 0 \end{cases}.$$

Clearly, $S_m \to T$ in $\mathcal{R}_{\mathfrak{N},\mathcal{O}^{\pm}}$ and $S_m \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ $\forall m$. Hence by Proposition 2.2.1, the corollary is proved. ∎

THEOREM 2.2.3. *Let $\eta_1, \eta_2 \in C_0^\infty(\mathbf{R})$ so that $\operatorname{supp} \eta_1 \subset \mathbf{R}^+$, $\operatorname{supp} \eta_2 \subset \mathbf{R}^-$, $\int \eta_1 = \int \eta_2 = 1$, $\mathcal{F}(\eta_1(e^t)e^t)(s) \neq 0$ and $\mathcal{F}(\eta_2(-e^t)e^t)(s) \neq 0$ for $s \in \mathbf{R}$. Let $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ have compact support. Assume that for all $\nu \in \mathcal{B}(\mathfrak{N})$,*

$$\lim_{n\to\infty} \langle T, n\eta_1(nt)\,dt \otimes \nu\rangle = \lim_{n\to\infty} \langle T, n\eta_2(nt)\,dt \otimes \nu\rangle. \tag{2.2.1}$$

Then T has a restriction to $\mathfrak{N}$ in the sense of Definition 1.3, whose value is given by (2.2.1) for all ν.

PROOF. By Theorem 2.2.1 and (2.2.1) we have that $T_{\mathfrak{N},\mathcal{O}^+}$ and $T_{\mathfrak{N},\mathcal{O}^-}$ exist and are equal. By Corollary 2.2.2, there is a sequence $S_m \in C_0^\infty(\mathbf{R}^{n+1})$ so that $S_m \to T$ in $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$. Hence, by Proposition 1.6, $T_{\mathfrak{N},\mathcal{O}}$ exists and $\langle T_{\mathfrak{N},\mathcal{O}}, \nu\rangle = \lim_m \langle (S_m)_{\mathfrak{N},\mathcal{O}}, \nu\rangle$ for all ν. Combining this with (2.2.1) and the fact that $S_m \to T$ in $\mathcal{R}_{\mathfrak{N},\mathcal{O}^{\pm}}$ completes the proof. ∎

Let $U = \{(y,s)\}$, $V = \{(x,t)\}$ be bounded open sets in $\mathbf{R}^{n+1}$, χ: $U \to V$ a diffeomorphism so that $\chi(y,0) \to (x,0)$ and assume that χ extends diffeomorphically to a neighborhood of U^c. Let $\mathfrak{N}_1$ be the slice $\{s = 0\}$ of U, $\mathfrak{N}_2$ the slice $\{t = 0\}$ of V. Let $\mathcal{O}_2 \subset V$ be open, $\mathcal{O}_2 \cap \mathfrak{N}_2 \neq \varnothing$. Let $\mathcal{O}_1 = \chi^{-1}(\mathcal{O}_2)$, and let $\bar{\chi}$: $\mathfrak{N}_1 \to \mathfrak{N}_2$ be the diffeomorphism induced by χ.

PROPOSITION 2.2.4. *Let $T \in \mathcal{R}_{\mathfrak{N}_2,\mathcal{O}_2}$, and assume that $T_{\mathfrak{N}_2,\mathcal{O}_2}$ exists. Then $\chi^* T \in \mathcal{R}_{\mathfrak{N}_1,\mathcal{O}_1}$ and has a restriction to $\mathfrak{N}_1$ given by:*

$$\langle (\chi^* T)_{\mathfrak{N}_1,\mathcal{O}_1}, \nu\rangle = \langle \bar{\chi}^* T_{\mathfrak{N}_2,\mathcal{O}_2}, \nu\rangle \qquad \textit{for all } \nu \in \mathcal{B}(\mathfrak{N}_1).$$

PROOF. Identical with the proof of Proposition 2.1.6. ∎

We now give a generalization of Theorem 2.2.3.

PROPOSITION 2.2.5. *Let $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}^+} \cap \mathcal{R}_{\mathfrak{N},\mathcal{O}^-}$ and assume that $T_{\mathfrak{N},\mathcal{O}^{\pm}}$ exist and are equal. Assume also, that for $\eta \in C_0^\infty(\mathbf{R})$, $\nu \in \mathcal{B}(\mathfrak{N})$, we have*

$$|\langle T, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\,dt \otimes \nu\rangle| \leqslant c_{\nu\eta} \qquad \textit{for all } \epsilon. \tag{2.2.2}$$

Then $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ and has a restriction to $\mathfrak{N}$ equal to its common boundary value.

PROOF. We need only show that $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$, then apply the proof of Theorem 2.2.3. Let $\eta \in C_0^\infty(\mathbf{R})$, $\eta = 1$ for $|t| < 1$, $\eta = 0$ for $|t| > 2$. Let $\mu = \varphi(x,t)\,dx\,dt \in \mathcal{B}(\mathcal{O})$. Then,

$$(2.2.3) \qquad |\langle T,\mu\rangle| \leqslant |\langle T,(1-\eta(\tfrac{t}{\epsilon}))\mu\rangle| + |\langle T,\eta(\tfrac{t}{\epsilon})\mu\rangle| = I + II.$$

As for I, we proceed as in the proof of Proposition 2.2.1 and obtain

$$(2.2.4) \qquad I \leqslant c \sum_{|\alpha|+j\leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha t^j \left(\frac{\partial}{\partial t}\right)^j \varphi \right\|_1 .$$

Identifying T with an element $D \in \mathcal{E}'(\mathbf{R}^{n+1})$, which we can since supp T is compact, we have that for all $\mu = \varphi(x,t)\,dx\,dt \in \mathcal{B}(\mathcal{O})$,

$$(2.2.5) \quad |\langle T,\mu\rangle| = |\langle D,\varphi\rangle| \leqslant c \sup \sum_{|\alpha|+j\leqslant M} \left| \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial t}\right)^j \varphi \right| \leqslant c \sum_{|\alpha|+j\leqslant M+1} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial t}\right)^j \varphi \right\|_1 .$$

By Taylor's formula, write

$$\varphi(x,t) = \varphi(x,0) + \sum_{j=1}^{K} \frac{\varphi_j(x,0)}{j!} t^j + R_{K+1}(x,t)$$

where $\varphi_j = (\partial/\partial t)^j \varphi$ and R_{K+1} has a $(K+1)$st order zero in t at $t = 0$. Then

$$|\langle T, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\varphi(x,t)\,dx\,dt - \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\varphi(x,0)\,dx\,dt\rangle| \leqslant \sum_{j=1}^{K} |\langle T, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})t^j\varphi_j(x,0)\,dx\,dt\rangle|$$
$$+ |\langle T, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})R_{K+1}(x,t)\,dx\,dt\rangle|.$$

Using (2.2.5), we have that

$$|\langle T, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})t^j\varphi_j(x,0)\,dx\,dt\rangle| \leqslant c\epsilon^j, \qquad |\langle T, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})R_{K+1}(x,t)\,dx\,dt\rangle| \leqslant c\epsilon \qquad \text{if } K > M.$$

Hence,

$$|\langle T, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\mu - \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\varphi(x,0)\,dx\,dt\rangle| \to 0 \text{ with } \epsilon.$$

Combining this with (2.2.2), we have

$$(2.2.6) \qquad |\langle T, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\mu\rangle| \leqslant c_{\eta\mu} \qquad \text{for all } \epsilon,\ \mu \in \mathcal{B}(\mathcal{O}).$$

Hence, by (2.2.6),

$$II = |\langle T, \eta(\frac{t}{\epsilon})\mu\rangle| \leqslant c_{\eta\mu}\epsilon \to 0 \text{ with } \epsilon.$$

Letting $\epsilon \to 0$ in (2.2.3) and combining the estimates for I and II, we obtain

$$|\langle T,\mu\rangle| \leqslant c \sum_{|\alpha|+j\leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha t^j \left(\frac{\partial}{\partial t}\right)^j \varphi \right\|_1 .$$

That is, $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$. ■

REMARK. The assumption $T_n, T \in \mathcal{R}_{\mathcal{N},\mathcal{O}}$ in Proposition 2.2.1 can be weakened to the following: For all $\nu \in \mathcal{B}(\mathcal{N})$, $\eta \in C_0^\infty(\mathbf{R})$,

$$|\langle T, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\, dt \otimes \nu\rangle \quad | \leqslant c_{\eta\nu} |\langle T_n, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\, dt \otimes \nu\rangle| \leqslant c_{\eta\nu n} \tag{2.2.7}$$

where $c_{\eta\nu}$ and $c_{\eta\nu n}$ are independent of ϵ.

The proof is the same as the proof of Proposition 2.2.1 except to estimate II, we follow the proof of Proposition 2.2.5.

PROPOSITION 2.2.6. *Let $T \in \mathcal{R}_{\mathcal{N},\mathcal{O}^+} \cap \mathcal{R}_{\mathcal{N},\mathcal{O}^-}$ and assume that $T_{\mathcal{N},\mathcal{O}^+}$ and $T_{\mathcal{N},\mathcal{O}^-}$ exist and are equal. Then there is an $S \in \mathcal{R}_{\mathcal{N},\mathcal{O}}$ such that $S = T$ on $\mathcal{O}\backslash\mathcal{N}$, and S has a restriction to $\mathcal{N}$ equal to the boundary values of T.*

PROOF. Let $\eta(t) = \zeta * \chi_0(\ln|t|)$ where ζ, χ_0 are as in the proof of Theorem 2.1.7. Then for all $t \neq 0$, $\sum_{k=-\infty}^{\infty} \eta(2^k t) = 1$. We define S as follows. Let $\mu = \varphi(x,t)\, dx\, dt \in \mathcal{B}(\mathcal{O})$. Define:

$$\langle S, \mu\rangle = \lim_{n\to\infty} \sum_{k=-\infty}^{n} \langle T, \eta(2^k t)\mu\rangle. \tag{2.2.8}$$

Imitating the proof of Theorem 2.1.7, we see that S is well defined and $S \in \mathcal{R}_{\mathcal{N},\mathcal{O}}$. Clearly, $S = T$ on $\mathcal{O}\backslash\mathcal{N}$. Hence, $S_{\mathcal{N},\mathcal{O}^+}$ and $S_{\mathcal{N},\mathcal{O}^-}$ exist and are equal. So by Corollary 2.2.2, there exists $\{S_m\} \in C_0^\infty(\mathcal{O})$ so that $S_m \to S$ in $\mathcal{R}_{\mathcal{N},\mathcal{O}}$. Thus by Proposition 1.6, $S_{\mathcal{N},\mathcal{O}}$ exists and equals the weak limit of $(S_m)_{\mathcal{N},\mathcal{O}}$. Clearly,

$$S_{\mathcal{N},\mathcal{O}^\pm} = T_{\mathcal{N},\mathcal{O}^\pm} = \text{weak}\lim (S_m)_{\mathcal{N},\mathcal{O}^\pm}.$$

Hence, $S_{\mathcal{N},\mathcal{O}} = T_{\mathcal{N},\mathcal{O}^\pm}$. ■

3. Poisson Type Integrals

Let $D \in \mathcal{D}'(\mathbf{R}^n)$, $\varphi \in C_0^\infty(\mathbf{R}^n)$, $\int\varphi = 1$. Then it is well known that $\varphi_\epsilon * D \to D$ in $\mathcal{D}'(\mathbf{R}^n)$ where

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^n}\varphi(\frac{x}{\epsilon}).$$

Let $\mathcal{O} \subset \mathbf{R}^n \times \mathbf{R}$ be defined by $\{(x,t)|t > 0\}$. Making use of the natural identification of distributions and generalized functions in this setting, we will define $\tilde{D} \in \mathcal{R}_{\mathcal{N},\mathcal{O}}$ so that $\tilde{D}_{\mathcal{N},\mathcal{O}} = D$. Here we identify $\mathcal{N} = \mathbf{R}^n = \{(x,0)\}$. For $\psi(x,t) \in C_0^\infty(\mathcal{O})$, we define;

$$\langle \tilde{D}, \psi\rangle = \int \langle D * t^{-n}\varphi(\frac{\cdot}{t}), \psi(\cdot,t)\rangle\, dt. \tag{2.3.1}$$

The integral is well defined since ψ has compact support and the integrand is a continuous function of t for $t \geqslant 0$. Clearly, $\tilde{D} \in \mathcal{D}'(\mathcal{O})$.

PROPOSITION 2.3.1. *Let $\tilde{D}$ be defined by (2.3.1). Then $\tilde{D} \in \mathcal{R}_{\mathcal{N},\mathcal{O}}$ and $\tilde{D}_{\mathcal{N},\mathcal{O}} = D$*

PROOF. Let $x_0 \in \mathbf{R}^n$, U a bounded neighborhood of x_0 in $\mathbf{R}^n \times \mathbf{R}$. Let $V = U \cap \mathcal{N}$. Then if $\operatorname{supp}\psi \subset U \cap \mathcal{O}$,

$$\langle \tilde{D}, \psi\rangle = \int \langle D * t^{-n}\varphi(\frac{\cdot}{t}), \psi(\cdot,t)\rangle\, dt = \int \langle D, \psi(\cdot,t) * t^{-n}\varphi(\frac{-\cdot}{t})\rangle\, dt. \tag{2.3.2}$$

Now since U is bounded, it is clear that for all $\psi \in C_0^\infty(U \cap \mathcal{O})$, that $\psi(\cdot,t) * t^{-n}\varphi(-\cdot/t)$ has bounded support as a function of x, uniformly in t. Let $W \subset \mathbf{R}^n$ be a bounded open set so that

$$\operatorname{supp}[\psi(\cdot,t) * t^{-n}\varphi(\frac{-\cdot}{t})] \subset W \qquad \text{for all } t, \qquad \text{for all } \psi \in C_0^\infty(U \cap \mathcal{O}).$$

Let $\chi \in C_0^\infty(\mathbf{R}^n)$, $\chi \equiv 1$ on W. Then since $\chi D \in \mathcal{E}'(\mathbf{R}^n)$, we have as in (2.2.5),

$$(2.3.3) \qquad |\langle \chi D, \xi\rangle| \leqslant c \sum_{|\alpha| \leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} \xi \right\|_1 \qquad \text{for all } \xi \in C_0^\infty(W).$$

Hence, from (2.3.2), (2.3.3),

$$|\langle \tilde{D}, \psi\rangle| \leqslant \int |\langle D, \psi(\cdot,t) * t^{-n}\varphi(\frac{-\cdot}{t})\rangle|\, dt = \int |\langle \chi D, \psi(\cdot,t) * t^{-n}\varphi(\frac{-\cdot}{t})\rangle|\, dt$$

$$\leqslant \int \sum_{|\alpha| \leqslant M} \left\| t^{-n}\varphi(\frac{-\cdot}{t}) * \left(\frac{\partial}{\partial x}\right)^{\alpha} \psi(\cdot,t) \right\|_1 dt \leqslant \|\varphi\|_1 \sum_{|\alpha| \leqslant M} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} \psi(x,t) \right\|_{L^1(\mathbf{R}^{n+1})}.$$

So $\tilde{D} \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$.

To prove that $\tilde{D}_{\mathfrak{N},\mathcal{O}} = D$, we will show that

$$(2.3.4) \qquad \lim_{\epsilon \to 0} \langle \tilde{D}, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\psi(x)\rangle = \langle D, \psi\rangle$$

for all $\psi \in C_0^\infty(V)$, $\eta \in C_0^\infty(\mathbf{R}^+)$, with $\int \eta = 1$. Then by Theorem 2.1.1, the result will follow. Let $\eta \in C_0^\infty(\mathbf{R}^+)$, $\psi \in C_0^\infty(V)$ be given. Then

$$(2.3.5) \quad \langle \tilde{D}, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\psi(x)\rangle = \int \langle D * t^{-n}\varphi(\frac{\cdot}{t}), \psi(x)\rangle \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\, dt = \int \langle D * (\epsilon u)^{-n}\varphi(\frac{\cdot}{\epsilon u}), \psi(x)\rangle \eta(u)\, du.$$

Now the integrand in (2.3.5) is absolutely integrable, uniformly bounded in ϵ and converges pointwise to $\langle D, \psi(x)\rangle \eta(u)$. Hence, by the Dominated Convergence Theorem,

$$\langle \tilde{D}, \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})\psi(x)\rangle \to \langle D, \psi\rangle \qquad \text{as } \epsilon \to 0$$

since $\int \eta(u)\, du = 1$. ■

CHAPTER III
CODIM $\mathfrak{N} > 1$

1. Boundary Values of Generalized Functions

Let $\mathfrak{M}$ have dimension $k + l$, $\mathfrak{N} \subset \mathfrak{M}$ an embedded submanifold of dimension k. In this section, we make the following assumptions on $\mathcal{O}$.

