AN ANALYSIS OF THE MULTIPLE MODEL ADAPTIVE CONTROL ALGORITHM

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SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
August, 1978

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Department of Electrical Engineering
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Submitted to the Department of Electrical Engineering and Computer Science on August 29, 1978 in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

ABSTRACT

The Multiple Model Adaptive Control algorithm has been used in applications of advanced control technology. However, in these applications, many undesirable characteristics of the method, such as high amplitude limit cycles, have been uncovered. In this thesis the basic types of behavior exhibited by the method are explored. This is done through the simulation and analysis of the method as applied to a sample system structure. This structure has been carefully chosen to exhibit the major phenomena of interest while remaining amenable to detailed analysis. Two major types of results are presented. First of all, detailed conditions for the existance of each of the types of behavior are developed for the special system structure under consideration. Of possibly greater significance are the qualitative insights which result from extrapolating the detailed conclusions to problems of more general structure. It is believed that the qualitative understandings developed in this thesis can form the basis for the introduction of design modifications (two of which are suggested in this thesis) and the development of a systematic methodology for the design of adaptive control systems using the Multiple Model Adaptive Control algorithms.

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ACKNOWLEDGEMENTS

I deeply appreciate the patient guidance and support provided by Professor Alan S. Willsky throughout my career at M.I.T. in various roles of counselor and thesis supervisor. I have greatly enjoyed working so closely with him. I would also like to thank Prof. Gunter Stein and Dr. Stanley B. Gershwin for their guidance and constructive suggestions as thesis readers. In addition, Prof. Michael Athans has provided considerable guidance and encouragement in the preliminary phase of this endeavor.

Of course, many other people have aided me by discussing various aspects of the problem. A few of these are: Mr. E.Y. Chow, Dr. J. Douglas Birdwell, Dr. David Castanon, Dr. Paul K. Houpt and Prof. Nils R. Sandell. I would also like to acknowledge the assistance I received from Ms. Margaret Flaherty in typing and from Mr. Arthur Giordani in overseeing the production of the figures.

Finally, I would like to thank my wife Sarah for her encouragement and patient understanding throughout the years of study. Without her this thesis would never have been completed.

This work was supported by the Office of Naval Research under Grant N00014-77-C-0224 and by NASA Ames Research Center under Grant NGL-22-009-124.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

In many applications of control theory, the dynamics of the plant are incompletely known at best. Furthermore, the dynamics are often time-varying and non-linear. In such an environment, control becomes a very difficult task and the problem of the optimal control of such systems remains unsolved. However, such systems need to be controlled and so a myriad of suboptimal schemes have emerged.

The many methods which have been proposed can, in general, be divided into two classes: the passive methods which rely on the robustness of a time-invariant feedback controller to maintain good performance and the active methods which involve changing the controllers as necessary. For example, much of the work of Horowitz [1,2] has been aimed at deriving a single, time-invariant control law which gives acceptable behavior for all plant parameter values (a passive approach). The work of Wong [3, 4] has similarly been aimed at analyzing the robustness properties of feedback controllers using a geometric approach, and Safonov [5] has derived robustness conditions for controllers when the parameter variations are due to a change in the operating point of a non-linear system. It should be pointed out that the ad hoc approach of increasing the plant noise design parameter (see Section 2.1) often mentioned for the standard Linear-Quadratic-Gaussian (LQG) problem [6] is also a passive method of overcoming plant uncertainty.

A major problem with such methods is that they are, by design, compromises. Performance for normal conditions is sacrificed in order to improve performance for other conditions. In the extreme, it may be impossible to maintain the desired performance for the full range of conditions using a fixed controller.

In contrast to the passive methods, the active methods make use of a time-varying controller. Thus, they employ mechanisms which force the controller to adapt to changes in the operating environment. This, at least in theory, improves performance under all operating conditions since the controller can be tuned to the actual, rather than the average or even worst case, conditions.

As previously mentioned, the optimal control of such systems is an unsolved problem. Thus ad hoc, suboptimal techniques have been proposed. However, partly because of the non-linearities (due to adaption) of such methods, they have not been subject to careful study regarding qualitative performance characteristics such as deterministic stability. In applications, many of these methods have exhibited difficulties which have been mitigated by further ad hoc modifications of the design [16, 23]. In general, these modifications were not the product of extensive, systematic analysis of the system's behavior and no general design methodology has emerged.

The research which is reported herein attempts to qualitatively and quantitatively examine the properties of one method of adaptive control which has been discussed in the literature, namely, the Multiple Model Adaptive Control (MMAC) method [7]. The MMAC method, which is discussed further in Chapter 2 of this thesis, has a very pleasing structure: a cascade of something which resembles a Maximum Aposteriori Probability (MAP) identifier [15] (basically a bank of Kalman Filters) and a bank of linear quadratic regulators. However, in use it has become clear that the MMAC method can exhibit unacceptable behavior - such as high amplitude limit cycles. In this thesis the major qualitative properties of the MMAC method are examined and the principle reasons for the unacceptable behavior explored. It is believed that a through understanding of the behavior will lead to guidelines for the modification of the design which will ultimately yield a general design methodology.

1.2 Background

The general area of adaptive control has received much attention recently. For example [26] and [27] both contain numerous references to a wide variety of approaches. The basic subject which is addressed is to generate a control u(t) for a system given by

$$\dot{\underline{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{u}(t) + \underline{\zeta}(t)$$

with observations

$$\underline{y}(t) = \underline{C}(t)\underline{x}(t) + \underline{\eta}(t) \quad .$$

The state $\underline{x}(t)$ is an n-vector while the input $\underline{u}(t)$ is an m-vector and $\underline{y}(t)$ is a p-vector. The vectors $\underline{\zeta}(t)$ and $\underline{\eta}(t)$ represent system uncertainties and observation noise respectively. The condition which introduces the most complexity is that the system matrix $\underline{A}$ (nxn), input matrix $\underline{B}$ (nxm) and output matrix $\underline{C}$ (pxn) are only incompletely known. The performance measure which is often used to judge such systems is a quadratic one, given by

$$J(\underline{u}) = \int_0^{\infty} [\underline{x}'(t)\underline{Q}\,\underline{x}(t) + u'(t)\underline{R}\,\underline{u}(t)]dt$$

where $\underline{Q}$ is an nxn positive semi-definite matrix and $\underline{R}$ is a mxm positive definite matrix.

The solution to this control problem has not been found and the work in [7] clearly indicates that the optimal system, assuming it can be found, will prove to be far too complex to implement. Thus, numerous suboptimal solutions have been proposed.

The MMAC method was largely inspired by the work of Magill [8] who was the first to examine a parallel -adaptive estimation algorithm in which the basic estimation is done by a bank of filters which are then coordinated by a centralized controller (see Figure 1.1). Further work in the area has been done by Lainiotis whose work has been summarized in [9]. The major thrust of this work has been aimed at adaptive estimation and parameter identification and not the control problem.

Many authors [10, 11, 7] have examined feedback controllers based on the structure of Magill's estimator, however. For example, Stein [10] has been able to derive upper and lower bounds on the cost in the optimal control problem and using the upper bound, has obtained a control law exhibiting a parallel structure. Saridis and Dao [11] have exploited Stein's lower bound to obtain a different control law. One major drawback of both algorithms is that they require significant on-line computation.

Willner [7] proposed the MMAC algorithm as discussed in this thesis and showed that it performed well in relation to both the

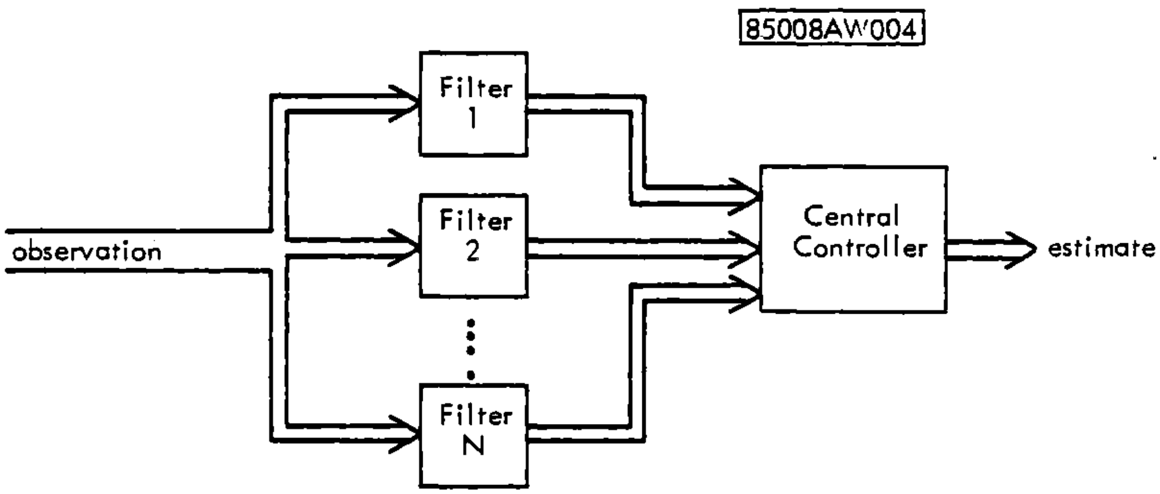

Fig. 1.1 Parallel-Adaptive Algorithm Structure

upper and lower bounds of Stein. Independently, Deshpande et al [12] arrived at the same algorithm as an ad hoc extension of the estimation/identification algorithm of Magill [8]. The method has been further considered by Lainiotis [13].

To our knowledge, no one has been successful in establishing any definitive properties on the behavior of the MMAC method. Specifically, the convergence properties of the identification problem (i.e., with the adaptive mechanism disabled) have only recently been proven. Both Hawkes and Moore [14] and Baram [15] have provided useful results, but both results do not hold in an adaptive situation (i.e., when the control law is a function of the system output).

The MMAC method has been applied to various settings. For example, in Athans et al. [16, 23] the method has been used to control the F-8 aircraft. Also, the F-8 controller of Stein et al. [17] can also be considered to be a multiple model design. Additional experience with and insight into the MMAC method has been gained by applying the estimation/identification algorithm to the detection of accidents on freeways [18].

To a great extent, the F-8 project [16, 23] has provided the motivation for our work. In that project, where the true system does not correspond exactly to any of the hypothesized models, several problems were encountered. First of all, the probabilities often oscillated rapidly between two models, exhibiting behavior very much like a limit cycle. This problem led to a design modification consisting of the insertion of low pass filters to slow the probability transitions.

A second problem which was encountered involved the choice of which models to include in the set of possible models. The aircraft, of course, responds in a continuous way to parameter changes, and model set selection was found to have a considerable effect on performance.

It is issues such as these which have motivated our study. Our goals have been to understand the characteristics of the MMAC method and develop a useful design methodology based on the MMAC algorithm.

1.3 Contributions of This Thesis

This thesis presents the results of a detailed study of the MMAC algorithm. The major conclusions of this study, which are detailed in Chapter 6, are of two basic types. First of all, there are specific conclusions which the analysis of Chapters 4 and 5 yield. However, since the analysis of these chapters relies heavily on a special case, the results are of relatively little direct applicability. However, they are indicative of the types of and basic causes for behavior which has been observed in more general situations. Thus, of possibly greater importance are the qualitative conclusions which result when the specific conclusions are extrapolated to general problem structures. These qualitative results are also detailed in Chapter 6.

The following are the major results of this thesis.

1. The neutral stability of the MMAC method is established.

2. Conditions which guarantee state convergence are derived.

3. The MMAC algorithm consists of a bank of Kalman Filters, each corresponding to a hypothesized linear time-invariant model of

the system generating the observations (referred to as the true system). The outputs of each filter are used to calculate the aposteriori probability that each filter matches the true system. The feedback control is then given by the probabilisticly weighted average of the controls calculated for each hypothesized model. (See Chapter 2 for a complete discussion). This results in a highly non-linear closed loop system. However, if the probability is constrained to be constant, then the closed loop system becomes linear time-invariant. In this thesis it is shown that even if the closed loop system is unstable for all constant values of the probability, the overall system may have a bounded (in fact "hyperbolicly stable" - see Section 4.6) response. For a special case, specific conditions are derived.

4. The effects of numerical roundoff are examined and a form of implementation which behaves well is proposed.

5. The specific results of this thesis can be used to predict the qualitative behavior of MMAC systems which have a more complex structure than the ones analyzed herein.

4.1 Overview of this Thesis

In Chapter 2, the MMAC algorithm is introduced in the form which will be used in the reaminder of this thesis. Both the continuous and discrete time versions of the algorithm are presented, although the discrete version is employed for the majority of the analysis.

In Chapter 3, the canonical problem which forms the focus for the entire thesis is introduced. This problem, about which various structural assumptions are made, has been carefully selected in order to provide a situation which retains many of the qualitative properties (such as the existence of oscillations) which have been observed in more general problems while simultaneously being amenable to detailed analysis. The remainder of the chapter contains a discussion of the various types of behavior which have been observed in simulations of the canonical problem along with sample simulation results to illustrate each behavior. The basic responses are shown to be of three types. In the first, termed "exponential" or "geometric", the states are geometrically stable. In this case, all of the states are decreasing for <u>all</u> constant values of the probability. The second type termed "oscillatory" results in the <u>probability</u> oscillating between zero and one, which in turn results in the states exhibiting an oscillatory behavior, alternately <u>increasing</u> and <u>decreasing</u> along with the probability. The third type termed "mixed", results in a behavior which, depending on the magnitude of the initial conditions, exhibits either an oscillatory or exponential behavior. These simulations have been used to motivate the analysis in the remainder of the thesis.

Chapter 4 contains the majority of the analysis of the MMAC method. In this chapter, each type of behavior described in Chapter 3 is analyzed in order to yield an understanding of the underlying causes of the behavior. This results in conditions which guarantee the existance of the exponential mode. Furthermore, various approximations are used to characterize the major modes of behavior which lead to conditions for the presence and absence of each behavior for the special case of the canonical problem. Beyond this, more general qualitative results are also presented.

In Chapter 5, various aspects of the problem of implementing the MMAC method using a digital computer are discussed. By far the most important of these is the modification to the analysis of the oscillatory behavior when the finite precision nature of the computer becomes a factor in the behavior. Also discussed are the effects of using each of the various forms of the equations as far as numerical accuracy is concerned. The chapter concludes with the proposal of a new form of the equations which is believed to allow the designer greater latitude in design without encountering numerical problems.

Chapter 6 contains a discussion of various ad hoc design modifications which have been proposed in order to overcome the shortcomings of the MMAC method. Also included there is a summary of the conclusions of this thesis as well as some suggestions for future research.

1.5 Notation

The following is a brief list of the notation employed in the thesis. Except for the notation for the components of the residual of a Kalman Filter, all are believed to be standard.

Matrices are represented by upper case letters which are underlined.

Vectors are represented by lower case letters which are underlined.

Scalars are represented by lower case letters (not underlined).

$\underline{A}'$	-	the transpose of $\underline{A}$.
$\|\|\underline{A}\|\|$	-	the norm of a matrix = Max $\lambda(\underline{A}'\underline{A})$
$\lambda(\underline{A})$	-	the set of eigenvalues of a matrix.
$N(\underline{A})$	-	the Null Space of a matrix.
$\|\|\underline{x}\|\|_{\theta^{-1}}$	-	the θ^{-1}-norm of a vector $\underline{x}$ given by $\sqrt{x'\theta^{-1}x}$ for θ^{-1} positive definite.
$\|\|\underline{x}\|\|$	-	the norm of a vector given by $\sqrt{x'x}$.
$\|a\|$	-	magnitude of a real or complex a.
$\Delta\underline{x}(k)$	-	change in $x(\cdot)$ from k-1 to k.
$\prod_{i=1}^{N}$	-	Product for i=1 to N
$\sum_{i=1}^{N}$	-	Sum for i = 1 to N
$\underline{x}$	-	<u>vector</u> of true state variables
x_i	-	<u>scalar</u> true state component i
$r_i^{(j)}$	-	<u>scalar</u> $j^{\underline{th}}$ component of $\underline{r}_i$
(a,b)	-	open interval between a and b.
[a,b]	-	closed interval between a and b.

CHAPTER 2

REVIEW OF THE MMAC METHOD

The purpose of the present chapter is to introduce the Multiple Model Adaptive Control (MMAC) algorithm. A full discussion will not be given as that is available from other sources [7, 12].

The MMAC algorithm is composed of two parts. The first, which performs an estimation/identification function, is similar to a Maximum Aposteriori Probability (MAP) algorithm [15] which is discussed in Section 2.1 for the discrete time case. The MAP algorithm is structured as a bank of Kalman Filters with some decision logic. The second part, which is cascaded with the MAP-like algorithm, is a control computation which is discussed in Sections 2.2 and 2.3 for the discrete time case.

The remaining sections of this chapter contain a development and discussion of the special forms of the equations for the MMAC algorithm which prove useful in the remaining chapters of this thesis.

2.1 Maximum Aposteriori Probability (MAP) Identification

2.1.1 The Kalman Filter (KF)

Assume that a linear, time-invariant (LTI) discrete time system is given by:

$$\underline{x}(k+1) = \underline{A}\,\underline{x}(k) + \underline{B}\,\underline{u}(k) + \underline{\zeta}(k) \tag{2.1a}$$

with observations:

$$\underline{y}(k) = \underline{C}\,\underline{x}(k) + \underline{\eta}(k) \tag{2.1b}$$

where $\underline{x}(k)$ is the n-dimensional state vector, $\underline{u}(k)$ is the m-dimensional control vector and $\underline{y}(k)$ is the p-dimensional observation vector. The noise sources $\underline{\zeta}(k)$ and $\underline{\eta}(k)$ are taken to be zero mean white Gaussian noises of covariances $\underline{\Xi}$ and $\underline{\psi}$ respectively. The matrices $\underline{A}$ (nxn), $\underline{B}$ (nxm), and $\underline{C}$ (pxn) are the system, input and output matrices respectively. We will use the notation

$$(\underline{A},\ \underline{B},\ \underline{C}) \tag{2.2}$$

to refer to the above system. The system $(\underline{A},\ \underline{B},\ \underline{C})$ which generates the observations $\underline{y}(k)$ will be called the <u>true system</u>.

In practice, the actual values of the matrices $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{\Xi}$ and $\underline{\psi}$ are unknown. However, estimates of these parameters are often available from a knowledge of the system. Thus, $(\underline{A}_i,\ \underline{B}_i,\ \underline{C}_i)$ will be used to denote the i^{th} <u>model</u> of the system, given by:

$$\underline{x}(k+1) = \underline{A}_i\underline{x}(k) + \underline{B}_i u(k) + \underline{\zeta}(k) \tag{2.3}$$

$$\underline{y}(k) = \underline{C}_i\underline{x}(k) + \underline{\eta}(k) .$$

For the purposes of the present study the values of $\underline{\psi}$ and $\underline{\Xi}$ will <u>not</u> vary from model to model, although extensions to that case could be considered.

It is well known that the steady state Kalman Filter (KF) [19, 20] which estimates the state x(k), based on the model

$$(\underline{A}_i,\ \underline{B}_i,\ \underline{C}_i)$$

is given by

$$\underline{\hat{x}}_i(k+1) = \underline{A}_i\underline{\hat{x}}_i(k) + \underline{B}_i\underline{u}(k) + \underline{H}_i\underline{r}_i(k+1)$$
$$\underline{r}(k+1) = \underline{y}(k+1) - \underline{C}_i[\underline{A}_i\underline{\hat{x}}_i(k) + \underline{B}_i\underline{u}(k)] \qquad (2.4)$$

where $\underline{H}_i$ is the Kalman gain for the $i^{\underline{th}}$ model:

$$\underline{H}_i = \underline{\Sigma}_i\underline{C}_i'\underline{\psi}^{-1} \qquad (2.5)$$

and $\underline{\Sigma}_i$ is the nxn solution to the familar Riccati equation

$$\underline{\Sigma}_i = [\underline{C}_i'\underline{\psi}^{-1}\underline{C}_i + (\underline{A}_i\underline{\Sigma}_i\underline{A}_i' + \underline{\Xi})^{-1}]^{-1} \ . \qquad (2.6)$$

The state estimate $\underline{\hat{x}}_i(\cdot)$ is n-dimensional, while the residual vector, $\underline{r}_i(\cdot)$, is p-dimensional.

2.1.2 Properties of the Kalman Filter

The Kalman Filter has many interesting properties, (see for example [19]), a few of which are useful in understanding the MMAC method. These are now discussed. A few definitions are useful.

Definition 1: The i^{th} Kalman Filter is said to be matched if the matrices used in the filter design (i.e., the model) and the matrices of the true system are identical; that is, if $\underline{A}_i = \underline{A}$, $\underline{B}_i = \underline{B}$ and $\underline{C}_i = \underline{C}$.

Definition 2: The Filter is called mismatched if it is not matched.

Property 1: If the i^{th} Kalman Filter is matched then in steady state:

$$E\{\underline{r}_i(k)\} = \underline{0} \qquad (2.7a)$$

$$E\left\{\underline{r}_i(k)\underline{\theta}_i^{-1}\underline{r}_i'(e)\right\} = \underline{I}\delta(k-e) \qquad (2.7b)$$

where $\delta(\cdot)$ is the impulse function, $E\{\cdot\}$ denotes expectation, and θ_i is given by

$$\underline{\theta}_i = \underline{\psi} + \underline{C}\,\underline{\Sigma}_i\underline{C}' \quad . \tag{2.7c}$$

Furthermore, if the i^{th} filter is matched, then $\hat{\underline{x}}_i(\cdot)$ gives the optimal estimate of the state $\underline{x}(\cdot)$.

Property 2: If the i^{th} Kalman Filter is matched, then the probability density function for the residual $\underline{r}_i(\cdot)$ is given by [20]:

$$p(\underline{r}_i) = \beta_i e^{-\frac{1}{2}[\underline{r}_i'\underline{\theta}_i^{-1}\underline{r}]} \tag{2.8a}$$

with

$$\beta_i = \left(\sqrt{(2\ )^m|\underline{\theta}_i|}\right)^{-1} \quad . \tag{2.8b}$$

Property 3: If the i^{th} Kalman Filter is mismatched then

$$E\{\underline{r}_i(k)\} = \overline{\underline{r}}_i(k) \tag{2.9a}$$

$$E\{(\underline{r}_i(k) - \overline{\underline{r}}_i(k))\underline{\theta}_i^{-1}(\underline{r}_i(\ell) - \overline{\underline{r}}_i(\ell))'\} = \overline{\underline{\theta}}_i(k-\ell) \tag{2.9b}$$

where $\overline{\underline{r}}$ in general is nonzero and $\overline{\underline{\theta}}_i$ is a function of the system and the noise covariances (see [28]) with, in general, $\overline{\underline{\theta}}_i(0) > \underline{I}$.

It follows that if two filters (one matched and one mismatched) are computed then

$$E\{\underline{r}_1'(k)\underline{\theta}_1^{-1}\underline{r}_1(k)\} < E\{\underline{r}_2'(k)\underline{\theta}_2^{-1}\underline{r}_2(k)\} \tag{2.10}$$

where $\underline{r}_1$ is from the matched filter and $\underline{r}_2$ is from a mismatched filter

[16]. Thus, if equation (2.8) is evaluated for both a matched (1) and a mismatched (2) KF, then

$$E\{p(\underline{r}_1)\} > E\{p(\underline{r}_2)\} \quad . \tag{2.11}$$

2.1.3 The Probability Equation

Assume that we are interested in identifying a dynamic system from its outputs and that the true system is known only to be one of N specified models. Baye's rule and Properties 1-3 imply [15, 7] that the probability that the i^{th} model matches the unknown system (P_i) is given by

$$P_i(k+1) = \frac{P_i(k)\ p(\underline{r}_i(k+1))}{\sum_{j=1}^{N} P_j(k)p(\underline{r}_j(k+1))} \tag{2.12}$$

where $p(r_i(k+1))$ is given by equation (2.8). The structure of the resulting system is shown in Figure 2.1.

2.2 Adaptive Control

If one knew with certainty which model matched the true system, it would be a simple matter to design a controller using any of the standard synthesis techniques. Therefore, one reasonable way to determine a control law for the unknown system is to probabilisticly weight the controls which would be used if one assumed that one of the models was correct. That is, let

$$\underline{u}(k) = \sum_{i=1}^{N} P_i(k)\underline{u}_i(k) \tag{2.13}$$

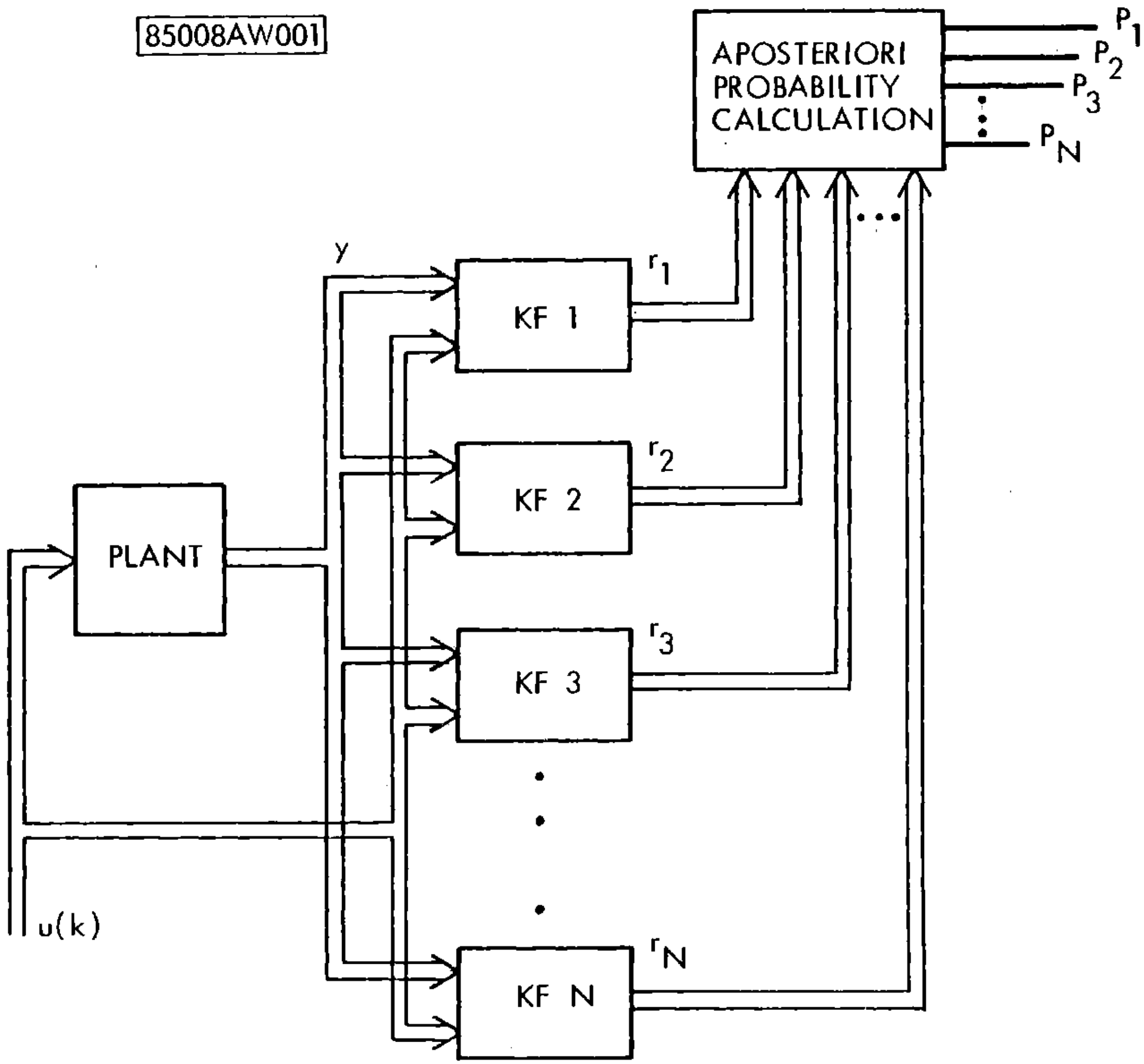

Fig. 2.1 Identification Structure

where $P_i(k)$ is given by equation (2.12) and $\underline{u}_i(k)$ is the control which one would apply if model i were assumed to be matched to the true system.

It will be assumed that

$$\underline{u}_i = \underline{G}_i\hat{\underline{x}}_i \tag{2.14}$$

although this is not necessary and is further discussed in Section 2.3. Figure 2.2 thus summarizes the Multiple Model Adaptive Control method.

2.3 The Control Law

The MMAC method as developed in the literature [7, 16] has assumed that the linear quadratic (LQ) methodology is used to design the controller. Thus, the feedback gains $\underline{G}_i$ of equation (2.14) are chosen to minimize

$$J(\underline{u}) = \sum_{k=1}^{\infty} (\underline{x}'(k)\underline{Q}\,\underline{x}(k) + \underline{u}'(k)\underline{R}\,\underline{u}(k)) \;. \tag{2.15}$$

It can be shown [21] that the optimal gains are given by

$$\underline{G}_i = [\underline{B}_i'\underline{K}_i\underline{B}_i + \underline{R}]^{-1}\;\underline{B}_i'\underline{K}_i\underline{A}_i \tag{2.16}$$

where $\underline{K}_i$ is the solution to the steady state Riccati equation

$$\underline{K}_i = \underline{Q}_i + \underline{A}_i'\underline{K}_i\underline{A}_i - \underline{A}_i'\underline{K}_i\underline{B}_i[\underline{B}_i'\underline{K}_i\underline{B}_i + \underline{R}]^{-1}\underline{B}_i\underline{K}_i\underline{A}_i \;. \tag{2.17}$$

Although to date, all references to MMAC have made use of the control law (2.16), this is not a necessary part of the method. Thus, any control law which gives good results for the respective model may be used. However, there is a strong interaction between control law choice and adaptive performance due to the feedback, about which very little is known.

