# A FATOU-TYPE THEOREM FOR HARMONIC FUNCTIONS ON SYMMETRIC SPACES[1]

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1. **Introduction.** The result to be proved in this article is that if $u$ is a bounded harmonic function on a symmetric space $X$ and $x_0$ any point in $X$ then $u$ has a limit along almost every geodesic in $X$ starting at $x_0$ (Theorem 2.3). In the case when $X$ is the unit disk with the non-Euclidean metric this result reduces to the classical Fatou theorem (for radial limits). When specialized to this case our proof is quite different from the usual one; in fact it corresponds to transforming the Poisson integral of the unit disk to that of the upper half-plane and using only a homogeneity property of the Poisson kernel. The kernel itself never enters into the proof.

2. **Harmonic functions on symmetric spaces.** Let $G$ be a semisimple connected Lie group with finite center, $K$ a maximal compact subgroup of $G$ and $\mathfrak{g}$ and $\mathfrak{k}$ their respective Lie algebras. Let $B$ denote the Killing form of $\mathfrak{g}$ and $\mathfrak{p}$ the corresponding orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Let Ad denote the adjoint representation of $G$. As usual we view $\mathfrak{p}$ as the tangent space to the symmetric space $X=G/K$ at the origin $o=\{K\}$ and accordingly give $X$ the $G$-invariant Riemannian structure induced by the restriction of $B$ to $\mathfrak{p}\times\mathfrak{p}$. Let $\Delta$ denote the corresponding Laplace-Beltrami operator.

Fix a maximal abelian subspace $\mathfrak{a}\subset\mathfrak{p}$ and let $M$ denote the centralizer of $\mathfrak{a}$ in $K$. If $\lambda$ is a linear function on $\mathfrak{a}$ and $\lambda\neq 0$ let $\mathfrak{g}_\lambda = \{X\in\mathfrak{g} \mid [H, X]=\lambda(H)X \text{ for all } H\in\mathfrak{a}\}$; $\lambda$ is called a restricted root if $\mathfrak{g}_\lambda\neq 0$. Let $\mathfrak{a}'$ denote the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$. Fix a Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$, i.e. a connected component of $\mathfrak{a}'$. A restricted root $\alpha$ is called positive (denoted $\alpha>0$) if its values on $\mathfrak{a}^+$ are positive. Let the linear function $\rho$ on $\mathfrak{a}$ be determined by $2\rho=\sum_{\alpha>0}(\dim \mathfrak{g}_\alpha)\alpha$ and denote the subalgebras $\sum_{\alpha>0}\mathfrak{g}_\alpha$ and $\sum_{\alpha>0}\mathfrak{g}_{-\alpha}$ of $\mathfrak{g}$ by $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ respectively. Let $N$ and $\bar{N}$ denote the corresponding analytic subgroups of $G$.

By a Weyl chamber in $\mathfrak{p}$ we understand a Weyl chamber in some maximal abelian subspace of $\mathfrak{p}$. The *boundary* of $X$ is defined as the set $B$ of all Weyl chambers in the tangent space $\mathfrak{p}$ to $X$ at $o$; since this boundary is via the map $kM\to\mathrm{Ad}(k)\mathfrak{a}^+$ identified with $K/M$, which by the Iwasawa decomposition $G=KAN$ equals $G/MAN$, this defi-

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nition of boundary is equivalent to Furstenberg's [2] (see also [6] and [4]). In particular the group $G$ acts transitively on $B$ as well as on $X$. The two actions will be denoted $(g, b)\to g(b)$ and $(g, x)\to g\cdot x$ ($g\in G$, $b\in B$, $x\in X$). Let $db$ denote the unique $K$-invariant measure on $B$ normalized by $\int_B db=1$. Then according to Furstenberg [2], the mapping $f\to u$ where

$$u(g\cdot o) = \int_B f(g(b))db \qquad (g \in G), \tag{1}$$

is a bijection of the set $L^\infty(B)$ of bounded measurable functions on $B$ onto the set of bounded solutions of Laplace's equation $\Delta u=0$ on $X$. The function $u$ in (1) is called the *Poisson integral* of $f$.