(3.1.0)
i) $\mathfrak{N} \subset \partial\mathcal{O}$,
ii) There exists $U = \{(x,y)\}$, a coordinate neighborhood of $\mathfrak{M}$, so that $U \cap \mathfrak{N} = \{(x,y) \mid y = 0\}$, and for each $x \in U \cap \mathfrak{N}$, there is a neighborhood V of x, an open truncated cone $W \subset \mathbf{R}^l$, so that $V \times W \subset \mathcal{O}$. W is a truncated cone if $\theta \in W$ implies $\tau\theta \in W$ for all $0 < \tau < \epsilon$, some ϵ.

By Proposition 1.7, we can assume without loss of generality that supp T is relatively compact and supp $T \subset U \times \{y| \quad |y| < \epsilon\}$. Combining this observation with (3.1.0), we use the following model:

(3.1.0)′
i) $\mathfrak{M} = \mathbf{R}^{k+l} \equiv \mathbf{R}^k \times \mathbf{R}^l$,
ii) $\mathfrak{N} \subset \mathbf{R}^k \equiv \mathbf{R}^k \times \{0\}$, is open
iii) $\mathfrak{N} \times W \subset \mathcal{O}$, where W is an open cone in $\mathbf{R}^l$,
iv) $\mathfrak{N} \subset \partial\mathcal{O}$,
v) $T \in \mathfrak{B}'(\mathcal{O})$, $(\text{supp } T)^c$ compact in $\mathfrak{M}$.

In this setting, there always exists permissible sequences for any $\nu \in \mathfrak{B}(\mathfrak{N})$. For example, if $\eta \in C_0^\infty(\mathbf{R}^+)$, $\int \eta(t)t^{l-1}\,dt = 1$, and $\{\zeta_n\} \subset C^\infty(S^{l-1})$, $\int_{S^{l-1}} \zeta_n\, d\sigma = 1$, supp $\zeta_n \subset S^{l-1} \cap W$, then

$$\omega_n = \nu \otimes n^l \eta(n|y|)\zeta_n\left(\frac{y}{|y|}\right) dy \tag{3.1.1}$$

converges to ν clearly, and supp $\omega_n \subset \mathcal{O}$. Also, if we require that

$$\int_{S^{l-1}} \left|\left(\frac{\partial}{\partial y'}\right)^\alpha \zeta_n\right| d\sigma \leqslant c_{|\alpha|} \qquad \text{for all } n.$$

then ω_n is clearly a permissible sequence as in Definition 1.2.

As in Chapter II, we want to justify the exclusive use of such sequences in applications. That is, we are interested in the following question:

Let $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$. If $\lim_n \langle T, \nu \otimes n^l \eta_0(n|y|)\zeta_n(y')\,dy\rangle$ exists for all $\nu \in \mathfrak{B}(\mathfrak{N})$, all $\{\zeta_n\}$ as above and some η_0, does T have boundary values in the sense of Definition 1.3? Here $y' = y/|y|$.

As before, for the answer to be affirmative, we need an assumption on the Fourier transform of η_0. Also, because of the geometry of the support of such sequences, we will need to modify $\mathcal{O}$.

Given $\mathcal{O}$ as in (3.1.0)′, we denote by $\mathcal{O}' \subset \mathcal{O}$ any open set of the form:

$$\mathcal{O}' = \mathfrak{N} \times V \tag{3.1.2}$$

where V is an open cone in $\mathbf{R}^l$ satisfying the following property:

(3.1.3) *There exists an open cone $W \subset \mathbf{R}^l$ so that $\mathfrak{N} \times W \subset \mathcal{O}$ and $V^c \backslash \{0\} \subset W$.*

Clearly, if we can take $W = \mathbf{R}^l \backslash \{0\}$ in (3.1.0)′, then we can take $\mathcal{O}' = \mathcal{O}$. We have then, the following:

THEOREM 3.1.1. *Let $\mathfrak{M}$, $\mathfrak{N}$, T, W and $\mathcal{O}$ be as in (3.1.0)′, $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$. Let $\eta_0 \in C_0^\infty(\mathbf{R}^+)$ satisfy:*

$$\int t^{l-1}\eta_0(t)\,dt = 1, \qquad \mathcal{F}(e^{lt}\eta(e^t))(s) \neq 0, \text{ for } s \in \mathbf{R}. \tag{3.1.4}$$

Assume that for all $\nu \in \mathfrak{B}(\mathfrak{N})$,

$$\lim_{n\to\infty} \langle T, \nu \otimes n^l \eta_0(n|y|)\zeta_n(y')\,dy\rangle \tag{3.1.5}$$

exists and has value independent of $\{\zeta_n\}$ where $\{\zeta_n\} \subset C^\infty(S^{l-1})$ satisfy:

$$\int \zeta_n\,dy' = 1, \qquad \operatorname{supp}\zeta_n \subset W \cap S^{l-1}, \qquad \int \left|\left(\frac{\partial}{\partial y'}\right)^\alpha \zeta_n\right| dy' \leqslant c_{|\alpha|} \quad \text{for all } n. \tag{3.1.6}$$

Then for any $\mathcal{O}' \subset \mathcal{O}$ as in (3.1.2), $T_{\mathfrak{N},\mathcal{O}'}$ exists. Also, if $\mathcal{O}'_1$, $\mathcal{O}'_2$ are two such sets, then $T_{\mathfrak{N},\mathcal{O}'_1} = T_{\mathfrak{N},\mathcal{O}'_2}$.

We will prove Theorem 3.1.1 following the same pattern as the proof of Theorem 2.1.1. But first, we express the seminorms in (1.1) in local spherical coordinates in y.

If $y_0 \in S^{l-1}$, let U' be any S^{l-1} neighborhood of y_0 diffeomorphic to an open set in $\mathbf{R}^{l-1}$. For example, let $U' = S^{l-1}\backslash\{y\}$ where $y \neq y_0$. Let Ψ: $(y'_1, \ldots, y'_l) \to (\theta_1(y'), \ldots, \theta_{l-1}(y'))$ denote this diffeomorphism. Then $\{(r = |y|, \theta_1, \ldots, \theta_{l-1})\}$ can be used as coordinates on the cone U generated by U'. By induction, one can show that on U,

$$\left(\frac{\partial}{\partial y}\right)^\beta = \sum_{|\alpha|+j\leqslant|\beta|} h_{\alpha,j}(y)\left(\frac{\partial}{\partial\theta}\right)^\alpha\left(\frac{\partial}{\partial r}\right)^j \tag{3.1.7}$$

where $h_{\alpha,j}(y)$ are homogeneous of degree $j - |\beta|$. Hence, on U, if $|\beta| = |\beta'|$, $r = |y|$, we have

$$y^\beta\left(\frac{\partial}{\partial y}\right)^{\beta'} = \sum_{|\alpha|+j\leqslant|\beta|} a_{\alpha,j}(y)\left(\frac{\partial}{\partial\theta}\right)^\alpha r^j\left(\frac{\partial}{\partial r}\right)^j \tag{3.1.8}$$

where $a_{\alpha,j}$ are homogeneous functions of degree 0. So if $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$, then locally,

$$|\langle T, \mu\rangle| \leqslant c \sum_{j+|\alpha|+|\beta|\leqslant M} \left\|\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial\theta}\right)^\beta r^j\left(\frac{\partial}{\partial r}\right)^j \varphi(x, r\theta)J(r,\theta)\right\|_1 \tag{3.1.9}$$

where $\mu = \varphi(x,y)\,dx\,dy$, $J(r,\theta) = F(\theta)r^{l-1}$ is the Jacobian of the transformation $(r,\theta) \to y$, $F \in C^\infty(\mathbf{R}^{l-1})$ and the L^1 norm is with respect to $dx\,d\theta\,dr$.

PROPOSITION 3.1.2. *Let $\eta_0 \in C_0^\infty(\mathbf{R}^+)$, $\{\zeta_n\} \subset C^\infty(S^{l-1})$, $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$ satisfy the assumptions of Theorem 3.1.1. Then for any $\eta \in C_0^\infty(\mathbf{R}^+)$ with $\int t^{l-1}\eta(t)\,dt = 1$, and any $\nu \in \mathfrak{B}(\mathfrak{N})$, we have that*

$$\lim_{n\to\infty} \langle T, \nu \otimes n^l \eta(n|y|)\zeta_n(y')\,dy\rangle \tag{3.1.10}$$

exists and is equal to the limit in (3.1.5).

PROOF. Applying Lemma 2.1.2 to $e^{lu}\eta(e^u)$, $e^{lu}\eta_0(e^u)$, we have that for each $N \in \mathbf{Z}^+$, and $\delta > 0$, there is a $\xi_1, \ldots, \xi_k, s_1, \ldots, s_k$ so that

$$\eta(e^u)e^{lu} = \sum_{i=1}^{k} \xi_i \eta_0(e^{u-s_i})e^{lu-ls_i} + r(u) \tag{3.1.11}$$

where $\|(d/du)^j r\|_1 < \delta, j = 0, \ldots, N$. And, since $\int t^{l-1}\eta = \int t^{l-1}\eta_0 = 1$, we have that $|1 - \sum_{i=1}^{k} \xi_i| < \delta$. Letting $e^u = nt$ in (3.1.11) we obtain

$$n^l\eta(nt)\zeta_n(y') = \sum_{i=1}^{k} \xi_i \eta_0\left(\frac{nt}{e^{s_i}}\right)\left(\frac{n}{e^{s_i}}\right)^l \zeta_n(y') + n^l g(t)\zeta_n(y') \tag{3.1.12}$$

where $g(t) = r(\ln[nt])/(nt)^l$. Hence by (3.1.9), if $\nu = \varphi(x)\,dx \in \mathfrak{B}(\mathfrak{N})$,

$$\begin{aligned} &\left|\left\langle T, \nu \otimes n^l\eta(n|y|)\zeta_n(y')\,dy - \sum_{i=1}^{k} \xi_i \nu \otimes \left(\frac{n}{e^{s_i}}\right)^l \eta_0\left(\frac{n|y|}{e^{s_i}}\right)\zeta_n(y')\,dy\right\rangle\right| \\ &\leqslant c \sum_{\substack{|\alpha|+|\beta|+j\leqslant M \\ i=1,2}} \left\|\left(\frac{\partial}{\partial x}\right)^\alpha \varphi\right\|_1 \left\|\left(\frac{\partial}{\partial \theta}\right)^\beta \varphi_i(\theta)\zeta_n(\theta)\right\|_1 \left\|t^{j+l-1}\left(\frac{d}{dt}\right)^j g(t)n^l\right\|_1 \end{aligned} \tag{3.1.13}$$

where $\{\varphi_1, \varphi_2\}$ is a partition of unity over S^{l-1} so we can use (3.1.9), and the L^1 norms are taken with respect to dx, $d\theta$ and dt respectively. Choosing N in Lemma 2.1.2 equal to M and letting $x = \ln[nt]$, we have

$$\left\|t^{j+l-1}\left(\frac{d}{dt}\right)^j g(t)n^l\right\|_1 \leqslant c \sum_{i=1}^{j} \left\|e^{lx}\left(\frac{d}{dx}\right)^i \left[\frac{r(x)}{e^{lx}}\right]\right\|_1 \leqslant c \sum_{i=1}^{j} \left\|\left(\frac{d}{dx}\right)^i r\right\|_1 \leqslant c\delta.$$

Hence by assumption (3.1.6) on $\{\zeta_n\}$, we have that (3.1.13) is bounded by $c\delta$. So

$$\begin{aligned} &\left[\overline{\lim_n} - \underline{\lim_n}\right]\left|\left\langle T, \nu \otimes n^l\eta(n|y|)\zeta_n(y')\,dy - \sum_{i=1}^{k} \xi_i \nu \otimes \left(\frac{n}{e^{s_i}}\right)\eta_0\left(\frac{n|y|}{e^{s_i}}\right)\zeta_n(y')\,dy\right\rangle\right| \\ &= \left[\overline{\lim_n} - \underline{\lim_n}\right]\left|\left\langle T, \nu \otimes n^l\eta(n|y|)\zeta_n(y')\,dy - \left(\sum_{i=1}^{k} \xi_i\right)\nu \otimes n^l\eta_0(n|y|)\zeta_n(y')\,dy\right\rangle\right| \leqslant c\delta. \end{aligned} \tag{3.1.14}$$

Hence, since $|1 - \sum_{i=1}^{k} \xi_i| < \delta$ and $|\langle T, \nu \otimes n^l\eta_0(n|y|)\zeta_n(y')\,dy\rangle| \leqslant c$, we have by (3.1.14) that

$$\begin{aligned} &\left[\overline{\lim_n} - \underline{\lim_n}\right]\left|\left\langle T, \nu \otimes n^l\eta(n|y|)\zeta_n(y')\,dy - \nu \otimes n^l\eta_0(n|y|)\zeta_n(y')\,dy\right\rangle\right| \\ &\leqslant \left[\overline{\lim_n} - \underline{\lim_n}\right]\left|\left\langle T, \nu \otimes n^l\eta(n|y|)\zeta_n(y')\,dy - \left(\sum_{i=1}^{k} \xi_i\right)\nu \otimes n^l\eta_0(n|y|)\zeta_n(y')\,dy\right\rangle\right| \\ &\quad + \left[\overline{\lim_n} - \underline{\lim_n}\right]\left|\left\langle T, \left(1 - \sum_{i=1}^{k} \xi_i\right)\nu \otimes n^l\eta_0(n|y|)\zeta_n(y')\,dy\right\rangle\right| \leqslant c\delta. \end{aligned} \tag{3.1.15}$$

Hence, the limit in (3.1.10) exists and equals the limit in (3.1.5). ■

In order to generalize Proposition 3.1.2 to more general permissible sequences, we construct a smooth sequence S_m so that $S_m \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}'}$, $\mathcal{O}'$ as before in (3.1.2). For this, let $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$, T

satisfy the hypotheses of Theorem 3.1.1. Let $\mathcal{O}'$ be any set given by (3.1.2). We consider two cases:

1) If $W \neq \mathbf{R}^l \backslash \{0\}$, then $W \cap S^{l-1} \neq S^{l-1}$. Hence, there is a C^∞ diffeomorphism, Ψ: $W \cap S^{l-1} \to \mathbf{R}^{l-1}$. Let $\zeta \in C_0^\infty(\mathbf{R}^{l-1})$, supp $\zeta \subset \{$neighborhood of 0$\}$, $\int \zeta\, dx = 1$. Let $\zeta_m(x) = m^{l-1}\zeta(mx)$, $h(z') = 1/F(\Psi(z'))$, where F is as in (3.1.9). Let $\eta \in C_0^\infty(\mathbf{R})$, $0 \notin \operatorname{supp} \eta$ and assume that $\int \eta(s)\, ds = 1$. Also, let $\varphi \in C_0^\infty(\mathbf{R}^k)$, $\int \varphi = 1$ and denote $m^k\varphi(mx)$ by $\varphi_m(x)$. Assume that ζ, η, φ are even functions. Finally, for $(x,y) \in \mathcal{O}'$, we define

$$(3.1.16) \qquad T'_m(x,y) = \left\langle T_{wz}, \varphi_m(x-w)\eta\left(m \ln \frac{|z|}{|y|}\right)\frac{m}{|z|^l}\zeta_m(\Psi(y') - \Psi(z'))h(z')\, dw\, dz\right\rangle$$

where as usual, $y' = y/|y|$, $z' = z/|z|$.

2) If $W = \mathbf{R}^l \backslash \{0\}$, let $\chi_1, \chi_2 \in C^\infty(S^{l-1})$ be a partition of unity of S^{l-1} so that supp $\chi_i \neq S^{l-1}$, $i = 1, 2$. Let Ψ_i be C^∞ diffeomorphisms from a neighborhood of supp χ_i to $\mathbf{R}^{l-1}$, $h_i(z') = 1/F_i(\Psi_i(z'))$, where F_i is as above only defined via Ψ_i. Let ζ, η, φ be as in 1). Then for $(x,y) \in \mathcal{O}$ we define

$$(3.1.17) \qquad T_m(x,y) = \sum_{i=1}^{2} \left\langle \chi_i T_{wz}, \varphi_m(x-w)\eta\left(m \ln \frac{|z|}{|y|}\right)\frac{m}{|z|^l}\zeta_m(\Psi_i(y') - \Psi_i(z'))h_i(z')\, dw\, dz\right\rangle.$$

We now show that T_m, T'_m has the required properties.

PROPOSITION 3.1.3. *Let $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ satisfy the hypotheses of Theorem 3.1.1. Then T_m [respectively T'_m] is C^∞ on $\mathcal{O}$ [resp. $\mathcal{O}'$], continuous to $\mathfrak{N} \times \{0\}$ and $T_m \to T$ in $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$*[tid! $T'_m \to T$ in $\mathcal{R}_{\mathfrak{N},\mathcal{O}'}$].