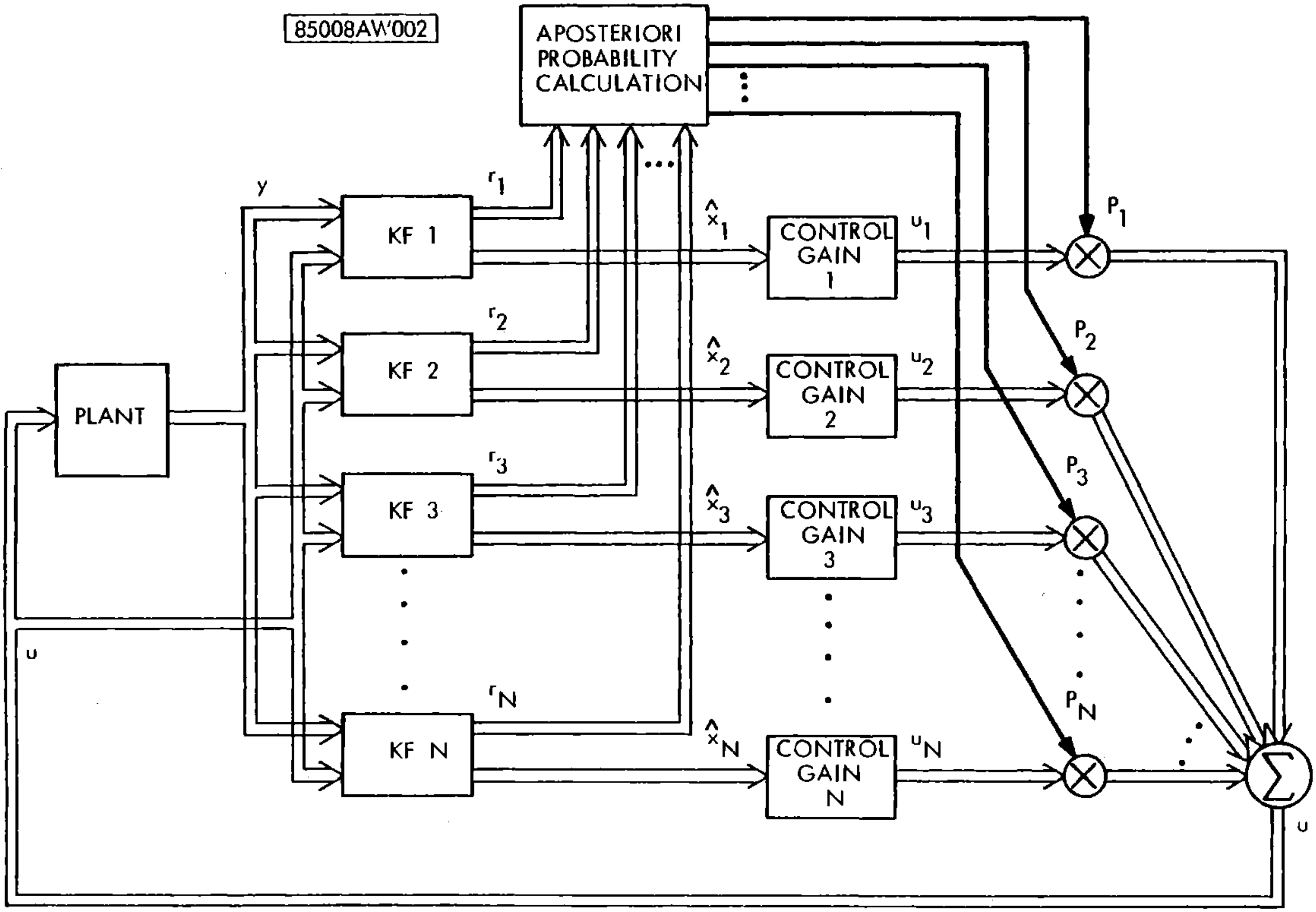

Fig. 2.2 Adaptive Control Structure

For the purposes of this study we will for the most part restrict our research to control laws based on the linear quadratic (Equation 2.15) form.

2.4 Comments on MMAC

A few comments are in order.

1) The MMAC algorithm, as shown by Willner [7] is suboptimal even for the problem originally posed. Willner was able to show that the algorithm is optimal for the final step of the dynamic programming algorithm [7] but was unable to continue the calculation backward in time.

2) As posed above, the true model is assumed to belong to a finite set of known models. Lainiotis [9, 13] has discussed the infinite set case but concludes that a finite approximation is then required for use in applications. Thus, for most real problems when the true model may take values from an infinite set, a further suboptimal approximation is required. The results of Baram [15] (which apply only to the open-loop case) may help in discretizing the model set.

2.5 Continuous Time MMAC

As developed in the preceeding sections, the MMAC method has been based on discrete time systems. For analysis purposes, it will be useful to consider the related continuous time problem, largely because of the simplified form of the probability equation.

The complete equations for the method will not be given here as they are the continuous time regulator and continuous time Kalman Filter equations [19, 6] with a set of equations for the probability of each model.

Dunn [22] has shown that if $P_i(t)$ is the probability of the $i^{\underline{th}}$ model at time t then

$$\dot{P}_i(t) = P_i(\underline{C}_i\underline{\hat{x}}_i - \sum_{j=1}^{N} P_j\underline{C}_j\underline{\hat{x}}_j)\underline{\theta}^{-1}(\underline{y} - \sum_{j=1}^{N} P_j\underline{C}_j\underline{\hat{x}}_j) \tag{2.18}$$

where $\underline{C}_j$, $\underline{\hat{x}}_i$ and $\underline{y}$ are defined analogously to the discrete time case and $\underline{\theta}$ is the observation noise covariance. The property which makes Equation (2.18) useful for analysis is the absence of either exponentials or a denominator. These equations are examined further in the two model, two state case in the following section.

2.6 A Special Case

Of special interest to the research at hand is the case when N=2 (i.e., the two model case) with full state observation. Furthermore, we shall assume that the input ($\underline{B}$) matrix is the identity.

The special case captures all essential features of the problem that we are interested in examining without adding unnecessary complexity, and it forms the central focus for this research.

2.6.1 Discrete Time Case

In the discrete time case, the true system is then given by

$$\begin{aligned} \underline{x}(k+1) &= \underline{A}\,\underline{x}(k) + \underline{u}(k) + \underline{\zeta}(k) \\ \underline{y}(k) &= \underline{x}(k) + \underline{\eta}(k) \end{aligned} \tag{2.19a}$$

with models

$$\text{Model 1:} \quad \begin{aligned} \underline{\hat{x}}_1(k+1) &= \underline{A}_1\underline{\hat{x}}_1(k) + \underline{u}(k) + \underline{\zeta}(k) \\ \underline{y}(k) &= \underline{\hat{x}}_1(k) + \underline{\eta}(k) \end{aligned} \tag{2.19b}$$

Model 2: $\hat{\underline{x}}_2(k+1) = \underline{A}_2\hat{\underline{x}}_2(k) + \underline{u}(k) + \underline{\zeta}(k)$

$$\underline{y}(k) = \hat{\underline{x}}_2(k) + \underline{\eta}(k) \quad . \tag{2.19c}$$

Thus, the MMAC method reduces to the following set of discrete time equations

$$
\begin{aligned}
\underline{x}(k+1) &= \underline{A}\,\underline{x}(k) + \underline{u}(k) + \underline{\zeta}(k) \\
\underline{y}(k) &= \underline{x}(k) + \underline{\eta}(k) \\
\hat{\underline{x}}_1(k+1) &= \underline{A}_1\hat{\underline{x}}_1(k) + \underline{u}(k) + \underline{H}_1\underline{r}_1(k+1) \\
\underline{r}_1(k+1) &= \underline{y}(k+1) - \underline{A}_1\hat{\underline{x}}_1(k) - \underline{u}(k) \\
\hat{\underline{x}}_2(k+1) &= \underline{A}_2\hat{\underline{x}}_2(k) + \underline{u}(k) + \underline{H}_2\underline{r}_2(k+1) \\
\underline{r}_2(k+1) &= \underline{y}(k+1) - \underline{A}_2\hat{\underline{x}}_2(k) - \underline{u}(k) \\
\underline{u}(k) &= -P_1\underline{G}_1\hat{\underline{x}}_1(k) - P_2\underline{G}_2\hat{\underline{x}}_2(k) \\
P_1(k+1) &= \frac{P_1(k)\;p(\underline{r}_1)}{P_1(k)p(\underline{r}_1) + P_2(k)p(\underline{r}_2)} \\
p(\underline{r}_i) &= \beta_i e^{-\frac{1}{2}[\underline{r}_i'\underline{\theta}_i^{-1}\underline{r}_i]} \quad .
\end{aligned}
\tag{2.20}
$$

The major goal of the research has been to understand such qualitative properties of the MMAC method as stability. Since the phenomenon which have been observed are largely due to the nonlinearities of the system and occur even in noise-free simulations, the noise terms $\zeta(\cdot)$ and $\eta(\cdot)$ add an unnecessary complexity. Thus, these terms are ignored in the remainder of this thesis. It should be noted that the KF's, designed assuming the noises to be present, are an integral part of the MMAC method and are therefore retained. Thus, the principle focus of the work is to examine the <u>deterministic</u> properties of Equation (2.20) and the corresponding continuous time system discussed in the next subsection. Since the sum of the probabilities of the models <u>must</u> always be 1, it is known that

$$P_2(k) = (1-P_1(k)) \qquad \forall k.$$

Then, rewriting Equation (2.20) in terms of the <u>residual</u> $\underline{r}_i(k)$ we see that the equations of the MMAC method can be summarized by

$$\begin{aligned}
\underline{x}(k+1) &= \underline{A}\,\underline{x}(k) + \underline{u}(k) \\
\underline{r}_1(k+1) &= (\underline{A}-\underline{A}_1)\underline{x}(k) + \underline{A}_1(\underline{I}-\underline{H}_1)\underline{r}_1(k) \\
\underline{r}_2(k+1) &= (\underline{A}-\underline{A}_2)\underline{x}(k) + \underline{A}_2(\underline{I}-\underline{H}_2)\underline{r}_2(k) \\
\underline{u}(k) &= -(P_1\underline{G}_1 + (1-P_1)\underline{G}_2)\underline{x}(k) + P_1\underline{G}_1(\underline{I}-\underline{H}_1)\underline{r}_1 + (1-P_1)\underline{G}_2(I-\underline{H}_2)\underline{r}_2 \\
P_1(k+1) &= \frac{P_1(k)\; p(\underline{r}_1)}{P_1(k)\; p(\underline{r}_1) + (1-P_1(k))\; p(\underline{r}_2)} \\
p(\underline{r}_i) &= \beta_i e^{-\frac{1}{2}[\underline{r}_i'\underline{\theta}^{-1}\underline{r}_i]}
\end{aligned} \tag{2.21}$$

It will be useful notationally to combine the state and residual equations into <u>one</u> vector equation. Therefore, we define the vector

$$\underline{w}'(k) = [\underline{x}'(k), \underline{r}_1'(k), \underline{r}_2'(k)]' \quad .$$

Thus, Equation (2.21) can be rewritten as

$$\underline{w}(k+1) = \begin{bmatrix} \underline{A}-P_1\underline{G}_1 - (1-P_1)\underline{G}_2 & P_1\underline{G}_1(\underline{I}-\underline{H}_1) & (1-P_1)\underline{G}_2(\underline{I}-\underline{H}_2) \\ (\underline{A}-\underline{A}_1) & \underline{A}_1(\underline{I}-\underline{H}_1) & \underline{0} \\ (\underline{A}-\underline{A}_2) & \underline{0} & \underline{A}_2(\underline{I}-\underline{H}_2) \end{bmatrix} \underline{w}(k) \quad (2.22b)$$

or

$$\underline{w}(k+1) = \tilde{\underline{A}}(P_1)\underline{w}(k)$$

with

$$P_1(k+1) = \frac{P_1(k)p(\underline{r}_1)}{P_1(k)\ p(\underline{r}_1) + (1-P_1(k))p(\underline{r}_2)} \quad . \quad (2.22b)$$

These equations, along with their continuous time counterparts, form the basis for the research which has been undertaken.

2.6.2 Continuous Time Case

Similar assumptions to those previously presented for the discrete time case can be made in the continuous time case. Here, we will again restrict our attention to the two model, state feedback case with B=I. Thus, the MMAC method reduces to the following equations

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{u}(t) + \underline{\zeta}(t)$$

$$\underline{y}(t) = \underline{r}(t) + \underline{\eta}(t)$$

$$\dot{\hat{\underline{x}}}_1(t) = \underline{A}_1\hat{\underline{x}}_1(t) + \underline{u}(t) + \underline{H}_1\underline{r}_1(t) \quad (2.23)$$

$$\underline{r}_1(t) = \underline{y}(t) - \hat{\underline{x}}_1(t)$$

$$\dot{\hat{\underline{x}}}_2(t) = \underline{A}_2\hat{\underline{x}}_2(t) + \underline{u}(t) + \underline{H}_2\underline{r}_2(t)$$

$$\underline{r}_2(t) = \underline{y}(t) - \hat{\underline{x}}_2(t)$$

$$\underline{u}(t) = -P_1 G_1 \hat{\underline{x}}_1(t) - P_2 \underline{G}_2 \hat{\underline{x}}_2(t)$$

$$\dot{P}_1(t) = P_1(t) P_2(t) [\hat{\underline{x}}_1(t) - \hat{\underline{x}}_2(t)]' \underline{\theta}^{-1} [\underline{x} - P_1(t)\hat{\underline{x}}_1 - (1-P_1(t))\hat{\underline{x}}_2(t)]$$

with $P_2(t) = 1-P_1(t)$.

For the same reasons as presented in the previous section the noise sources $\underline{\zeta}(t)$ and $\underline{\eta}(t)$ will again be ignored for the majority of the proposed research. It should be noted that under this condition, the residual $r_i(t)$ exactly equals the estimation error $x(t) - \hat{x}_i(t)$. Thus, equations (2.23) can be rewritten as

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{u}(t)$$

$$\dot{\underline{r}}_1(t) = (\underline{A}_1 - \underline{H}_1)\underline{r}_1(t) + (\underline{A} - \underline{A}_1)\underline{x}(t) \qquad (2.24)$$

$$\dot{\underline{r}}_2(t) = (\underline{A}_2 - \underline{H}_2)\underline{r}_2(t) + (\underline{A} - \underline{A}_2)\underline{x}(t)$$

$$\underline{u}(t) = -P_1(t)\underline{G}_1(\underline{x}(t) - \underline{r}_1(t)) - (1-P_1(t))\,\underline{G}_2(\underline{x}(t) - \underline{r}_2(t))$$

$$\dot{P}_1(t) = P_1(t)[1-P_1(t)][\underline{r}_2(t) - \underline{r}_1(t)]'\,\underline{\theta}^{-1}[P_1(t)\underline{r}_1(t) + (1-P_1(t))\underline{r}_2(t)] .$$

Thus, if we again combine equations by defining the variable

$$\underline{w}(t) = \begin{bmatrix} \underline{x}(t) \\ \underline{r}_1(t) \\ \underline{r}_2(t) \end{bmatrix} .$$

Equation (2.24) can be rewritten as

$$\dot{\underline{w}}(t) = \begin{bmatrix} [\underline{A} - P_1\underline{G}_1 - (1-P_1)\underline{G}_2] & P_1\underline{G}_1 & (1-P_1)\underline{G}_2 \\ (\underline{A}-\underline{A}_1) & \underline{A}_1-\underline{H}_1 & \underline{0} \\ (\underline{A}-\underline{A}_2) & \underline{0} & \underline{A}_2-\underline{H}_2 \end{bmatrix} \underline{w}(t) \quad (2.25a)$$

or $\dot{\underline{w}}(t) = \tilde{\underline{A}}(P_1)\underline{w}(t)$

with (2.25b)

$$\dot{P}_1(t) = P_1(t)[1-P_1(t)]]\underline{r}_2(t) - \underline{r}_1(t)]\ \underline{\theta}^{-1}[P_1(t)\underline{r}_1(t) + (1-P_1(t))\underline{r}_2(t)] .$$

2.7 A Change of Variable

A change of variable which proves useful in the sequel is to let

$$q = 2P_1 - 1 . \quad (2.26)$$

Making this change in Equation (2.22) yields

$$\underline{w}(k+1) = \tilde{\underline{A}}(q)\ \underline{w}(k) \quad (2.27)$$

$$q(k+1) = \frac{(1+q)p(\underline{r}_1) - (1-q)p(\underline{r}_2)}{(1+q)p(\underline{r}_1) + (1-q)p(\underline{r}_2)}$$

where

$$\tilde{\underline{A}}(q) = \begin{bmatrix} \underline{A} - \frac{1}{2}(1+q)\underline{G}_1 - \frac{1}{2}(1-q)\underline{G}_2 & \frac{1}{2}(1+q)\underline{G}_1(\underline{I}-\underline{H}_1) & \frac{1}{2}(1-q)\underline{G}_2(\underline{I}-\underline{H}_2) \\ \underline{A} - \underline{A}_1 & \underline{A}_1(\underline{I}-\underline{H}_1) & \underline{0} \\ \underline{A} - \underline{A}_2 & \underline{0} & \underline{A}_2(\underline{I}-\underline{H}_2) \end{bmatrix} .$$

In continuous time the corresponding change to Equation (2.25) yields

$$\dot{\underline{w}}(t) = \tilde{\underline{A}}(q)\,\underline{w}(t)$$
$$\dot{q}(t) = \frac{1}{4}(1-q^2)[\underline{r}_2-\underline{r}_1]'\underline{\theta}^{-1}[(1+q)\underline{r}_1 + (1-q)\underline{r}_2] \tag{2.28}$$

where

$$\tilde{\underline{A}}(q) = \left[\begin{array}{c:c:c} \underline{A} - \frac{1}{2}(1+q)\underline{G}_1 - \frac{1}{2}(1-q)\underline{G}_2 & \frac{1}{2}(1+q)\underline{G}_1 & \frac{1}{2}(1-q)\underline{G}_2 \\ \underline{A} - \underline{A}_1 & \underline{A}_1 - \underline{H}_1 & \underline{0} \\ \underline{A} - \underline{A}_2 & \underline{0} & \underline{A}_2 - \underline{H}_2 \end{array}\right].$$

Note that the same basic notation is used for the $\tilde{\underline{A}}$ matrices of Equations (2.22), (2.25), (2.27) and (2.28). However, no confusion should arise as the meaning should always be clear from context.

2.8 A Useful Definition

A further concept which will prove useful can be seen by considering either the continuous or discrete time problems as summarized by either Equation (2.25) or (2.22)

$$\text{Continuous time:} \quad \dot{\underline{w}}(t) = \tilde{\underline{A}}(P_1)\underline{w}(t) \tag{2.25}$$

$$\text{Discrete time:} \quad \underline{w}(k+1) = \tilde{\underline{A}}(P_1)\underline{w}(k) \tag{2.22}$$

where

$$\tilde{\underline{A}}(P_1) = \begin{cases} \text{Continous time:} & \left[\begin{array}{c|c|c} \underline{A}-P_1\underline{G}_1-(1-P_1)\underline{G}_2 & P_1\underline{G}_1 & (1-P_1)\underline{G}_2 \\ \underline{A}-\underline{A}_1 & \underline{A}_1-\underline{H}_1 & \underline{0} \\ \underline{A}-\underline{A}_2 & \underline{0} & \underline{A}_2-\underline{H}_2 \end{array}\right] \\ \text{Discrete time:} & \left[\begin{array}{c|c|c} \underline{A}-P_1\underline{G}_1-(1-P_1)\underline{G}_2 & P_1\underline{G}_1(\underline{I}-\underline{H}_1) & (1-P_1)\underline{G}_2(\underline{I}-\underline{H}_2) \\ \underline{A}-\underline{A}_1 & \underline{A}_1(\underline{I}-\underline{H}_1) & \underline{0} \\ \underline{A}-\underline{A}_2 & \underline{0} & \underline{A}_2(\underline{I}-\underline{H}_2) \end{array}\right] \end{cases}$$

Thus, it is clear that if $P_1(\cdot)$ is held constant, then Equations (2.25) and (2.22) describe linear, time-invariant dynamical systems in the variable $\underline{w}' = [\underline{x}', \underline{r}_1', \underline{r}_2']'$. Thus, Equations (2.25) and (2.22) with P constant will be referred to as the linear, time invariant system for fixed P, or LTI for fixed P. As will be seen in Chapters 4 and 5, many of the major properties of the MMAC method can be expressed in terms of the properties of the LTI system for fixed P as a function of P.

A second important point to note is that in applications and simulations of the MMAC system, another parameter becomes important due to the finite precision usually available for computation. From the equations of this chapter, it is clear that the MMAC method has the property that

$$P_i(k) = 0 \Longrightarrow P_i(\cdot) = 0 \text{ for all future times.}$$

Such a situation is usually to be avoided, as it reduces the flexibility of the algorithm in a number of applications. For example, the parameter of the true model are often changing in adaptive control situations, and one would like to require P_i to be nonzero for all time so that the MMAC algorithm can respond to such changes. Thus, one almost always applies an additional constraint on the probability such as

$$P_i(k) \geq P_{lim} \qquad \forall i, \forall k .$$

This has been done throughout this study with a value of $P_{lim} = 10^{-50}$. The effect of such a limit is examined in detail in Section 5.1.

CHAPTER 3

QUALITATIVE RESULTS

In order to guide and motivate the research, we have examined a problem consisting of a system with two independent states and two models. This system, while simple, captures many of the basic issues which are important to the method and sheds light on the fundamental problems involved with the MMAC design. The problem is formulated in the next section of this chapter. The remaining sections contain simulations which demonstrate the various types of behavior which have been observed. A discussion of the important properties is included.

3.1 Problem Formulation

In most applications of an adaptive control algorithm, a detailed analysis of the behavior of the algorithm has proved intractable. This is especially true for the MMAC algorithm. However, in the case of MMAC, it is possible to find a simple example problem that lends itself to analysis and simultaneously maintains the basic properties exhibited in simulations of the more general systems. Thus, for the purposes of this thesis, a sample problem structure has been chosen which displays what we feel are the critical characteristics of the method and which is also amenable to detailed analysis.

The chosen true system to be controlled is given by:

$$\underline{x}(k+1) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \underline{x}(k) + \underline{u}(k) + \underline{\zeta}(k) \tag{3.1}$$

$$\underline{y}(k) = \underline{x}(k) + \underline{\eta}(k)$$

or a continuous time system of the same structure and $\underline{A}$ matrix

$$\dot{\underline{x}}(t) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \underline{x}(t) + \underline{u}(t) + \zeta(t) \qquad (3.2)$$

$$\underline{y}(t) = \underline{x}(t) + \underline{\eta}(t)$$

where a takes on values in the range [0, 2]. The discrete time system is useful for simulation studies and most of the analysis. However, use of the continuous time system provides greater insight for the linearization results in Section 4.1. The set of models for either the discrete or continuous time case is given by

Model 1: $(\underline{A}_1, \underline{I}, \underline{I})$

Model 2: $(\underline{A}_2, \underline{I}, \underline{I})$

with

$$\underline{A}_1 = \begin{bmatrix} a & 0 \\ 0 & \hat{a} \end{bmatrix} \qquad \underline{A}_2 = \begin{bmatrix} \hat{a} & 0 \\ 0 & a \end{bmatrix}$$

where $\hat{a}$ takes on various values from 0 to 1.5. The parameters a and $\hat{a}$ are varied to obtain different responses from the overall system.

For the purposes of KF design, the noise sources $\underline{\zeta}(\cdot)$ and $\underline{\eta}(\cdot)$ are assumed to be zero mean, white and Gaussian with covariances of $I\delta(\cdot)$ where $\underline{I}$ is the 2x2 identity matrix. This structure results in diagonal Kalman Gain matrices such that

$$\underline{H}_1 = \begin{bmatrix} h & 0 \\ 0 & \hat{h} \end{bmatrix} \qquad \underline{H}_2 = \begin{bmatrix} \hat{h} & 0 \\ 0 & h \end{bmatrix}$$

where h is the gain corresponding to a and $\hat{h}$ is the gain corresponding to $\hat{a}$. Further, the control law weight $\underline{Q}_i$ and $\underline{R}_i$ are chosen diagonal such that

$$\underline{R}_1 = \underline{R}_2 = \underline{I}$$

$$\underline{Q}_1 = \begin{bmatrix} q & 0 \\ 0 & \hat{q} \end{bmatrix} \qquad \underline{Q}_2 = \begin{bmatrix} \hat{q} & 0 \\ 0 & q \end{bmatrix} .$$

This results in a diagonal control gain matrix

$$\underline{G}_1 = \begin{bmatrix} g & 0 \\ 0 & \hat{g} \end{bmatrix} \qquad \underline{G}_2 = \begin{bmatrix} \hat{g} & 0 \\ 0 & g \end{bmatrix} .$$

The structure described above will be referred to as the canonical problem.

The emphasis of this study has been on the examination of such qualitative properties as stability. Thus, we ignore the noise sources both in the simulation and in the analysis of the properties of the MMAC method. That is, in both the analysis and simulations, $\zeta(\cdot)$ and $\eta(\cdot)$ are set equal to zero. Note, however, that the KF's designed with the assumed non-zero noise sources are retained. It is clear that noise can have a major effect on a system [29]. However, as we will see, many of the properties of the MMAC method are due to the nonlinearities of the probability equation. Thus, it is our feeling that an anlysis of this noise-free case is of considerable importance.

A few comments are in order concerning the the choice of the example problem. First of all, the system is chosen to be the simplest possible and still capture the important phenomena. Thus, we selected a two state, two model system. The choice of model is, admittedly, somewhat extreme in the degree of symmetry and mismatch between the models and the true system. However, this choice has been deliberately made in order to investigate phenomena which have been observed in actual applications [23]. Thus, it is felt that the analysis of this problem has yielded insight into the more general case.

3.2 Basic Responses

Various types of responses have been observed in simulations of the system presented in Section 3.1. Table 3.1 details the parameters used for each simulation. The remainder of this chapter presents examples of each of the major modes along with a discussion of the important characteristics of each. The simulations, which have been performed on an IBM 370 computer using double precision FORTRAN, have been used to motivate and guide the analysis of Chapters 4 and 5. In fact, the types of analysis described in these chapters and, in particular in Section 4.6, were to a large degree suggested by the simulations discussed in this chapter.

For each simulation presented, three plots have been included. The first is a plot of the probability of model one (P_1) versus time. The second is a plot of the two true states x_1 and x_2 versus time. The third is a plot of the quantity $\ln(x_1 x_2)$. This quantity has been found to be indicative of the stability of the closed loop, nonlinear system. This variable is further discussed in Section 4.6 where it is linked with the concept of "hyperbolic stability".

Case #	a	$\hat{a}$	q	g	$\hat{q}$	$\hat{g}$	h	$\hat{h}$	Figures
1a	2	0	1.02	1.62	1	0	.809	.5	3.2
1b	2	0	0	1.5	1	0	.809	.5	3.3
1c	2	0	*	1.4	1	0	.809	.5	3.4
2	1.5	1.0	1	1.09	1	.618	.7245	.62	3.5,6
3	.9	0	1	.538	1	0	.597	.5	3.1

TABLE 3.1

Parameters of Sample Cases

*This value of the control can not result from an LQ design.

3.2.1 Exponential Mode

The first type of behavior is one in which the states of the true system increase or decrease in an exponential manner (see Figure 3.1). This type of behavior appears to arise in two situations. First, examination of the equations of Chapter 2 indicate that if the KF residuals $\underline{r}_1$ and $\underline{r}_2$ are equal with the true state components equal to each other, then $P_1(t) = 0$ or equivalently, $P_1(k+1) = P_1(k)$. Thus, if the system is symmetrically initialized (that is, the two true states are equal with $\underline{r}_1 = \underline{r}_2 = \underline{0}$ and $P_1(0) = .5$) then $P_1(t) = .5 \; \forall t$ or equivalently $P_1(k) = .5 \; \forall k$. The closed loop system is then time invariant and stability analysis for the resulting LTI system follows as usual. Note that the resulting system can be exponentially stable, neutrally stable or exponentially unstable depending on the control gains $\underline{G}_1$ and $\underline{G}_2$. Although this is clearly a singular condition, it nonetheless is important from an applications point of view because one commonly attempts to initialize the system with equal probability and with $\underline{r}_1 = \underline{r}_2$ (i.e., $\hat{\underline{x}}_1 = \hat{\underline{x}}_2$). An analysis of this mode is presented in Section 4.1.