If $g\in G$ let $k(g)\in K$, $H(g)\in\mathfrak{a}$ be determined by $g=k(g)\exp H(g)n$ ($n\in N$). Observe that if $g^h$ denotes $hgh^{-1}$ for $h\in G$ then $k(\bar{n}^m)=k(\bar{n})^m$, $H(\bar{n}^m)=H(\bar{n})$ for $\bar{n}\in\overline{N}$, $m\in M$. According to Harish-Chandra [3, Lemma 44], the mapping $\bar{n}\to k(\bar{n})M$ is a bijection of $\overline{N}$ onto a subset of $K/M$ whose complement is of lower dimension and if $f$ is a continuous function on $B$, then

$$\int_B f(b)db = \int_{\overline{N}} f(k(\bar{n})M)\exp(-2\rho(H(\bar{n})))d\bar{n} \tag{2}$$

for a suitably normalized Haar measure $d\bar{n}$ on $\overline{N}$. If $a\in A$ we have $ak(\bar{n})MAN=k(\bar{n}^a)MAN$ whence

$$a(k(\bar{n})M) = k(\bar{n}^a)M \tag{3}$$

so the action of $a$ on the boundary corresponds to the conjugation $\bar{n}\to\bar{n}^a$ on $\overline{N}$.

Let $E_1, \cdots, E_r$ be a basis of $\bar{\mathfrak{n}}$ such that each $E_i$ lies in some $\mathfrak{g}_{-\alpha}$, say $\mathfrak{g}_{-\alpha_i}$. Since the map exp: $\bar{\mathfrak{n}}\to\overline{N}$ is a bijection we can, for each $H\in\mathfrak{a}^+$, consider the function $\bar{n}\to|\bar{n}|_H$ defined as follows: If $\bar{n}=\exp(\sum_1^r a_iE_i)$ ($a_i\in R$) we put

$$|\bar{n}|_H=\operatorname*{Max}_{1\leq i\leq r}(|a_i|^{1/\alpha_i(H)})$$

Since

$$\bar{n}^{\exp tH} = \exp\left(\sum_1^r a_i\exp(-\alpha_i(H)t)E_i\right) \tag{4}$$

we have

$$|\bar{n}^{\exp tH}|_H = e^{-t}|\bar{n}|_H \qquad \text{for } \bar{n}\in\overline{N},\ t\in R,\ H\in\mathfrak{a}^+. \tag{5}$$

For $r>0$ let $B_{H,r}$ denote the set $\{\bar{n}\in\overline{N} \mid |\bar{n}|_H<r\}$ and let $V_{H,r}$ denote the volume of $B_{H,r}$ (with respect to the Haar measure on $\overline{N}$).

LEMMA 2.1. *Let $f\in L^\infty(B)$ and $u$ the Poisson integral* (1) *of $f$. Put $F(\bar{n})=f(k(\bar{n})M)$ for $\bar{n}\in\overline{N}$. Fix $\bar{n}_0\in\overline{N}$ and $H\in\mathfrak{a}^+$ and assume*

$$\frac{1}{V_{H,r}}\int_{B_{H,r}} \left| F(\bar{n}_0\bar{n}) - F(\bar{n}_0)\right| d\bar{n} \to 0 \tag{6}$$

*for $r\to 0$. Then*

$$\lim_{t\to+\infty} u(k(\bar{n}_0)\exp tH(\cdot o)) = f(k(\bar{n}_0)M).$$

PROOF. By the Iwasawa decomposition we can write $\bar{n}_0=k(\bar{n}_0)\cdot(a_1n_1)^{-1}$ $(a_1\in A,\ n_1\in N)$ so

$$u(k(\bar{n}_0)\exp tH\cdot o) = u(\bar{n}_0a_1n_1\exp tH\cdot o) = u(\bar{n}_0\exp tHa_1n_1^{\exp(-tH)}\cdot o).$$

But $G=A\overline{N}K$ so $n_1^{\exp(-tH)}=a(t)\bar{n}(t)k(t)$, each factor tending to $e$ as $t\to+\infty$. If $H_t\in\mathfrak{a}$ is determined by

$$\exp tH_t = \exp tHa_1a(t)$$

we have

$$u(k(\bar{n}_0)\exp tH\cdot o) = u(\bar{n}_0\bar{n}(t)^{\exp tH_t}\exp tH_t\cdot o).$$