PROOF. Without loss of generality, assume supp $\eta \subset \{t \mid 1 \leqslant |t| \leqslant 2\}$. Then for any y, we have that

$$\operatorname{supp} \eta\left(m \ln \frac{|z|}{|y|}\right) \cdot \frac{m}{|z|^l} \subset \{z \mid\, |y|e^{-2m} \leqslant |z| \leqslant |y|e^{2m}\},$$

and supp $\zeta_m(\Psi_i(y') - \Psi_i(z'))$ is contained in an S^{l-1}-neighborhood of y' whose diameter decreases to 0 as $m \to \infty$. Hence, since $(\operatorname{supp} T)^c$ is compact, $T_m(x,y)$ is well defined and C^∞ on $\mathcal{O}$ since we can differentiate under the 'integral.' As for T'_m, note that since $\Psi(V \cap S^{l-1})$ is compactly contained in $\Psi(W \cap S^{l-1})$, we have that for m large, if $(x,y) \in \mathcal{O}'$,

$$\operatorname{supp}\left[\varphi_m(x-w)\eta\left(m \ln \frac{|z|}{|y|}\right)\frac{m}{|z|^l}\zeta_m(\Psi(y') - \Psi(z'))h(z')\, dw\, dz\right] \subset \mathfrak{N} \times W.$$

Hence, $T'_m(x,y)$ is well defined and C^∞ on $\mathcal{O}'$.

To show that T'_m is continuous to $\mathfrak{N} \times \{0\}$, we first show that for each x, m,

$$(3.1.18) \qquad T'_m(x,0) \equiv \lim_{\substack{y \to 0 \\ y \in \mathcal{O}}} T'_m(x,y) \qquad \text{exists}.$$

For this, let x, m be fixed. Define

$$\mu_y(w,z) = \varphi_m(x-w)\eta\left(m \ln \frac{|z|}{|y|}\right)\frac{m}{|z|^l}\zeta_m(\Psi(y') - \Psi(z'))h(z')\, dw\, dz.$$

Then $T'_m(x,y) = \langle T, \mu_y(w,z)\rangle$ and if we let $\bar{\eta}(t) = \eta(m \ln t)m/t^l$, $t > 0$, we have that $\bar{\eta} \in C_0^\infty(\mathbf{R}^+)$ and $\int t^{l-1}\bar{\eta}(t)\, dt = 1$. Note that we define $\bar{\eta} = 0$ for $t \leqslant 0$. Also, if we let $\bar{\zeta}_y(z') = \zeta_m(\Psi(y') - \Psi(z'))h(z')$, then for m large enough, supp $\bar{\zeta}_y(z') \subset W$ for all $y \in V$. Now for each y, $\int_{S^{l-1}} \bar{\zeta}_y(z')\,dz' = 1$ and for each α we have

$$\int_{S^{l-1}} \left|\left(\frac{\partial}{\partial z'}\right)^{\alpha} \bar{\zeta}_y(z')\right| dz' \leqslant c_{|\alpha|}$$

independently of y. Hence,

$$\mu_y(w,z) = \varphi_m(x-w)\bar{\eta}\left(\frac{|z|}{|y|}\right)\frac{1}{|y|^l}\bar{\zeta}_y(z')\,dw\,dz$$

is easily seen to be a permissible sequence converging to $\varphi_m(x-w)\,dw$ as $y \to 0$, so by Proposition 3.1.2, $T'_m(x,0)$ exists. Now

$$\frac{\partial}{\partial x_i} T'_m(x,y) = \left\langle T, \frac{\partial}{\partial x_i}[\varphi_m(x-w)]\eta\left(m \ln \frac{|z|}{|y|}\right)\frac{m}{|z|^l}\zeta_m(\Psi(y') - \Psi(z'))h(z')\,dw\,dz\right\rangle.$$

Hence, using (3.1.9), a calculation shows that

$$\left|\left(\frac{\partial}{\partial x_i}\right)T'_m(x,y)\right| \leqslant c_m \qquad \text{for all } y,\ x. \tag{3.1.19}$$

So if $(x,y) \in \mathcal{O}'$, $(x_0,0) \in \mathfrak{N}\times\{0\}$, we have by (3.1.19), that

$$\begin{aligned}|T'_m(x,y) - T'_m(x_0,0)| &\leqslant |T'_m(x,y) - T'_m(x_0,y)| + |T'_m(x_0,y) - T'_m(x_0,0)| \\ &\leqslant \sum_{i=1}^{k} \left|\frac{\partial}{\partial x_i} T'_m(x,y)\right| |x - x_0| + |T'_m(x_0,y) - T'_m(x_0,0)| \to 0\end{aligned}$$

as $(x,y) \to (x_0,0)$.

Hence, $T'_m(x,y)$ is continuous to $\mathfrak{N}\times\{0\}$ and $T'_m(x,0)$ is given by (3.1.18). A similar argument proves the same statement of $T_m(x,y)$.

Finally, we show that $T'_m \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}'}$. A similar argument will show that $T_m \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}}$. For this, let $\mu = \psi(x,y)\,dx\,dy \in \mathfrak{B}(\mathcal{O}')$. Then since ψ has compact support, we have that,

$$\begin{aligned}\int T'_m(x,y)\psi(x,y)\,dx\,dy &= \left\langle T, \left[\int \psi(x,y)\varphi_m(x-w)\eta\left(m \ln \frac{|z|}{|y|}\right)\frac{m}{|z|^l}\right.\right. \\ &\quad \left.\left.\times\, \zeta_m \Psi(y') - \Psi(z'))\,dx\,dy\right]h(z')\,dw\,dz\right\rangle.\end{aligned} \tag{3.1.20}$$

For m sufficiently large,

$$\operatorname{supp}\left[\int \psi(x,y)\varphi_m(x-w)\eta\left(m \ln \frac{|z|}{|y|}\right)\frac{m}{|z|^l}\zeta_m(\Psi(y') - \Psi(z'))h(z')\,dx\,dy\right] \subset \mathcal{O}'.$$

Hence, since $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}'}$ we have that

$$\begin{aligned}|\langle T'_m - T, \mu\rangle| &\leqslant c \sum_{\substack{|\alpha|+|\beta|\leqslant M \\ |\beta|=|\beta'|}} \int \left|\left(\frac{\partial}{\partial w}\right)^{\alpha} z^{\beta}\left(\frac{\partial}{\partial z}\right)^{\beta'}\right. \\ &\quad \left[\int \psi(x,y)\varphi_m(x-w)\eta\left(m \ln \frac{|z|}{|y|}\right)\frac{m}{|z|^l}\zeta_m(\Psi(y') - \Psi(z'))h(z')\,dx\,dy\right. \\ &\quad \left.\left. -\, \psi(w,z)\right]\right| dw\,dz.\end{aligned} \tag{3.1.21}$$

Applying (3.1.9) to (3.1.21), and letting $|y| = e^s$, $|z| = e^t$, $u = \Psi(y')$, $v = \Psi(z')$ we have that (3.1.21) is bounded by

$$c \sum_{|\alpha|+|\beta|+j \leqslant M} \int \left| \left(\frac{\partial}{\partial w}\right)^{\alpha} \left(\frac{\partial}{\partial v}\right)^{\beta} \left(\frac{\partial}{\partial t}\right)^{j} \right.$$

(3.1.22)
$$[\int \varphi_m(x-w)\eta_m(s-t)\zeta_m(u-v)\psi(x, e^s\Psi^{-1}(u))F(u)e^{ls}\, dx\, ds\, du$$
$$\left. - \psi(w, e^t\Psi^{-1}(v))F(v)e^{lt}] \right| dw\, dt\, dv$$

where $\eta_m(s) = m\eta(ms)$ and F is defined as in (3.1.9). Now since φ, η and ζ are even functions, we can apply Lemma 2.1.5 to obtain that (3.1.22) is bounded by

$$(3.1.23) \qquad c_m \sum_{|\alpha|+|\beta|+j \leqslant M+1} \int \left| \left(\frac{\partial}{\partial w}\right)^{\alpha} \left(\frac{\partial}{\partial v}\right)^{\beta} \left(\frac{\partial}{\partial t}\right)^{j} [\psi(w, e^t\Psi^{-1}(v))F(v)e^{lt}] \right| dw\, dt\, dv$$

where $c_m \to 0$ as $m \to \infty$. Letting $|z| = e^t$, we have by induction that

$$\left(\frac{\partial}{\partial t}\right)^{j} = \sum_{i=1}^{j} c_{ij} |z|^i \left(\frac{\partial}{\partial |z|}\right)^{i}.$$

Hence, substituting $|z| = e^t$ and observing that $(\partial/\partial t)^i[f(t)e^{lt}] = [\sum_{k=0}^{i} c_{ik}(\partial/\partial t)^k f]e^{lt}$, we have that (3.1.23) is bounded by

$$(3.1.24) \qquad c_m \sum_{|\alpha|+|\beta|+j \leqslant M+1} \int \left| \left(\frac{\partial}{\partial w}\right)^{\alpha} |z|^{j+l-1} \left(\frac{\partial}{\partial |z|}\right)^{j} \left(\frac{\partial}{\partial v}\right)^{\beta} [\psi(w, |z|\Psi^{-1}(v))F(v)]\, dw\, d|z|\, dv. \right.$$

Now

$$\left(\frac{\partial}{\partial v}\right)^{\beta} = \sum_{|\gamma| \leqslant |\beta|} |z|^{|\gamma|} c_{\gamma}(v) \left(\frac{\partial}{\partial z}\right)^{\gamma}$$

where $c_{\gamma} \in C^{\infty}(\mathbf{R}^{l-1})$. Hence, the integral in (3.1.24) is dominated by

$$(3.1.25) \qquad c_m \sum_{\substack{|\alpha|+|\gamma|+j \leqslant M+1 \\ k=j+|\gamma|}} \int \left| \left(\frac{\partial}{\partial w}\right)^{\alpha} |z|^{k+l-1} \left(\frac{\partial}{\partial |z|}\right)^{j} \left(\frac{\partial}{\partial z}\right)^{\gamma} [\psi(w, |z|\Psi^{-1}(v))F(v)] \right| dw\, d|z|\, dv.$$

Finally, letting $v = \Psi(z')$ and noting that

$$\left(\frac{\partial}{\partial |z|}\right)^{j} = \sum_{|\beta|=j} c_{\beta} z^{\beta} \left(\frac{\partial}{\partial z}\right)^{\beta}; \qquad |z| \leqslant c \sum_{i=1}^{l} |z_i|$$

and $F \in C^{\infty}(\mathbf{R}^{l-1})$, we see that the integral in (3.1.35) is dominated by

$$(3.1.26) \qquad c_m \sum_{\substack{|\alpha|+|\beta| \leqslant M+1 \\ |\beta|=|\beta'|}} \left\| \left(\frac{\partial}{\partial w}\right)^{\alpha} z^{\beta} \left(\frac{\partial}{\partial z}\right)^{\beta'} \psi \right\|_1 .$$

Hence, $T'_m \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}'}$ as claimed, completing the proof of the proposition. ■

Henceforth, we will denote by T_m the approximation constructed in Proposition 3.1.3. Choose $S_m \in C_0^\infty(\mathbf{R}^{k+l})$ so that

$$\sup|S_m - T_m| < \frac{1}{m} \tag{3.1.27}$$

on $\mathcal{O}'$ or $\mathcal{O}$ depending on W in (3.1.3). Clearly then, $S_m \to T$ in $\mathcal{R}_{\mathfrak{N},\mathcal{O}'}$ (respectively $\mathcal{R}_{\mathfrak{N},\mathcal{O}}$).

PROOF OF THEOREM 3.1.1. Using S_m in (3.1.27), the proof is the same as that of Theorem 2.1.1. ■

Let $U = \{(x,y)\}$, $V = \{(w,z)\}$ be bounded open sets in $\mathbf{R}^{k+l}$ or coordinate neighborhoods on two manifolds. Let $\mathfrak{N}_1$ be the slice $\{y = 0\}$ of U, $\mathfrak{N}_2$ the slice $\{z = 0\}$ of V. Let χ: $U \to V$ be a diffeomorphism, χ: $\mathfrak{N}_1 \to \mathfrak{N}_2$, that extends diffeomorphically to a neighborhood of U^c and let $\chi|_{\mathfrak{N}_1} = \bar{\chi}$ be the diffeomorphism, $\bar{\chi}$: $\mathfrak{N}_1 \to \mathfrak{N}_2$. Let $\mathcal{O}_2 \subset V$ satisfy (3.1.0), $\mathcal{O}_1 = \chi^{-2}(\mathcal{O}_1)$. Let $T \in \mathcal{B}'(\mathcal{O}_2)$. Then if $\mu = \varphi(x,y)\,dx\,dy \in \mathcal{B}(\mathcal{O}_1)$, we define as before:

$$\langle\chi^* T, \mu\rangle = \left\langle T, \varphi(\chi^{-1}(w,z))\left|\frac{\partial\chi^{-1}}{\partial(w,z)}\right| dw\,dz\right\rangle. \tag{3.1.28}$$

By Proposition 1.1, if $T \in \mathcal{R}_{\mathfrak{N}_2,\mathcal{O}_2}$, then $\chi^* T \in \mathcal{R}_{\mathfrak{N}_1,\mathcal{O}_1}$.

PROPOSITION 3.1.4. *In the above notation, if $T_{\mathfrak{N}_2,\mathcal{O}_2}$ exists, then $(\chi^* T)_{\mathfrak{N}_1,\mathcal{O}_1}$ exists, and for all $\nu \in \mathcal{B}(\mathfrak{N}_1)$,*

$$\langle(\chi^* T)_{\mathfrak{N}_1,\mathcal{O}_1}, \nu\rangle = \langle\bar{\chi}^* T_{\mathfrak{N}_2,\mathcal{O}_2}, \nu\rangle. \tag{3.1.29}$$

PROOF. The same as the proof of Proposition 2.1.6. ■

2. Restrictions of Generalized Functions

It's now relatively easy to obtain a theorem for restrictions corresponding to Theorem 3.1.1. We assume that $\mathcal{O} \subset \mathbf{R}^{k+l}$ is open, $\mathbf{R}^{k+l} = \{(x,y)\}$ and $\mathcal{O} \cap \{(x,0)\} = \mathfrak{N}$. Let $\mathcal{O}_0 = \mathcal{O}\backslash\mathfrak{N}$.

THEOREM 3.2.1. *Let $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$. Let $\eta_0 \in C_0^\infty(\mathbf{R}^+)$ satisfy (3.1.2). Assume that for all $\nu \in \mathcal{B}(\mathfrak{N})$,*

$$\lim_{n\to\infty} \langle T, \nu \otimes n^l\eta_0(n|y|)\zeta_n(y')\,dy\rangle \tag{3.2.1}$$

exists and has value independent of $\{\zeta_n\}$ where $\{\zeta_n\}$ satisfy:

$$\int_{S^{l-1}} \zeta_n\,d\sigma = 1, \qquad \zeta_n \in C^\infty(S^{l-1}), \qquad \int_{S^{l-1}} \left|\left(\frac{\partial}{\partial y'}\right)^\alpha \zeta_n\right| d\sigma \leqslant c_{|\alpha|} \qquad \textit{for all } n. \tag{3.2.2}$$

Then T has a restriction to $\mathfrak{N}$ in the sense of Definition 1.3 whose value is given by (3.2.1) for all ν.