A somewhat similar type of behavior occurs when the LTI system for fixed P is <u>stable</u> for all P. This, of course, is a fairly restrictive condition which in essence requires extreme robustness of each controller. However, such a condition <u>does</u> imply exponential stability. This occurs <u>independently</u> of how the probability behaves. Note that this is a nontrivial result since $\underline{\tilde{A}}(P(k))$ is time-varying because $P(k)$ is. Section 4.2 contains the analysis of this situation and Figure 3.1 is an example of the type of simulation results obtained.

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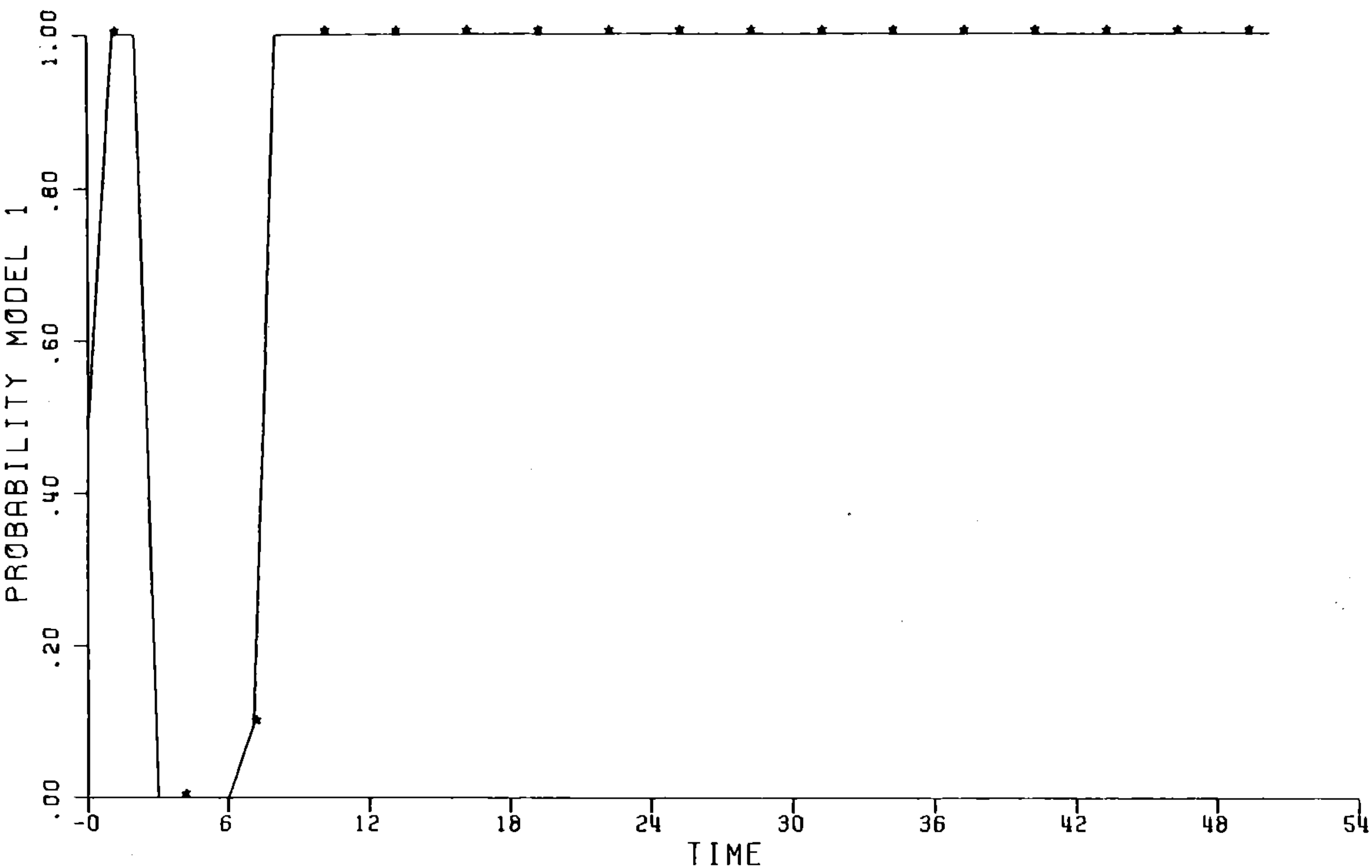

Fig. 3.1 Universally Stable Simulation (Case 3)

a) Probability of Model 1

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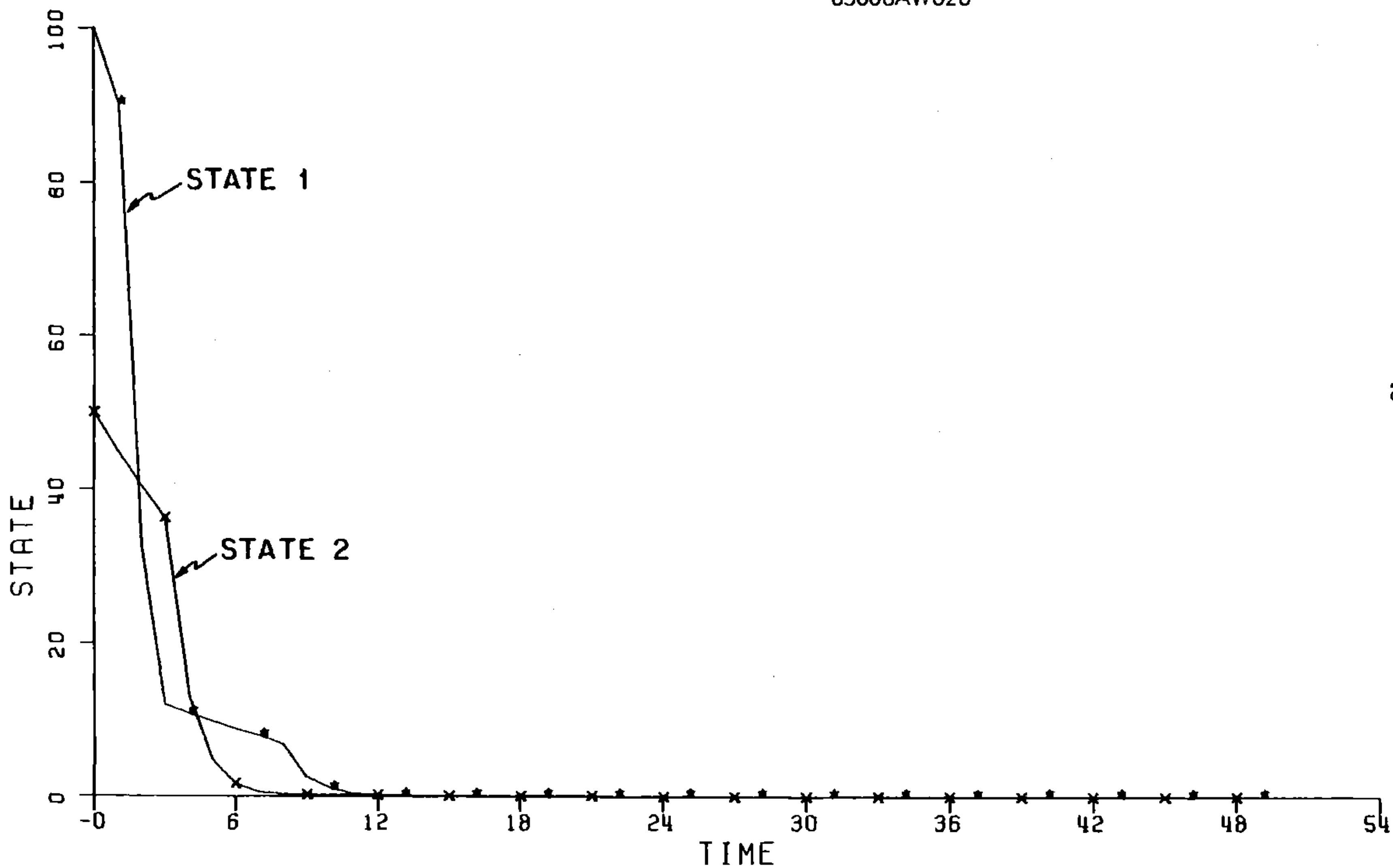

Fig. 3.1 Universally Stable Simulation (Case 3)

b) True States

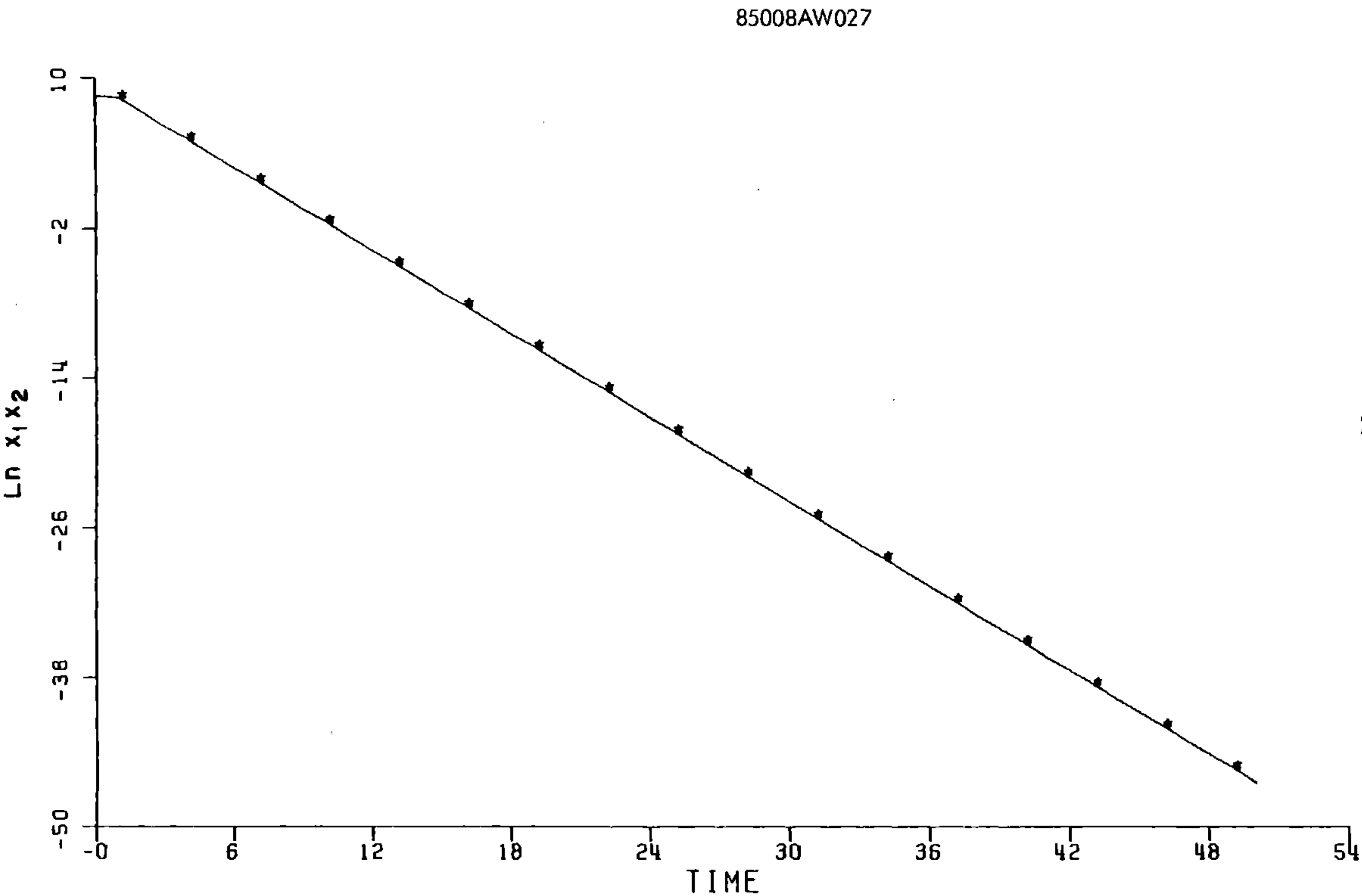

Fig. 3.1 Universally Stable Simulation
(Case 3)

c) Ln x_1x_2

3.2.2 Oscillatory Reponse

Probably the most unusual behavior which has been observed both in the current work and in applications of the MMAC algorithm [23] has been an oscillatory response in which the probability jumps between near-zero and near-one* in what appears to be (but strictly speaking is not) a limit cycle.

Figures 3.2 through 3.4 are examples of this type of behavior. Figure 3.2 is a case in which the peaks of the state trajectories are approximately constant while the period of oscillation increases. Figure 3.3 is a case in which both the peaks and the period are constant while in Figure 3.4, the system is unstable. These three cases are obtained by changing the value of the control gain. Note that each gain would yield stable behavior for the model used in its design.

It is interesting to note that the states of the system are also highly oscillatory. It might be expected that the plant dynamics would smooth the rapid probability transitions to form a smoother "average" state response. However, as shown in Chapter 4, the oscillatory state behavior is a direct consequence of the model mismatch problem in the MMAC algorithm.

The reasons for this oscillatory behavior, which are discussed in Sections 4.4 through 4.6 and again in Section 5.1, are closely related to the fact that neither of the hypothesized models individually yields

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*This is the one set of simulations in which the lower limit on the probability (see Section 2.8) of 10^{-50} is achieved. Section 5.1 discusses the analysis in this case in detail.

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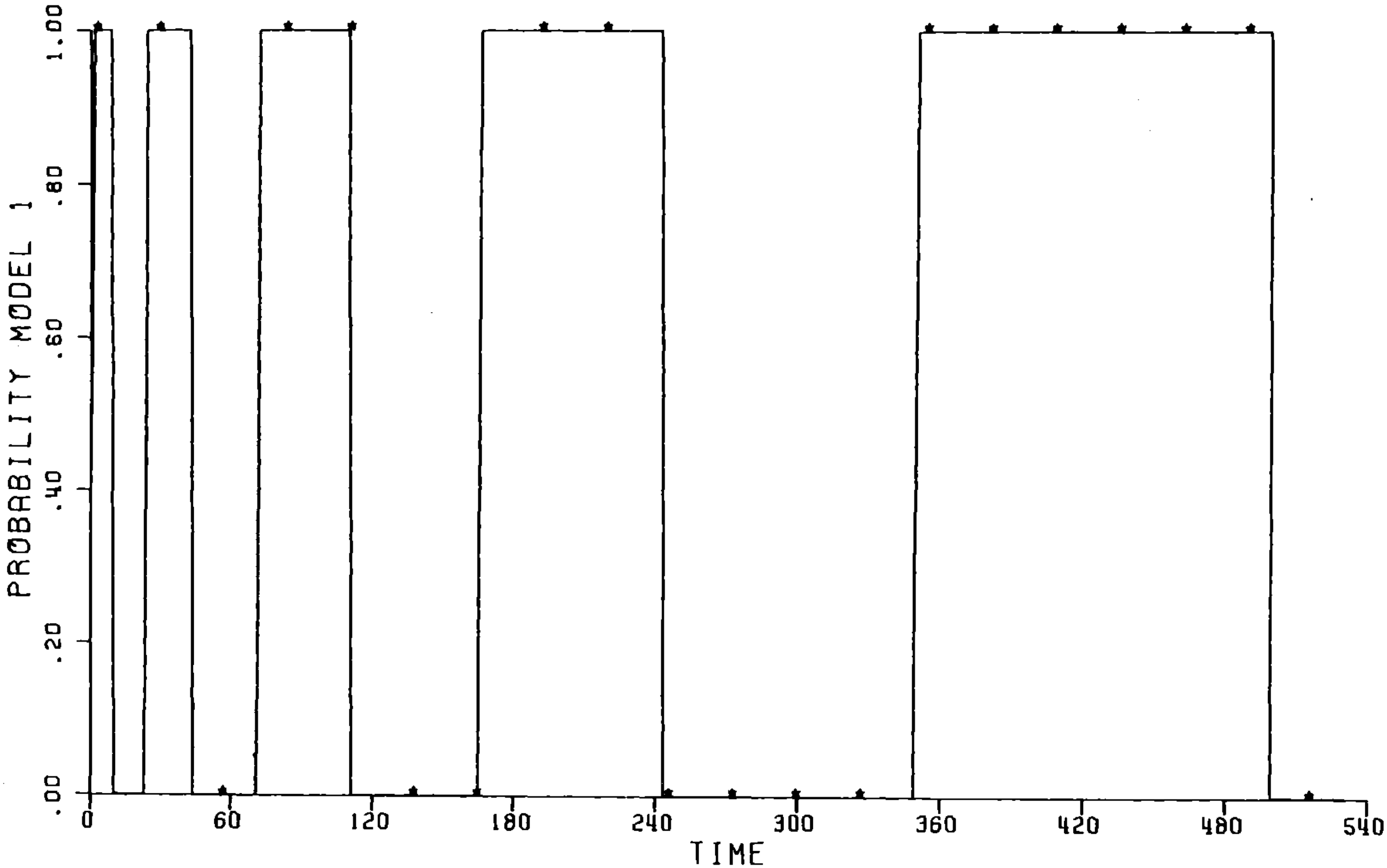

Fig. 3.2 Stable Oscillation
(Case 1a)

a) Probability of Model 1

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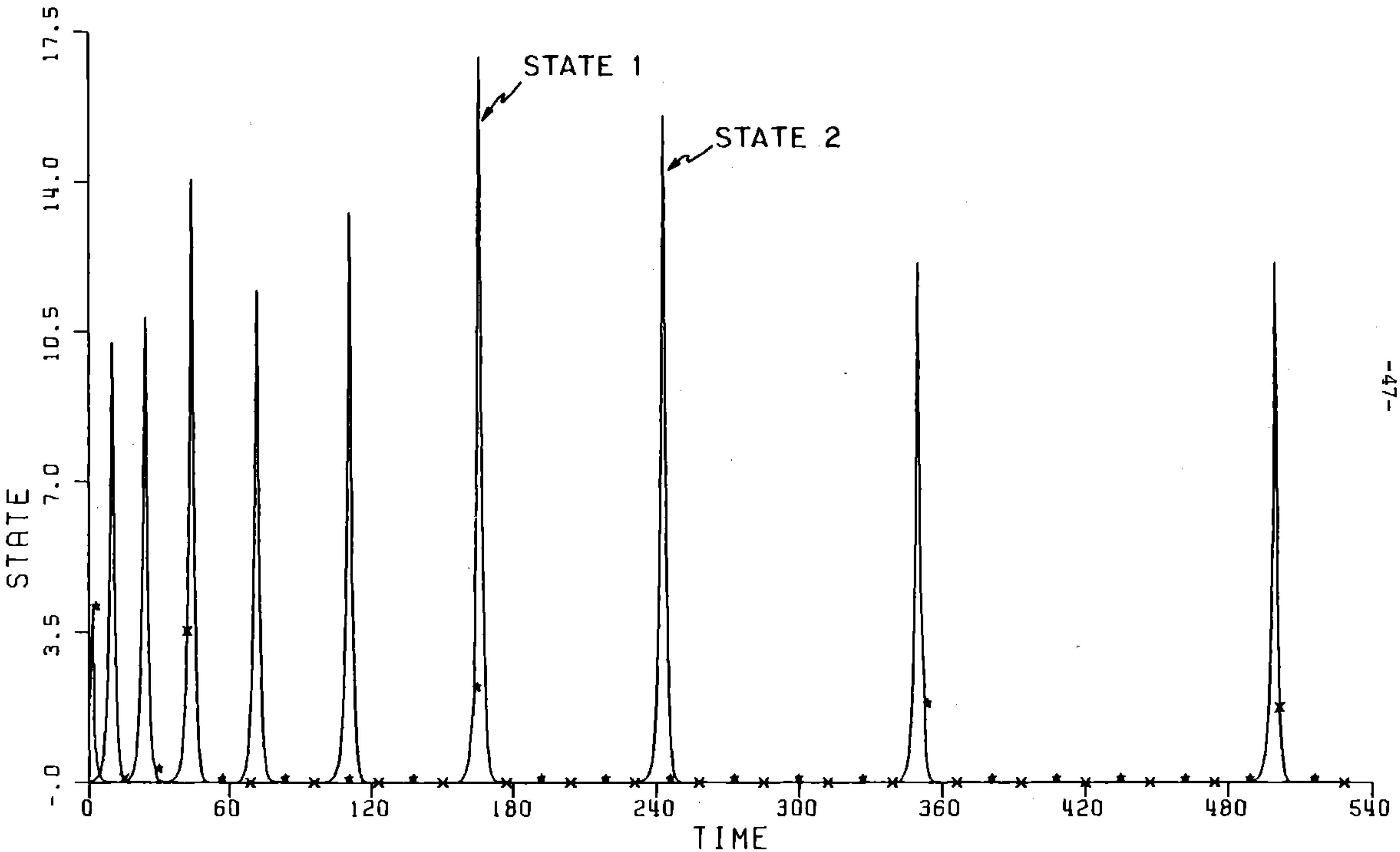

Fig. 3.2 Stable Oscillation
(Case 1a)

b) True States

85008AW033

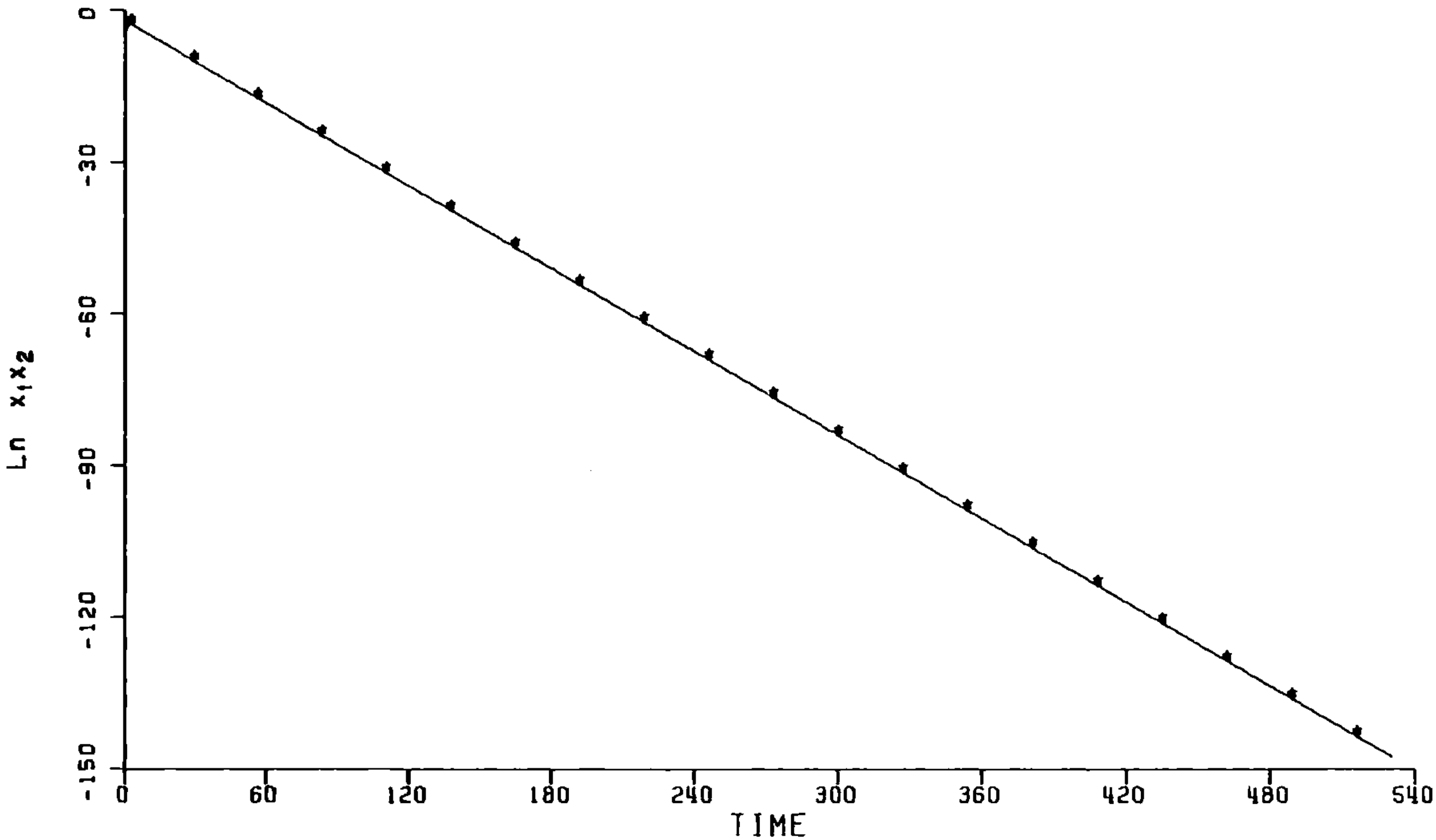

Fig. 3.2 Stable Oscillation
(Case 1a)

c) Ln x_1x_2

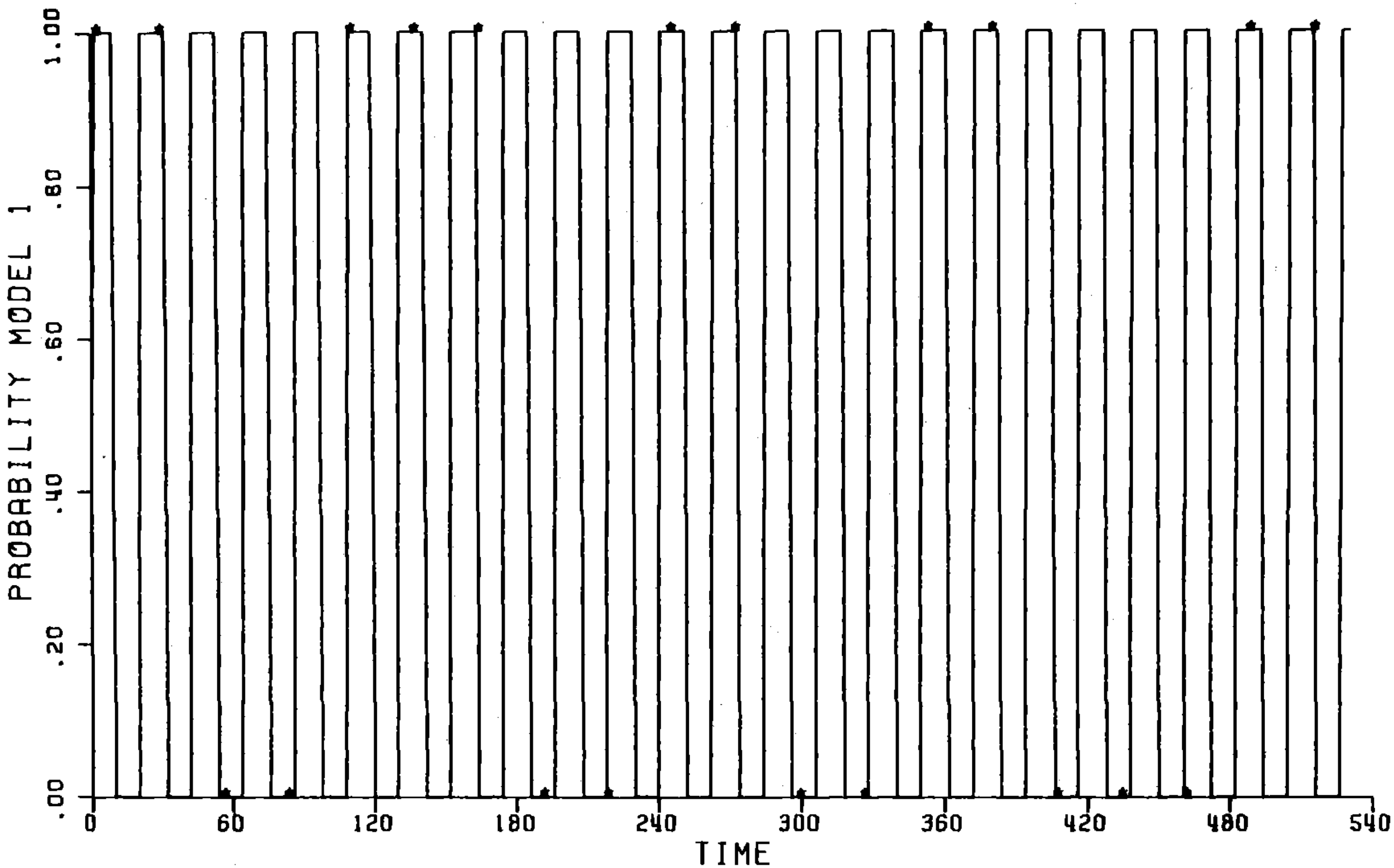

Fig. 3.3 Neutrally Stable Oscillation
(Case 1b)

a) Probability of Model 1

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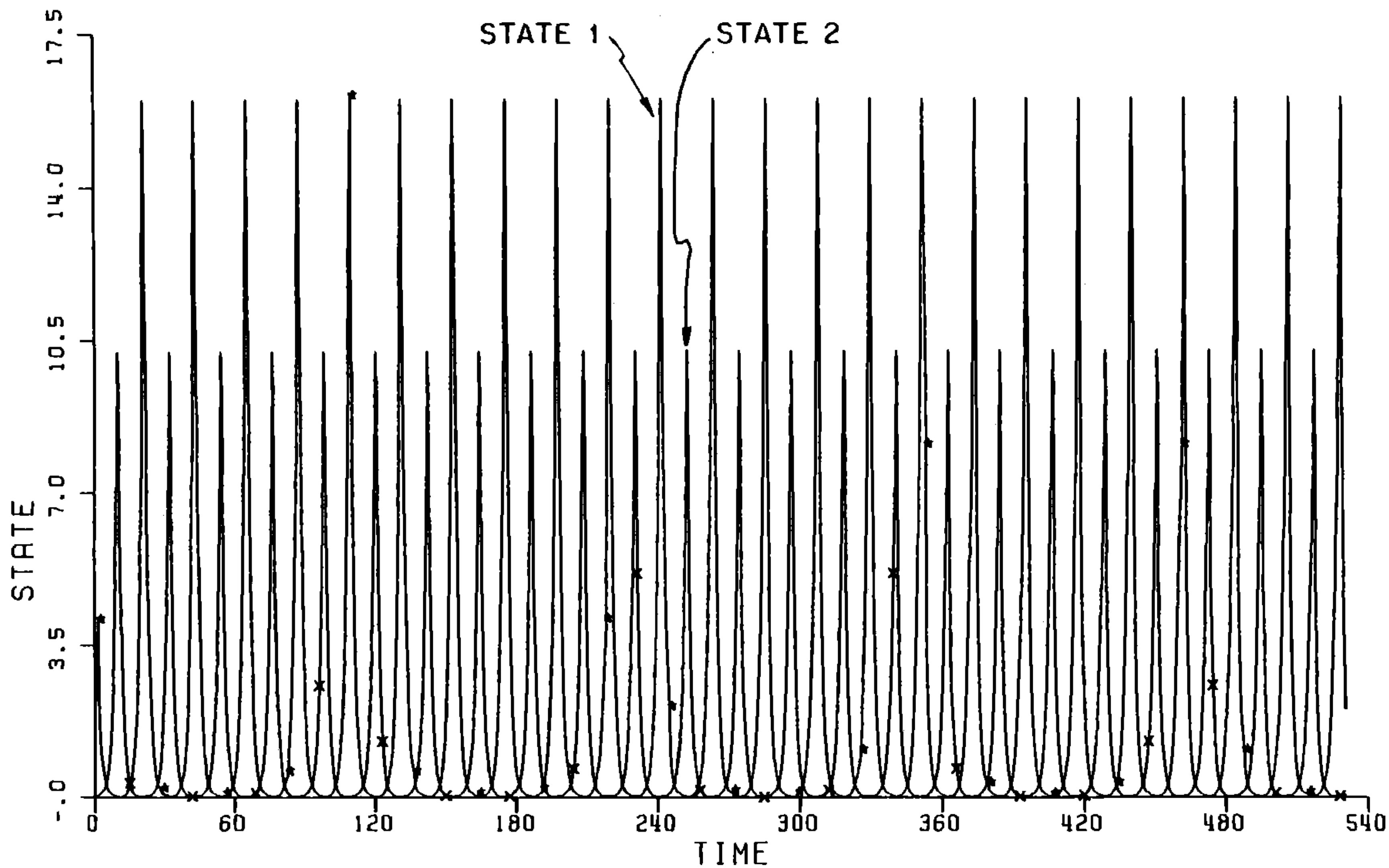