The function $f'(b)=f(\bar{n}_0\bar{n}(t)^{\exp tH_t}(b))$ has Poisson integral $u'(x)=u(\bar{n}_0\bar{n}(t)^{\exp tH_t}\cdot x)$; using (1) on $u'$ and $f'$ with $g=\exp tH_t$ we get from (2) and (3)

$$u(k(\bar{n}_0)\exp tH\cdot o) - f(k(\bar{n}_0)M)$$
$$= \int_{\overline{N}}(F(\bar{n}_0\bar{n}(t)^{\exp tH_t}\bar{n}^{\exp tH_t}) - F(\bar{n}_0))\exp(-2\rho(H(\bar{n})))d\bar{n}$$

so

$$\begin{aligned}&\left| u(k(\bar{n}_0)\exp tH\cdot o) - f(k(\bar{n}_0)M)\right| \\ &\quad\leqq \int_{\overline{N}}\left| F(\bar{n}_0\bar{n}^{\exp tH_t}) - F(\bar{n}_0)\right| \exp(-2\rho(H(\bar{n}(t)^{-1}\bar{n})))d\bar{n}.\end{aligned}\tag{7}$$

Now if $c>0$ let $\overline{N}_c$ denote the "square"

$$\overline{N}_c = \left\{\exp\left(\sum_1^r a_iE_i\right)\Big|\ |a_i| \leqq c,\ 1\leqq i\leqq r\right\}.$$

The integral on the right in (7) equals the sum

$$\begin{aligned}(8)\qquad &\int_{\bar N_c} \left| F(\bar n_0 \bar n^{\exp tH_t}) - F(\bar n_0)\right| \exp(-2\rho(H(\bar n(t)^{-1}\bar n)))d\bar n \\ &+ \int_{\bar N - \bar N_c} \left| F(\bar n_0 \bar n^{\exp tH_t}) - F(\bar n_0)\right| \exp(-2\rho(H(\bar n(t)^{-1}\bar n)))d\bar n.\end{aligned}$$

Since $\rho(H(\bar n)) \geqq 0$ for all $\bar n \in \bar N$ ([3, p. 287]) and since the mapping $\bar n \rightarrow \bar n^{\exp H}$ has Jacobian $\exp(-2\rho(H))$ (cf. (4)) we see that

$$\begin{aligned}(9)\qquad &\int_{\bar N_c} \left| F(\bar n_0 \bar n^{\exp tH_t}) - F(\bar n_0)\right| \exp(-2\rho(H(\bar n(t)^{-1}\bar n)))d\bar n \\ &\leqq \exp(2\rho(tH_t)) \int_{\bar N_c^{\exp tH_t}} \left| F(\bar n_0 \bar n) - F(\bar n_0)\right| d\bar n.\end{aligned}$$

Now $\bar n \in \bar N_c{}^{\exp tH_t}$ if and only if

$$\bar n = \exp\left(\sum a_i e^{-\alpha_i(tH_t)} E_i\right) \quad \text{where } |a_i| \leqq c$$

and $tH_t - tH$ is bounded (for fixed $\bar n_0$ and $H$). It follows that

$$\bar N_c^{\exp tH_t} \subset B_{H,de^{-t}} \qquad \text{for all } t \geqq 0,$$

$d = d(H, \bar n_0, c)$ being a constant. But since the map exp: $\bar{\mathfrak{n}} \rightarrow \bar N$ is measure-preserving it is clear that

$$V_{H,de^{-t}} = \exp(-2\rho(H)t)d_1 \qquad t \geqq 0$$

where $d_1 = d_1(H, \bar n_0, c)$ is another constant. Also

$$\exp(2\rho(tH_t)) \leqq \exp(2\rho(tH))d_2$$

where $d_2(H, \bar n_0)$ is a constant. Thus the right hand side of (9) can be majorized for all $t \geqq 0$:

$$\begin{aligned}(10)\qquad &\exp 2\rho(tH_t) \int_{\bar N_c^{\exp tH_t}} \left| F(\bar n_0 \bar n) - F(\bar n_0)\right| d\bar n \\ &\leqq d_3 \frac{1}{V_{H,de^{-t}}} \int_{B_{H,de^{-t}}} \left| F(\bar n_0 \bar n) - F(\bar n_0)\right| d\bar n\end{aligned}$$

where $d$ and $d_3$ are constants depending on $H$, $\bar n_0$ and $c$.