PROOF. Clearly, $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}_0}$ and by Theorem 3.1.1, $T_{\mathfrak{N},\mathcal{O}_0}$ exists. Let $\{\mu_n\} \subset \mathcal{B}(\mathcal{O})$ be a permissible sequence converging to $\nu \in \mathcal{B}(\mathfrak{N})$. Say $\mu_n = \varphi_n(x,y)\,dx\,dy$, $\nu = \varphi(x)\,dx$. Let $\chi \in C^\infty(\mathbf{R})$ so that $\chi = 1$ for $t > 1$, $\chi = 0$ for $t < 1/2$. Then for all $\mu \in \mathcal{B}(\mathcal{O})$,

$$\langle T, \chi(\tfrac{1}{\epsilon}|y|)\mu\rangle \to \langle T, \mu\rangle \qquad \text{as } \epsilon \to 0. \tag{3.2.3}$$

To see this, note that since $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$,

$$\left|\left\langle T,\left(1-\chi\left(\frac{|y|}{\epsilon}\right)\right)\mu\right\rangle\right| \leqslant c \sum_{\substack{|\alpha|+|\beta|\leqslant M \\ |\beta|=|\beta'|}} \left\|\left(\frac{\partial}{\partial x}\right)^{\alpha} y^{\beta}\left(\frac{\partial}{\partial y}\right)^{\beta'}\left[\left(1-\chi\left(\frac{|y|}{\epsilon}\right)\right)\varphi\right]\right\|_1$$

$$\leqslant c \sum_{|\alpha|+|\beta|\leqslant M} \left\|\left(1-\chi\left(\frac{|y|}{\epsilon}\right)\right)\left(\frac{\partial}{\partial x}\right)^{\alpha} y^{\beta}\left(\frac{\partial}{\partial y}\right)^{\beta'}\varphi\right\|_1$$

$$+ c \sum_{\substack{|\alpha|+|\beta|+|\gamma|\leqslant M \\ |\gamma|>0}} \left\|\left[y^{\gamma}\left(\frac{\partial}{\partial y}\right)^{\gamma'}\left(1-\chi\left(\frac{|y|}{\epsilon}\right)\right)\right]\right.$$

$$\times\left.\left[\left(\frac{\partial}{\partial x}\right)^{\alpha} y^{\beta}\left(\frac{\partial}{\partial y}\right)^{\beta'}\varphi\right]\right\|_1 = I + II.$$

Clearly, $I \to 0$ as $\epsilon \to 0$. Writing $y^{\gamma}(\partial/\partial y)^{\gamma'}$ by (3.1.8), differentiating, and letting $|y| = \epsilon r$, we obtain that

$$II \leqslant c\epsilon \sum_{\substack{|\alpha|+|\beta|\leqslant m \\ 0<j<m}} \int_{\frac{1}{2}}^{1} \iint r^{j}|\chi^{j}(r)\varphi_{\alpha\beta\beta'}(x,\epsilon r y')|(\epsilon r)^{l-1}\,dx\,dy'\,dr$$

$$\text{where } \varphi_{\alpha\beta\beta'} = \left(\frac{\partial}{\partial x}\right)^{\alpha} y^{\beta}\left(\frac{\partial}{\partial y}\right)^{\beta'}\varphi.$$

Hence, $II \to 0$ with ϵ. That is, $\langle T, \chi(|y|/\epsilon)\mu\rangle \to \langle T,\mu\rangle$ as $\epsilon \to 0$. So for each n, choose ϵ_n with

$$\left|\langle T,\mu_n\rangle - \left\langle T,\chi\left(\frac{|y|}{\epsilon_n}\right)\mu_n\right\rangle\right| < \frac{1}{n}.$$

Then $\lim_n\langle T,\mu_n\rangle$ exists if and only if $\lim_n\langle T,\chi(|y|/\epsilon_n)\mu_n\rangle$ exists. But a calculation shows that $\chi(|y|/\epsilon_n)\mu_n$ is a permissible sequence on $\mathcal{O}_0$ converging to ν. Hence, $\lim_n\langle T,\chi(|y|/\epsilon_n)\mu_n\rangle$ exists and equals the limit in (3.2.1). Therefore, the same is true of $\lim_n\langle T,\mu_n\rangle$. That is, $T_{\mathfrak{N},\mathcal{O}}$ exists. ■

PROPOSITION 3.2.2. *Let T_n, $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$ and assume that $T_n \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}_0}$. Then $T_n \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}}$.*

PROOF. The same as the proof of Proposition 2.2.1 only we estimate I in that proof as II was estimated in the preceding proof. ■

COROLLARY 3.2.3. *Let $T \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$ and assume that $T_{\mathfrak{N},\mathcal{O}_0}$ exists. Then there exists $\{S_m\} \subset C_0^{\infty}(\mathcal{O})$ so that $S_m \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}}$.*

PROOF. Choose $\{S_m\} \subset C_0^{\infty}(\mathcal{O})$ as in (3.1.27) where $S_m \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}_0}$. $\{S_m\}$ exists since $T_{\mathfrak{N},\mathcal{O}_0}$ does. Then by Proposition 3.2.2, we have that $S_m \to T$ in $\mathfrak{R}_{\mathfrak{N},\mathcal{O}}$. ■

Let $U = \{(x,y)\}$, $V = \{(w,z)\}$ be bounded open sets in $\mathbf{R}^{k+l}$, χ *a* C^{∞} diffeomorphism, $\chi\colon U \to V$ so that $\chi\colon (x,0) \to (w,0)$. Assume χ extends diffeomorphically to a neighborhood of U^c. Let $\mathfrak{N}_1$, respectively $\mathfrak{N}_2$, be the slice $\{y = 0\}$ of U, respectively $\{z = 0\}$ of V. Let $\bar{\chi}$ be the induced diffeomorphism, $\bar{\chi}\colon \mathfrak{N}_1 \to \mathfrak{N}_2$. Let $\mathcal{O}_2 \subset V$ so that $\mathfrak{N}_2 \cap \mathcal{O}_2 \neq \varnothing$. Let $\mathcal{O}_1 = \chi^{-1}(\mathcal{O}_2)$.

PROPOSITION 3.2.4. *Using the above notation, let $T \in \mathfrak{R}_{\mathfrak{N}_2,\mathcal{O}_2}$ and assume $T_{\mathfrak{N}_2,\mathcal{O}_2}$ exists. Then $(\chi^* T)_{\mathfrak{N}_1,\mathcal{O}_1}$ exists and*

$$\langle(\chi^* T)_{\mathfrak{N}_1,\mathcal{O}_1},\nu\rangle = \langle\bar{\chi}^* T_{\mathfrak{N}_2,\mathcal{O}_2},\nu\rangle \qquad \text{for all } \nu \in \mathfrak{B}(\mathfrak{N}_1). \tag{3.2.4}$$

PROOF. Identical to the proof of Proposition 2.1.6. ■

The proofs of the following two Propositions are the same as those of Proposition 2.2.5 and Proposition 2.2.6 respectively, and are omitted.

PROPOSITION 3.2.5. *Let* $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}_0}$ *and assume that for all* $\nu \in \mathcal{B}(\mathfrak{N})$, $\eta \in C_0^\infty(\mathbf{R}^l)$ *we have that*

$$|\langle T, \nu \otimes \frac{1}{\epsilon^l}\eta(\frac{y}{\epsilon})\,dy\rangle| \leqslant c_{\nu\eta} \qquad \text{for all } \epsilon. \tag{3.2.5}$$

Then $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$. *Further if* $T_{\mathfrak{N},\mathcal{O}_0}$ *exists, so does* $T_{\mathfrak{N},\mathcal{O}}$ *and* $T_{\mathfrak{N},\mathcal{O}} = T_{\mathfrak{N},\mathcal{O}_0}$.

PROPOSITION 3.2.6. *Let* $T \in \mathcal{R}_{\mathfrak{N},\mathcal{O}_0}$ *and assume that* $T_{\mathfrak{N},\mathcal{O}_0}$ *exists. Then there exists an* $S \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ *so that* $S = T$ *on* $\mathcal{O}_0$, $S_{\mathfrak{N},\mathcal{O}}$ *exists and equals* $T_{\mathfrak{N},\mathcal{O}_0}$.

CHAPTER IV
EXISTENCE OF RESTRICTIONS AND PRODUCTS

1. A Refinement of the Wave-Front Set of a Distribution

Let X be a C^{∞} second countable manifold. We recall the definition of the wave front set of a distribution $D \in \mathcal{D}'(X)$ and several of its properties.

DEFINITION 4.1.1. *Let $X \subset \mathbf{R}^n$ be open, $D \in \mathcal{D}'(X)$. Then the wave front set of D, denoted $WF(D)$, is defined as the complement in $X \times (\mathbf{R}^n \backslash \{0\})$ of*

$$\{(x_0,\xi_0) \mid \textit{there exists neighborhoods } U_{x_0}, V_{\xi_0} \textit{ so that for all } \Phi \in C_0^{\infty}(U_{x_0}), \textit{ for all } N \in \mathbf{Z}^+, (\Phi D)\hat{}\,(\tau\xi) = \mathcal{O}(\tau^{-N}) \textit{ uniformly in } \xi \in V_{\xi_0}\}. \tag{4.1.1}$$

It is clear that

$$\text{sing supp } D = \pi(WF(D)) \tag{4.1.2}$$

where $\pi\colon X \times (\mathbf{R}^n \backslash \{0\}) \to X$, $\pi(x,\xi) \to x$ and sing supp D is defined as the complement in X of

$$\{x \mid \exists U_x \text{ with } D \in C^{\infty}(U)\}. \tag{4.1.3}$$

Also, it is clear that $WF(D)$ is a closed cone in $X \times (\mathbf{R}^n \backslash \{0\})$ where by conic, we mean:

$$(x_0,\xi_0) \in WF(D) \Rightarrow (x_0,\tau\xi_0) \in WF(D) \qquad \text{for all } \tau \in \mathbf{R}^+. \tag{4.1.4}$$

PROPOSITION 4.1.1. $(x_0,\xi_0) \notin WF(D)$ *if and only if for all real valued functions $\psi(x,a) \in C^{\infty}(\mathbf{R}^n \times \mathbf{R}^p)$ so that $d_x\psi(x_0,a_0) = \xi_0$, there are neighborhoods U_{x_0}, A_{a_0} so that for all $\Phi \in C_0^{\infty}(U)$, all N,*

$$\langle D, \Phi e^{-i\tau\psi(\cdot,a)}\rangle = \mathcal{O}(\tau^{-N}) \tag{4.1.5}$$

uniformly in $a \in A_0$.

PROOF. See [1]

Hence, if X is a manifold, $D \in \mathcal{D}'(X)$, Proposition 4.1.1 is taken as the definition of $WF(D)$. This definition is clearly coordinate invariant and agrees with Definition 4.1.1 if $X \subset \mathbf{R}^n$. With this definition, it is natural to consider $WF(D)$ as a closed conic subset of $T^*X \backslash \{0\}$. That is, the cotangent bundle of X minus the zero section.

The objective of the rest of this section will be to refine the $WF(D)$ into orders of decay and then show this refinement to be coordinate invariant. We begin with:

DEFINITION 4.1.2. *Let $X \subset \mathbf{R}^n$ be open, $x_0 \in X$, $V \subset \mathbf{R}^n$ a cone. Let $D \in \mathcal{D}'(X)$, $k \in \mathbf{R}$. We define the order of D at x_0 on V to be less than or equal to k, denoted*

$$\operatorname{Ord}_{x_0,V} D \leqslant k \tag{4.1.6}$$

if there exists a neighborhood U of x_0, an open conic neighborhood W of V so that $V^c\backslash\{0\} \subset W$ and for all $\varphi \in C_0^\infty(U)$ we have

$$|(\varphi D)\hat{}\,(\xi)| \leqslant c(1+|\xi|)^k \qquad \textit{uniformly on } W. \tag{4.1.7}$$

We say that $\operatorname{Ord}_{x_0,V} D = k$ *if*

$$k = \inf\{k' | \operatorname{Ord}_{x_0,V} D \leqslant k'\}. \tag{4.1.8}$$

If $\xi_0 \in \mathbf{R}^n$, *we define*

$$\operatorname{Ord}_{x_0,\xi_0} D = \operatorname{Ord}_{x_0,V} D \qquad \textit{where } V = \{\tau\xi_0 | \tau \in \mathbf{R}^+\}. \tag{4.1.9}$$

If $\operatorname{Ord}_{x_0,V} D \leqslant k$ *for all* $k \in \mathbf{R}$, *we say that* $\operatorname{Ord}_{x_0,V} D = -\infty$.

Note that $\operatorname{Ord}_{x_0,V} D = k$ does not imply that $\operatorname{Ord}_{x_0,V} D \leqslant k$. Also, if $V \subset V'$, we clearly have that

$$\operatorname{Ord}_{x_0,V} D \leqslant \operatorname{Ord}_{x_0,V'} D \qquad \text{for all } x_0. \tag{4.1.10}$$

Letting $g_D(x,\xi) = \operatorname{Ord}_{x,\xi} D$, we see that for each $D \in \mathcal{D}'(X)$, $g_D(x,\xi)$ is an upper semicontinuous function on $X \times \mathbf{R}^n$. That is, for each λ,

$$\{(x,\xi)|\ |g_D(x,\xi)| < \lambda\} \text{ is open in } X \times \mathbf{R}^n.$$

Also, if $D \in \mathcal{E}'(X)$,

$$\operatorname{Ord}_{x_0,\mathbf{R}^n} D \leqslant k \qquad \text{for some } k \text{ independent of } x_0. \tag{4.1.11}$$

Finally, if $D \in L_k^1(X)$,

$$\operatorname{Ord}_{x_0,\mathbf{R}^n} D \leqslant -k \qquad \text{for all } x_0. \tag{4.1.12}$$

PROPOSITION 4.1.2. *Let h, g be measurable functions on $\mathbf{R}^n$ so that $|g| \leqslant c(1+|x|)^k$, $|h| \leqslant c(1+|x|)^N$. Then if $-N$ is sufficiently large,*

$$|(h * g)(x)| \leqslant c(1+|x|)^k. \tag{4.1.13}$$

PROOF. If $k \geqslant 0$, we can choose $N = -k-n-1$ and

$$\begin{aligned}|(h*g)(x)| &\leqslant \int (1+|x-y|)^k(1+|y|)^{-k-n-1}\,dy \leqslant (1+|x|)^k \int (1+|y|)^{-n-1}\,dy \\ &\leqslant c(1+|x|)^k.\end{aligned}$$

If $k < 0$,

$$\begin{aligned}|(h*g)(x)| &\leqslant \int_{|y|\leqslant|x|/2} (1+|x-y|)^k(1+|y|)^N\,dy + \int_{|y|\geqslant|x|/2} (1+|x-y|)^k(1+|y|)^N\,dy \\ &= I + II.\end{aligned}$$

If $|y| \leqslant |x|/2$, then $|x-y| \geqslant |x|/2$ so

$$I \leqslant c(1+|x|)^k \int (1+|y|)^N \leqslant c(1+|x|)^k \qquad \text{if } N = -n-1.$$

And, in II, since $(1+|x-y|)^k \leqslant 1, \quad (1+|y|)^k \leqslant c(1+|x|)^k$,

$$II \leqslant (1+|x|)^k \int (1+|y|)^{N-k}\,dy \leqslant c(1+|x|)^k \qquad \text{if } N-k \leqslant -n-1.$$

Combining the estimates above, completes the proof. ■

COROLLARY 4.1.3. *Let $D \in \mathfrak{D}'(X)$, $x_0 \in X$. Then*

$$\operatorname{Ord}_{x_0,\mathbf{R}^n} D \leqslant k, \qquad \textit{for some } k \in \mathbf{R}. \tag{4.1.14}$$

PROOF. Let $U \subset X$ be any bounded neighborhood of x_0. Let $\chi \in C_0^\infty(\mathbf{R}^n)$, $\chi \equiv 1$ on U. Then $\chi D \in \mathcal{E}'(X)$, so by (4.1.11),

$$|(\chi D)\hat{}\,(\xi)| \leqslant c(1+|\xi|)^k \qquad \text{for some } k, \qquad \text{all } \xi.$$

Now for all $\varphi \in C_0^\infty(U)$,

$$|(\varphi D)\hat{}\,(\xi)| = |(\varphi\chi D)\hat{}\,(\xi)| = |\hat{\varphi} * (\chi D)\hat{}\,(\xi)| \leqslant c(1+|\xi|)^k$$

by Proposition 4.1.2 since $\hat{\varphi} \in \mathcal{S}(\mathbf{R}^n)$. ■

The following two propositions will be useful in showing how Definition 4.1.1 is affected by coordinate changes.

PROPOSITION 4.1.4. *Let $\overline{\varphi}$: $\mathbf{R}^n \to \mathbf{R}^n$ be smooth, and assume that $[\partial\overline{\varphi}/\partial x]_{x_0} = [0]$. Then there is a neighborhood U of x_0 so that for all $\varphi \in C_0^\infty(U)$,*

$$\int \varphi(x) e^{i\tau(x+\overline{\varphi}(x))\cdot\xi'}\,dx = \mathcal{O}(\tau^{-N}) \tag{4.1.15}$$

for all N, uniformly in $\xi' \in S^{n-1}$.