Fig. 3.3 Neutrally Stable Oscillation
(Case 1b)

b) True States

85008AW035

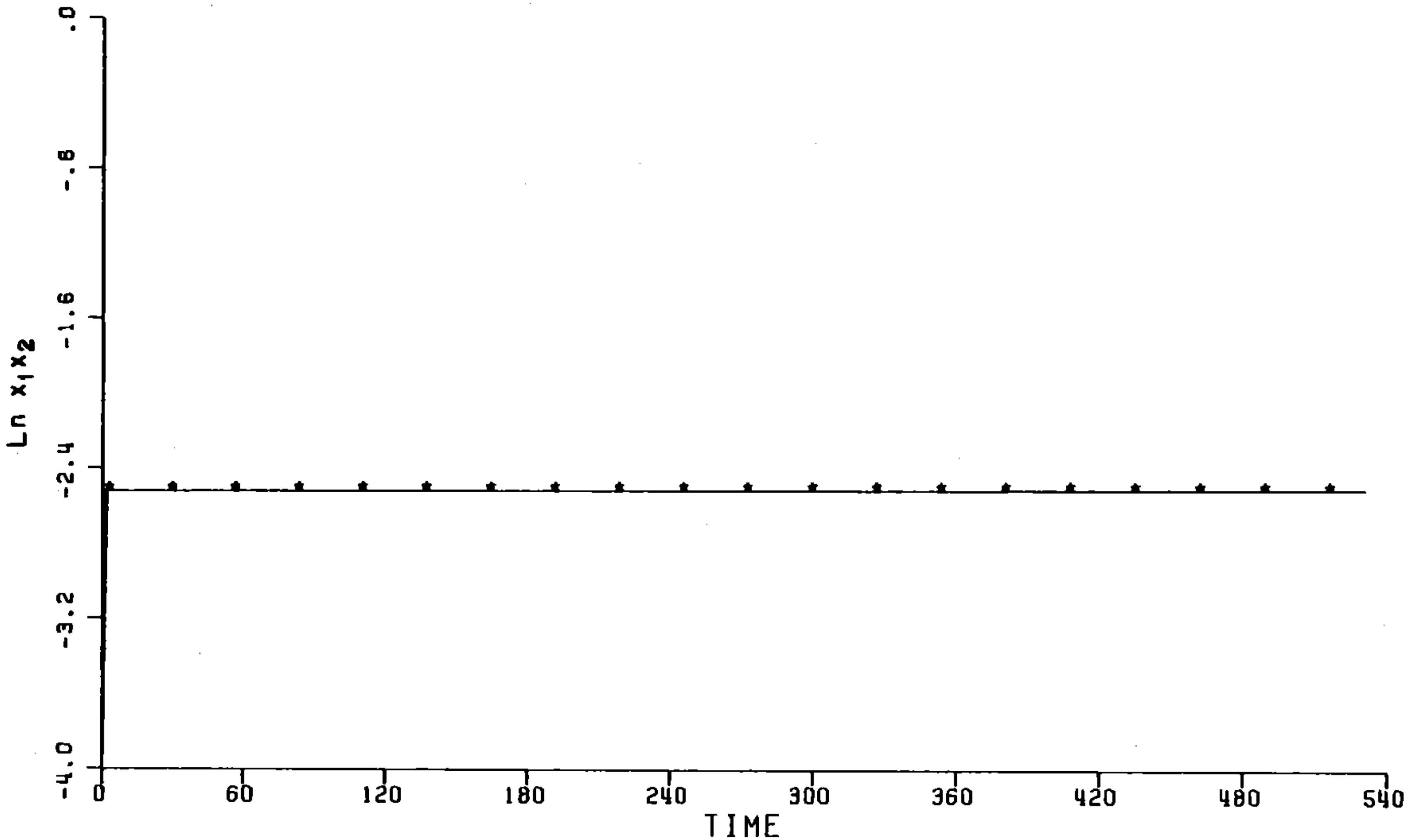

Fig. 3.3 Neutrally Stable Oscillation
(Case 1b)

c) Ln x_1x_2

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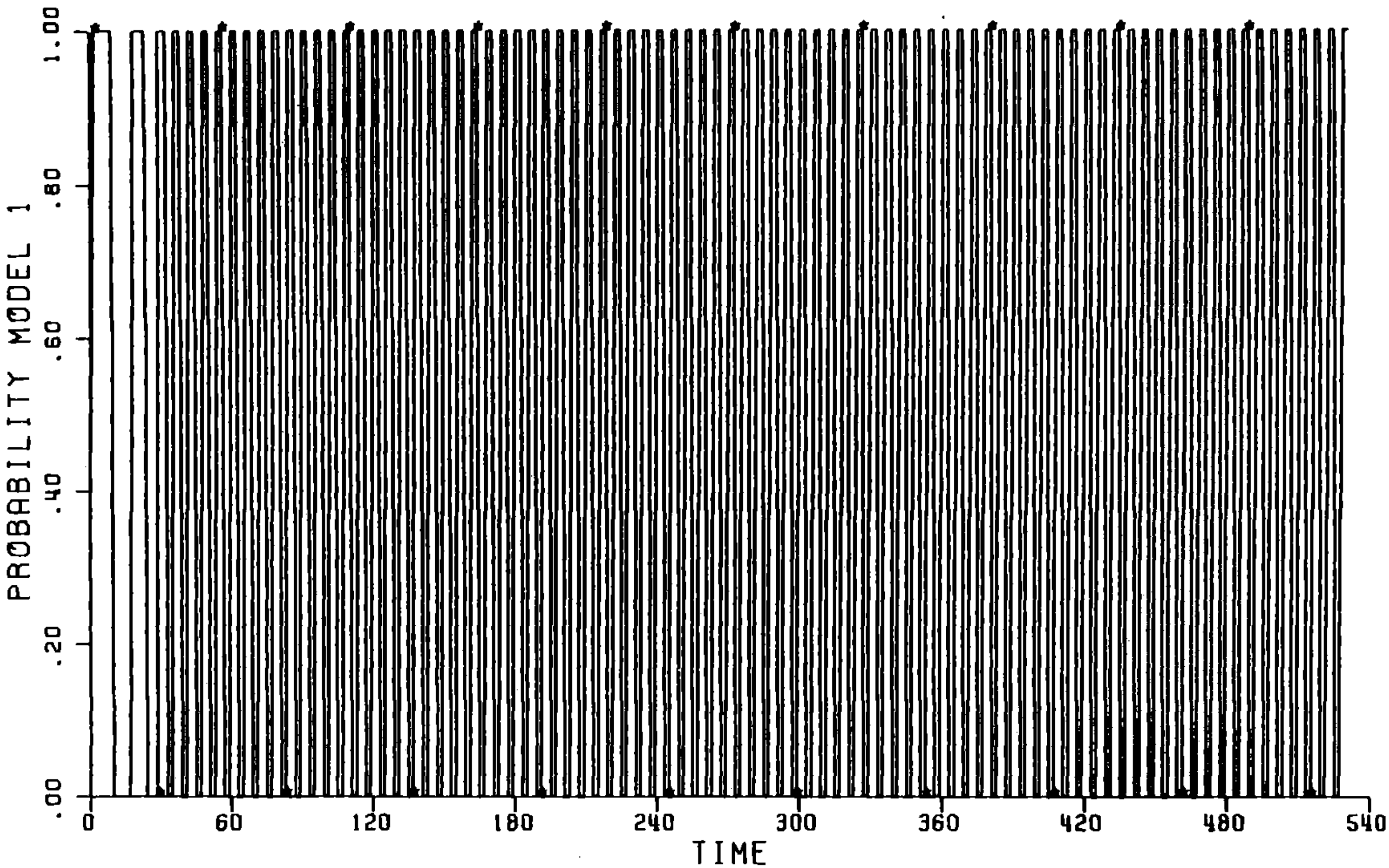


Fig. 3.4 Unstable Oscillation
(Case 1c)

a) Probability of Model 1

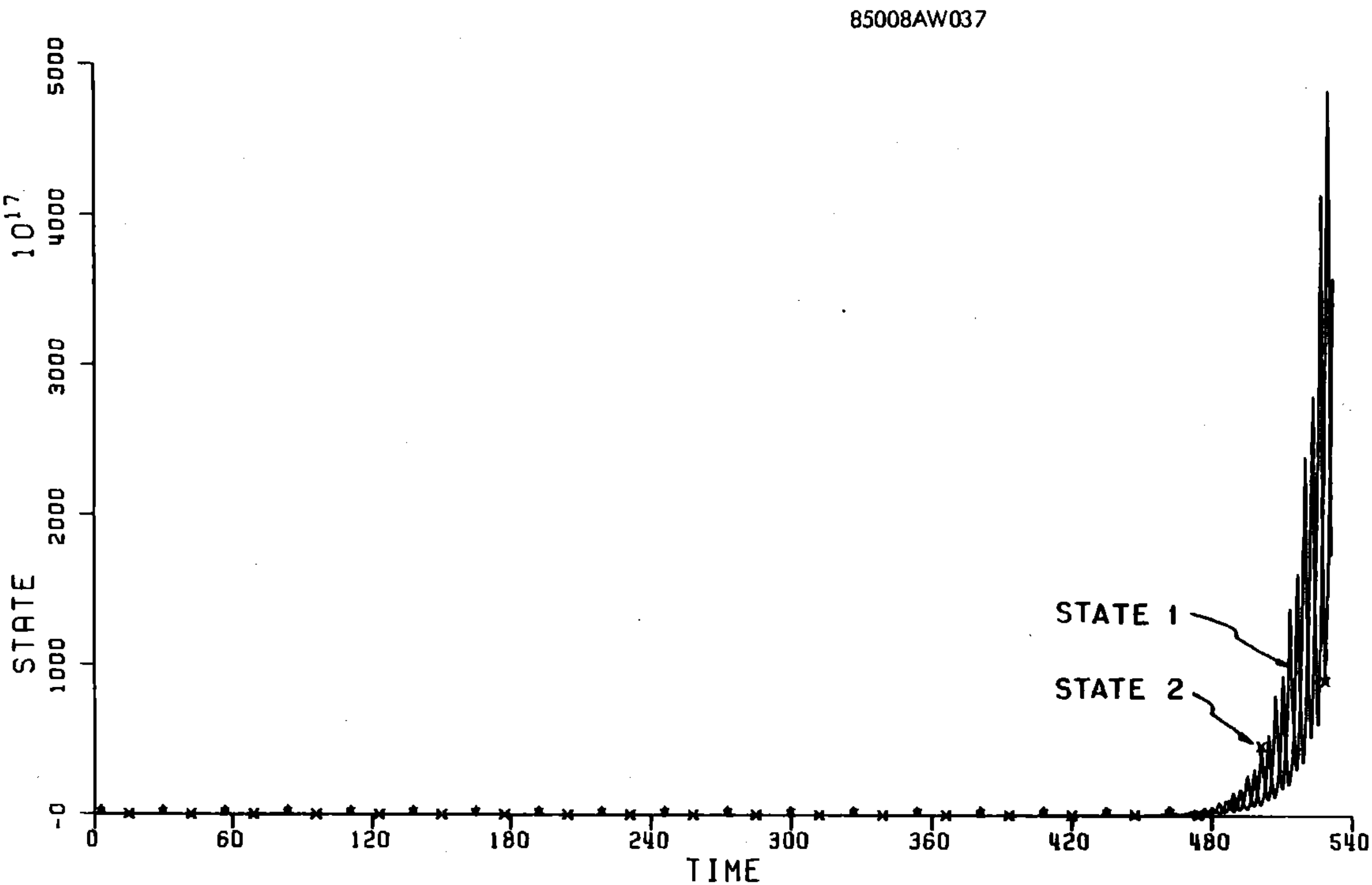


Fig. 3.4 Unstable Oscillation
(Case 1c)

b) True States

85008AW038

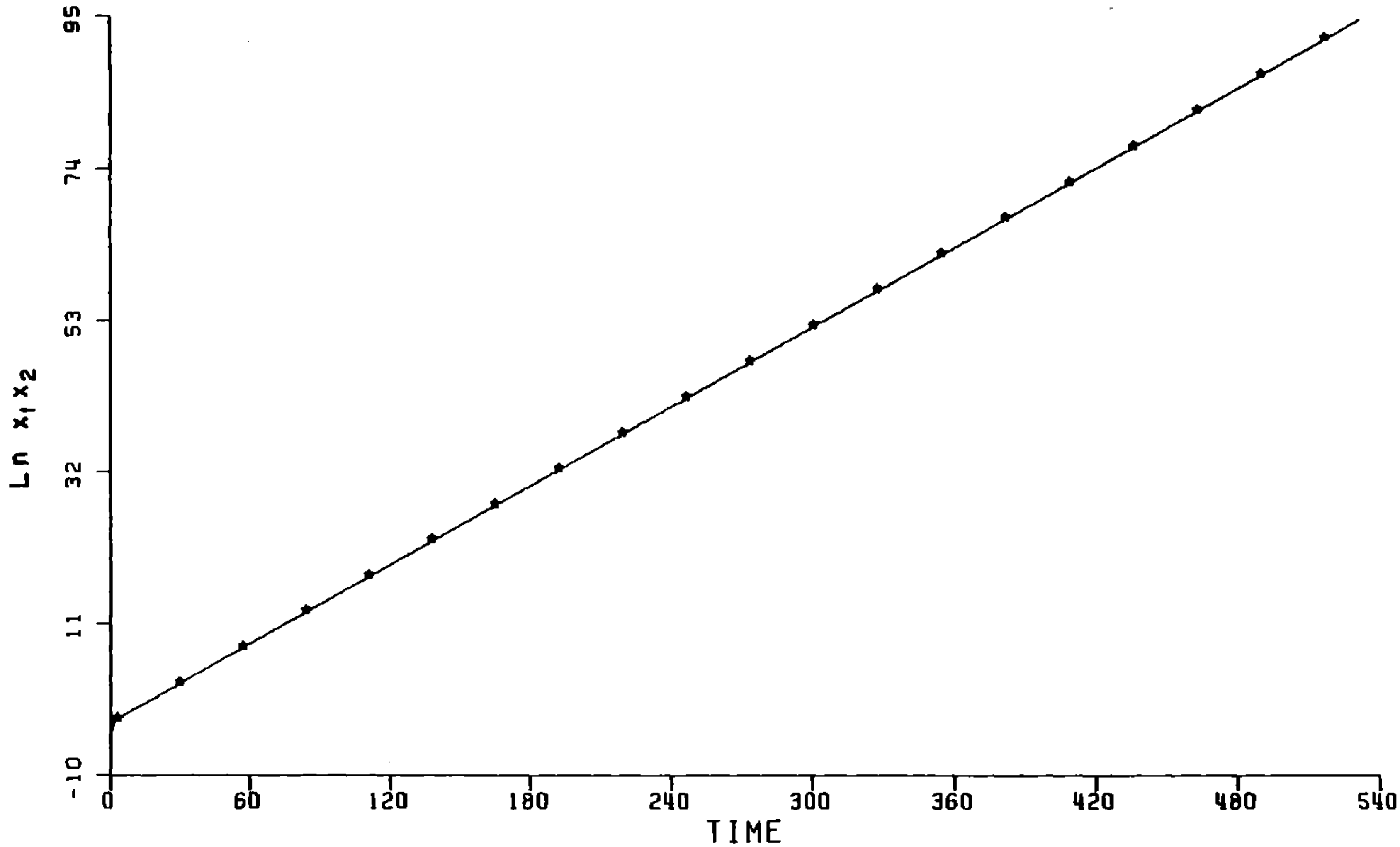


Fig. 3.4 Unstable Oscillation
(Case 1c)

c) Ln x_1x_2

a control which stabilizes the true system. Basically, the adaptive control law then attempts to achieve stability by alternately controlling each mode of the system. This results in unusual behavior which may be bounded (Figure 3.2 and 3.3) or unstable (Figure 3.4). Chapters 4 and 5 contains analysis which gives conditions for each type of behavior. The principle conclusion is that stability will result when the controller at any time stabilizes some modes faster than it destabilizes the remainder. Thus, for the two state problem, a state must be reduced more when the controller stabilizes it than it is increased when the control results in unstable behavior for that state.

One variable which appears useful in characterizing the above condition is the product of the true states, that is, the quantity x_1x_2. This variable is plotted for each simulation (i.e., see Figure 3.2(c)). A complete analysis of the properties of this variable is given in Section 4.5. While not strictly a Lyapunov function, examination of this variable provides a type of analysis which permits a characterization of the stability behavior of the MMAC method. Furthermore, it captures the important observed characteristics of the simulations. For example, the analysis of Section 4.6 uses this variable to predict the stability of the three cases of Figures 3.2 - 3.4. The connection with stability can best be seen by considering a plot of x_1 versus x_2. Figure 3.5(a) is such a plot corresponding to the simulation of Figure 3.2. The state trajectory for this example tends to look like a family of hyperbolas

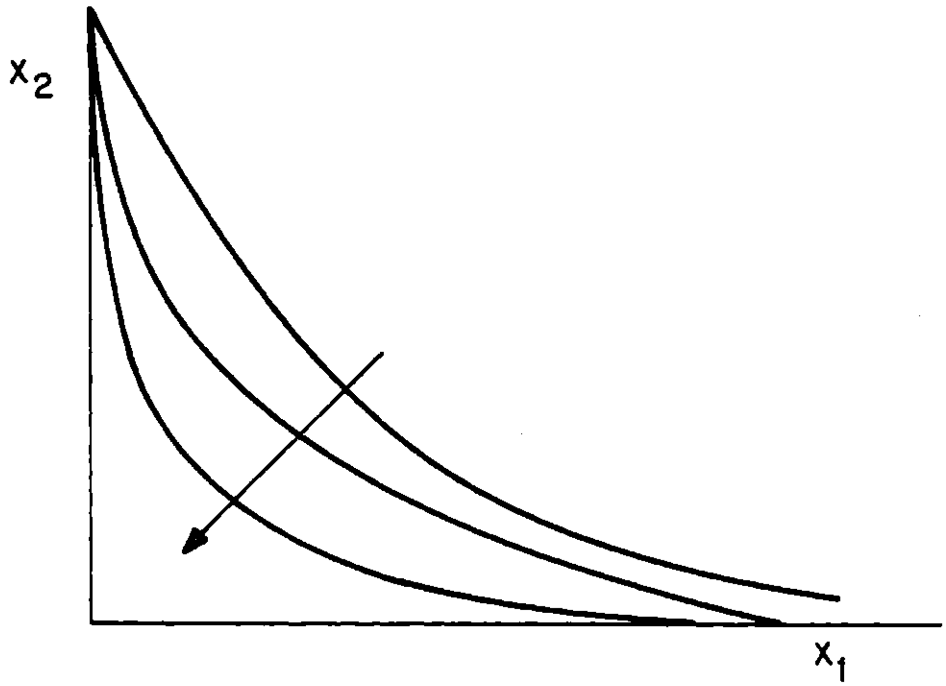
$$x_1x_2 = e^{bt}$$

If the overall system is bounded, the trajectory then changes as in Figure 3.5(a). Likewise, unbounded behavior occurs whenever b is greater than zero (see Figure 3.5(b)).

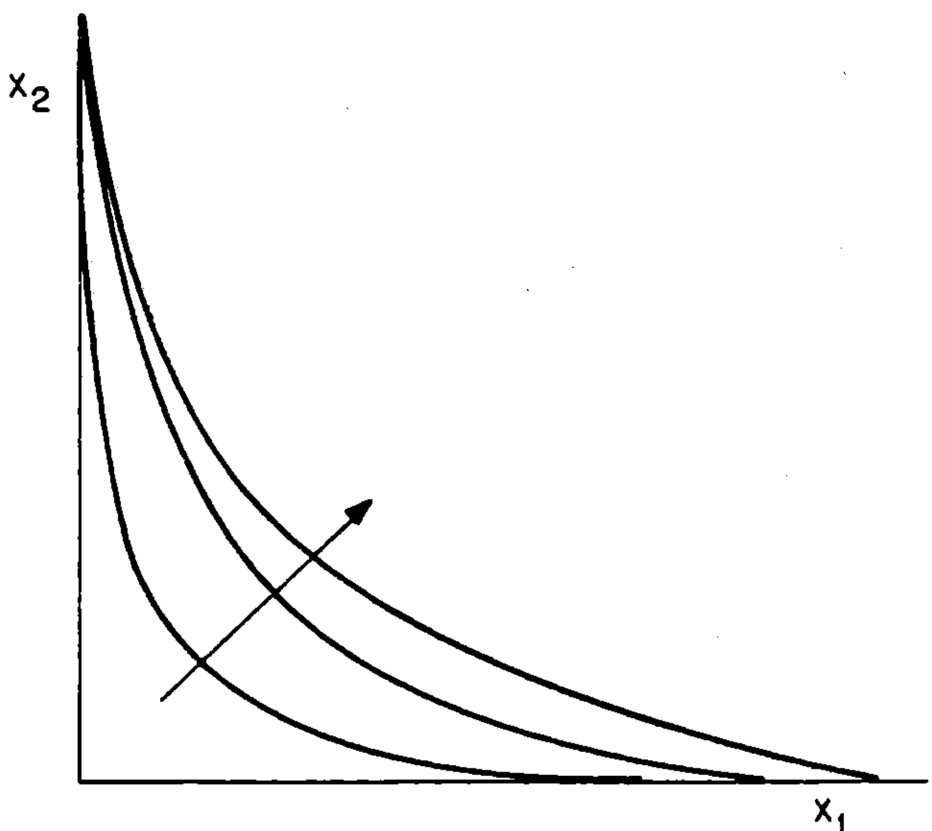
It should be noted that simply bounding the product of the states is not sufficient to guarantee the boundedness of the individual states since the states could still go to infinity along a hyperbola. However, an analysis of the probability equation (Sections 4.4 and 5.1) provides bounds on the values of the state at which probability transitions will occur. This leads to the conclusion that the peaks of the state trajectories will be bounded whenever the product of the states is bounded. Thus when a constant value of P results in a stabilizing control, the system may still be bounded but not asymtotically stable. Further, when the system is bounded and b is less than zero, then the period of oscillation is increasing since the stable mode decreases more than the unstable mode increases and the peaks of the curves are constant resulting in the unstable mode having to increase more on each cycle of the probability.

3.2.3 Mixed Case

When the LTI system for fixed P is stable for some P but not for all P, depending upon the initial conditions, we can obtain simulations which exhibit characteristics of either of the preceding types of behavior in that either an oscillatory or an exponential response may be observed depending on the initial conditions. For large initial conditions the response will initially be oscillatory but usually will



(a) Stable



(b) Unstable

Fig. 3.5 Phase-Plane Plot

finally decay to a constant value of P.* Furthermore, this limiting value of P will be such that $\tilde{\underline{A}}(P)$ is a stable matrix. Figure 3.6 is a simulation of the Case 2 configuration with large initial conditions.

When the initial conditions are small, a second type of response occurs. Assume that $\tilde{A}(P)$ is a stable matrix for $P \in [1/2-\varepsilon, 1/2+\varepsilon]$ and unstable otherwise. If $P(0) = 1/2$, then there exists some non-zero $\underline{w}(0)$ such that $||\underline{w}(k)||$ decays sufficiently quickly so that the resulting change in P does not take it beyond $1/2 \pm \varepsilon$. This is a direct consequence of the fact derived in Section 4.1 that the change in the probability is proportional to $||\underline{w}||^2$ while the change in $\underline{w}$ is proportional only to $||\underline{w}||$. Figure 3.7 is a simulation of the Case 2 configuration for small initial conditions. Note that the probability merely makes a small jump and that it shows little tendency to return to 1/2. Section 4.7 details a procedure which can be used to estimate the range of $\underline{w}(0)$ which results in this non-oscillatory behavior.

3.3 Summary

The preceeding sections have given an overview of the types of behavior which the MMAC method can produce. Table 3.2 summarizes the major characteristics of each. Each type of behavior requires a different form of analysis in order to understand the dominant effects. This analysis is given in Chapters 4 and 5.

Exponential behavior occurs primarily when the basic characteristics of the closed-loop system are <u>independent</u> of the probability, either

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*Whether or not the oscillations always die out in this case remains an open question.