On the other hand, if $\| \ \|_\infty$ denotes the uniform norm on $\bar N$ the second term in (8) is majorized by

$$\begin{aligned}(11)\qquad &2\|F\|_\infty \int_{\bar N - \bar N_c} \exp\left(-2\rho(H(\bar n(t)^{-1}\bar n))\right)d\bar n \\ &= 2\|F\|_\infty \left(1 - \int_{\bar n(t)\bar N_c} \exp\left(-2\rho(H(\bar n))\right)d\bar n\right).\end{aligned}$$

Now given $\epsilon>0$ we first choose $c$ so large that

$$2\|F\|_\infty\left(1-\int_{\overline{N}_{c/2}}\exp(-2\rho(H(\bar{n})))d\bar{n}\right)<\epsilon/2;$$

since $\bar{n}(t)\to e$ for $t\to+\infty$ we can choose $t_1$ such that $\bar{n}(t)\overline{N}_c\supset\overline{N}_{c/2}$ for $t\geqq t_1$. Then the expression in (11) is $<\epsilon/2$ for $t\geqq t_1$; by our assumption (6) we can choose $t_2$ such that the right hand side of (10) is $<\epsilon/2$ for $t>t_2$. In view of (7) and (8) this proves the lemma.

The next lemma shows that, for a fixed $H$, the assumption of Lemma 2.1 actually holds for almost all $\bar{n}_0\in\overline{N}$.

LEMMA 2.2. *Let $F\in L^\infty(\overline{N})$ and fix $H\in\mathfrak{a}^+$. Then*

$$\lim_{r\to 0}\frac{1}{V_{H,r}}\int_{B_{H,r}}\left|F(\bar{n}_0\bar{n})-F(\bar{n}_0)\right|d\bar{n}=0 \tag{12}$$

*for almost all $\bar{n}_0\in\overline{N}$.*

The proof of this result is essentially in the literature: In [1] Edwards and Hewitt give all the necessary arguments for the case of a discrete sequence tending to 0 and everything they do remains trivially valid in the case $r\to 0$. The result in the exact form required here was also proved by E. M. Stein independently of [1] (cf. his expository article [6]).

THEOREM 2.3. *Let $u$ be a bounded solution of Laplace's equation $\Delta u=0$ on the symmetric space $X$. Then for almost all geodesics $\gamma(t)$ starting at $o$*

$$\lim_{t\to\infty}u(\gamma(t))\quad \textit{exists.} \tag{13}$$

PROOF. Let $S^+=\{H\in\mathfrak{a}^+|\ B(H,H)=1\}$. Then the mapping $(kM,H)\to\mathrm{Ad}(k)H$ is a bijection of $(K/M)\times S^+$ onto a subset of the unit sphere $S$ in $\mathfrak{p}$ whose complement has lower dimension. Since $\dim(K/M-k(\overline{N})/M)<\dim K/M$ the mapping $(\bar{n},H)\to\mathrm{Ad}(k(\bar{n}))H$ is a bijection of $\overline{N}\times S^+$ onto a subset of $S$ whose complement in $S$ has lower dimension. If $\overline{N}_H$ denotes the set of $\bar{n}_0$ for which (12) holds (with $F(\bar{n})=f(k(\bar{n})M)$) and if $S_0=\bigcup_{H\in S^+}\mathrm{Ad}(k(\overline{N}_H))H$ it follows from the Fubini theorem that $S-S_0$ is a null set. This concludes the proof.

REMARKS. (i) If $f$ is continuous the limit relation

$$\lim_{t\to+\infty}u(k\exp tH\cdot o)=f(kM)\qquad (H\in\mathfrak{a}^+,\ kM\in K/M)$$

follows immediately from (1), (2) and (3), by use of the dominated convergence theorem. (See also [4, Theorem 18.3.2.]) In particular, $u$ has the same limit along all geodesics from $o$ which lie in the same Weyl chamber in $\mathfrak{p}$.

(ii) In the case when $X$ has rank one ($\dim \mathfrak{a} = 1$) A. W. Knapp [5] has proved (13), even under the weaker assumption that $f \in L^1(B)$.

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