PROOF. Let

$$g_i(x,\xi') = \frac{\partial}{\partial x_i}[(x+\overline{\varphi}(x))\cdot\xi'] = \xi_i' + \sum_{j=1}^n \frac{\partial\overline{\varphi}_j}{\partial x_i}\xi_j'.$$

Since $\partial\overline{\varphi}_j/\partial x_i = 0$ for all i, j when $x = x_0$, it is clear that $\sum_{i=1}^n g_i^2(x,\xi') = 1 + h(x,\xi')$ where $|h(x,\xi')|$ can be made arbitrarily small, uniformly in $\xi' \in S^{n-1}$, by taking $|x-x_0|$ small. Choose U so that $\sum_{i=1}^n g_i^2 \neq 0$ for $x \in U$, $\xi' \in S^{n-1}$. Define

$$L = \sum_{i=1}^n \frac{g_i(x,\xi')}{\sum_j g_j^2(x,\xi')}\frac{\partial}{\partial x_i}. \tag{4.1.16}$$

Then L is well defined on $U \times S^{n-1}$ and $L[(x+\overline{\varphi}(x))\cdot\xi'] = 1$. Hence if $\varphi \in C_0^\infty(U)$,

$$\int \varphi(x) e^{i\tau(x+\overline{\varphi}(x))\cdot\xi'}\,dx = \frac{1}{i\tau}\int \varphi(x) L[e^{i\tau(x+\overline{\varphi}(x))\cdot\xi'}]\,dx = \frac{1}{i\tau}\int [L'\varphi] e^{i\tau(x+\overline{\varphi}(x))\cdot\xi'}\,dx$$

$$= \frac{1}{(i\tau)^N}\int [(L')^N\varphi] e^{i\tau(x+\overline{\varphi}(x))\cdot\xi'}\,dx.$$

Hence, $|\int \varphi(x) e^{i\tau(x+\overline{\varphi}(x))\cdot\xi'}\,dx| \leqslant c_N \tau^{-N}$ for all N, uniformly in $\xi' \in S^{n-1}$ where c_N depends on $\sup_U |(L')^N \varphi|$. ■

PROPOSITION 4.1.5. *Let f be a measurable function on $\mathbf{R}^n$ satisfying $|f(\xi)| \leqslant c(1+|\xi|)^M$ some M. Assume that on an open cone $W \subset \mathbf{R}^n$, $|f(\xi)| \leqslant c(1+|\xi|)^k$ uniformly. Let g be a measurable function on $\mathbf{R}^n$ so that $|g(\xi)| \leqslant c(1+|\xi|)^N$. Then if $-N$ is sufficiently large, we have that*

$$|(g * f)(\xi)| \leqslant c(1+|\xi|)^k \tag{4.1.17}$$

uniformly on V, where V is any cone satisfying $V^c \backslash \{0\} \subset W$.

PROOF. Clearly, since $V^c \backslash \{0\} \subset W$, we have that there exists $c > 0$ so that if $\xi \in V$, then $\{x|\ |x-\xi| < c|\xi|\}$ is contained in W. Let U be an open conic neighborhood of V defined by:

$$U = \{x|\ |x-\xi| < c|\xi| \quad \text{for some } \xi \in V\}.$$

Then

$$|(g * f)(\xi)| \leqslant \int_U |g(x-\xi) f(x)|\,dx + \int_{\mathbf{R}^n \backslash U} |g(x-\xi) f(x)|\,dx = I + II.$$

Now since $U \subset W$, we have that

$$I \leqslant \int_{\mathbf{R}^n} |g(x-\xi)|(1+|x|)^k\,dx \leqslant c(1+|\xi|)^k \qquad \text{for all } \xi$$

if N is chosen as in Proposition 4.1.2. Now for II, note that if $\xi \in V$, $x \in \mathbf{R}^n \backslash U$ then $|x-\xi| \geqslant c|\xi|$. Hence, since $k - M \leqslant 0$,

$$\begin{aligned} II &\leqslant \int (1+|x-\xi|)^N (1+|x|)^M\,dx \leqslant c(1+|\xi|)^{k-M} \int (1+|\xi-x|)^{N+M-k}(1+|x|)^M\,dx \\ &\leqslant c(1+|\xi|)^{k-M}(1+|\xi|)^M \\ &= c(1+|\xi|)^k \qquad \text{for } \xi \in V \end{aligned}$$

by Proposition 4.1.2 if we choose N appropriately. Combining estimates I & II completes the proof. ■

We will use Proposition 4.1.4 and Proposition 4.1.5 to see how the order of a distribution behaves under coordinate changes. First, we remove the presence of a norm $|\ |$ in Definition 4.1.2.

DEFINITION 4.1.3. *Let V be a cone in $\mathbf{R}^n$. We say that a relatively compact set $U' \subset \mathbf{R}^n \backslash \{0\}$ is a generating neighborhood for V if $U = \{\tau\xi | \xi \in U', \tau \in \mathbf{R}^+\}$ is an open conic neighborhood for V.*

Clearly, in Definition 4.1.2 we can require that

$$(\varphi D)^{\hat{}}(\tau\xi') = \mathcal{O}(\tau^k) \tag{4.1.18}$$

uniformly for $\xi' \in U'_V$ where U'_V is a generating neighborhood for V.

THEOREM 4.1.6. *Let X, Y be open in $\mathbf{R}^n$, Φ a diffeomorphism, $\Phi\colon X \to Y$ so that $\Phi(\overline{x}) = \overline{y}$. Let $d\Phi_{\overline{x}}$ denote the Jacobian of Φ at $\overline{x}$. Let $D \in \mathcal{D}'(Y)$, V a closed cone in $\mathbf{R}^n$, $W = (d\Phi_{\overline{x}})^t V$. Then*

$$\operatorname{Ord}_{\overline{y},V} D = \operatorname{Ord}_{\overline{x},W} \Phi_* D. \tag{4.1.19}$$

PROOF. Let Φ_i be the ith coordinate function of Φ. Expanding by Taylor's formula about $x = \bar{x}$,

$$\Phi_i(x) = \Phi_i(\bar{x}) + \sum_{j=1}^{n} \frac{\partial \phi_i(\bar{x})}{\partial x_j}(x - \bar{x})_j + h_i(x) \tag{4.1.20}$$

where $(x - \bar{x})_j = x_j - \bar{x}_j$; $h_i \in C^\infty$, and $\partial h_i/\partial x_j = 0$ at $x = \bar{x}$ for all i, j. Hence

$$\Phi(x) = [\Phi(\bar{x}) - (d\Phi_{\bar{x}})\bar{x}] + (d\Phi_{\bar{x}})x + h(x).$$

Letting $y = \Phi(x)$, we get

$$y = [\bar{y} - (d\Phi_{\bar{x}})\Phi^{-1}(\bar{y})] + (d\Phi_{\bar{x}})\Phi^{-1}(y) + \bar{\varphi}(y) \tag{4.1.21}$$

where $\bar{\varphi}(y) = h \circ \Phi^{-1}(y)$ has the property that $\partial\bar{\varphi}_i/\partial y_j = 0$ for all i, j at $y = \bar{y}$. Let $k \in \mathbf{R}$ so that $\mathrm{Ord}_{\bar{y},V} D \leqslant k$. Then by Definition 4.1.2, there is a neighborhood $U_{\bar{y}}$ of $\bar{y}$, a generating neighborhood U'_V of V so that

$$|\langle D, \varphi(y)e^{i\tau(y\cdot\xi')}\rangle| = \mathcal{O}(\tau^k) \tag{4.1.22}$$

uniformly in $\xi' \in U'_V$ for all $\varphi \in C_0^\infty(U_{\bar{y}})$. We define $U_{\bar{x}} \subset \Phi^{-1}(U_{\bar{y}})$ shortly. Let $U'_W \subset (d\Phi_{\bar{x}})^t U'_V$ so that if U_W is the cone generated by U'_W, then

$$W \subset U_W \quad \text{and} \quad U_W^c \subset (d\Phi_{\bar{x}})^t U_V \backslash (0). \tag{4.1.23}$$

We will show that

$$|\langle \Phi_* D, \psi(x)e^{i\tau(x\cdot\nu')}\rangle| = \mathcal{O}(\tau^k) \tag{4.1.24}$$

uniformly in $\nu' \in U'_W$ for all $\psi \in C_0^\infty(U_{\bar{x}})$.

Now by definition,

$$\langle \Phi_* D, \psi(x)e^{i\tau(x\cdot\nu')}\rangle = \langle D, \psi(\phi^{-1}(y))e^{i\tau\Phi^{-1}(y)\cdot\nu'}\rangle. \tag{4.1.25}$$

Now for any choice of $U_{\bar{x}} \subset \Phi^{-1}(U_{\bar{y}})$, $\psi(y) \equiv \psi(\Phi^{-1}(y)) \in C_0^\infty(U_{\bar{y}})$. Also, for any $\nu' \in U'_W$, there is a $\xi' \in U'_V$ so that $\nu' = (d\Phi_{\bar{x}})^t\xi'$. Hence, (4.1.25) becomes

$$\langle D, \psi(y)e^{i\tau(d\Phi_{\bar{x}})\Phi^{-1}(y)\cdot\xi'}\rangle. \tag{4.1.26}$$

Let $\varphi \in C_0^\infty(U_{\bar{y}})$ so that $\varphi \equiv 1$ on supp ψ. Substituting (4.1.21) into (4.1.26) we obtain

$$\begin{aligned} &e^{i\tau[(d\Phi_{\bar{x}})\Phi^{-1}(\bar{y})-\bar{y}]\cdot\xi'}\langle D, \psi(y)e^{i\tau(y-\bar{\varphi}(y))\cdot\xi'}\rangle = e^{i\tau[(d\Phi_{\bar{x}})\Phi^{-1}(\bar{y})-\bar{y}]\cdot\xi'}\langle[\varphi D][\psi(y)e^{-i\tau\bar{\varphi}(y)\cdot\xi'}], e^{i\tau(y\cdot\xi')}\rangle \\ &\qquad = e^{i\tau[(d\Phi_{\bar{x}})\Phi^{-1}(y)-\bar{y}]\cdot\xi'}[\varphi D]^\wedge * [\psi(\cdot)e^{-i\tau\bar{\varphi}(\cdot)\cdot\xi'}]^\wedge(\tau\xi'). \end{aligned} \tag{4.1.27}$$

Now

$$(\varphi D)^\wedge(\tau\xi') = \mathcal{O}(\tau^k) \tag{4.1.28}$$

uniformly in $\xi' \in U'_V$ by (4.1.22). Also, by Proposition 4.1.4, if $U \subset U_{\bar{y}}$ is small enough, $\bar{y} \in U$,

$$[\psi(\cdot)e^{-i\tau\bar{\varphi}(\cdot)\cdot\xi'}]^\wedge(\tau\xi') = \mathcal{O}(\tau^{-N}) \qquad \forall N \tag{4.1.29}$$

uniformly in $\xi' \in S^{n-1}$ for any $\psi \in C_0^\infty(U)$. Hence, by assumption (4.1.23) on U'_W, we can choose N large, apply Proposition 4.1.5 to (4.1.27) to obtain that $\forall \psi \in C_0^\infty(U_{\bar{x}})$, where $U_{\bar{x}} \subset \Phi^{-1}(U)$,

$$\langle \Phi_* D, \psi(x) e^{i\tau(x \cdot \nu')} \rangle = \mathcal{O}(\tau^k) \tag{4.1.30}$$

uniformly in $\nu' \in U'_W$. Hence,

$$\mathrm{Ord}_{\bar{x},W} \Phi_* D \leqslant \mathrm{Ord}_{\bar{y},V} D. \tag{4.1.31}$$

So if $\mathrm{Ord}_{\bar{y},V} D = -\infty$, we are done. Otherwise, by the same argument, we can show that

$$\mathrm{Ord}_{\bar{y},(d\Phi_{\bar{y}}^{-1})W} (\Phi^{-1})_* (\Phi_* D) \leqslant \mathrm{Ord}_{\bar{x},W} (\Phi_* D). \tag{4.1.32}$$

Noting that $(\Phi^{-1})_*(\Phi_* D) = D$, $(d\Phi_{\bar{y}}^{-1})W = V$ yields

$$\mathrm{Ord}_{\bar{y},V} D \leqslant \mathrm{Ord}_{\bar{x},W} (\Phi_* D). \tag{4.1.33}$$

Combining (4.1.32) with (4.1.33) proves (4.1.19). ■

We now define the order of a distribution $D \in \mathcal{D}'(X)$ where X is a manifold.

Definition 4.1.4. *Let $D \in \mathcal{D}'(X)$, X a manifold, $x_0 \in X$, V a closed cone in $(T^*X \backslash \{0\})_{x_0}$. We say that*

$$\mathrm{Ord}_{x_0,V} D \leqslant k, \qquad k \in \mathbf{R}$$

if there exists a coordinate neighborhood $U_{x_0} = \{(x)\}$, a generating neighborhood U'_V for V so that for all $\varphi \in C_0^\infty(U_{x_0})$,

$$\langle \varphi D, e^{i\tau(x \cdot \xi')} \rangle = \mathcal{O}(\tau^k) \tag{4.1.34}$$

uniformly in $\xi' \in U'_V$. We define $\mathrm{Ord}_{x_0,V} D = k$ *etc. as in Definition 4.1.2.*

By Theorem 4.1.6, the order of a distribution is well defined, independently of the coordinate system chosen. That is, if $\mathrm{Ord}_{x_0,V} D \leqslant k$ in one coordinate system, the same is true in any coordinate system.

Definition 4.1.5. *Let $k \in \mathbf{R}$, $x_0 \in X$, $\xi_0 \in (T^*X \backslash \{0\})_{x_0}$, $D \in \mathcal{D}'(X)$. We say that (x_0, ξ_0) is in the k-wave front set of D, denoted*

$$(x_0, \xi_0) \in WF_k D \tag{4.1.35}$$

if $\mathrm{Ord}_{x_0,\xi_0} D \geqslant k$. That is, if $\mathrm{Ord}_{x_0,\xi_0} T = k'$ where $k' \geqslant k$. We let $WF_\infty D = \cup_k WF_k D$.

Clearly, $WF_k D$ is conic for $-\infty < k \leqslant \infty$, and since $\mathrm{Ord}_{x,\xi} D$ is an uppersemicontinuous of (x, ξ), we see that for $-\infty < k < \infty$, $WF_k D$ is a closed set in $T^*X \backslash \{0\}$.

Also, if $(x_0, \xi_0) \in WF_k D$, then $(x_0, \xi_0) \in WF(D)$. Hence

$$WF_\infty D = \bigcup_k WF_k D \subset WF(D). \tag{4.1.36}$$

Also, it is clear that

$$(T^*X \backslash \{0\}) \backslash (WF(D)) \subset \{(x_0, \xi_0) | \mathrm{Ord}_{x_0,\xi_0} D = -\infty\}. \tag{4.1.37}$$

However, we may have that

$$\{(x_0, \xi_0) | \mathrm{Ord}_{x_0,\xi_0} D = -\infty\} \cap WF(D) \neq \emptyset.$$

To see this, we construct the following example.

Given $b > a > 0$, let

$$f_{a,b}(x) = \begin{cases} (1/|x| - a) + (1/b - |x|) & \text{if } a < |x| < b, \\ 0 & \text{otherwise}. \end{cases}$$

Let $f^k_{a,b}(x)$ be the kth primitive of $f_{a,b}(x)$ on $\{x| \quad a < |x| < b\}$. For example,

$$f^2_{a,b}(x) = (|x| - a)[\ln(|x| - a) - 1] + (b - |x|)[\ln(b - |x|) - 1]$$

is continuous on $\{x|a \leqslant |x| \leqslant b\}$. Here we define $f^k_{a,b}(x) = 0$ outside of $\{x|a < |x| < b\}$. Clearly,

$$f^k_{a,b}(x) \in L^1_{k-1}(I_{a,b}) \backslash L^1_k(I_{a,b}) \qquad \text{for } k \geqslant 1$$

where $I_{a,b} = \{x|a < |x| < b\}$. Define,

$$F(x) = \begin{cases} f^n_{1/n+1,1/n}(x) & \text{if } x \in I_{1/n+1,1/n} \\ 0 & \text{otherwise}. \end{cases}$$

Then for each $n \geqslant 1$,

$$F(x) \in L^1_{n-1}(|x| < \frac{1}{n}) \backslash L^1_n(|x| < \frac{1}{n}).$$

Clearly then, $F(x)$ is not C^∞ in any neighborhood of $x = 0$, so

$$WF(F(x))_0 \neq \emptyset.$$

However, if $\varphi \in C^\infty_0(|x| < 1/n)$, then $(\varphi F)\hat{} = \hat{\varphi} * \hat{F} = \mathcal{O}(|\xi|)^{1-n}$ by Proposition 4.1.2 since $\hat{\varphi} \in \mathcal{S}(\mathbf{R})$. Hence

$$\mathrm{Ord}_{0,\mathbf{R}^n} F = -\infty.$$

With this in mind, we give the following.

DEFINITION 4.1.6. *We define* $WF_{-\infty} D = \{(x_0, \xi_0) | \mathrm{Ord}_{x_0,\xi_0} D = -\infty\} \cap WF(D)$.