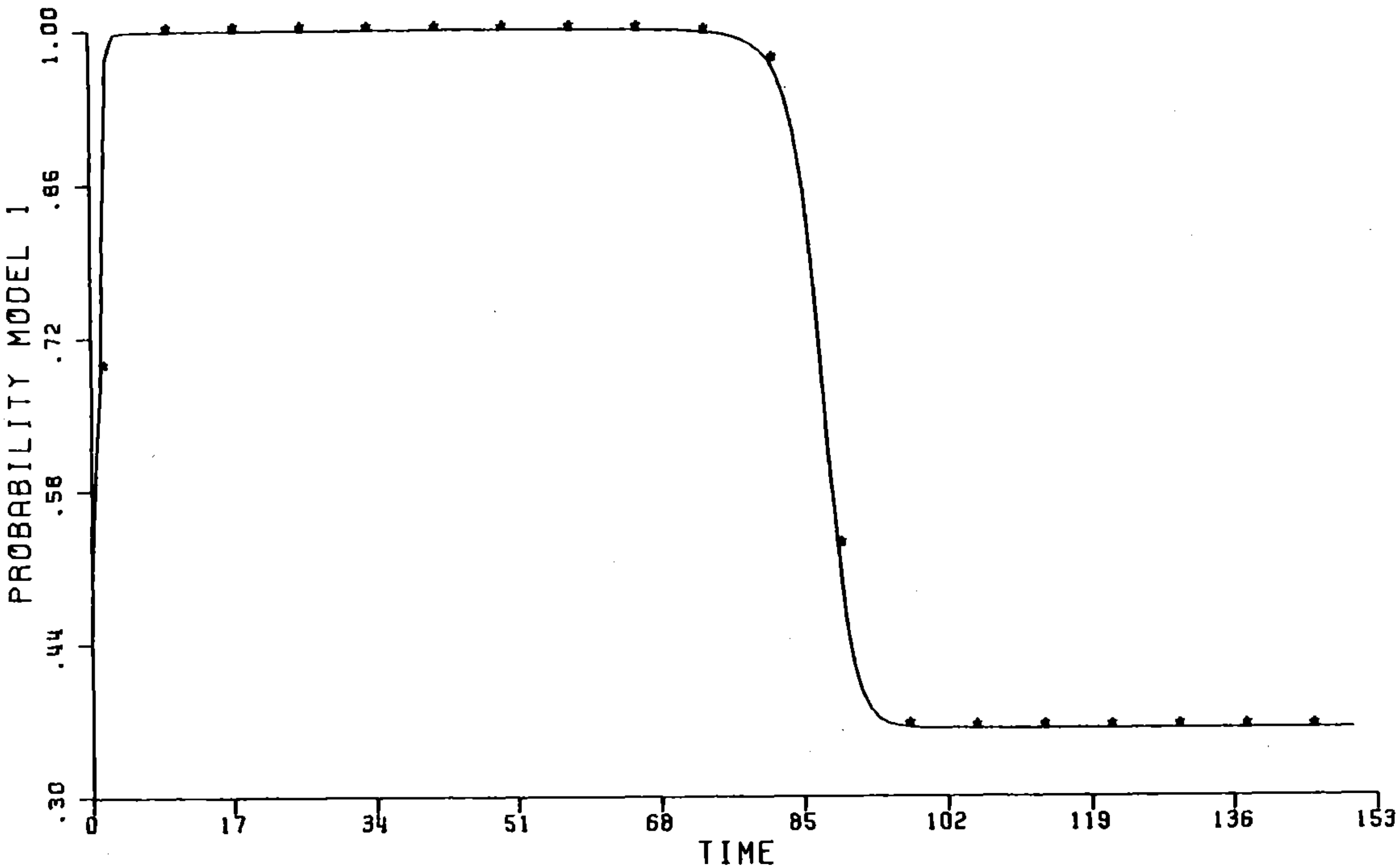

Fig. 3.6 Domain of Attraction - Large Initial Conditions (Case 2)

a) Probability of Model 1

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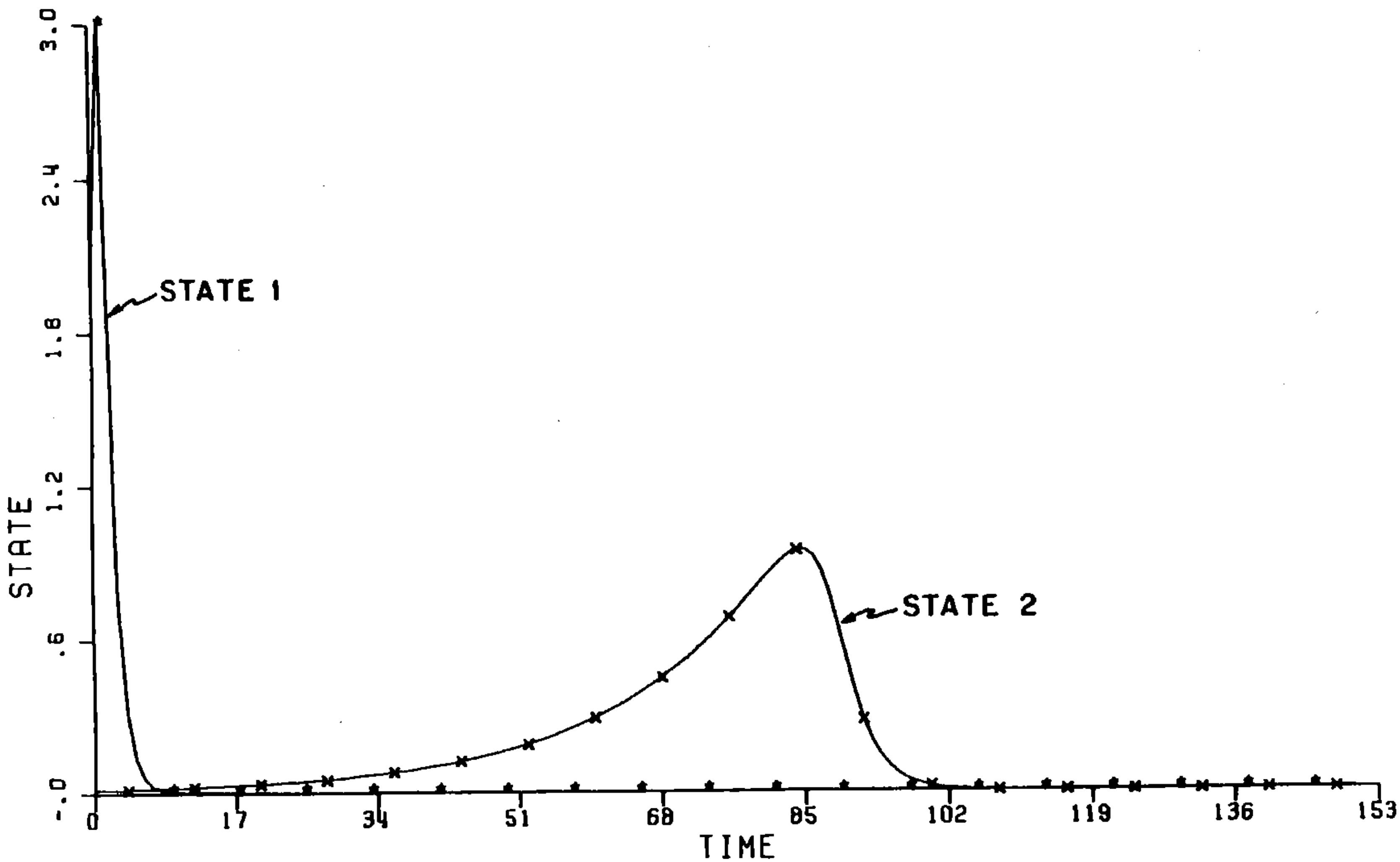

Fig. 3.6 Domain of Attraction - Large Initial Conditions (Case 2)

b) True States

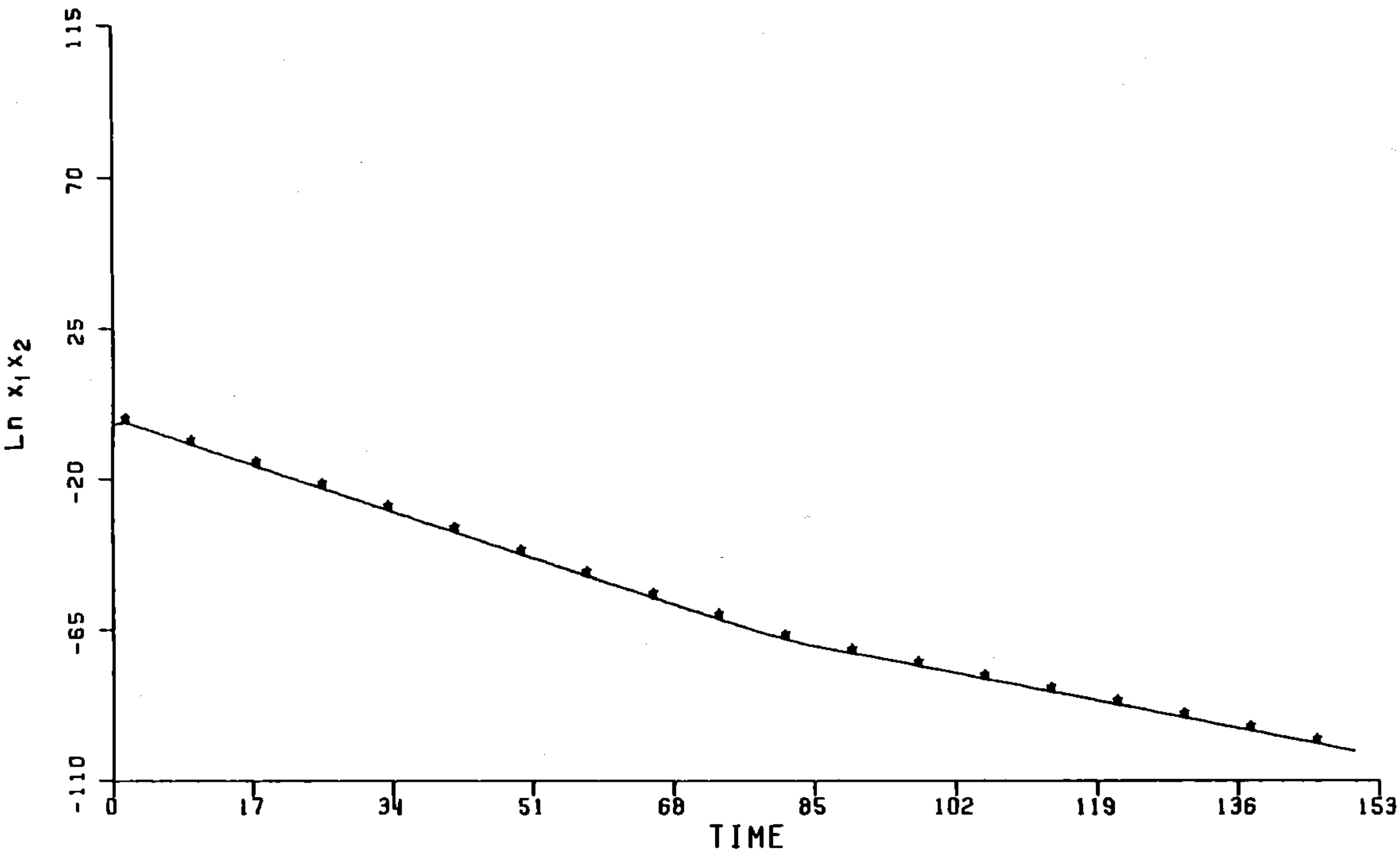

Fig. 3.6 Domain of Attraction - Large Initial Conditions (Case 2)

c) Ln x_1x_2

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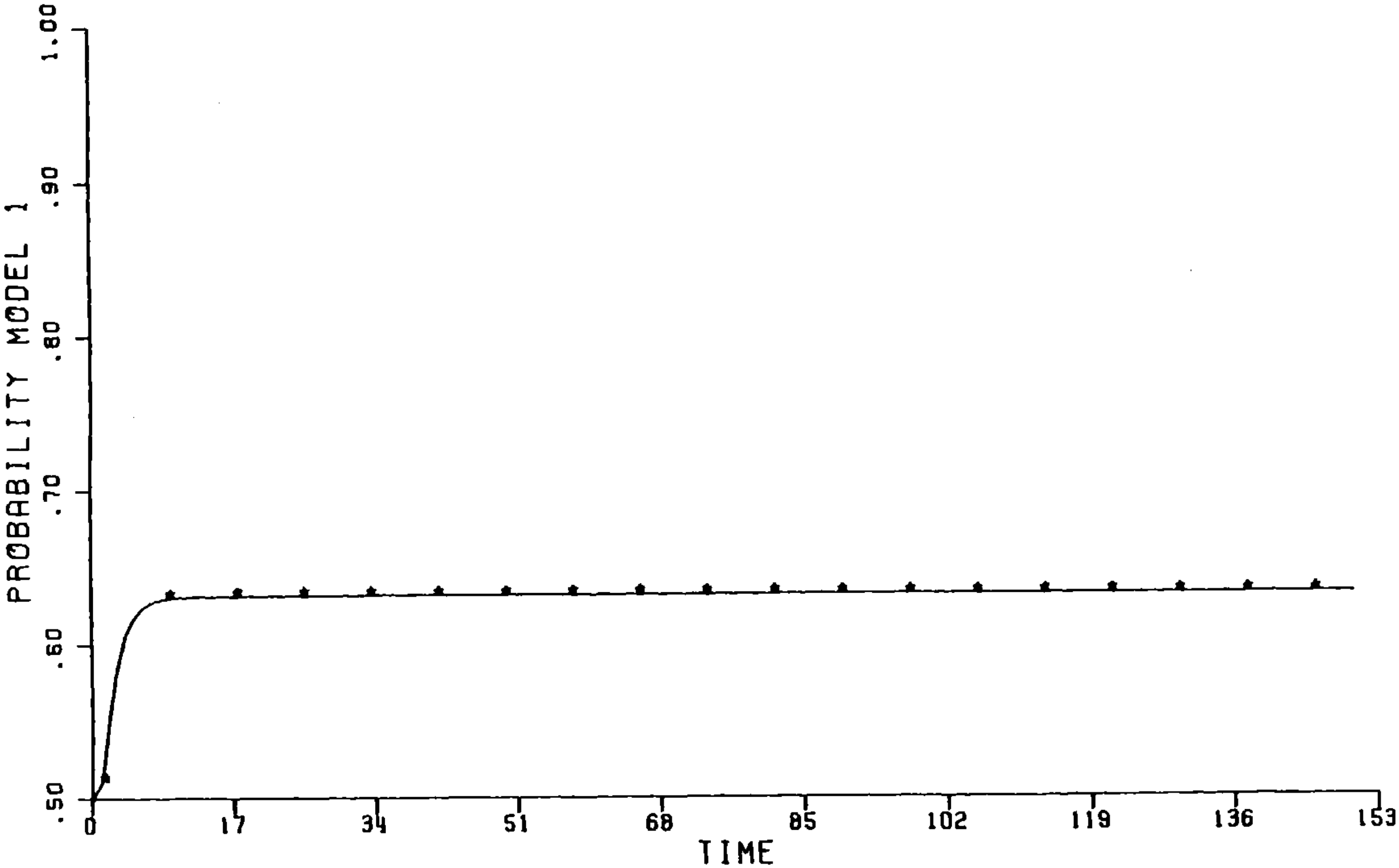

Fig. 3.7 Domain of Attraction - Small Initial Conditions (Case 2)

a) Probability of Model 1

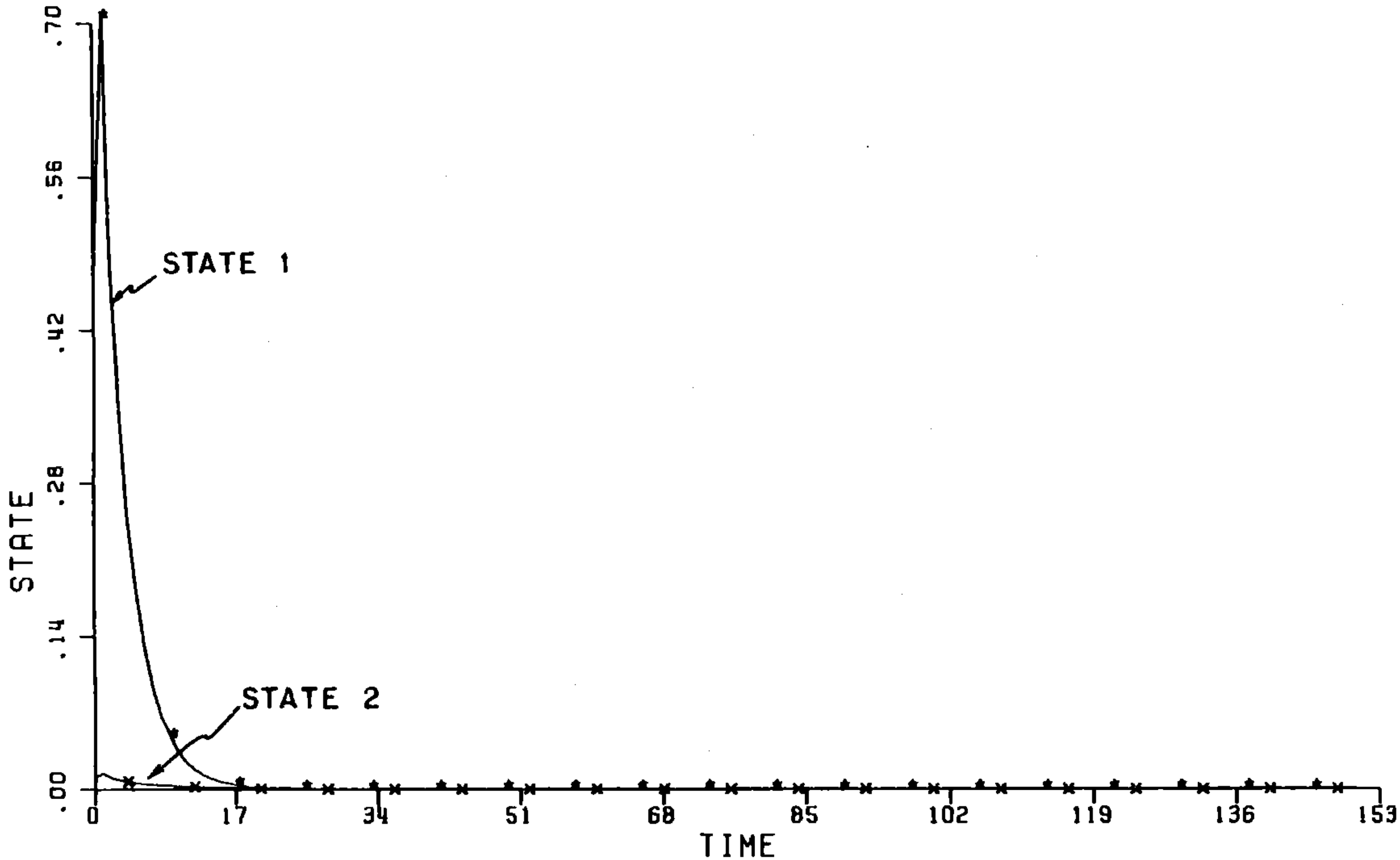

Fig. 3.7 Domain of Attraction - Small Initial Conditions (Case 2)

b) True States

85008AW044

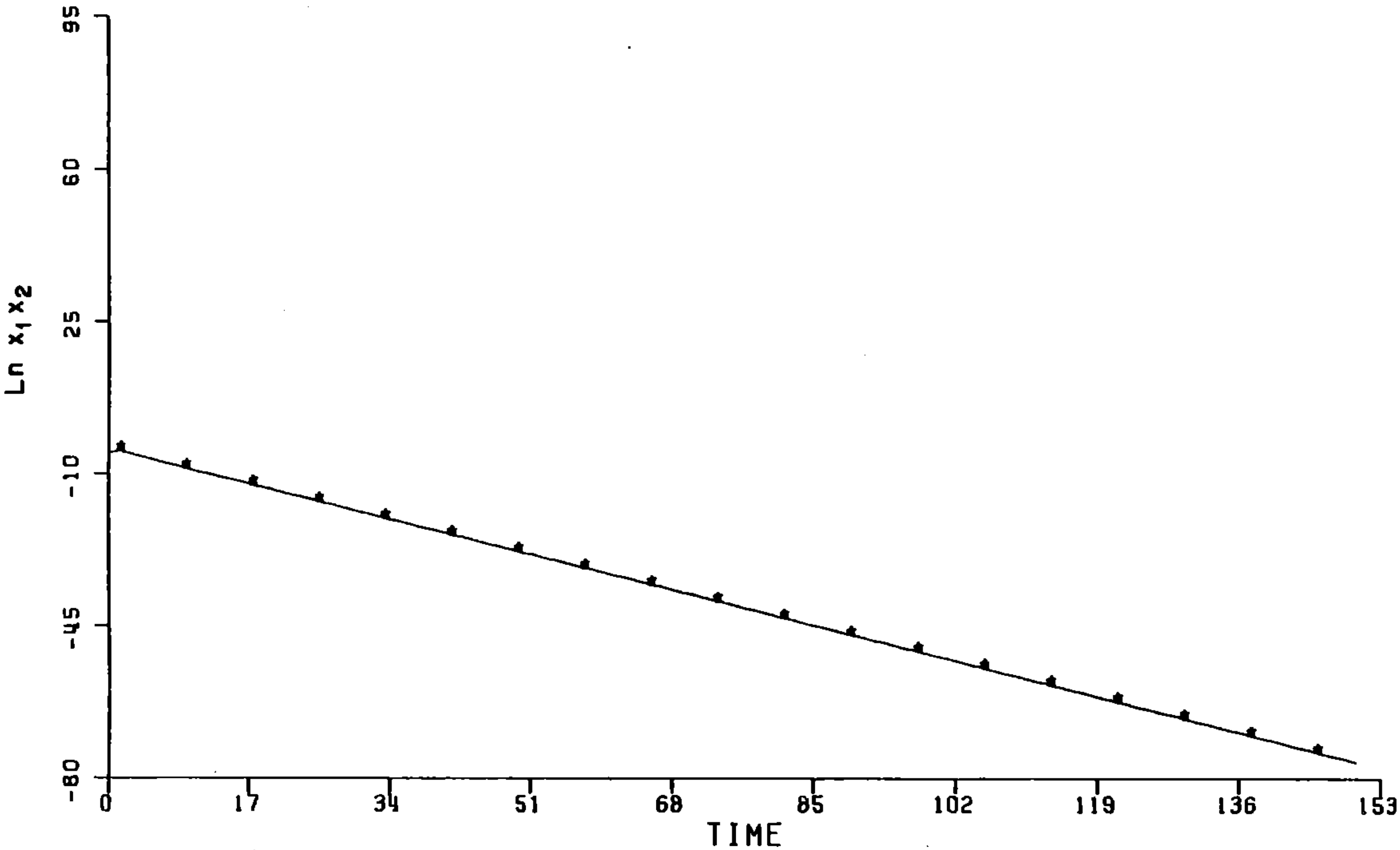

Fig. 3.7 Domain of Attraction - Small Initial Conditions (Case 2)

c) Ln $x_1 x_2$

Section	Fig.	Values of P yielding Stable LTI System	b_i	bounded ?	period of oscillation
3.2.1	3.1	All	decreasing	Yes	NA
3.2.3	3.5 3.6	Some	decreasing	Yes	increasing
3.2.2	3.4	None	increasing	No	*
3.2.2	3.3	None	constant	Yes	constant
3.2.2	3.2	None	decreasing	Yes	increasing

N A = resulting behavior not periodic

* = period approaches a constant

TABLE 3.2

Principle Modes of Response

because the probability is held constant by the dynamics or because all states either increase or decrease regardless of the value of P. Sections 4.1 through 4.3 provide significant insight in this case. Section 4.1 contains the analysis of the local behavior, concluding that the probability equation is neutrally stable because the rate of change of the probability is proportional to the square of the system states. Section 4.2 considers the only case in which exponentially stable behavior can be guaranteed. This is shown to occur when $\underline{\tilde{A}}(P)$ is a stable matrix for all values of the probability. Section 4.3 provides an asymtotic result when one of the models matches the true system.

The oscillatory mode of behavior is analyzed in detail in Sections 4.4 through 4.6 and includes criteria as to when the overall system will be stable or unstable. It is shown that the important parameter determining stability is the relationship between the stabilizing time constants of each state. Thus, if the controller stabilizes one state faster than it destabilizes the other, the overall system will be stable. Section 4.4 contains a detailed analysis of the probability equation which shows why state oscillations are an integral part of the oscillatory behavior. Finally, Section 4.7 discusses the reasons for the mixed type of behavior. It concludes with a procedure for determining the range of initial conditions such that non-oscillatory behavior will occur.

CHAPTER 4

STABILITY ANALYSIS

It is the purpose of the present chapter to provide the analysis necessary to understand the basic stability properties and qualitative behavior of the MMAC method. At present, there is no single stability result which totally describes these properties. Thus, it becomes necessary to examine several approaches, each of which adds to the overall picture, but with none providing a whole view. Furthermore, it often is necessary to combine the results of differing methods to deduce a single property. An example of this is the combining of the Lyapunov results of Section 4.5 with the analysis of the behavior of the probability equation in Sections 4.4 and 4.5 to arrive at a stability result which neither type of analysis alone could provide.

Chapter 3 contains simulations of carefully selected special cases of systems controlled by the MMAC algorithm. The purpose of the present chapter is to attempt to provide an understanding of the properties of the MMAC method by first noting various features from the simulations and then, guided by the simulations, attempting to understand the features by analyzing some special cases. This results in some specific conclusions for the special cases and, more importantly, it yields considerable insight into the qualitative behavior of the MMAC system in more general situations. The major conclusions of this chapter are:

1) At best, the MMAC system is neutrally stable about an equilibrium point in that the probability has no tendency to return to its initial

value following a perturbation. This is not an unexpected property for an adaptive controller.

2) If $\underline{\tilde{A}}(P)$ is an unstable matrix for $P = 1/2$ then the special case of Section 3.1 results in the probability oscillating.

3) This oscillatory behavior results in the states being either bounded or unstable. Specific conditions for two special cases are given. Qualitatively, the requirement for the boundedness of the states is that the modes of $\underline{\tilde{A}}(P)$ which are unstable for $P = 1 (P=0)$ must be dominated by the stability of the same mode when $P = 0$ $(P=1)$ in that they must grow slower for $P = 1$ $(P=0)$ than they decay for $P = 0$ $(P=1)$.

4) The oscillations observed in Section 3.2 may be bounded even if no constant value of P results in a stabilizing control. The controller then attempts to achieve stability by alternately controlling each mode of the system. This alternating of controls also occurs for large initial conditions when $\underline{\tilde{A}}(P)$ is a stable matrix for some but not all values of P.

As seen in Section 3.2, two major types of behavior have been observed in simulations: oscillatory and exponential (non-oscillatory). The analysis of the present chapter is aimed at understanding each. Thus, Sections 4.1 through 4.3 deal largely with the exponential behavior while Sections 4.4 through 4.6 consider the case in which the probability oscillates. Section 4.7 considers the case in which either type of behavior can occur and contains a discussion of the initial conditions that lead to each type of behavior. A detailed overview of the chapter follows.

Section 4.1 through 4.3 contain preliminary material dealing with the exponential case. For example, the neutral stability of the probability exhibited in the simulations of Figures 3.6 and 3.7 is derived in Section 4.1. The other major conclusion of the section is that changes in the probability are proportional to the norm of the state squared while changes in the states are only proportional to the state to the first power. Thus, for small values of the state, the state tends to change faster than the probability. This further explains the switching behavior exhibited in the oscillatory responses in Section 3.2. In Section 3.2.1, the case in which both models result in stabilizing controls for all values of the probabilities has been simulated. It is shown that under some conditions the probability dynamics can effectively be ignored in determining stability. Section 4.2 contains the analysis of this case and proves that $\underline{\tilde{A}}(P)$ must be a stable matrix for <u>all</u> P in order to guarantee that the states will be exponentially stable for all initial conditions.

Section 3.2.2 includes simulations in which the probability tends to look like a switch, taking on values near one or zero but seldom in between. Section 4.4 contains a detailed analysis of the probability equation which leads to an understanding as to <u>why</u> the probability jumps so rapidly. Sections 4.5 and 4.6 continue this analysis to present two methods of analysis for ascertaining when the oscillatory behavior will be bounded or unstable. These sections, which result in basically the same criteria, take different points of view; Section 4.5 contains approximations to the general solution over several time intervals while

Section 4.6 employs a function which resembles a Lyapunov function.

From Section 3.2.3, it is clear that the size of the initial conditions partly determines whether the probability will oscillate or not. In Section 4.7 the situation is analyzed in detail and a procedure developed which allows for the determination of the limits of the initial conditions which result in the probability not oscillating.

The majority of the analysis in this chapter has been performed in the discrete time domain. In many ways this is to be preferred as this is the form in which the method is most often implemented in practice. However, some of the analysis is done using the continuous time version because the results are simpler and lend themselves to interpretation more easily. No claim is made as to the complete equivalence of the two forms. In fact, one could expect some differences between them due to the different assumptions on the availability of sensor data (i.e., discretely versus continuously available). However, the qualitative conclusions of one are applicable to the other, as can be seen by the examination of the two sets of equations. It should also be born in mind that the analysis of this chapter is aimed at understanding the phenomena exhibited in the simulations of Chapter 3. Thus, unless stated, it can be assumed that the canonical structure of Section 3.1 is under consideration. However, the structure of the canonical problem has been carefully selected to accentuate certain types of behavior observed in more general MMAC simulations [16, 23]. Thus, the intuition and qualitative results are believed to provide significant insight into the behavior for more general systems.

It should be noted that throughout this thesis, the terms "oscillatory", "non-oscillatory" and "exponential" refer to qualitative behavior of the probability. Although not observed in the simulations in Chapter 3, it is possible to have the LTI system for fixed P display a second-order response and it is important to note that this is not considered an oscillatory response for the purposes of this thesis.

4.1 Linearized Analysis

In Chapter 3, simulation results have indicated that the probability may tend to be neutrally stable in that it shows no tendency to return to its initial value following a perturbation. Greater understanding of the causes and consequences of this can be gained by examining the local behavior of the states and the probability about an equilibrium point. Examination of Equation (2.22) or (2.28) shows that the MMAC algorithm results in a closed loop system which has a set of equilibrium points: any point of the form $\underline{w}=0$, with any value of P is an equilibrium point since when $\underline{w}=0$, there is no information in the system that would lead to a change in P.* The value of the state about which the present analysis is performed is $\underline{w}=0$, $P=1/2$. The reasons for considering this point in preference to others stems largely from the high degree of symmetry inherent in the problem formulation given in Section 3.1. For example, the range of P for which the LTI system for fixed P is stable

*Similar reasoning indicates that neutral stability is a general property of any adaptive controller.

is easily seen to be symmetric about $P = 1/2$ (i.e., about $q = 0$).