Hence,

$$\bigcup_{k \geqslant -\infty} WF_k D \subset WF(D). \tag{4.1.38}$$

Also, by definition,

$$WF(D) \subset \bigcup_{k \geqslant -\infty} WF_k D. \tag{4.1.39}$$

Combining (4.1.38) with (4.1.39) we have

$$WF(D) = \bigcup_{k \geqslant -\infty} WF_k D. \tag{4.1.40}$$

Now given k, if $\varphi \in C^N_0(X)$, $N = N(k)$ is large, $\varphi(x_0) \neq 0$, then by (4.1.11) and Proposition 4.1.5 we have that

$$WF_k(\varphi D)_{x_0} = (WF_k D)_{x_0}. \tag{4.1.41}$$

If $\varphi \in C_0^\infty(X)$, $\varphi(x_0) \neq 0$, then (4.3.23) is true for all k. Now let $\pi\colon T^*X \to X$ be the usual projection; $(x,\xi) \to x$.

DEFINITION 4.1.7. *Let $D \in \mathfrak{D}'(X)$, $-\infty < k < \infty$. We define the singular k-support of D by:*

$$\text{(4.1.42)}\qquad \begin{aligned} \operatorname{sing}_k \operatorname{supp} D &= \left[\cup\, \pi(WF_{k'} D)\right]^c \qquad \text{where } k' > k,\\ \operatorname{sing}_{-\infty} \operatorname{supp} D &= \Big[\bigcup_{k \geqslant -\infty} \pi(WF_k D)\Big]^c. \end{aligned}$$

Note that if $\operatorname{Ord}_{x_0,\xi_0} T = k$, where $\xi_0 \in (T^*X\backslash\{0\})_{x_0}$, then x_0 may or may not be in the $\operatorname{sing}_k \operatorname{supp} T$. However, if $\operatorname{Ord}_{x_0,V} T \leqslant k$ where $V = (T^*X\backslash\{0\})_{x_0}$, then there is a neighborhood U_{x_0} so that $\operatorname{Ord}_{x,V} T \leqslant k$ for all $x \in U_{x_0}$. Hence, $x_0 \notin \operatorname{sing}_k \operatorname{supp} T$.

If $W \subset (T^*X\backslash\{0\})_{x_0}$ is a closed cone, we will sometimes write that $\operatorname{Ord}_{x_0,W} T \nleqslant k$ where we will mean either

i) $\operatorname{Ord}_{x_0,W} T = k'$ where $k' > k$, or

ii) $\operatorname{Ord}_{x_0,W} T = k$ but we do not have that $\operatorname{Ord}_{x_0,W} \leqslant k$.

Then if $\operatorname{Ord}_{x_0,W} T \nleqslant k$, we see that $x_0 \in \operatorname{sing}_k \operatorname{supp} T$.

Clearly,

$$\text{(4.1.43)}\qquad \operatorname{sing}_k \operatorname{supp} D \subset \operatorname{sing} \operatorname{supp} D \qquad \text{for all } k.$$

And, by (4.1.40) and (4.1.2),

$$\text{(4.1.44)}\qquad \operatorname{sing}_{-\infty} \operatorname{supp} D = \operatorname{sing} \operatorname{supp} D.$$

PROPOSITION 4.1.7. *Let $k \in \mathbf{Z}^+ \cup \{0\}$. Then*

$$\text{(4.1.45)}\quad \operatorname{sing}_{-k} \operatorname{supp} D \subset X\backslash\{x \mid\ \exists U_x \text{ with } \ \psi D \in C_0^k(X) \quad \text{for all } \psi \in C_0^\infty(U)\}.$$

PROOF. Given x_0, assume that there exists U_{x_0} with

$$\psi D \in C_0^k(X) \qquad \forall \psi \in C_0^\infty(U_{x_0}).$$

Then in any coordinate system on U_{x_0}, $\psi D \in L_k^1(U_{x_0})$. Hence by (4.1.12),

$$\operatorname{Ord}_{x_0,W} D \leqslant -k, \qquad \text{where } W = (T^*X\backslash\{0\})_{x_0}.$$

So, by the remarks following Definition 4.1.7,

$$x_0 \in X\backslash(\operatorname{sing}_{-k} \operatorname{supp} D). \quad \blacksquare$$

PROPOSITION 4.1.8. *Let $\Phi\colon X \to Y$ be a C^∞ diffeomorphism, $D \in \mathfrak{D}'(Y)$, $\overline{\Phi}^*$ denote the induced mapping of $T^*Y \to T^*X$. Then*

$$\text{(4.1.46)}\qquad WF_k(\Phi_* D) = \overline{\Phi}^*(WF_k D) \qquad -\infty \leqslant k < \infty.$$

Hence

$$\text{(4.1.47)}\qquad \operatorname{sing}_k \operatorname{supp}(\Phi_* D) = \Phi^{-1}(\operatorname{sing}_k \operatorname{supp} D) \qquad -\infty \leqslant k < \infty.$$

PROOF. Immediate from Theorem 4.1.6 and the above definitions. ■

2. Existence of Restrictions of Generalized Functions

Let $T \in \mathcal{B}'(\mathfrak{M})$, μ a nowhere vanishing density on $\mathfrak{M}$.

DEFINITION 4.2.1. *Let $k \in \mathbf{R} \cup \{-\infty\}$. We define $WF_k(T)$ by*

$$WF_k(T) = WF_k(T\mu) \tag{4.2.0}$$

where $T\mu \in \mathfrak{D}'(\mathfrak{M})$ is defined by, $\langle T\mu, \varphi\rangle = \langle T, \varphi\mu\rangle$ for all $\varphi \in C_0^\infty(\mathfrak{M})$.

The above definition makes sense, since if μ_1, μ_2 are two such choices of nowhere vanishing densities, then $\mu_1 = f\mu_2$ where $f \in C^\infty(\mathfrak{M})$ and $f \neq 0$ for all $x \in \mathfrak{M}$. Hence, $T\mu_1 = fT\mu_2$. So by (4.1.41), $WF_k(T\mu_1) = WF_k(T\mu_2)$ for all k.

Our purpose in this section will be to derive a sufficient condition for restrictability of a generalized function T in terms of its k-wave front sets.

In the following, we will use the convention set forth in Chapter I. That is, in local coordinates, we will identify T with $T\mu$. So if $\mu_1 \in \mathcal{B}(\mathcal{O})$, $T \in \mathcal{B}'(\mathcal{O})$, $\mu_1 = \varphi\mu$, we will identify $\langle T, \mu_1\rangle$ with $\langle T\mu, \varphi\rangle$. For notational convenience, we will denote $T\mu$ by D.

Let $\mathfrak{N}$ be an embedded submanifold of $\mathfrak{M}$. Say $\dim \mathfrak{M} = k + l$, $\dim \mathfrak{N} = k$.

DEFINITION 4.2.1. *The conormal bundle of $\mathfrak{N}$, denoted $N^*(\mathfrak{N})$ is*

$$\{(z,\eta) \in T^*(\mathfrak{M})\backslash\{0\} \mid \ z \in \mathfrak{N}, \quad \eta|_{T_z^*(\mathfrak{N})} = 0\}. \tag{4.2.1}$$

In local coordinates, if $U = \{(x_1, \dots, x_k, y_1, \dots, y_l)\}$ so that $V \equiv U \cap \mathfrak{N} = \{(x,y) \mid y = 0\}$, then $T_x(\mathfrak{N})$ is spanned by $\{\partial/\partial x_i | i = 1, \dots, k\}$ clearly. Hence, $N^*(\mathfrak{N})$ over V is spanned by $\{dy_1, \dots, dy_l\}$.

THEOREM 4.2.1. *Let $T \in \mathcal{B}'(\mathfrak{M})$, $\mathfrak{N} \subset \mathfrak{M}$ as above. Assume that for each $x \in \mathfrak{N}$,*

$$\operatorname{Ord}_{x, N_x^*(\mathfrak{N})} T < -l. \tag{4.2.2}$$

Then $T \in \mathcal{R}_{\mathfrak{N},\mathfrak{M}}$ and $T_{\mathfrak{N},\mathfrak{M}}$ exists.

Note that (4.2.2) can be written as;

$$WF_{-l}T \cap N^*(\mathfrak{N}) = \varnothing. \tag{4.2.3}$$

REMARK. If $WF(D) \cap N^*(\mathfrak{N}) = \varnothing$, $D \in \mathfrak{D}'(\mathfrak{M})$, the existence of a restriction for D was shown in [1] using properties of wave front sets.

PROOF. We first show that $T \in \mathcal{R}_{\mathfrak{N},\mathfrak{M}}$. Let $x_0 \in \mathfrak{N}$ be given, $U_{x_0} \subset \mathfrak{M}$ as in Definition 4.1.4. Choose $U \subset U_{x_0}$ with coordinate functions $\{(x,y)\}$ so that $U \cap \mathfrak{N} = \{(x,y) | y = 0\}$. Let $\psi \in C_0^\infty(U_{x_0})$ so that $\psi \equiv 1$ on U. Then for all $\mu = \varphi(x,y)\,dx\,dy \in \mathcal{B}(U)$,

$$\langle T, \mu\rangle = \langle \psi D, \varphi\rangle. \tag{4.2.4}$$

Now

$$\begin{aligned}\langle \psi D, \varphi\rangle &= \int (\psi D)\hat{}\,(\xi,\eta)\check{\varphi}(\xi,\eta)\,d\xi\,d\eta \\ &= \int_{|\eta|<\epsilon|\xi|} (\psi D)\hat{}\,(\xi,\eta)\check{\varphi}(\xi,\eta)\,d\xi\,d\eta + \int_{|\eta|\geqslant\epsilon|\xi|} (\psi D)\hat{}\,(\xi,\eta)\check{\varphi}(\xi,\eta)\,d\xi\,d\eta = I + II.\end{aligned}$$

In these coordinates, we can identify $N_{x_0}(\mathfrak{N})$ with $\{(0,\eta)\}$. Hence, for any $\epsilon > 0$,

$$\{(\xi,\eta) \mid |\eta| > \epsilon|\xi|\} \tag{4.2.5}$$

is a conic neighborhood of $N_{x_0}(\mathfrak{N})$ in $T^*\mathfrak{M}\backslash\{0\}$. Choose ϵ so that

$$|(\psi D)\hat{}\,(\xi,\eta)| \leqslant c(1+|\xi|^2+|\eta|^2)^K, \qquad 2K < -l \tag{4.2.6}$$

on $\{(\xi,\eta) \mid |\eta| > \epsilon|\xi|\}$. This can be done by assumption (4.2.2) and Definition 4.1.4. With this ϵ, we estimate I and II. Now since $\psi D \in \mathcal{E}'(\mathfrak{M})$, we can find M so that $|(\psi D)\hat{}\,| \leqslant c(1+|\xi|^2+|\eta|^2)^M$. Hence

$$\begin{aligned} |I| &\leqslant \int_{|\eta|<\epsilon|\xi|} (1+|\xi|^2+|\eta|^2)^M |\check{\varphi}(\xi,\eta)|\, d\xi\, d\eta \\ &\leqslant \sup|(1+|\xi|^2)^N \check{\varphi}| \int_{|\eta|<\epsilon|\xi|} (1+|\xi|^2)^{-N}(1+|\xi|^2+|\eta|^2)^M\, d\xi\, d\eta. \end{aligned} \tag{4.2.7}$$

On the domain of integration, $(1+|\xi|^2)^{-N} < c(1+|\eta|^2)^{-N}$. Hence, (4.2.7) is bounded by

$$\sup|(1+|\xi|^2)^{2N}\check{\varphi}| \int (1+|\xi|^2)^{-N}(1+|\eta|^2)^{-N}(1+|\xi|^2+|\eta|^2)^M\, d\xi\, d\eta. \tag{4.2.8}$$

Choosing N large, the integral in (4.2.8) is convergent and we obtain that I is bounded by

(4.2.9)

$$c \sup|(1+|\xi|^2)^{2N}\check{\varphi}| = c \sup|[(1+\Delta_x)^{2N}\varphi]\check{}\,| \leqslant c\|(1+\Delta_x)^{2N}\varphi\|_1 \leqslant c \sum_{|\alpha|\leqslant 4N} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} \varphi(x,y) \right\|_1$$

where $\Delta_x = -\sum_{i=1}^{k} (\partial/\partial x_i)^2$. Using (4.2.6), we have that II is bounded in absolute value by

$$\sup|(1+|\xi|^2)^N\check{\varphi}| \int (1+|\eta|^2+|\xi|^2)^K(1+|\xi|^2)^{-N}\, d\xi\, d\eta \tag{4.2.10}$$

where $2K < -l$, N is arbitrary. Letting $\eta_i = (1+|\xi|^2)^{\frac{1}{2}}\zeta_i$, we have that

$$\int (1+|\xi|^2+|\eta|^2)^K d\eta = (1+|\xi|^2)^{K+\frac{1}{2}} \int (1+|\zeta|^2)^K d\zeta \leqslant c(1+|\xi|^2)^{K+\frac{1}{2}} \tag{4.2.11}$$

since $2K < -l$. Hence, inserting (4.2.11) into (4.2.10), we have that

$$\begin{aligned} |II| &\leqslant c \sup|[(1+\Delta_x)^N\varphi]\check{}\,| \int (1+|\xi|^2)^{-N+K+\frac{1}{2}}\, d\xi \\ &\leqslant c \sum_{|\alpha|\leqslant 2N} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} \varphi(x,y) \right\|_1 \qquad \text{for } N \text{ large}. \end{aligned} \tag{4.2.12}$$

Combining (4.2.12) with (4.2.9) proves that $T \in \mathfrak{R}_{\mathfrak{N},\mathfrak{M}}$.

To prove the existence of $T_{\mathfrak{N},\mathfrak{M}}$, let $\nu = \varphi(x)\,dx \in \mathfrak{B}(U \cap \mathfrak{N})$. Let $\eta \in C_0^\infty(\mathbf{R}^+)$ so that $\int t^{l-1}\eta(t)\,dt = 1$. Let $\{\zeta_m\} \subset C^\infty(S^{l-1})$ satisfy (3.1.6). Then clearly,

$$\mu_m = \varphi(x) m^l \eta(m|y|)\zeta_m(y')\, dx\, dy \tag{4.2.13}$$

is a permissible sequence converging to ν. Then

$$\langle T,\mu_m\rangle = \int (\psi D)\hat{}\,(\xi,\eta)\check{\varphi}(\xi) g_m(\eta)\, d\xi\, d\eta \tag{4.2.14}$$

where $g_m(\eta)$ is the inverse Fourier transform of $m^l\eta(m|y|)\zeta_m(y')$. Clearly, $g_m(\eta) \to 1$ pointwise as $m \to \infty$. Also, $|g_m| \leqslant \|m^l\eta(m|y|)\zeta_m(y')\|_1 \leqslant c$ for all m by (3.1.6) where the L^1 norm is with respect to dy. Hence, the integrand in (4.2.14) is uniformly bounded, uniformly integrable by the above estimates for I and II, and converges pointwise as $m \to \infty$ to $(\psi D)\hat{}\,(\xi,\eta)\check{\varphi}(\xi)$. Hence, applying the Dominated Convergence Theorem proves the existence of $\lim_m\langle T,\mu_m\rangle$. So, by Theorem 3.2.1, $T_{\mathfrak{N},\mathfrak{M}}$ exists and equals this limit. ■

THEOREM 4.2.2. *Let* $T \in \mathcal{B}'(\mathfrak{M})$, $\mathfrak{N} \subset \mathfrak{M}$ *embedded so that* $\dim \mathfrak{M} = k + l$, $\dim \mathfrak{N} = k$. *Assume that for some* $x_0 \in \mathfrak{N}$,

$$\operatorname{Ord}_{x_0,T^*_{x_0(\mathfrak{N})}} T \leqslant h < -l. \tag{4.2.15}$$

By Theorem 4.2.2, there is a neighborhood $\mathcal{O}$ *of* x_0 *in* M *so that* $T_{\mathfrak{N},\mathcal{O}}$ *exists. Then*

$$\operatorname{Ord}_{x_0,T^*_{x_0(\mathfrak{N})}}(T_{\mathfrak{N},\mathcal{O}}) \leqslant h + l. \tag{4.2.16}$$

PROOF. Let $U = \{(x,y)\}$ be a coordinate neighborbood of x_0 in $\mathfrak{M}$ so that

i) $U \subset \mathcal{O}$

ii) $U \cap \mathfrak{N} = \{(x,y) \mid y = 0\}$

iii) $\forall\psi \in C_0^\infty(U)$,

$$|\langle D, \psi(x,y)e^{i(x\cdot\xi+y\cdot\eta)}\rangle| \leqslant c(1 + |\xi| + |\eta|)^h \tag{4.2.17}$$

for all ξ, η where $h < -l$ and D is a distribution associated with T.

That U exists is clear by (4.2.15) and Theorem 4.1.6. Choose $V \subset U \cap \mathfrak{N}$ so that $x_0 \in V$, $V^c \subset U$. Let $\psi \in C_0^\infty(U)$ so that $\psi \equiv 1$ on V. Let $D_{\mathfrak{N},\mathcal{O}}$ be a distribution on $\mathfrak{N} \cap \mathcal{O}$ associated with $T_{\mathfrak{N},\mathcal{O}}$. Let $\eta \in C_0^\infty(\mathbf{R}^l)$ so that $\int \eta = 1$. Then if $\varphi \in C_0^\infty(V)$,

$$\langle \varphi D_{\mathfrak{N},\mathcal{O}}, e^{ix\cdot\xi}\rangle = \lim_{\epsilon\to 0}\langle \varphi D, \frac{1}{\epsilon^l}\eta(\frac{y}{\epsilon})e^{ix\cdot\xi}\rangle. \tag{4.2.18}$$

Hence,

$$|\langle \varphi D_{\mathfrak{N},\mathcal{O}}, e^{ix\cdot\xi}\rangle| \leqslant \varlimsup_{\epsilon}|\langle \varphi D, \frac{1}{\epsilon^l}\eta(\frac{y}{\epsilon})e^{ix\cdot\xi}\rangle|. \tag{4.2.19}$$

Now

$$\begin{aligned} |\langle \varphi D, \frac{1}{\epsilon^l}\eta(\frac{y}{\epsilon})e^{ix\cdot\xi}\rangle| &= |\langle \psi D, \frac{1}{\epsilon^l}\eta(\frac{y}{\epsilon})\varphi(x)e^{ix\cdot\xi}\rangle| = |\langle (\psi D)\hat{}\,(x,y), (\frac{1}{\epsilon^l}\eta(\frac{y}{\epsilon})\varphi(x)e^{ix\cdot\xi})\check{}\,(x,y)\rangle| \\ &\leqslant \int |(\psi D)\hat{}\,(x,y)\check{\eta}(\epsilon y)\check{\varphi}(x+\xi)|\,dx\,dy \\ &\leqslant \int (1 + |x| + |y|)^h|\check{\eta}(\epsilon y)\check{\varphi}(x+\xi)|\,dx\,dy \end{aligned} \tag{4.2.20}$$

by (4.2.17). Letting $y_i = (1 + |x|)\zeta_i$, (4.2.20) is equal to

$$\int (1 + |x|)^{h+l}|\check{\varphi}(x+\xi)| \int (1 + |\zeta|)^h\check{\eta}(\epsilon[1 + |x|]\zeta)\,d\zeta\,dx. \tag{4.2.21}$$

Now since $h < -l$, we can apply the Dominated Convergence theorem to obtain that

$$\varlimsup_{\epsilon}\int (1 + |\zeta|)^h\check{\eta}(\epsilon[1 + |x|]\zeta)\,d\zeta \leqslant c. \tag{4.2.22}$$

Hence, by (4.2.22), (4.2.21) and (4.2.19) we obtain, since $\check{\varphi} \in \mathcal{S}(\mathbf{R}^k)$, that $\forall N \in \mathbf{Z}^+$,

$$|\langle \varphi D_{\mathfrak{N},\mathcal{O}}, e^{ix\cdot\xi}\rangle| \leqslant c_N \int (1 + |x|)^{h+l}(1 + |x + \xi|)^{-N}\, dx \tag{4.2.23}$$

for all ξ. Applying Proposition 4.1.2 to the integral in (4.2.23), we have for N sufficiently large, that for each $\varphi \in C_0^\infty(V)$, there is a $c \in \mathbf{R}$ so that

$$|\langle \varphi D_{\mathfrak{N},\mathcal{O}}, e^{ix\cdot\xi}\rangle| \leqslant c(1 + |\xi|)^{h+l}. \tag{4.2.24}$$

That is,

$$\operatorname{Ord}_{x_0, T^*_{x_0(\mathfrak{N})}}(T_{\mathfrak{N},\mathcal{O}}) \leqslant h + l. \quad \blacksquare \tag{4.2.25}$$

COROLLARY 4.2.3. *Let $T \in \mathfrak{B}'(\mathfrak{M})$ satisfy (4.2.15) for all $x_0 \in \mathfrak{N}$. Then for every $h < 0$,*

$$\operatorname{sing}_h \operatorname{supp}(T_{\mathfrak{N},\mathfrak{M}}) \subset (\operatorname{sing}_{(h-l)} \operatorname{supp} T) \cap \mathfrak{N}. \tag{4.2.26}$$

PROOF. Let $x_0 \in \operatorname{sing}_h \operatorname{supp}(T_{\mathfrak{N},\mathfrak{M}})$. By the remarks following Definition 4.1.7 we have that

$$\operatorname{Ord}_{x_0, T^*_{x_0(\mathfrak{N})}}(T_{\mathfrak{N},\mathfrak{M}}) \nleqslant h.$$

Hence, by Theorem 4.2.2,

$$\operatorname{Ord}_{x_0, T^*_{x_0(\mathfrak{N})}} T \nleqslant h - l.$$

So by the same remarks, $x_0 \in \operatorname{sing}_{(h-l)} \operatorname{supp} T$. $\blacksquare$

3. Products of Generalized Functions

In this section, we discuss sufficient conditions on generalized functions in order that their product is well defined.

Let X be a second countable C^∞ manifold. Let $m \in \mathbf{Z}^+$, $\mathfrak{M} = \prod_{i=1}^m X_i$ where $X_i = X$ for all i, and $\mathfrak{M}$ is given the usual product structure. Let $\mathfrak{N} \subset \mathfrak{M}$ be defined by: $\mathfrak{N} = \{(x_1, \ldots, x_m) \cdot | x_i = x_j \ \forall i,j\}$. Note that x_i in general will be an n-tuples of coordinates if $\dim X = n$. Let $T_i \in \mathfrak{B}'(X)$, $i = 1, \ldots, m$. Clearly, $\otimes_{i=1}^m T_i \in \mathfrak{B}'(\mathfrak{M})$, where $\otimes_{i=1}^m T_i$ is the tensor product.

DEFINITION 4.3.1. *Assume $\otimes_{i=1}^m T_i \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$ where $\mathcal{O}$ is a neighborhood of $\mathfrak{N}$ in $\mathfrak{M}$. We define the generalized function, $\prod_{i=1}^m T_i$ by*

$$\prod_{i=1}^m T_i = \left(\bigotimes_{i=1}^m T_i\right)_{\mathfrak{N},\mathcal{O}} \tag{4.3.1}$$

when the restriction exists.

Clearly, if $T_i \in C^\infty(X)$, then $\otimes_{i=1}^m T_i \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$, $(\otimes_{i=1}^m T_i)_{\mathfrak{N},\mathcal{O}}$ exists and is equal to the product of T_i in the ordinary sense. From Theorem 4.3.1, it will be clear that if $T_i \in C^\infty(X)$ $\forall i \neq j$, and $T_j \in \mathfrak{B}'(X)$, then the above statement is also true. Also, it is clear that if $\bigcap_{i=1}^m \operatorname{supp} T_i = \varnothing$, then $\otimes_{i=1}^m T_i \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$ for any $\mathcal{O}$, and $(\otimes_{i=1}^m T_i)_{\mathfrak{N},\mathcal{O}}$ is well defined to be 0.

Let $T_i \in \mathfrak{B}'(X)$, $\Gamma_i = WF(T_i)$, $i = 1, \ldots, m$. Let $\Gamma_i^0 \subset T^*(X)$ be defined by

$$\Gamma_i^0 = \{(x,0) | x \in \text{supp } T_i\}. \tag{4.3.2}$$

Clearly, by definition,

$$WF(T_1 \otimes T_2) \subset \Gamma_1 \times \Gamma_2 \cup \Gamma_1 \times \Gamma_2^0 \cup \Gamma_1^0 \times \Gamma_2. \tag{4.3.3}$$

By induction, and (4.3.3), we obtain that

$$WF\Big(\bigotimes_{i=1}^{m} T_i\Big) \subset \cup\Big(\prod_{i=1}^{m} \Gamma_i'\Big) \tag{4.3.4}$$

where the union is over all $2^m - 1$ combinations of $\Gamma_i' = \Gamma_i$ or $\Gamma_i' = \Gamma_i^0$ where we exclude $\prod_{i=1}^m \Gamma_i^0$. Now, given cones $\Gamma_i \subset T^*X\backslash\{0\}$, $i = 1, \dots, m$, we define:

$$\sum_{i=1}^{m} \Gamma_i = \{(x, \xi_1 + \cdots + \xi_m) | \quad (x, \xi_i) \in \Gamma_i \qquad \forall i\}. \tag{4.3.5}$$

THEOREM 4.3.1. *Let $T_i \in \mathcal{B}'(X)$, $\Gamma_i = WF(T_i)$, $i = 1, \dots, m$. Assume that for every $k \leqslant m$ and every k-tuple $(i_1, \dots, i_k)$, $i_j \neq i_\ell$ $\forall j, \ell$, we have that*

$$\sum_{j=1}^{k} \Gamma_{i_j} \subset T^*X\backslash\{0\}. \tag{4.3.6}$$

Then $\otimes_{i=1}^m T_i \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ where $\mathfrak{N}$ is the diagonal of $\mathfrak{M} = \prod_{i=1}^m X$, and $\mathcal{O}$ is any neighborhood of $\mathfrak{N}$. Further, $\otimes_{i=1}^m T_i$ has a restriction to $\mathfrak{N}$.

That is, $\prod_{i=1}^m T_i \in \mathcal{B}'(X)$ is well defined by Definition 4.3.1.

REMARK. It is shown in [1] that if $D, E \in \mathcal{D}'(X)$, $WF(D) + WF(E) \subset T^*X\backslash\{0\}$, then $DE \in \mathcal{D}'(x)$ is well defined by:

$$DE = \Phi_*(D \otimes E)$$

where Φ: $X \to X \times X$ given by: $x \to (x,x)$.

PROOF. Combining (4.3.4) with assumption (4.3.6), we see that:

$$\Big\{(x_1, \dots, x_m, \xi_1, \dots, \xi_m) \in T^*(\mathfrak{M}) | \; x_i = x_j \qquad \forall i,j, \qquad \sum_{i=1}^{m} \xi_i = 0\Big\} \tag{4.3.7}$$

does not intersect $WF(\otimes_{i=1}^m T_i)$. To prove that $\otimes_{i=1}^m T_i \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$, we must show the following. Given $\mathbf{X}_0 \in \mathfrak{N}$, $\exists M \in Z^+$, and a coordinate neighborhood U of $\mathbf{X}_0$ with coordinate function $\{(u_1, \dots, u_m)\}$, where $U \cap N = \{(u_1, \dots, u_m) | u_i = 0, i \geqslant 2\}$, so that $\forall \quad \mu = \varphi(u)\, du \in \mathcal{B}(U)$,

$$\Big|\Big\langle \bigotimes_{i=1}^{m} T_i, \mu \Big\rangle\Big| \leqslant c \sum_{\substack{|\alpha|+|\beta| \leqslant m \\ |\beta| = |\beta'|}} \Big\| \Big(\frac{\partial}{\partial u_1}\Big)^{\alpha} (u_2, \dots, u_m)^{\beta} \Big(\frac{\partial}{\partial(u_2, \dots, u_m)}\Big)^{\beta'} \varphi \Big\|_1. \tag{4.3.8}$$

We will, in fact, show (4.3.8) with all $\beta = 0$. To do this, let $\mathbf{X}_0$ be given. Then if $\Xi^i = (\xi_1^i, \dots, \xi_m^i) \in [T^*(\mathfrak{M})\backslash\{0\}]_{\mathbf{X}_0}$, and $\sum_{j=1}^m \xi_j^i = 0$, we have by (4.3.7) and Definition 4.2.1 that $\exists$ neighborhoods U_i of $\mathbf{X}_0$, V_i of Ξ^i so that for all $\varphi \in C_0^\infty(U_i)$,

$$\Big|\Big\langle \bigotimes_{i=1}^{m} D_i, \varphi(x_1, \dots, x_m) e^{i\tau(x\cdot\xi)} \Big\rangle\Big| = \mathcal{O}(\tau^{-N}) \tag{4.3.9}$$

for all N uniformly in V_i. In (4.3.9), we identify $T_i \in \mathcal{B}'(X)$ with $D_i \in \mathcal{D}'(X)$. For each

$\Xi^i \in S^*_{\mathbf{X}_0}(\mathfrak{M})$, the cosphere bundle over $\mathbf{X}_0$, let U_i, V_i be given. Since $S^*_{\mathbf{X}_0}(\mathfrak{M})$ is compact, and $\{(\xi_1,\ldots,\xi_m) \in S^*_{\mathbf{X}_0} | \ \sum_{i=1}^m \xi_i = 0\}$ is closed in $S^*_{\mathbf{X}_0}(\mathfrak{M})$, we conclude that there is $\{\Xi^1,\ldots,\Xi^N\} \subset S^*_{\mathbf{X}_0}(\mathfrak{M})$ and associated U_i, V_i $i = 1,\ldots,N$ so that $\cup_{i=1}^N V_i$ covers $\{(\xi_1,\ldots,\xi_m) \in S^* | \ \sum_{i=1}^m \xi_i = 0\}$. Let $W = \cap_{i=1}^N U_i$, $V = \cup_{i=1}^N V_i$. Then if $\varphi \in C_0^\infty(W)$,

$$\left|\left\langle \bigotimes_{i=1}^m D_i, \varphi e^{i\tau(x\cdot\xi)}\right\rangle\right| = \mathcal{O}(\tau^{-N}) \tag{4.3.10}$$

for all $\mathfrak{M}$ uniformly in V.

Writing $\mathbf{X}_0 = (x_0,\ldots,x_0) \in \mathfrak{M}$, where $x_0 \in X$, let U_{x_0} be a coordinate neighborhood of x_0 in X, say with coordinate functions $\{(y_1,\ldots,y_n)\}$ so that $(\prod_{i=1}^m U_{x_0})^c \subset W$. Let $U \subset \prod_{i=1}^m U_{x_0}$ have coordinate functions $\{(u_1,\ldots,u_m)\}$ where $u_1 = (y_{11},\ldots,y_{1n})$, $u_j = (y_{11} - y_{j1},\ldots,y_{1n} - y_{jn}) j \geqslant 2$. Here the set $\{(y_{k1},\ldots,y_{kn})\}$ are the coordinates on the kth U_{x_0}. Clearly, $U \cap \mathfrak{M}$ is the slice $\{(u_1,\ldots,u_m) | \ u_j = 0 \ \ j \geqslant 2\}$ of U.

Let $\Phi \in C_0^\infty(W)$ so that $\Phi \equiv 1$ on U. Then if $\varphi \in C_0^\infty(U)$,

(4.3.11)

$$\begin{aligned}
\left|\left\langle \bigotimes_{i=1}^m D_i, \varphi(u_1,\ldots,u_m)\right\rangle\right| &= \left|\left\langle \Phi(y_1,\ldots,y_m)\left(\bigotimes_{i=1}^m D_i\right), \varphi(y_1, y_1 - y_2,\ldots,y_1 - y_m)\right\rangle\right| \\
&= \left|\int \left[\Phi\left[\bigotimes_{i=1}^m D_i\right]\right]^\wedge (\xi_1,\ldots,\xi_m)\check{\varphi}\left(\sum_{i=1}^m \xi_i, -\xi_2,\ldots,-\xi_m\right) d\Xi\right| \\
&\leqslant \int_{|\sum_{i=1}^m \xi_i| > \epsilon \sum_{i=1}^m |\xi_i|} \left|\left[\Phi\left[\bigotimes_{i=1}^m D_i\right]\right]^\wedge (\xi)\check{\varphi}\left(\sum_{i=1}^m \xi_i, -\xi_2,\ldots,-\xi_m\right)\right| d\Xi \\
&\quad + \int_{|\sum_{i=1}^m \xi_i| < \epsilon \sum_{i=1}^m |\xi_i|} \left|\left[\Phi\left[\bigotimes_{i=1}^m D_i\right]\right]^\wedge (\xi)\check{\varphi}\left(\sum_{i=1}^m \xi_i, -\xi_2,\ldots,-\xi_m\right)\right| d\Xi \\
&= I_\epsilon + II_\epsilon .
\end{aligned}$$

In the above, we abuse notation by writing y_1 for $(y_{11},\ldots,y_{1n})$, etc. Now $\Phi[\otimes_{i=1}^m D_i] \in \mathcal{E}'(\mathfrak{M})$. Hence

$$\left|\left[\Phi\left[\bigotimes_{i=1}^m D_i\right]\right]^\wedge (\xi_1,\ldots,\xi_m)\right| \leqslant c(1 + |\xi_1| + \cdots + |\xi_m|)^M, \text{ some } M. \tag{4.3.12}$$

So, for any ϵ,

$$\begin{aligned}
I_\epsilon &\leqslant c\int_{|\sum_{i=1}^m \xi_i| > \epsilon \sum_{i=1}^m |\xi_i|} (1 + |\xi_1| + \cdots + |\xi_m|)^M \left|\check{\varphi}\left(\sum_{i=1}^m \xi_i, -\xi_2,\ldots,-\xi_m\right)\right| d\Xi \\
&\leqslant \sup\left|\left(1 + \left|\sum_{i=1}^m \xi_i\right|^2\right)^K \check{\varphi}\left(\sum_{i=1}^m \xi_i, -\xi_2,\ldots,-\xi_m\right)\right| \\
&\quad \times \int_{|\sum_{i=1}^m \xi_i| > \epsilon \sum_{i=1}^m |\xi_i|} \left(1 + \sum_{i=1}^m |\xi_i|\right)^M \left(1 + \left|\sum_{i=1}^m \xi_i\right|^2\right)^{-K} d\Xi.
\end{aligned}$$

But, on the domain of integration,

$$\left(1 + \left|\sum_{i=1}^m \xi_i\right|^2\right)^{-K} \leqslant c_\epsilon\left(1 + \left[\sum_{i=1}^m |\xi_i|\right]^2\right)^{-K}.$$

Hence,

$$
(4.3.13)\quad \begin{aligned} I_\epsilon &\leqslant c_\epsilon \sup |[(1+\Delta_{u_1})^K \varphi(u_1,\ldots,u_m)]^\vee| \int \left(1+\sum_{i=1}^m |\xi_i|\right)^M \left(1+\left[\sum_{i=1}^m |\xi_i|\right]^2\right)^{-K} d\Xi \\ &\leqslant c_\epsilon \sum_{|\alpha|\leqslant 2K} \left\| \left(\frac{\partial}{\partial u_1}\right)^\alpha \varphi(u_1,\ldots,u_m)\right\|_1 \end{aligned}
$$

for K large. Here, Δ_{u_1} is the Laplacin in u_1.