In order to lend clarity to the results, the continuous time equations are considered. Examination of the discrete time equations results in similar conclusions. The continuous time algorithm is rewritten as

$$\dot{\underline{w}}(t) = \tilde{\underline{A}}(q)\ \underline{w}(t) \tag{2.28}$$

$$q(t) = \frac{1}{4}(1-q^2)[r_2-\underline{r}_1]'\ \underline{\theta}^{-1}[(1+q)\underline{r}_1 + (1-q)\underline{r}_2]$$

$$= \frac{1}{4}(1-q^2)\ \underline{w}'[\underline{Q}_1-q\underline{Q}_2]\underline{w}$$

where

$$\tilde{\underline{A}}(q) = \begin{bmatrix} \underline{A}-\frac{1}{2}(1+q)\underline{G}_1-\frac{1}{2}(1-q)\underline{G}_2 & \frac{1}{2}(1+q)\underline{G}_1 & \frac{1}{2}(1-q)\underline{G}_2 \\ \underline{A}-\underline{A}_1 & \underline{A}_1-\underline{H}_1 & 0 \\ \underline{A}-\underline{A}_2 & 0 & \underline{A}-\underline{H}_2 \end{bmatrix}.$$

Simple calculations then lead to the linearized system equation

$$\begin{bmatrix} \dot{\Delta w} \\ \dot{\Delta q} \end{bmatrix} = \begin{bmatrix} \tilde{\underline{A}}(q_0) & \tilde{\tilde{\underline{A}}}\underline{w}_0 \\ \frac{1}{2}(1-q_0^2)\ \dot{\underline{w}}_0'[\underline{Q}_1-q_0\underline{Q}_2] & \frac{1}{4}\ \underline{w}_0'[3q_0^2\underline{Q}_2-2q_0\underline{Q}_1-\underline{Q}_2]\underline{w}_0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix} \tag{4.1}$$

with

$$\tilde{\tilde{\underline{A}}} = \begin{bmatrix} \frac{1}{2}(\underline{G}_2-\underline{G}_1) & \frac{1}{2}\underline{G}_1 & -\frac{1}{2}\underline{G}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{Q}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\underline{\theta}^{-1} & 0 \\ 0 & 0 & \underline{\theta}^{-1} \end{bmatrix} \qquad \underline{Q}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \underline{\theta}^{-1} & \underline{\theta}^{-1} \\ 0 & -\underline{\theta}^{-1} & \underline{\theta}^{-1} \end{bmatrix}$$

where the linearization is done about the values $[\underline{w}_0, q_0]$. Note that if $\underline{w}_0 = 0$, then the linearized system reduces to

$$\begin{bmatrix} \Delta\dot{w} \\ \hline \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} \tilde{\underline{A}}(q_0) & \underline{0} \\ \underline{0} & 0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \hline \Delta q \end{bmatrix} . \tag{4.2}$$

Thus, the system is, at best, neutrally stable to first order (i.e., the linearized system has a zero eigenvalue). This should not be particularly surprising; when the state is zero, there is no need or basis for changing the identification results. It should be further noted that if $\tilde{\underline{A}}(q_0)$ is an unstable matrix, then the resulting linearized equations are <u>unstable</u>. Thus, in order to have asymtotic non-oscillatory behavior in a system subject to perturbations, it is necessary that there exists a value of q_0 such that $\tilde{\underline{A}}(q_0)$ is a <u>stable</u> matrix.

Equation (2.28) can also be rewritten exactly as:

$$\underline{\dot{w}}(t) = \tilde{\underline{A}}(0)\ \underline{w}(t) + \tilde{\tilde{\underline{A}}}\underline{w}(t)\,q(t) \tag{4.3a}$$

$$\dot{q}(t) = \frac{1}{4}\,\underline{w}'[\underline{Q}_1 - \underline{Q}_2 q - \underline{Q}_1 q^2 + \underline{Q}_2 q^3]\underline{w} . \tag{4.3b}$$

(This is the full Taylor series expansion about w=0, q=0). A few observations can now be made. First of all, for q near zero, the rate of change of the state is proportional to $||w||$ while the rate of change of the probability is proportional to $||w||^2$ - hence the neutral stability of the linearized system. Thus, the probability tends to change slowly for small values of w. Furthermore, again for q small, the higher order terms in q can be ignored so Equation (4.3b) can be approximated by

$$\dot{q}(t) \approx \frac{1}{4}\left[||\underline{r}_2||^2_{\theta_2^{-1}} - ||\underline{r}_1||^2_{\theta_1^{-1}}\right] \quad . \tag{4.4}$$

Thus, the changes in q are such that q increases if $||\underline{r}_2|| > ||\underline{r}_1||$ and decreases if $||\underline{r}_2|| < ||\underline{r}_1||$, which agrees well with the intuitive notions of how an adaptive controller should behave. Furthermore, when $\underline{r}_1 = \underline{r}_2$, there is no information as to which model provides the better match to the true system and so no change in the identification results can be made. It should be noted that this may result in exponentially unstable behavior when $\tilde{\underline{A}}(q_0=0)$ is an unstable matrix and the system is initialized with $q(0) = q_0 = 0$ and, for example, $\underline{r}_1(0) = \underline{r}_2(0) = \underline{0}$. This is an important special case since in practice one often attempts to initialize an MMAC system with just such initial conditions. Equation (4.4) also illustrates the neutral stability of the probability.

In summary, if $\tilde{\underline{A}}(0)$ is an unstable matrix, the equilibrium $\underline{w} = 0$, $q = 0$ is unstable and small perturbations in the state cause a divergence from the equilibrium which will most likely result in the oscillatory behavior of the probability (resulting in either bounded or unstable state responses) observed in Figures 3.2 through 3.4. However, if $\tilde{A}(0)$ is a stable matrix, then two modes of behavior are possible. For perturbations large enough that $\tilde{\underline{A}}(q)$ becomes an unstable matrix, at least temporary oscillations most likely occur as seen in Figure 3.6. For smaller perturbations, the state, $\underline{w}$, will return to zero and the probability will simply move to a new value which is such that $\tilde{\underline{A}}(q)$ is stable. This occurs since, by Equation (4.4), the probability has no tendency to return to zero unless $||r_2||_{\underline{\theta}_2^{-1}} - ||r_1||^2_{\underline{\theta}_1^{-1}}$ changes sign. This is precisely the behavior observed in Figure 3.7.

The next section discusses the conditions such that the latter type of non-oscillatory behavior occurs for all initial conditions. This can be viewed as an extension of the results of this section for the case in which $\tilde{A}(P)$ is stable for all P. Following an analysis of the stability of the oscillatory behavior, attention again is focused on the linearized system in Section 4.7 where a procedure is derived for determining how large the above perturbations can be and still have non-oscillatory behavior guaranteed.

4.2 Universal Stability

For any adaptive control algorithm one would at least like to be able to conclude that the overall system is asymptoticly stable about the point $\underline{w}=0$ in spite of any uncertainties about the true system. As seen in the previous section, the MMAC method always results in a system with neutral stability in the probability. However, it is also shown that it is possible for the states to locally converge to zero despite the behavior of the probability. The present section discusses one case in which global asymptotic convergence of the states can be demonstrated.

Consider the discrete time MMAC system given by Equation (2.22):

$$\underline{w}(k+1) = \tilde{\underline{A}}(P)\,\underline{w}(k)$$

$$P(k+1) = \frac{P(k)\,p(\underline{r}_1)}{P(k)p(\underline{r}_1) + (1-P(k))\,p(\underline{r}_2)} \tag{2.22}$$

We now make the following <u>assumption</u>, which will be termed <u>Universal Stability</u>: Assume $||\tilde{\underline{A}}(P)|| \leq \varepsilon < 1 \quad \forall\, P \in [0,1]$ where $||\tilde{\underline{A}}||$ is the norm of $\tilde{\underline{A}}$. It should be noted that this is a fairly restrictive assumption as it requires that the control law associated with <u>each</u> hypothesized model stabilize the true system. In general, this requires that the condition $||\tilde{\underline{A}}(P)|| < 1$ be explicitly tested for all values of P. However, as discussed in Section 4.3 in some cases such as when the true system and all of the hypothesized models are diagonal, it becomes sufficient to perform the test for extreme values of P only. When the assumption is valid, the following theorem is useful.

<u>Theorem</u>: Under the assumption of universal stability, $\underline{w}(k) \to 0$ geometrically as $k\to\infty$.

Proof: By assumption, $||\tilde{\underline{A}}(p_1)|| \leq \varepsilon < 1 \quad \forall\, P_1 \in [0,1]$. Thus, taking norms in Equation (2.22) yields

$$||\underline{w}(k+1)|| = ||\tilde{\underline{A}}(P_1)\underline{w}(k)|| \leq ||\tilde{\underline{A}}(P_1)||\ ||\underline{w}(k)|| \qquad (4.5)$$

$$\leq \varepsilon\ ||\underline{w}(k)|| \quad .$$

Thus, $||\underline{w}(k)|| \leq \varepsilon^k||\underline{w}(0)||$. Since by assumption $\varepsilon < 1$, the conclusion follows. ▲

It should be noted that by a suitable redefinition of the matrix $\tilde{\underline{A}}(P)$, this theorem can be extended to the N-model case where $\underline{P}$ becomes a vector of probabilities such that $||\underline{P}|| < 1$ and each P_i is non-negative.

The value of this theorem in applications is, of course, severely limited by the assumption on the stability of $\tilde{\underline{A}}(P_1)$. However, as demonstrated by the analysis in the remainder of this chapter and the simulations of Section 3.2, a condition such as this is necessary to guarantee geometric convergence. Any weakening of the hypothesis admits the possibility of at least transitory oscillatory responses.

4.3 An Asymptotic Result

As discussed in Chapter 1, the MMAC method was developed with the implicit assumption that one member of the set of hypothesized models exactly matches the true system. Under this assumption, Baram [15] has shown that the identification algorithm (i.e., the value of the probability when all feedback gains are set to $\underline{0}$) converges to the matched model when the input is ergodic. However, he also shows that when deterministic inputs are used and none of the models match the true system the convergence properties are indeterminate - the model to which the probability converges is a function of the input. Furthermore, his results require the ergodicity of the residual which clearly is not guaranteed when the probabilisticly weighted control is applied*. Thus, no general convergence result has been derived for the closed loop adaptive situation.

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*Ljung et al [24, 25] have considered closed loop identification for some specific model structures and identification methods. However, we have not attempted to apply their results to the MMAC method; this remains as a topic for the future.

However, the following theorem provides some insight into the expected behavior. It should be pointed out that no assumption as to the number of models is made.

Theorem: Assume the following for a discrete time MMAC System:

(1) There are N hypothesized models.

(2) Model #1 exactly matches the true system.

(3) All models (and therefore by (2) the true system) have diagonal A-matrices with C=B=I (as in Chapter 3).

(4) $\underline{r}_1(0) = 0$

(5) $P_1(0) > 0$

Then, $||x(k)|| \to 0$ as $k \to \infty$.

Proof: Since $\underline{r}_1(0) = 0$, and model 1 matches the true system, $\underline{r}_1(k) = 0 \; \forall k$. Thus, $P_1(k)$ is non-decreasing and either:

Case 1) $P_1(k) \to 1$ as $k \to \infty$ or

Case 2) $\exists \varepsilon > 0 \ni 1-P_1(k) > \varepsilon \quad \forall k$.
(equivalent, $\exists i \ni P_i(k) > 0$.)

Consider Case 1:

$P_1(k) \to 1 \Rightarrow \exists K \ni k > K \Rightarrow \underline{\tilde{A}}(P(k))$ is a stable matrix since $\underline{\tilde{A}}(1)$ is stable by hypothesis. This in turn implies the convergence of $||x||$.

Consider Case 2:

$1-P(k) > \varepsilon \quad \forall k$ together with $P(k)$ non-decreasing implies $\nexists$ at least one other model such that its residual approaches zero since otherwise $P_1(k)$ would increase to 1. Assume for the moment that there is only one such other model, denoted by model i. (That is, assume that only $\underline{r}_1(\cdot)$ and $\underline{r}_i$ approach zero.) Due to the use of <u>full state</u> observations in the canonical problem, the equation for $\underline{r}_i(\cdot)$ can be rewritten from Equation (2.22) as

$$\underline{r}_i(k+1) = \underline{A}_i(\underline{I}-\underline{H})\,\underline{r}_i(k) + (\underline{A}-\underline{A}_i)\underline{x}(k)$$

Since $\underline{r}_i(k) \to 0$, $\underline{x}(k)$ must approach an element of $N(A-A_i)$.* The dynamic equation for $x(k)$ is now rewritten as

$$\underline{x}(k+1) = \underline{B}_0(k)\underline{x}(k) + \sum_{j=1}^{N} P_j(k)\underline{G}_j(I-\underline{H}_j)\underline{r}_j(k)$$

where

$$\underline{B}_0(k) = [P_1(k)(\underline{A}-\underline{G}_1) + (1-P_1(k))(\underline{A}-\underline{A}_1) + (1-P(k))(\underline{A}_i-\underline{G}_i) - \sum_{\substack{j=2 \\ j\neq i}}^{N} P_j(k)\underline{G}_j] \; .$$

If $\underline{r}_j(\cdot)$, $j\neq 1$ or i, is bounded, then, since $P_j(k) \to 0$, $P_jG_j(I-K_j)r_j(t) \to 0$. If $\underline{r}_j(\cdot)$, $j\neq 1$ or i, $\to\infty$, then, since $P_j \to 0$ as $e^{-r_j' r_j}$, $P_j\,\underline{r}_j \to 0$. This implies that $\sum_{j=1}^{N} P_j(k)\underline{G}_j(\underline{I}-\underline{H}_j)\underline{r}_j(k) \to \underline{0}$. Thus, consider the un-undriven system

$$\underline{x}(k+1) = \underline{B}_0(k)\underline{x}(k)$$

which, since $P_1(\infty)$, $P_i(\infty)$ exist, can be rewritten as

---

*$N(\cdot)$ represents the null space of a matrix.

$$\underline{x}(k+1) = [P_1(\infty)(\underline{A}-\underline{G}_1) + (1-P_1(\infty))(\underline{A}_i-\underline{G}_i)]\,\underline{x}(k)$$

$$+ [(P_1(k) - P_1(\infty))(\underline{A}-\underline{G}_1) - (P_1(k)-P_1(\infty))(\underline{A}_i-\underline{G}_i)]\,\underline{x}(k)$$

$$+ (1-P_1(k))(\underline{A}-\underline{A}_i)\,\underline{x}(k) \quad .$$

$(\underline{A}-\underline{G}_1)$ and $(\underline{A}_i-\underline{G}_i)$ are stable by design. Then since they are also <u>diagonal</u>, the convex combination of stable matrices is stable. Thus, there exists a $b<1 \ni$

$$||[P_1(\infty)(\underline{A}-\underline{G}_1) + (1-P_1(\infty))(\underline{A}_i-\underline{G}_i)]\,\underline{x}(k)|| \leq b||\underline{x}(k)|| \quad .$$

Then, for some $\varepsilon \ni b+\varepsilon < 1$ and any given δ, we can find a $K \ni$

$$||[(P_1(k)-P_1(\infty))(\underline{A}-\underline{G}_1)-(P_1(k)-P_1(\infty))(\underline{A}_i-\underline{G}_i)]\,\underline{x}(k)|| \leq \varepsilon\, ||\underline{x}(k)||$$

(since $P_1(k) \to P_1(\infty)$) and

$$||(1-P_1(k))(\underline{A}-\underline{A}_i)\,\underline{x}(k)|| \leq \delta$$

(since $\underline{x}(k) \to N(A-A_i)$). Combining these yields

$$||\underline{x}(k+1)|| \leq (b+\varepsilon)\, ||\underline{x}(k)|| + \delta \qquad \forall k \geq K$$

or, using the variation of constant formula,

$$||\underline{x}(k)|| \leq (b+\varepsilon)^{k-K}\, ||\underline{x}(K)|| + \sum_{j=K}^{k-1} (b+\varepsilon)^{k-1-j}\delta \qquad \forall k \geq K$$

or $$||\underline{x}(k)|| \leq (b+\varepsilon)^{k-K}\, ||\underline{x}(K)|| + \frac{\delta - (b+\varepsilon)^{k}\delta^{K}}{1-(b+\varepsilon)} \quad ,$$

Taking the limit as $k \to \infty$

$$\lim_{k\to\infty} ||\underline{x}(k)|| \leq 0 + \frac{\delta}{1-(b+\varepsilon)}$$

since K was selected such that $(b+\varepsilon) < 1$. But δ was arbitrary. Thus $\lim_{k\to\infty} ||\underline{x}(k)|| = 0$. The case in which more than two models have non-vanishing probabilities follows similarly.

▲

Some comments on this theorem are important.

1. Assumption (5) merely guarantees that the probability of model 1 can change since

$$P_i(0) = 0 \Longrightarrow P(k) = 0 \quad \forall k \ .$$

2. Assumption (4) has been made for convenience. It is used to guarantee that $P_1(\cdot)$ is non-decreasing, thus preventing oscillations in the probability due to $||\underline{r}_1(\cdot)|| - ||\underline{r}_i(\cdot)||$ changing sign.

3. Assumption (3) is, of course, very restrictive. It is necessary to enable one to conclude stability when a convex combination of stabilizing controls is applied. Given a non-diagonal matrix A and controllers $\underline{G}_1$ and $\underline{G}_2$ with

$$|\lambda(\underline{A}-\underline{G}_i)| < 1 \ , \qquad i=1,\ 2$$

it can be shown by counterexample that

$$\underline{A}-b\underline{G}_1-(1-b)\underline{G}_2 \qquad b\in[0,1] \tag{4.6}$$

is <u>not</u> necessarily a stable matrix. It <u>may</u> be possible to overcome this problem by using the fact that $\underline{r}_1(k) \to 0$ as $k \to \infty$ which restricts the kinds of state interactions which can occur in Expression (4.6). It should be pointed out that assumption (3) can be relaxed to any situation in which the convex combination of stable matrices is stable. For example, if $\underline{A}$ and all of the $\underline{A}_i$ are in Jordan form with identical Jordan structure, then our result holds. This is equivalent to assuming that the shapes of the modes of the system are known but that the eigenvalues are not. This is less restrictive than the diagonal case.

4. Note that no claim is made about the convergence of the probability even when one model matches the true system. This is, of course, another example of the need for persistently exciting inputs in order to guarantee the convergence of identification algorithms [36].

4.4 An Analysis of the Probability Equation

The simulations of Section 3.2 have indicated that the behavior of the probability often resembles the output of a switch, alternately taking on values near zero and near one but seldom in between. By now, it should be clear that this property is largely determined by the equation for the probability, Equation (2.22b). Therefore, the present section contains a detailed examination of the characteristics of this equation. The principal conclusion is that the equation for the probability can be rewritten as a scalar, static nonlinearity and a summation (the log likelihood ratio). This decomposition aids the analysis since attention can be focused on each part separately. Thus, this section examines the characteristics of the nonlinearity and shows that the switch-like behavior of the probability is largely due to this nonlinearity. Section 4.5 continues the analysis by examining in detail how the true system and set of models affect the log likelihood ratio and therefore, by the analysis of this section, the probability and ultimately the closed loop behavior.

Equation (2.22b) is repeated for convenience:

$$P(k+1) = \frac{P(k)p(\underline{r}_1)}{P(k)p(\underline{r}_1) + (1-P(k))p(\underline{r}_2)} \tag{2.22b}$$

where $p(r_i) = \beta_i e^{-\frac{1}{2}[\underline{r}_i' \underline{\theta}_i^{-1} \underline{r}_i]}$. This can be rewritten as

$$P(k+1) = \frac{P(k)\gamma(k+1)}{P(k)\gamma(k+1) + (1-P(k))} \tag{4.7}$$

where $\gamma(k)$ is the instantaneous likelihood ratio

$$\gamma(k) = \frac{p(r_1(k))}{p(r_2(k))} = \frac{\beta_1 e^{-\frac{1}{2}[\underline{r}_1'\underline{\theta}_1^{-1}\underline{r}_1]}}{\beta_2 e^{-\frac{1}{2}[\underline{r}_2'\underline{\theta}_2^{-1}\underline{r}_2]}}$$

$$= \tilde{\beta}\, e^{-\frac{1}{2}[\underline{r}_1'\underline{\theta}_1^{-1}\underline{r}_1 - \underline{r}_2'\underline{\theta}_2^{-1}\underline{r}_2]} \quad . \tag{4.8}$$

It is now possible to rewrite Equation (4.7) such that the probability does <u>not</u> appear recursively:

$$P(k) = \frac{P(0) \prod_{i=1}^{k} \gamma(k)}{P(0) \prod_{i=1}^{k} \gamma(k) + (1-P(0))} \quad . \tag{4.9}$$

Finally, let

$$\alpha(k) = \sum_{i=1}^{k} [\underline{r}_1'\underline{\theta}_1^{-1}\underline{r}_1 - \underline{r}_2'\underline{\theta}_2^{-1}\underline{r}_2] \quad . \tag{4.10}$$

Thus, Equation (4.9) becomes

$$P(k) = \frac{P(0)\tilde{\beta}e^{-\frac{1}{2}\alpha(k)}}{P(0)\tilde{\beta}e^{-\frac{1}{2}\alpha(k)} + (1-P(0))} \quad . \tag{4.11}$$

Note that $\alpha(k)$ is the <u>Log Likelihood Ratio.</u> Thus, Equation (4.11) provides the connection between the Log Likelihood Ratio and the probability. Figure 4.1a is a plot of $P(k)$ versus $\alpha(k)$ for a few values of $P(0)$ and $\tilde{\beta} = 1$. Figure 4.1b is a detail of the same curve for $P(0) = 1/2$. <u>Define</u> α_s to be the value of $\alpha(\cdot)$ for which $P(\cdot) = 1/2$. It is then clear that

$$\begin{aligned} \alpha(\cdot) \gg \alpha_s &\Rightarrow P(\cdot) \approx 0 \\ \alpha(\cdot) \ll \alpha_s &\Rightarrow P(\cdot) \approx 1 \quad . \end{aligned} \tag{4.12}$$

Thus, α_s will be called the <u>switch point</u> for $\alpha(\cdot)$. Equation (4.11) can be solved to give

$$\alpha_s = -2 \ln\left[\frac{1-P_0}{P_0\beta}\right] \quad . \tag{4.13}$$

Now, consider Equation (4.11) evaluated in the vicinity of α_s. For example, if

$$\alpha(\cdot) = \alpha_s + 2\varepsilon$$

where ε is any positive or negative number we see that the resulting probability is

$$P(\cdot) = \frac{e^{-\varepsilon}}{1 + e^{-\varepsilon}}$$

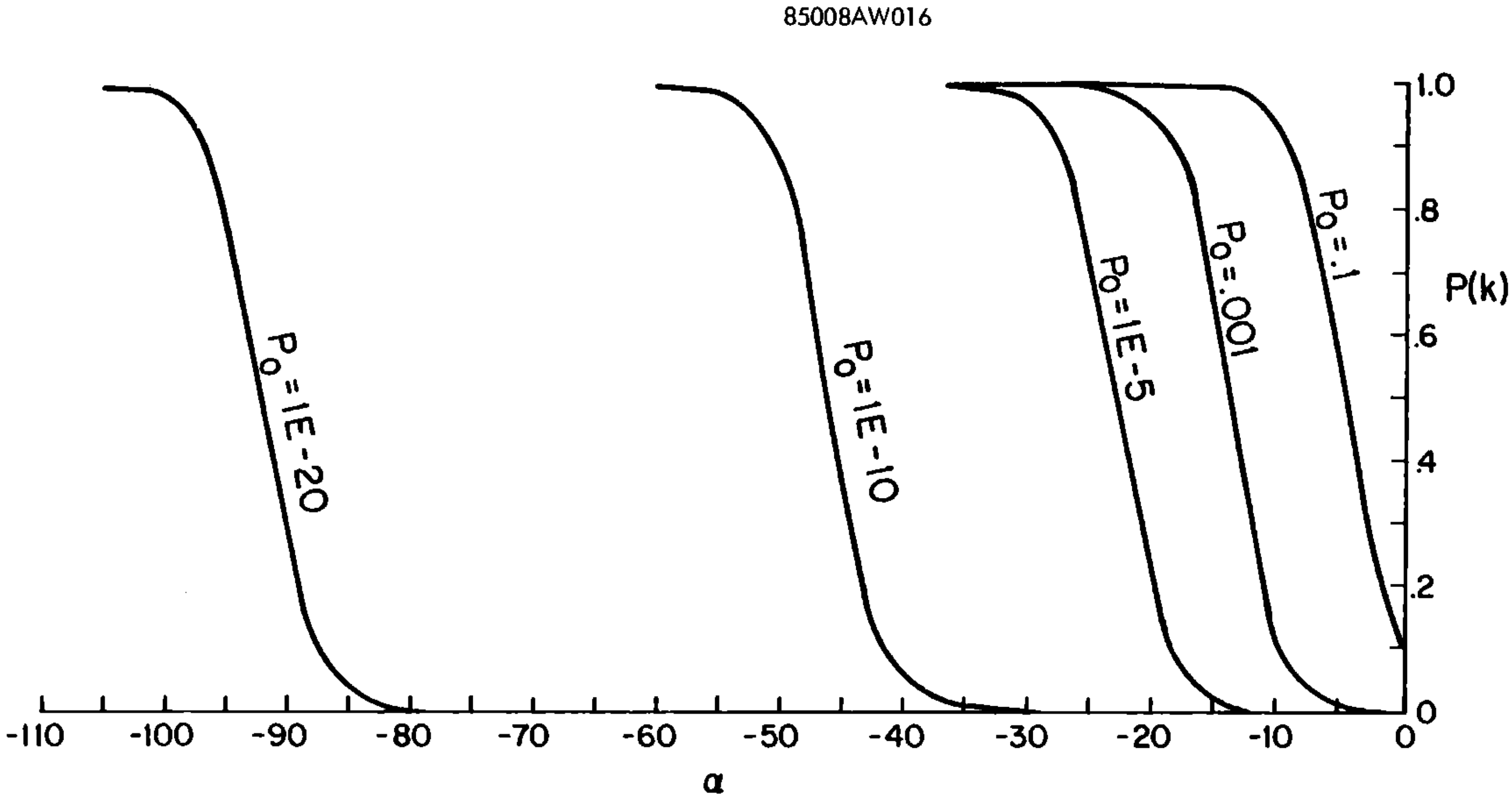

Fig. 4.1 Plot of P(·) versus α(·)

a) Various Values of P_0

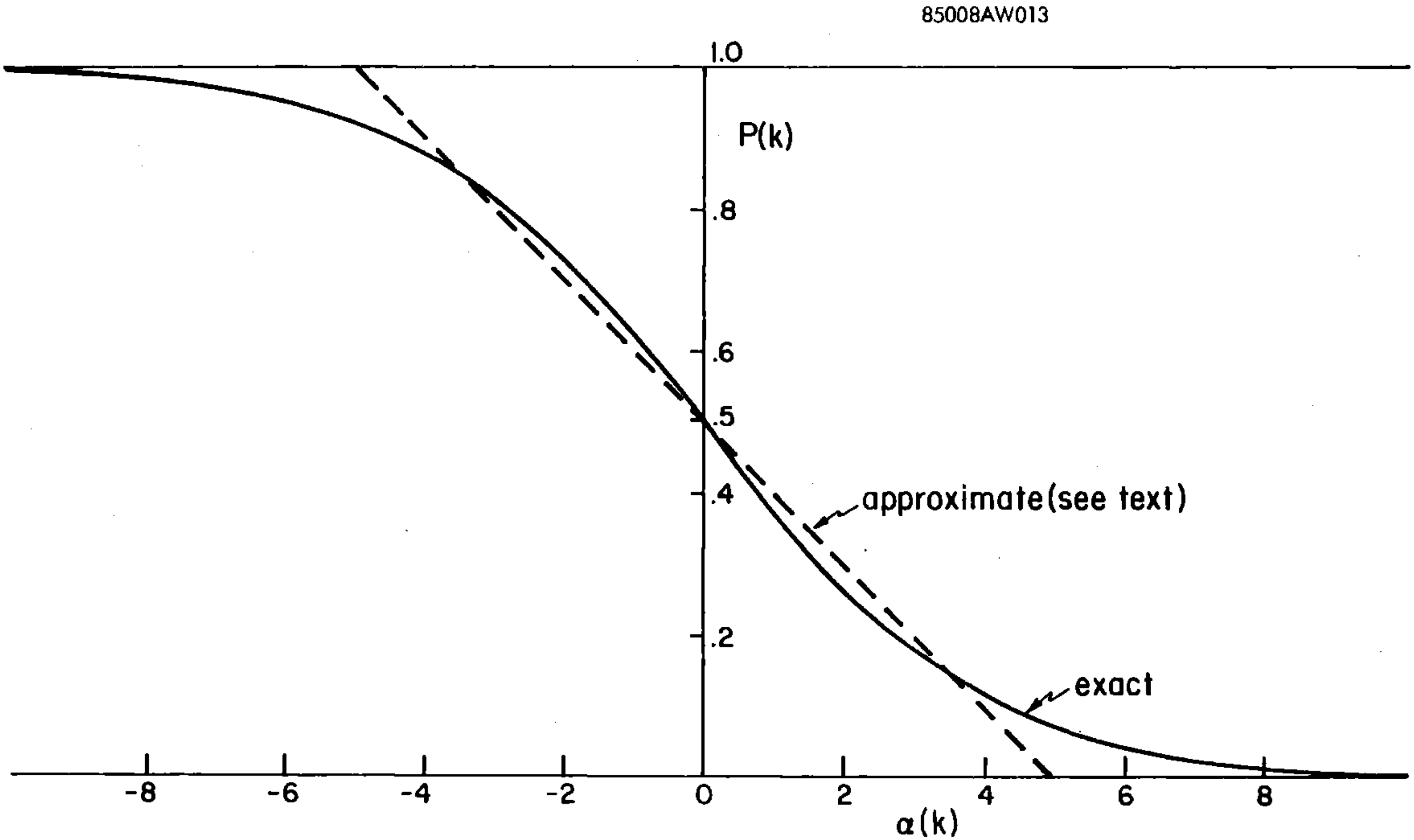

Fig. 4.1 Plot of P(•) versus α(•)

b) Detail of $P_0 = 1/2$

which is totally indpendent of <u>any</u> parameters of the MMAC method. Finally, simple calculations reveal that the curve of $P(\cdot)$ versus $\alpha(\cdot)$ is anti-symmetric about the point α_s. Thus we see that the switch point α_s is determined solely by the apriori information $P(0)$ and $\tilde{\beta}$ and that <u>no</u> parameters of either the true system or the models (except for the assumed noise covariances contained in $\tilde{\beta}$) affect the switching characteristics of the system other than by determining $\alpha(k)$. Various approximations to $P(\cdot)$ can thus be calculated from $\alpha(\cdot)$. One example, shown is the dotted line in Figure 4.1b, is

$$P(k) = \begin{cases} 1 & \alpha(\cdot) < \alpha_s - 5 \\ 0 & \alpha(\cdot) > \alpha_s + 5 \\ .5-(\alpha(\cdot) - \alpha_s)/10 & \text{otherwise.} \end{cases} \tag{4.14}$$

Examination of Equation (4.11) or Figure 4.1 reveals the reason for the zero-one type behavior so often noted in the simulation results in Chapter 3. Only for values of $\alpha(\cdot)$ such that

$$|\alpha(\cdot) - \alpha_s| < 5$$

will an intermediate value of P occur. As seen in Equation (4.10) (see also Section 4.1), $\alpha(\cdot)$ is proportional to the <u>square</u> of the norm of $\underline{r}_i(\cdot)$. Thus, for $\underline{r}_i(\cdot)$ large, $\alpha(\cdot)$ tends to change at a rate <u>twice</u> as large as that for $||\underline{r}_i(\cdot)||$ and may never fall in the range.