To estimate II_ϵ, we make the following observation. If ϵ is small enough,

$$
(4.3.14)\quad \left\{(\xi_1,\ldots,\xi_m) \in S^*_{\mathbf{x}_0}(\mathfrak{N}) \;\middle|\; \left|\sum_{i=1}^m \xi_i\right| < \epsilon\right\} \subset V.
$$

Extending conically, we see by (4.3.14) that if ϵ is small enough,

$$
(4.3.15)\quad \left\{(\xi_1,\ldots,\xi_m) \;\middle|\; \left|\sum_{i=1}^m \xi_i\right| < \epsilon \sum_{i=1}^m |\xi_i|\right\} \subset V.
$$

Hence, for this ϵ, we have by (4.3.10) that

$$
(4.3.16)\quad II_\epsilon \leqslant \left|\sup \check{\varphi}\left(\sum_{i=1}^m \xi_i, -\xi_2,\ldots,-\xi_m\right)\right| \int \left(1+\sum_{i=1}^m |\xi_i|\right)^{-K} d\Xi \qquad \text{for all } K.
$$

Choosing $K > mn$, we obtain

$$
(4.3.17)\quad II_\epsilon \leqslant c\left|\sup \check{\varphi}\left(\sum_{i=1}^m \xi_i,\ldots,-\xi_m\right)\right| \leqslant c\|\varphi(u_1,\ldots,u_m)\|_1 .
$$

Combining (4.3.13), (4.3.17) with (4.3.11) proves (4.3.8). That is, $\otimes_{i=1}^m T_i \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$.

The proof that $(\otimes_{i=1}^m T_i)_{\mathfrak{N},\mathcal{O}}$ exists is then the same as in the proof of Theorem 4.2.1. ■

When $m = 2$, Theorem 4.3.1 can be improved by using k-wave front sets. First, if $\Gamma \subset T^*(X)$ we define

$$
(4.3.18)\quad -\Gamma = \{(x,-\xi)\,|\,(x,\xi) \in \Gamma\}
$$

THEOREM 4.3.2. *Let* $T_1, T_2 \in \mathfrak{B}'(X)$, $\dim X = n$. *Let* $\Gamma_i = WF(T_i)$. *Also, let*

$$
(4.3.19)\quad V_1(x_0) = [\Gamma_1 \cap (-\Gamma_2)]_{x_0}, \qquad V_2(x_0) = [(-\Gamma_1) \cap \Gamma_2]_{x_0}.
$$

Assume that for each $x_0 \in X$,

$$
(4.3.20)\quad \operatorname{Ord}_{x_0,V_1(x_0)} T_1 + \operatorname{Ord}_{x_0,V_2(x_0)} T_2 < -n.
$$

Then $T_1 \otimes T_2 \in \mathfrak{R}_{\mathfrak{N},\mathcal{O}}$ *where* $\mathfrak{N}$ *is the diagonal in* $\mathfrak{M} = X \times X$, *and* $\mathcal{O}$ *is a neighborhood of* $\mathfrak{N}$. *Further,* $(T_1 \otimes T_2)_{\mathfrak{N},\mathcal{O}}$ *is well defined. That is,* $T_1 T_2 \in \mathfrak{B}'(X)$ *is well defined by Definition 4.3.1.*

REMARKS. 1) Note that (4.3.20) can be stated as: If $WF_k(T_1)_{x_0} \cap V_1(x_0) \neq \varnothing$ and $WF_l(T_2)_{x_0} \cap V_2(x_0) \neq \varnothing$, then $k + l < -n$.

2) Clearly, $V_1 = -V_2$, so $V_1 = \varnothing$ if and only if $V_2 = \varnothing$. To be consistent with Theorem 4.3.1, we define

$$
\operatorname{Ord}_{x_0,\varnothing} T = -\infty.
$$

PROOF. Let $(x_0, x_0) \in \mathfrak{N}$ be given. Let U_1, U_2 be neighborhoods of $x_0 \in X$ as in Definition 4.1.2 corresponding to $V_1(x_0)$, $V_2(x_0)$ respectively. Let $U \subset U_1 \times U_2$ so that there exists $\psi_1 \in C_0^\infty(U_1)$, $\psi_2 \in C_0^\infty(U_2)$ with $\psi_1\psi_2 \equiv 1$ on U. Let U_1, U_2 have coordinate functions $\{(x_1, \ldots, x_n)\}$, $\{(y_1, \ldots, y_n)\}$ respectively. Let U be given coordinates $\{(u, v)\}$ where $u_i = x_i$, $v_i = x_i - y_i$ for all i. Then if $\mu = \varphi(u, v)\,du\,dv \in \mathfrak{B}(U)$,

(4.3.21)

$$|\langle T_1 \otimes T_2, \mu\rangle| = |\langle(\psi_1 D_1) \otimes (\psi_2 D_2), \varphi(x, x - y)\rangle| \leqslant \int |(\psi_1 D_1)\hat{}\,(\xi)(\psi_2 D_2)\hat{}\,(\eta)\check{\varphi}(\xi + \eta, -\eta)|\,d\xi\,d\eta$$

where D_i is a distribution associated with T_i. We break the domain of integration in (4.3.21) into three parts;

$$N_1 = \{(\xi, \eta)|\ |\xi + \eta| > \epsilon(|\xi| + |\eta|)\},$$
$$N_2 = \{(\xi, \eta)|\ |\xi + \eta| < \epsilon(|\xi| + |\eta|)\} \cap \{WFT_1 \times WFT_2\},$$
$$N_3 = \{(\xi, \eta)|\ |\xi + \eta| < \epsilon(|\xi| + |\eta|)\} \cap \{WFT_1 \times WFT_2\}',$$

where $\{WFT_1 \times WFT_2\}' = \mathbf{R}^n \times \mathbf{R}^n \backslash \{WFT_1 \times WFT_2\}$. As in the proof of Theorem 4.3.1, the integral over N_1 is bounded by

$$c_\epsilon \sum_{|\alpha| \leqslant k} \left\| \left(\frac{\partial}{\partial u}\right)^\alpha \varphi(u, v) \right\|_1 \tag{4.3.22}$$

for some k. Next, consider N_2' where we define

$$N_2' = N_2 \cap (S^{n-1} \times S^{n-1}) = \{(\xi, \eta) \in (WFT_1 \times WFT_2) \cap (S^{n-1} \times S^{n-1})|\ |\xi + \eta| < 2\epsilon\}.$$

Now if ξ_0 is not in an S^{n-1}-neighborhood of $WFT_1 \cap (-WFT_2) \cap S^{n-1}$, then for ϵ small, $(\xi_0, \eta) \notin N_2'$ for all $\eta \in WFT_2 \cap S^{n-1}$. Similarly, if η_0 is not in an S^{n-1}-neighborhood of $(-WFT_1) \cap WFT_2 \cap S^{n-1}$, then for ϵ small, $(\xi, \eta_0) \notin N_2'$ for all $\xi \in WFT_1 \cap S^{n-1}$. Hence, if $(\xi_0, \eta_0) \in N_2'$, we must have that:

ξ_0 is in an S^{n-1}-neighborhood of $WFT_1 \cap (-WFT_2) \cap S^{n-1}$;

η_0 is in an S^{n-1}-neighborhood of $(-WFT_1) \cap WFT_2 \cap S^{n-1}$.

Let W_{V_1}, W_{V_2} be the open conic neighborhoods of V_1, V_2 as in Definition 4.1.2. Let $W'_{V_i} = W_{V_i} \cap S^{n-1}$, $i = 1, 2$. Then the above comments imply that $N_2' \subset W'_{V_1} \times W'_{V_2}$ for ϵ small. Extending conically, we have that $N_2 \subset W_{V_1} \times W_{V_2}$. Hence, the integrand over N_2 is bounded by

$$\sup|(1 + |\xi + \eta|^2)^K \check{\varphi}(\xi + \eta, -\eta)| \int_{|\xi+\eta| < \epsilon(|\xi| + |\eta|)} (1 + |\xi|)^a (1 + |\eta|)^b (1 + |\xi + \eta|^2)^{-K}\,d\xi\,d\eta \tag{4.3.23}$$

where $a + b < -n$, K to be chosen. We estimate now, the integral in (4.3.23). Letting $\xi + \eta = u$, $\xi = v$ we obtain

$$\int_{|\xi+\eta| < \epsilon(|\xi| + |\eta|)} (1 + |\xi|)^a (1 + |\eta|)^b (1 + |\xi + \eta|^2)^{-K}\,d\xi\,d\eta$$
$$\leqslant \int (1 + |v|)^a \left[\int \frac{(1 + |u - v|)^b}{(1 + |u|^2)^K}\,du\right] dv \leqslant c \int (1 + |v|)^a (1 + |v|)^b\,dv \leqslant c$$

since $a + b < -n$ and K can be chosen large enough to apply Proposition 4.1.2. Using this in

(4.2.23), the integral over N_2 is bounded by

$$(4.3.24)\qquad c\sum_{|\alpha|\leqslant 2K}\left\|\left(\frac{\partial}{\partial u}\right)^{\alpha}\varphi(u,v)\right\|_1.$$

Now, for the integral over N_3, note that at least one of $(\psi_1 T_1)^\wedge(\xi)$, $(\psi_2 T_2)^\wedge(\eta)$ is rapidly decreasing, so this integral is bounded by an integral like (4.3.23) excepting that $a + b < k$ for all k. Hence, as above, this integral is bounded by (4.3.24). Combining (4.3.24) and (4.3.22) with (4.3.21), we obtain

$$(4.3.25)\qquad |\langle T_1 \otimes T_2, \mu\rangle| \leqslant c\sum_{|\alpha|\leqslant k}\left\|\left(\frac{\partial}{\partial u}\right)^{\alpha}\varphi(u,v)\right\|_1 \qquad \text{for all } \mu \in \mathcal{B}(U).$$

That is, $T_1 \otimes T_2 \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$.

To prove that $(T_1 \otimes T_2)_{\mathfrak{N},\mathcal{O}}$ exists, we follow the proof of Theorem 4.2.1. ∎

We now consider two algebraic properties of products as defined by (4.3.1). As for commutativity, we have the following.

THEOREM 4.3.3. *Let $T_i \in \mathcal{B}'(X)$, $i = 1, \ldots, m$ and assume that $\otimes_{i=1}^m T_i \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$ where $\mathcal{O}$ is a neighborhood in $\mathfrak{M}$ of the diagonal $\mathfrak{N}$. Then for any permutation π: $(1, \ldots, m) \to (\pi(1), \ldots, \pi(m))$ we have that*

$$(4.3.26)\qquad \bigotimes_{i=1}^{m} T_{\pi(i)} \in \mathcal{R}_{\mathfrak{N},\mathcal{O}'}$$

where $\mathcal{O}' \subset \mathcal{O}$. Further, assume that $\otimes_{i=1}^m T_i$ has a restriction to $\mathfrak{N}$ equal to T. Then $\otimes_{i=1}^m T_{\pi(i)}$ also has a restriction to $\mathfrak{N}$ and

$$(4.3.27)\qquad \left(\bigotimes_{i=1}^{m} T_{\pi(i)}\right)_{\mathfrak{N},\mathcal{O}'} = T.$$

PROOF. Let Φ: $\prod_{i=1}^m X_{\pi(i)} \to \prod_{i=1}^m X_i$ be the diffeomorphism defined by:

$$(4.3.28)\qquad \Phi: (x_{\pi(1)}, \ldots, x_{\pi(m)}) \to (x_1, \ldots, x_m).$$

By Proposition 1.1, if $\otimes_{i=1}^m T_i \in \mathcal{R}_{\mathfrak{N},\mathcal{O}}$, then $\Phi^*[\otimes_{i=1}^m T_i] = \otimes_{i=1}^m T_{\pi(i)} \in \mathcal{R}_{\mathfrak{N},\mathcal{O}'}$ where $\mathcal{O}' \subset \mathcal{O}$. Here we used that Φ: $\mathfrak{N} \to \mathfrak{N}$. Hence (4.3.26). Now by Proposition 3.2.4, we have that the existence of $(\otimes_{i=1}^m T_i)_{\mathfrak{N},\mathcal{O}}$ implies the existence of $[\Phi^*(\otimes_{i=1}^m T_i)]_{\mathfrak{N},\mathcal{O}'} = (\otimes_{i=1}^m T_{\pi(i)})_{\mathfrak{N},\mathcal{O}'}$. Further, we have that

$$(4.3.29)\qquad \left[\Phi^*\left(\bigotimes_{i=1}^{m} T_i\right)\right]_{\mathfrak{N},\mathcal{O}'} = \overline{\Phi}^*\left[\left(\bigotimes_{i=1}^{m} T_i\right)_{\mathfrak{N},\mathcal{O}}\right]$$

where $\overline{\Phi}$ is the induced diffeomorphism, $\overline{\Phi}$: $\mathfrak{N} \to \mathfrak{N}$. Noting that $\overline{\Phi}$ = identity, we conclude from (4.3.29) that

$$(4.3.30)\qquad \left(\bigotimes_{i=1}^{m} T_{\pi(i)}\right)_{\mathfrak{N},\mathcal{O}'} = \left(\bigotimes_{i=1}^{m} T_i\right)_{\mathfrak{N},\mathcal{O}}. \quad ∎$$

As for distributivity, we only have the problem of existence. That is, from the existence of $S(\sum_{i=1}^m T_i)$, we cannot conclude the existence of ST_j for any j. The converse, however, is easily seen to be true and we have the following.

PROPOSITION 4.3.4. *Let $S, T_i \in \mathcal{B}'(X)$, $i = 1, \ldots, m$. Assume that ST_j exists for all j. Then $S(\sum_{i=1}^m T_i)$ exists. Further*

$$S\left(\sum_{i=1}^{m} T_i\right) = \sum_{i=1}^{m} ST_i. \tag{4.3.31}$$

PROOF. Noting that

$$S \otimes \left(\sum_{i=1}^{m} T_i\right) = \sum_{i=1}^{m} (S \otimes T_i), \tag{4.3.32}$$

the proof is trivial. ■

REFERENCE

(1) Hörmander, L., Fourier Integral Operators I, *Acta Math.* **127** (1971) 79—183.

BIOGRAPHICAL SKETCH

Robert R. Reitano was born on May 6, 1950 in Lawrence, Massachusetts. After graduating from high school in 1967, he went to the University of Massachusetts in Boston where he graduated with a B.A. in 1971 with Senior Honors in mathematics.

In June 1972, he received an M.A. from the University of Massachusetts at Amherst. During that year, he was supported by a Teaching Assistantship and Staff Scholarship.

In September of that year, he became a part-time instructor at the University of Massachusetts at Boston, a position he still holds.

In January of 1974, he became a full time graduate student at Massachusetts Institute of Technology with Professor Alberto P. Calderón as thesis advisor. During much of this stay at M.I.T., he was supported by a Teaching Assistantship and Staff Scholarship.

On October 10, 1976, he will marry Carol L. Santacroce of Everett and will remain at the University of Massachusetts in Boston for the 1976-77 academic year.