One potential advantage of using Equation (4.11) in a MMAC controller is that a wealth of information has been accumulated about the behavior of the log likelihood ratio [34, 35]. Thus, use of this approach allows full use of this information while still permitting a simple control to be calculated.

The approximation (4.14) is also useful for characterizing the length of the half-periods of the oscillations observed in Section 3.2. By the definition of α_s, it is clear that jumps in the probability (i.e., from 0 to 1 or from 1 to 0) correspond to transitions of $\alpha(\cdot)$ through α_s. It is thus possible to calculate a bound on the half-period of an oscillation. Define T_1 to be the time of a transition in the probability (assume from $P=0$ to $P=1$) and (T_1+T_2) to be the next transition. (see Figure 4.2). Thus, by the above analysis and the definition of α_s

$$\alpha(T_1) = \alpha_s = \alpha(T_1 + T_2) \quad . \tag{4.15}$$

By the definition of $\alpha(\cdot)$

$$\begin{aligned} &\sum_{i=1}^{T_1} [\underline{r}_1'(i)\underline{\theta}_1^{-1}\underline{r}_1(i) - \underline{r}_2'(i)\underline{\theta}_2^{-1}\underline{r}_2(i)] \\ &= \sum_{i=1}^{T_2+T_1} [\underline{r}_1'(i)\underline{\theta}_1^{-1}\underline{r}_1(i) - \underline{r}_2'(i)\underline{\theta}_2^{-1}\underline{r}_2(i)] \end{aligned} \tag{4.16}$$

or

$$\sum_{i=T_1+1}^{T_2+T_1} [\underline{r}_1'(i)\underline{\theta}_1^{-1}\underline{r}_1(i) - \underline{r}_2'(i)\underline{\theta}_2^{-1}\underline{r}_2(i)] = 0 \ . \tag{4.17}$$

Equation (4.17) thus provides a condition which T_1 and T_2 must satisfy. This equation is explored further in the next section where an approximation for $\underline{r}_i(\cdot)$ is employed.

This section has presented a detailed analysis of the probability equation (2.22). It has been shown that the equation can be broken down into a static nonlinearity and a summation (the log likelihood ratio). Attention has then been focused on the static nonlinearity.

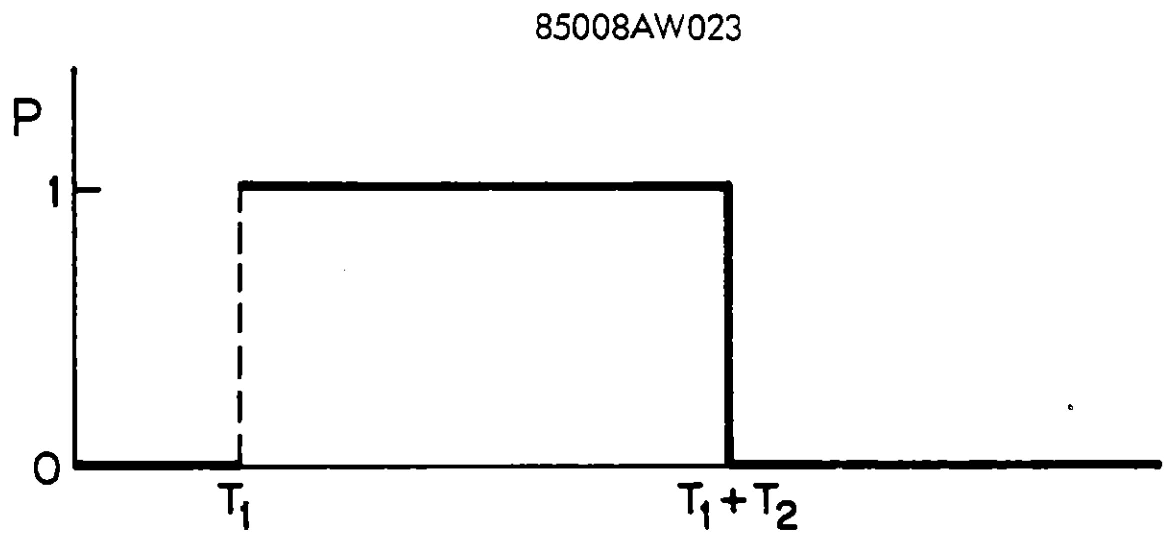

Fig. 4.2 Switching Times

The next section continues the analysis by considering the behavior of $\alpha(\cdot)$ for one class of systems.

4.5 An Analysis of the Oscillatory Behavior

Many of the simulations of Section 3.2 display an oscillatory behavior involving all states and the probability. The preceeding section contains an analysis of the equation for the probability of a model, one of the conclusions of which is an observation that Equation (2.22b) can be divided into two parts; one containing a static nonlinearity (4.10) and a relatively simple summation

$$\alpha(k) = \sum_{i=1}^{k} ||\underline{r}_1(k)||^2_{\underline{\theta}_1^{-1}} - ||\underline{r}_2(k)||^2_{\underline{\theta}_2^{-1}}$$
$$= \sum_{i=1}^{k} \underline{r}_1(k)\theta_1^{-1}\underline{r}_1(k) - \underline{r}_2(k)\theta_2^{-1}\underline{r}_2(k) \quad . \qquad (4.15)$$

Furthermore, the approximations

$$\alpha(\cdot) >> \alpha_s \Rightarrow P(\cdot) \approx 0$$
$$\alpha(\cdot) << \alpha_s \Rightarrow P(\cdot) \approx 1 \qquad (4.11)$$

have been found to be very good for $|\alpha(\cdot) - \alpha_s|$ greater than about 5 where α_s can be determined solely in terms of $P(0)$ and $\tilde{\beta}$. Note that for the present study we can take $\alpha_s = 0$ since we can assume that the system started in the remote past in which case $P(0)$ can be chosen arbitrarily.

The present section continues the analysis of the oscillatory behavior of the probability by considering the behavior of the variable $\alpha(k)$ for the canonical problem of Section 3.1. The approach taken is to isolate the basic modes of the state of the canonical problem by partitioning w in a new fashion. This partitioning emphasizes the two basic modes inherent in the structure, namely those associated with true state #1 and those associated with true state #2. Because of the diagonal nature of the canonical system, this partitioning allows $\alpha(\cdot)$ to be bounded by simple exponentials which are then analyzed.

In this section extensive use has been made of the observation from Chapter 3 that during periods of oscillatory behavior the probability tends to be virtually constant for long periods of time and then abruptly changes. This square-wave like behavior, clearly seen in for example Figure 3.2, is a key element of the approximations employed in this section.

Recall from Chapter 2 that

$$\underline{w}(k+1) = \underline{\tilde{A}}(P)\underline{w}(k) \tag{2.22}$$

where $\underline{w}'(\cdot) = [\underline{x}'(\cdot) \quad \underline{r}_1'(\cdot) \quad \underline{r}_2'(\cdot)]$. As seen in Section 3.2, during periods of oscillatory behavior, each component of $\underline{w}(\cdot)$ is alternately stable and then unstable. It thus is natural to regroup the states such that components which are simultaneously stable are grouped together. To this end, define $\underline{y}_1(\cdot)$ to be those states of $\underline{w}$ which correspond to the first component of the true state and $\underline{y}_2(\cdot)$ those which correspond to the second. That is,

$$\underline{y}_1(\cdot) = \begin{bmatrix} x_1 & (\cdot) \\ r_1^{(1)} & (\cdot) \\ r_2^{(1)} & (\cdot) \end{bmatrix} \quad ; \quad \underline{y}_2(\cdot) = \begin{bmatrix} x_2 & (\cdot) \\ r_1^{(2)} & (\cdot) \\ r_2^{(2)} & (\cdot) \end{bmatrix}$$

where $r_j^{(i)}(\cdot)$ is the $i^{\underline{th}}$ component of the residual of the KF for the $j^{\underline{th}}$ model and $x_i(\cdot)$ is the $i^{\underline{th}}$ component of the true state $\underline{x}$. Using this decomposition Equation (2.22) can be rewritten as

$$\begin{bmatrix} \underline{y}_1(k+1) \\ \underline{y}_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{\underline{A}}_1(P) & \underline{0} \\ \underline{0} & \tilde{\underline{A}}_2(P) \end{bmatrix} \begin{bmatrix} \underline{y}_1(k) \\ \underline{y}_2(k) \end{bmatrix} \tag{4.18}$$

where $\tilde{\underline{A}}_1(P)$ and $\tilde{\underline{A}}_2(P)$ contain the appropriate elements of $\tilde{\underline{A}}(P)$. The block diagonal nature of Equation (4.18) is a <u>direct</u> consequence of the assumption that the true system and each model is diagonal. If $\underline{\phi}_1$ and $\underline{\phi}_2$ are defined to be the appropriate partitioned versions of $\underline{\theta}_1^{-1}$ and $\underline{\theta}_2^{-1}$, Equation (4.15) becomes

$$\alpha(k) = \left[\sum_{i=1}^{k} \underline{y}_2'(0) \prod_{j=1}^{i} \tilde{\underline{A}}_2'(P(j))\underline{\phi}_2 \prod_{j=1}^{i} \tilde{\underline{A}}_2(P(j))\underline{y}_2(0) - \right. \tag{4.19}$$

$$\left. \underline{y}_1'(0) \prod_{j=1}^{i} \tilde{\underline{A}}_1(P(j))\underline{\phi}_1 \prod_{j=1}^{i} \tilde{\underline{A}}_1(P(j))\underline{y}_1(0) \right] .$$

We will consider the initial conditions shown in Figure 4.3(a) in which $\alpha(0) < 0$ (but $\alpha(-1) > 0$), $||\underline{y}_1(0)||^2 > ||\underline{y}_2(0)||^2$ and $P = 1$. Note that this corresponds to the case in which a change in probability (from $P = 0$ to $P = 1$) has just occurred. The equation (4.19) and the simulations of Section 3.2 indicate that if oscillatory behavior occurs, the trajectories of $||\underline{y}_i(\cdot)||^2$, $\alpha(\cdot)$ and $P(\cdot)$ must be as sketched in Figure 4.3, since $||\underline{y}_1(\cdot)||^2$ is decreasing by design for $P(\cdot) \approx 1$.

It will be assumed for the present analysis that P can be closely approximated by either 0 or 1. As seen in the previous section, this corresponds to assuming that $|\alpha(\cdot) - \alpha_s|$ is greater than about 5. From the simulations of Chapter 3, this is a reasonable approximation during periods of oscillatory behavior such as are being analyzed here.* Furthermore, it will be assumed that both K.F.'s have been correctly initialized (equivalently, have been running since the remote past) so that the matched K.F. residuals are zero.

The remainder of this section is concerned with characterizing the peak excursions of $||y_i||$ (see Figure 4.3) and also the half-period T_1. This is done by approximating the behavior of $\alpha(\cdot)$ after first approximating the behaviors of $\underline{y}_i'(0)\underline{\tilde{A}}_i'^k\underline{\phi}_i\underline{\tilde{A}}_i^k(1)\underline{y}_i(0)$ (see Equation 4.19). It is clear that each of these terms must behave as exponentials for $P = 1$; that is

$$||\underline{y}_i(k)||^2 \leq a_i^k \; ||\underline{y}_i(0)||^2 \tag{4.20}$$

where $a_i = ||\underline{\tilde{A}}_i(1)||^2$. Thus

$$\alpha(k) \approx \sum_{i=0}^{k} [a_2^i \; ||\underline{y}_2(0)||^2\sigma_2 - a_1^i \; ||\underline{y}_1(0)||^2\sigma_1] \tag{4.21}$$

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*When the probability is forced to either zero or one (by, for example, use of a maximum likelihood type control rather than the probabilistically weighted one) then this is not an approximation.

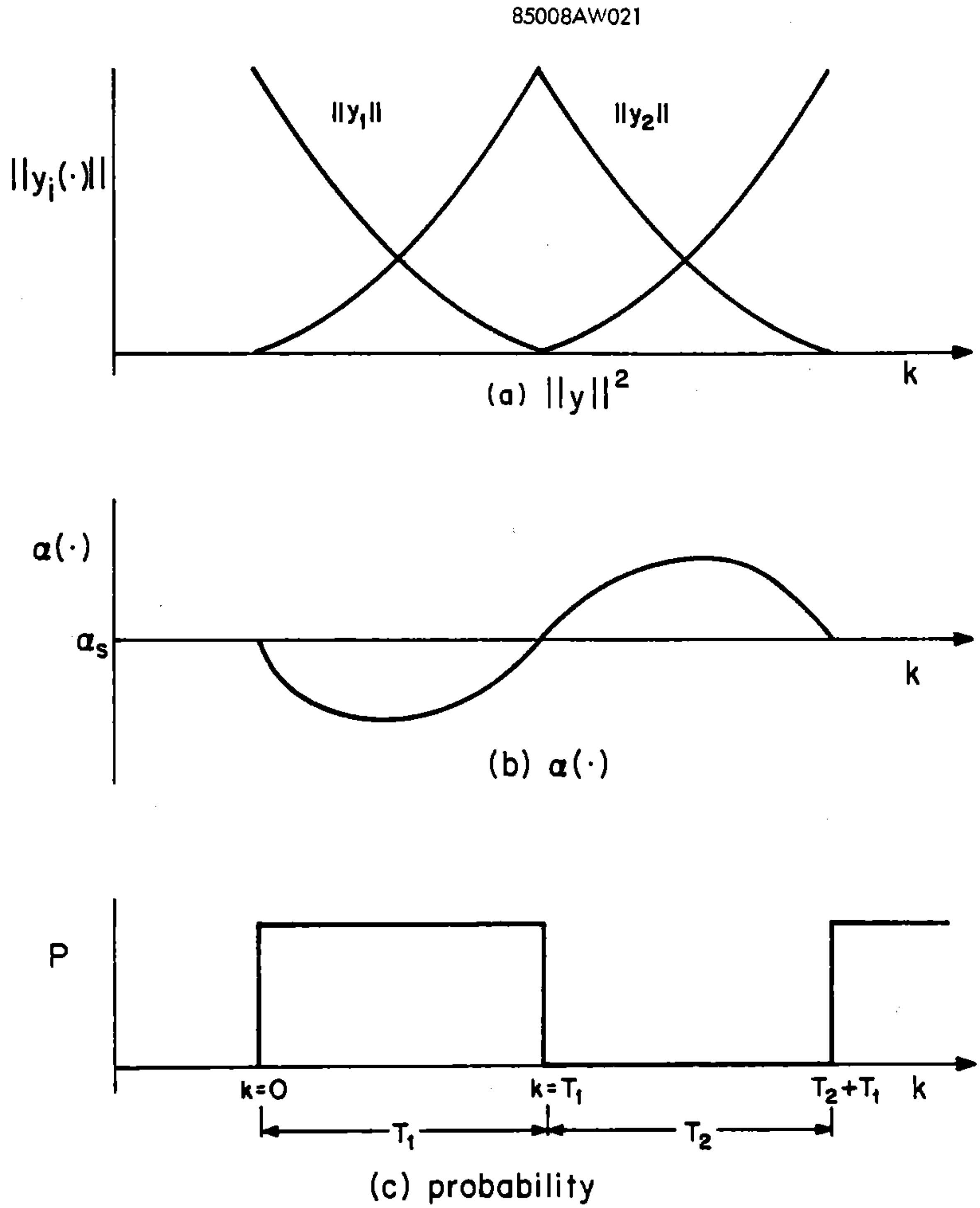

Fig. 4.3 Typical Oscillation

where σ_i = the maximum eigenvalue of ϕ_i.* A few comments are necessary on this approximation. First of all, note that for $P = 1$ it can be shown that

$$\begin{aligned} a_1 &< 1 \\ a_2 &> 1 \end{aligned} \tag{4.22}$$

by the constructions of $\underline{\tilde{A}}_i(\cdot)$. Furthermore, as noted earlier,

$$||\underline{y}_2(0)||^2 << ||\underline{y}_1(0)||^2 \quad .$$

Thus, during the initial part of the interval, the term in $y_1(\cdot)$ dominates while later the term in $y_2(\cdot)$ dominates. We are interested in determining the half-period, i.e., when the probability switches from 1 to 0. As seen in Section 4.4, this occurs when $\alpha(\cdot) = 0$. Thus, an approximation to this condition is to find T_1 such that

$$\sum_{i=0}^{T_1} a_2^i \; ||\underline{y}_2(0)||^2 \approx \sum_{i=0}^{T_1} a_1^i \; ||\underline{y}_1(0)||^2 \quad . \tag{4.23}$$

It should be pointed out that the term in $\underline{y}_2$ leads to an underestimation of T_1 while the term in $\underline{y}_1$ leads to an overestimation. Thus, Equation (4.23) does not yield either an upper or lower bound. However, it is felt that this is still of value since the analysis yields insight into the type of behavior observed in simulations. Furthermore, although sometimes not of use, the T_1 of Equation (4.23) does lie between the upper and lower bound.

*By the symmetry of the canonical problem, note that $\sigma_1 = \sigma_2$.

Evaluating the summations in Equation (4.23) yields

$$||y_1(0)||^2 \left[\frac{a_1^{(T_1+1)} - 1}{a_1 - 1}\right] \approx ||\underline{y}_2(0)||^2 \left[\frac{a_2^{(T_1+1)} - 1}{a_2 - 1}\right] \qquad (4.24)$$

where by the assumption that on the interval $k=0$ to T_1, $P \approx 1$,

$$a_1 = ||\underline{\tilde{A}}_1(1)|| < 1$$

$$a_2 = ||\underline{\tilde{A}}_2(1)|| > 1$$

and use has been made of the fact that $\sigma_1 = \sigma_2$. Equation (4.24) could be solved numerically to arrive at an approximation of T_1 although no closed-form solution is possible.

Further insight into the behavior can be obtained by invoking another assumption. Since

$$a_1 < 1$$

and $\quad a_2 > 1$

we assume

$$a_1^{T_1+1} \ll a_1$$
$$a_2^{T_1+1} \gg a_2 \qquad (4.25)$$

This is clearly reasonable since T_1 depends on a_1 and a_2 in such a way that a_1 closer to 1 yields T_1 larger. Thus, although (4.25) is an assumption, it is very reasonable. Using this, Equation (4.24) becomes

$$||\underline{y}_1(0)||^2 \frac{1}{(1-a_1)} \approx ||\underline{y}_2(0)||^2 \frac{a_2^{T_1+1}}{a_2-1} \tag{4.26}$$

or, using Equation (4.20)

$$||\underline{y}_2(T_1)||^2 \approx ||\underline{y}_1(0)||^2 \frac{(a_2-1)}{a_2(1-a_1)} \quad . \tag{4.27}$$

Defining

$$R = \frac{(a_2-1)}{a_2(1-a_1)} \tag{4.28}$$

yields

$$||\underline{y}_2(T_1)||^2 \approx R\, ||\underline{y}_1(0)||^2 \; . \tag{4.29}$$

Thus, if $R > 1$, the value of $||y_i(\cdot)||$ when P switches increases with each cycle and $||y_i(\cdot)|| \to \infty$ (i.e., the closed loop system is unstable). Similarly, if $R < 1$, then peaks of $||y_i(\cdot)||$ decrease with each cycle.

An interesting interpretation of these results can now be given by noting that

$$a_1 a_2 > 1 \Longrightarrow R > 1 \tag{4.30a}$$

$$a_1 a_2 < 1 \Longrightarrow R < 1 \tag{4.30b}$$

$$a_1 a_2 = 1 \Longrightarrow R = 1 \tag{4.30c}$$

and that a_1 is the slowest decaying mode of the stable matrix $\tilde{\underline{A}}_1(1)$ and a_2 is the fastest growing (most unstable) mode of $\tilde{\underline{A}}_2(1)$. Thus the closed loop system will be stable if the slowest decaying stable mode of the matching subsystem $(\tilde{\underline{A}}_1(1))$ dominates the unstable modes of the mismatched system in the sense that the product of the appropriate norms is less than 1.

Equation (4.26) can also be rewritten as

$$a_2^{T_1} \approx \frac{||\underline{y}_1(0)||^2}{||\underline{y}_2(0)||^2} R \quad . \qquad (4.31)$$

Thus, the following approximation to T_1 can be given

$$T_1 \approx \ln R\left[\frac{||\underline{y}_1(0)||^2}{||\underline{y}_2(0)||^2}\right] / \ln a_2 \quad . \qquad (4.32)$$

An interesting phenomena can be seen by considering multiple periods. Thus, by analogy with Equation (4.32)

$$T_2 \approx \ln\left[\frac{R||\underline{y}_2(T_1)||^2}{||\underline{y}_1(T_1)||^2}\right] / \ln a_2 \quad .$$

But it can be shown that

$$\frac{||\underline{y}_2(T_1)||^2}{||\underline{y}_1(T_1)||^2} \approx \frac{R^2}{a_1^{T_1} a_2^{T_1}} \frac{||\underline{y}_1(0)||^2}{||\underline{y}_2(0)||^2} \quad .$$

This results in the conclusion that if the initial period is sufficiently long, then subsequent periods will be of increasing length if $a_1 a_2 < 1$.

The above analysis results in the conclusions that $a_1 a_2$ is the key quantity in determining the stability of the closed loop system. If R is less than one, then the peaks of the state trajectories decrease and the period increases. However, if R is greater than one, instability results. These conclusions agree well with the simulation results of Chapter 3. The only point of discrepancy is in the simulation of Figure 3.2. In that case, the peaks appear to remain constant even though calculations reveal that stability results. Close examination of the results of the simulation have revealed that in this case the lower limit on the probability (see Chapter 3) is consistently achieved. As shown in Chapter 5, this somewhat modifies the behavior resulting in uniform peaks even when the system is stable. Further discussion of this important case is deferred to Chapter 5.

In summary, the stability of the MMAC algorithm is determined by the relation between the growth rate of the most unstable mode of the mismatched subsystem and the rate of decay of the slowest stable mode of the matched subsystem. This leads to the qualitative conclusion that stability results when the stable modes are "faster" than the unstable modes. This concept is further emphasized in the following section where an alternate view of the problem leads to a similar result.

4.6 Quasi-Lyapunov Analysis

An alternate view of the stability results given in the previous section can be obtained by employing an analysis which closely resembles the methods of Lyapunov. The results obtained compliment those of Section 4.5 and given increased insight into the behavior of the MMAC method.

The approach of this section is first to explore the use of the "normal" quadratic Lyapunov Function. This is used to introduce the ideas and also to demonstrate the drawbacks of this standard approach. This leads to the introduction and motivation of a new Lyapunov-like function which is then used to derived a stability result which emphasizes the role of the individual control gains. This function is motivated largely by the simulation results in which the controller is seen to alternately attempt to control each mode of the system. Thus, each state tends to be piece-wise exponential (see Chapter 3). This results in a phase-plane plot, such as seen in Figure 3.5 which resembles a hyperbola. It is this observation which led to the function investigated in the major part of this section.

First, consider the function

$$v(k) = \underline{w}'(k)\ \underline{w}(k) \tag{4.33}$$

where w(k) is governed by Equation (2.22)

$$\underline{w}(k+1) = \underline{\tilde{A}}(P)\ \underline{w}(k) \quad . \tag{2.22}$$

Thus, v(k+1) can be written as

$$v(k+1) = \underline{w}'(k)\underline{\tilde{A}}'(P\ (k))\underline{\tilde{A}}(P(k))\underline{w}(k) \quad . \tag{4.34}$$

In order to prove stability, it is required to show that

$$v(k) > v(k+1) \quad . \tag{4.35}$$

Unfortunately, Equation (4.34) depends strongly on P and, thus stability is difficult to show. The one case in which it is possible to determine stability is when $||\tilde{A}(P)|| < 1 \ \ \forall\, P$. However, this case has been well discussed in Section 4.2 and so will not be repeated.

An improved result is obtained if attention is first restricted to the diagonal case. Thus, <u>assume</u> that the canonical system of Section 3.1 is under consideration. Define the function

$$v(k) = x_1(k)x_2(k) \tag{4.36}$$

or

$$v(k) = \frac{1}{2}\underline{w}'(k)\tilde{I}\underline{w}(k) \tag{4.37}$$

where

$$\tilde{I} = \left[\begin{array}{cc|c} 0 & 1 & \underline{0} \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{\underline{0}} & \underline{0} \end{array}\right] .$$

A definition is useful in the sequel: The system (2.22) is <u>hyperbolically stable</u> if $v(k) \to 0$. The choice of $v(k)$ is heavily motivated by the simulation results in Chapter 3. As discussed in Section 3.2, the phase-plane plot of x_1 vs. x_2 resembles a hyperbola (truncated for large excursions). Figure 3.5 graphically depicts this phenomenon. A second motivation stems from the analysis in the previous sections of this chapter in which each state tends to be governed by an equation such as

$$||x_i(k+1)|| = a^k||x_i(k)|| \tag{4.38}$$

for periods of time. This, of course, is due to the fact that the probability tends to be piece-wise constant due to the nonlinearities analyzed in Section 4.4.

It should be pointed out that the function in Equation (4.36) is <u>not</u> a valid Lyapunov function. First of all, $v(k)$ can be either positive or negative depending on the initial conditions. Secondly, $v(k) = 0$ does <u>not</u> imply that x_1 is even <u>bounded</u>! Neither of these invalidate the analysis, however. A slight redefinition of $v(\cdot)$ would overcome the first. This is not explicitly done here because it is not necessary for the problem at hand and would complicate the analysis. The bounding of the state has already been indicated in previous sections and thus is provided by alternate means. Furthermore, even if a bound were not available, the analysis would yield insight into how the system behaves when each mode is alternately controlled.

Some useful notation for the diagonal system is now needed. Recall from Section 3.1 that

$$\underline{A} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} ; \quad \underline{A}_1 = \begin{bmatrix} a & 0 \\ 0 & \hat{a} \end{bmatrix} ; \quad \underline{A}_2 = \begin{bmatrix} \hat{a} & 0 \\ 0 & a \end{bmatrix}$$

Define

$$\underline{G}_1 = \begin{bmatrix} g & 0 \\ 0 & \hat{g} \end{bmatrix} ; \quad \underline{G}_2 = \begin{bmatrix} \hat{g} & 0 \\ 0 & g \end{bmatrix}$$

and

$$\underline{H}_1 \begin{bmatrix} h & 0 \\ 0 & \hat{h} \end{bmatrix} ; \quad H_2 = \begin{bmatrix} \hat{h} & 0 \\ 0 & h \end{bmatrix} .$$

Temporarily, assume that $\hat{g} = 0$.* Then Equation (4.37) becomes

$$v(k+1) = \underline{w}'(k)\underline{\tilde{A}}'(P)\underline{\tilde{I}}\underline{A}(P)\underline{w}(k) \tag{4.39}$$

which, when written out term by term becomes

$$v(k+1) = [a(a-g) + Pg^2 - P^2g^2]\ v(k) \tag{4.40}$$

or $v(k+1) = A(P)\ v(k)$. Note that $A(P)$, which is plotted in Figure 4.4,

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*This corresponds to Case 1 in Chapter 3.

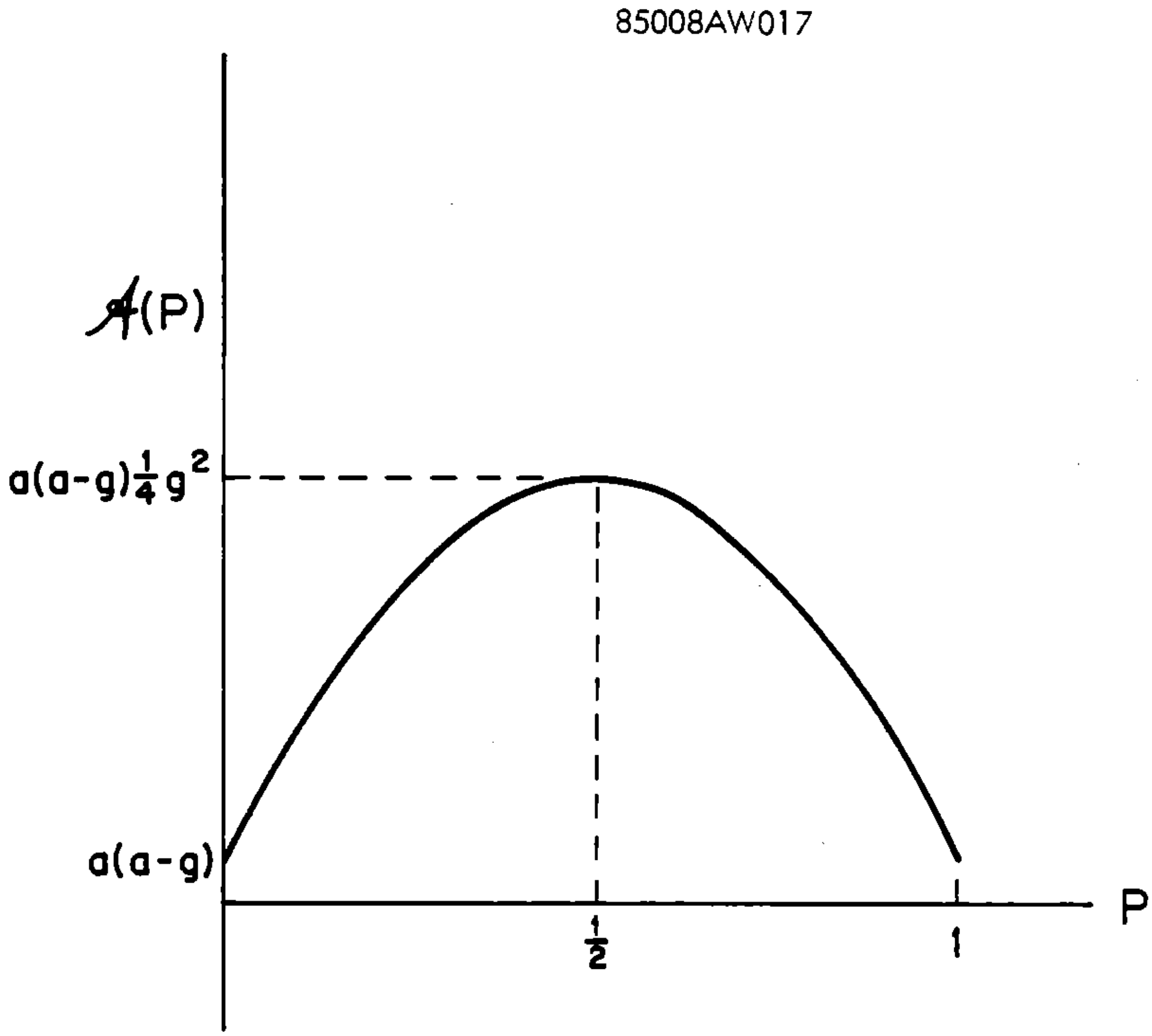

Fig. 4.4 Sketch of A(P)

is a convex quadratic function of P. The following two theorems then follow:

Theorem: Given the canonical MMAC system of Section 3.1 with $\hat{g}=0$, the overall system is hyperbolically stable (i.e., $x_1 x_2 \to 0$) if the following two conditions are met.

$$(1) \quad |a(a-g)| < 1 \tag{4.41}$$

and

$$(2) \quad |a(a-g) + \frac{1}{4} g^2| < 1 \; . \tag{4.42}$$

This plus the bounding results of Section 4.5 yields stability.

Theorem: Given the canonical MMAC system of Section 3.1 with $\hat{g}=0$, the overall system is unstable if

$$(1) \quad a(a-g) > 1 \tag{4.43}$$

or if

$$(2) \quad a(a-g) + \frac{1}{4} g^2 < -1 \; . \tag{4.44}$$

A few comments can now be made.

1. Conditions (4.41) and (4.42) essentially require that v(k) be decreasing for all P. Note that this can be true independent of whether $\tilde{A}(P)$ is stable. In fact, for the Case 1 example of Chapter 3, $\tilde{A}(P)$ is unstable $\forall P$ but v(k) is still decreasing.

2. Theorems such as the above can easily be derived for the relaxed case in which the two elements of the true system matrix are not equal. The curve for A(P) is then not symmetric about P = 1/2 but

similar analysis can still be done. The results do not significantly increase our understanding and so are not presented.

(3) Note that Condition (4.43) is consistent with the results summarized by Equation (4.30) of the previous section. The added condition of (4.42) results from relaxing the assumption in Section 4.5 that P is always "near" zero or one. Further insight results from plotting the Conditions (4.41) and (4.42). The shaded area in Figure 4.5 is the set of a and g for which Condition (4.41) is satisfied and the crosshatched area is where Condition (4.42) is satisfied. The points corresponding to Cases 1a, b and c from Section 3.2 are plotted.

A few comments are in order. First, as evidenced by the simulation results, it <u>may</u> be sufficient for hyperbolic stability that Condition (4.41) be satisfied. In that case, if P oscillates between zero and one (which the simulations indicate is likely), stability will result. Further, if the probability is forced to be either zero or one by using the control for the <u>most likely</u> model rather than the probabilistically weighted control, then Condition (4.41) is sufficient to guarantee stability.

Another interesting observation is that for this special case, the smallest control gain which yields $v(\cdot)$ decreasing for $P=0$ or 1 (i.e. the <u>lower</u> edge of the shaded area in Figure 4.5) exactly corresponds to the value of the control gains for the LQ problem with a state cost of <u>zero</u> [31, 32]. This implies that for this <u>special case</u> use of the LQ methodology <u>guarantees</u> that Condition (4.41) will be satisfied. This is believed to be due to the robustness properties of the LQ design

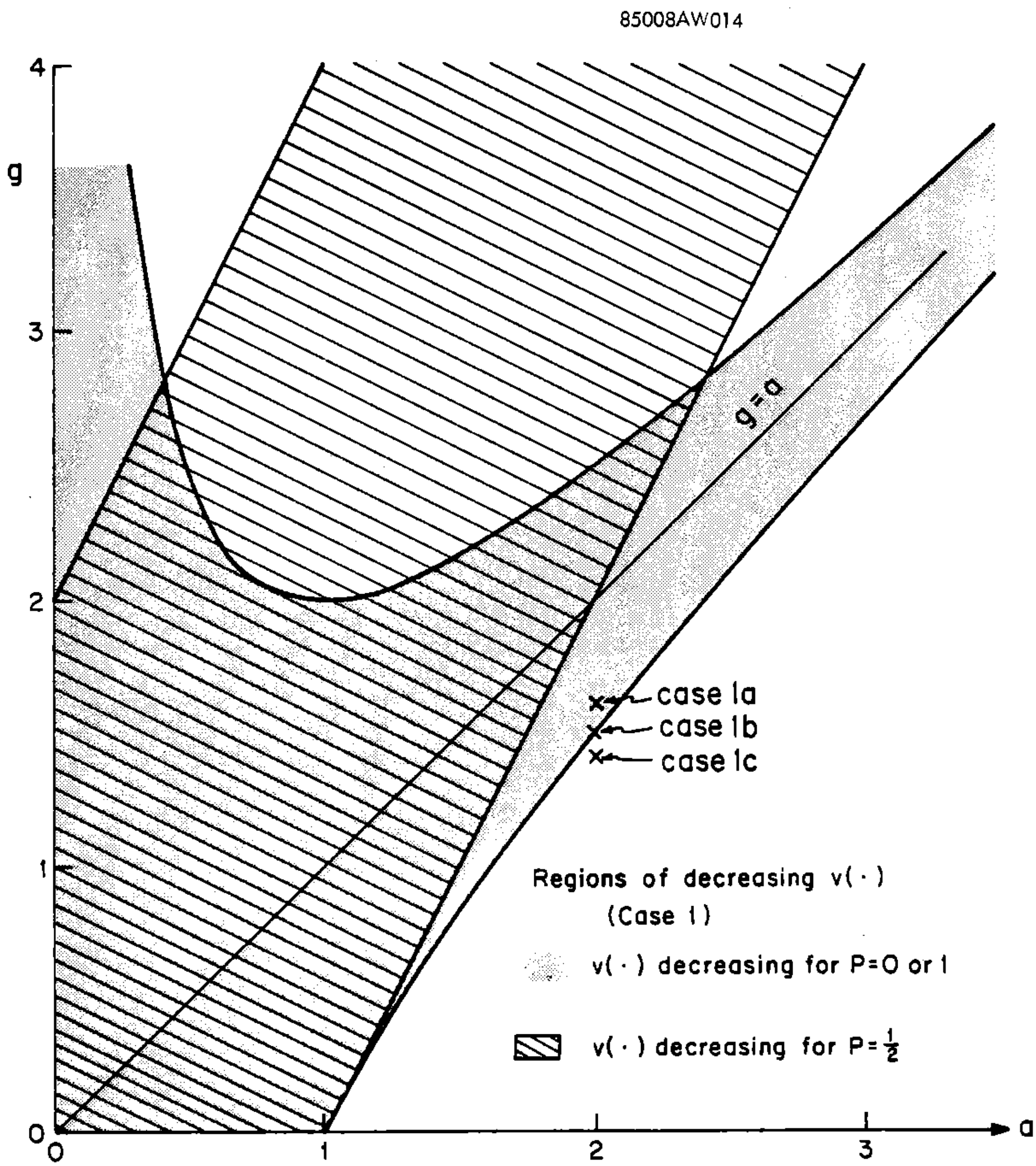

Fig. 4.5 Regions of Decreasing v(·)

technique discussed in [5, 33]. The implications to more general MMAC structures remains an open question. However, note that there are many control gains (<u>not</u> generated by an LQ methodology) which yield <u>unstable</u> behavior. For example, the matched system is stable for all control gains

$$g \in (1,2) \quad .$$

However, from Figure 4.5, if the chosen control gain is

$$g \in (1, 1.5)$$

then the resulting MMAC system is unstable.

A second point to note from Figure 4.5 is that for $a > 2$, <u>no</u> linear controller will result in $v(k)$ decreasing for $P = 1/2$. Under these conditions, the only stable behavior possible must be of the oscillatory type.

The removal of the assumption that $\hat{g} = 0$ results in the analysis becoming intractable. For example, when <u>evaluated</u> at $P = 1$, the equation corresponding to (4.40) becomes

$$\begin{aligned} v(k+1) = {} & (a-\hat{g})\ (a-g)\ v(k) + \\ & + (a-\hat{g})\ g(1-h) r_1^{(1)} x_2 \\ & + (a-\hat{g})\ \hat{g}(1-\hat{h}) r_1^{(2)} x_1 \\ & + g^2 (1-h)^2\ r_1^{(2)} r_1^{(1)} \quad . \end{aligned} \tag{4.45}$$

If it is assumed that the matched KF state has been correctly initialized (i.e., $r_1^{(1)} = 0$) this becomes

$$v(k+1) = (a-\hat{g})\ (a-g)v(k) + (a-g)\hat{g}(1-\hat{h})r_1^{(2)}x_1 \tag{4.46}$$

where the mismatched component of the KF now enters to modify v(k+1). Equation (4.46) has evaded further analysis except in a few obvious special cases (such as $\hat{k}=1$) which will not be discussed.

This section has attempted to accomplish two goals. First of all, one interesting special case has been examined to the point that both stability and instability results have been presented. These results complement those of the previous section and provide an alternate view of the mechanisms underlying the stability of the MMAC method. A possibly more valuable contribution has been to point out the need for careful design of each individual controller and the set of available models.

4.7 Domain of Attraction

In Section 4.1, it is demonstrated that if $\tilde{\underline{A}}(P)$ is a stable matrix for $P=1/2$, then the system linearized about $P=1/2$ is neutrally stable in that the probability has no tendency to return to 1/2. This is seen to be due to the fact that if $\underline{w} = 0$, then there is neither the need nor the basis for changing the probability. Equivalently, the equilibrium set is $\{\underline{w} = 0$ with any $P\}$. This is sketched in Figure 4.6. The stability of each equilibrium point has already been examined using a linearization approach. One important question which remains is to determine the set of initial conditions such that the system will return to an equilibrium point without first oscillating. (The analysis of Section 4.4 through 4.6

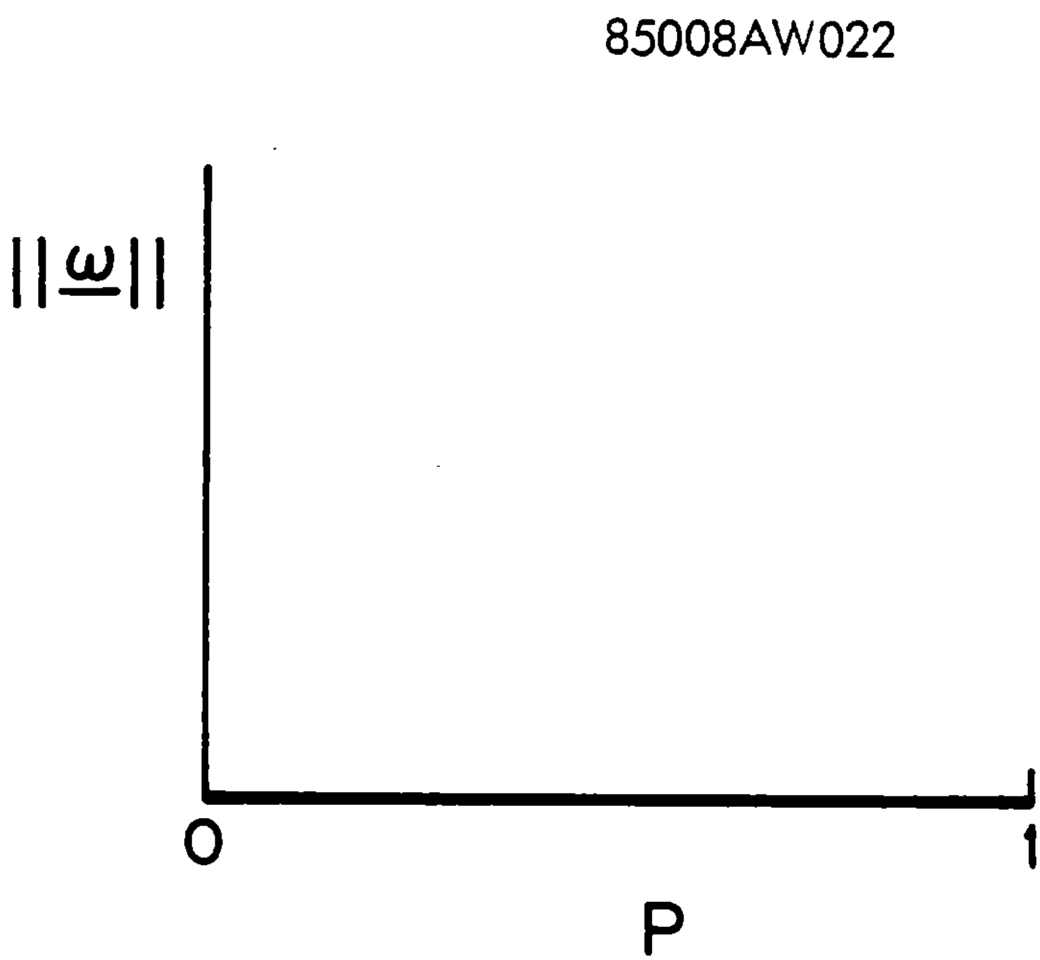

Fig. 4.6 Set of Equilibrium States

have indicated that stable oscillations may result in $\underline{w}$ approaching the equilibrium set and it should be noted that this is not the case under consideration here).

The importance of this analysis is best seen by considering the simulations of Case 2 in Chapter 3. There it is seen that if $\underline{\tilde{A}}(P)$ is a stable matrix for $P = 1/2$ but not for $P = 0$ or 1 then small initial condition perturbations result in small changes in $\underline{w}(\cdot)$ and $P(\cdot)$ whereas large perturbations result in $P(\cdot)$ going to zero or one and then oscillating at least for a while. This section continues the analysis of Section 4.1 and results in a procedure for determining a bound on $||w(0)||$ such that the probability does not oscillate.

Consider the equations from Section 3.6.1

$$\underline{w}(k+1) = \underline{\tilde{A}}(P)\ \underline{w}(k)$$
$$P(k+1) = \frac{P(k)\ p(\underline{r}_1)}{P(k)p(\underline{r}_1) + (1-P(k))p(\underline{r}_2)} \quad . \tag{2.22}$$

We will assume throughout most of this section that $P(0) = 1/2$ (i.e., $\alpha_s = 0$)* and that $\tilde{A}(\cdot)$ is such that there exists an $\varepsilon < 1/2$ such that

$$||\underline{\tilde{A}}(P)|| < 1 \qquad \forall P \ \varepsilon\ (1/2 - \varepsilon,\ 1/2 + \varepsilon)$$
$$||\underline{\tilde{A}}(P)|| \geq 1 \qquad \text{otherwise.} \tag{4.47}$$

It can be shown that this is the only case in which exponential behavior has not been investigated. In Section 4.2, the case of $||\tilde{A}(P)|| < 1 \ \ \forall P$ has been investigated, and if $||\underline{A}(P)|| > 1\ \forall P$ then Section 4.1 has

*As in Section 4.4 we assume $\tilde{\beta} = 1$.

shown that oscillatory behavior (either stable or unstable) must result. Assumption (4.47) represents the examination of a previously unstudied case. As seen in previous sections, $\underline{\tilde{A}}(P(\cdot))$ is a function of $\alpha(\cdot)$ which in turn is a function of the data (state) generated by $\underline{\tilde{A}}(P(\cdot))$. Thus, there is a close connection between bounds on $\alpha(\cdot)$ and bounds on $||\underline{\tilde{A}}(P(\cdot))||$. The basic approach of this section is to bound $\alpha(\cdot)$ in such a way that $||\underline{\tilde{A}}(P(\cdot))||$ <u>remains</u> a stable matrix. This bound on $\alpha(\cdot)$ is then translated to a bound on $||\underline{w}_0||$. That is, we bound the size of the initial conditions $\underline{w}_0$ so that $\underline{w}(k)$ is small enough so that in turn $\alpha(\cdot)$ remains within its bound. This is accomplished as follows. From Equation (4.48) it clear that

$$P \in (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon) \Rightarrow ||\underline{\tilde{A}}(P)|| < 1 \qquad (4.48)$$

and that $P \in (\frac{1}{2} - \varepsilon_1, \frac{1}{2} + \varepsilon_1) \Rightarrow ||\underline{\tilde{A}}(P)|| < a < 1$ for some $\varepsilon_1 < \varepsilon$. (Refer to Figure 4.7.) From Section 4.4, each value of $P(\cdot)$ maps to a value of $\alpha(\cdot)$ as shown in Figure 4.7 and thus attention can be focused on $\alpha(\cdot)$. Define δ to be the value of $\alpha(\cdot)$ such that $P = 1/2 - \varepsilon$ and δ_1 as the value of $\alpha(\cdot)$ for $P = 1/2 - \varepsilon_1$.

It is shown in Appendix A that

$$|\alpha(k)| \leq \frac{a_1 ||\underline{w}_0||^2}{(1 - a_1)\sigma} \qquad \forall k \qquad (4.49)$$

where a_1 is $\max_{\alpha(\cdot)} ||\underline{\tilde{A}}(P(k))||$* and σ is related to the residual weighting matrices of $\underline{\theta}_1$ and $\underline{\theta}_2$. Thus, one may choose an $\alpha(\cdot) = \delta_1$ for some δ_1,

*The maximum is achieved for the maximum value of $\alpha(\cdot)$ for the canonical problem.

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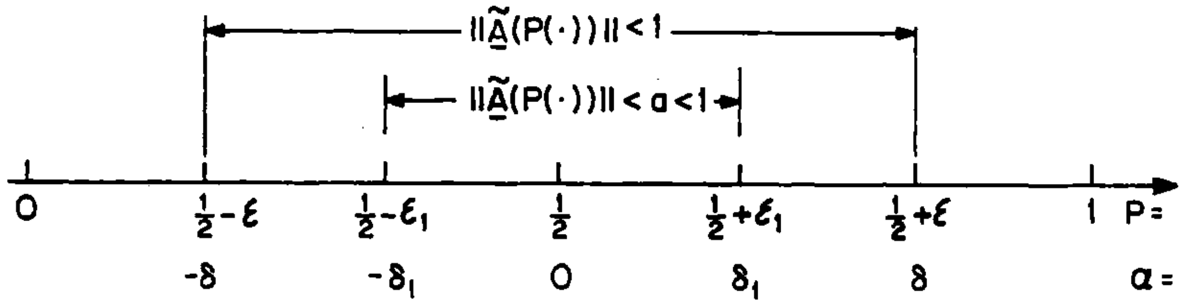

Fig. 4.7 Stability Regions for $\tilde{\underline{A}}(P)$

compute the value of $a < 1$, and finally use the equality from Equation (4.49) to find a $||\underline{w}_0||^2$ such that $\alpha(k)$ is bounded as necessary. By changing δ_1 and repeating the procedure, a different value of $||\underline{w}_0||^2$ is obtained which may be either greater or less than the first. The detailed development of the algorithm is given in Appendix A. The resulting procedure may be formalized as follows:

Step 1: Find ε such that Condition (4.48) is satisfied. This in general will require an iterative procedure for determining $||\tilde{A}(P)||$ for various values of P.

Step 2: Calculate δ from Equation (A.1).

Step 3: Choose an $0 < \alpha_1 < \delta$.

Step 4: Calculate P_1 corresponding to α_1 from Equation (4.11). Finally, compute $a = ||A(P_1)||^2$.* Note that by the selection of α in Step 3, a is less than 1.

Step 5: Calculate $||\underline{w}||^2$ from

$$||\underline{w}||^2 = \frac{\alpha\sigma(1-a)}{a} \tag{4.50}$$

Repeat steps 3 through 5 for different values of α_1 to maximize $||\underline{w}||$. Note that a maximum exists since all functions used above are continuous and $||\underline{w}||$ is minimized for $\alpha = 0$ and for $\alpha = \delta$. The best way to understand

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*Due to the nature of the canonical problem, $||A(P)||^2$ is maximized for extreme values of P. In general, greater care is needed in order to assure $||A(P)||^2 \leq a$ for all P in the region of interest.

the procedure is to verify that $||\underline{w}||$ in Equation (4.50) results in exponential behavior. To do this it is sufficient to show that $||w_0|| = ||w|| \Rightarrow |\alpha(k)| < \delta \quad \forall k$. But by construction

$$|\alpha(k)| \leq \alpha \quad \forall k$$

and so $\alpha < \delta$ by Step 3 finishes the result for $P_0 = 1/2$.

As an example of the use of this algorithm, Table 4.1 demonstrates its application to the conditions of the Case 2 simulations from Chapter 3. The table contains the evaluation of $||\underline{w}_0||^2$ for various values of α_1. It is thus possible to conclude that if $P(0) = 1/2$ and $||\underline{w}(0)||^2 \leq .236$ then exponential behavior will result. Simulations confirm this observation and indicate that $||\underline{w}(0)||^2 \approx .8$ results in P just reaching the boundary of the stability range of $\underline{\tilde{A}}(P)$.

It is possible to extend the procedure to $P_0 \in (1/2 - \varepsilon \;\; 1/2 + \varepsilon)$ as follows. Given P_0, compute α_0 from Equation (4.11). This effectively reduces the amount $\alpha(\cdot)$ can change and still have $\underline{\tilde{A}}(P(\cdot))$ stable. Thus replace δ in Steps 2 through 5 by $\delta - |\alpha_0|$. It thus becomes possible to compute a complete region in $P_0 \times ||\underline{w}_0||$ space such that a nonoscillatory response results. Figure 4.8 depicts such a region. It should be pointed out that the resulting set is not exhaustive in that points outside the set may result in nonoscillatory behavior. This is due largely to the conservative nature of Inequality (4.49) which uses a worst case estimate of $||\underline{\tilde{A}}(P(k))||$ for all k. Thus, $\alpha(k)$ tends to change slower than predicted by Inequality (4.49). The procedure does, however, yield a useful <u>lower bound</u> to the full set of initial conditions which yield nonoscillatory responses.

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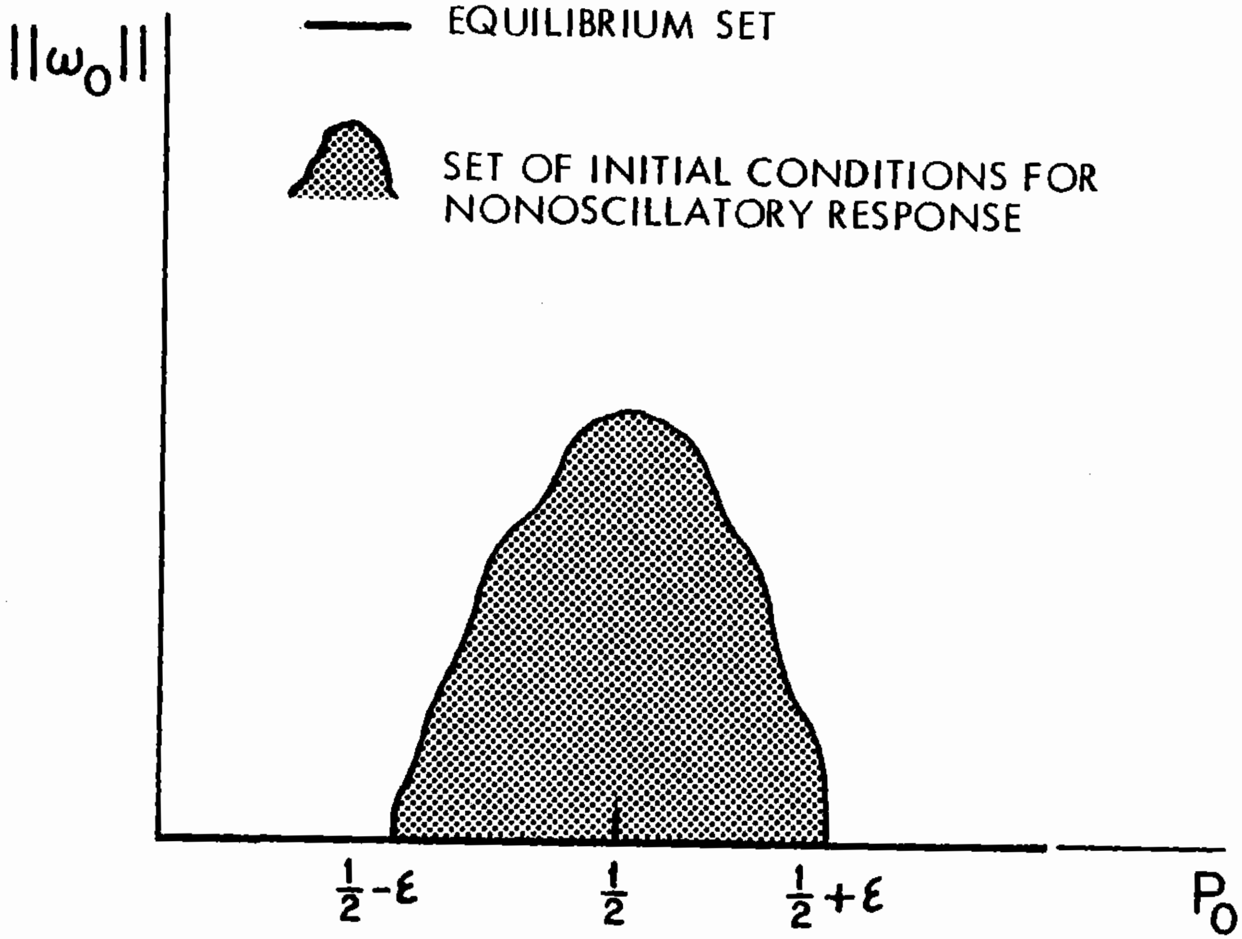

Fig. 4.8 Set of Initial Conditions for Exponential Response.

α_1	P_1	a	$\|\|w\|\|^2$
2.20	.25	.969	.1135
1.69	.3	.931	.2022
1.238	.35	.894	.236
.811	.4	.859	.2147
.401	.45	.826	.13625

TABLE 4.1

Example of Range of $||\underline{w}(0)||$ for Exponential Behavior

4.8 Summary

This chapter has presented a collection of analyses of the MMAC method. It has relied heavily on the special cases given in Section 3.1 and as such, any conclusions with respect to more general cases must be viewed with some caution. However, these special cases have been carefully selected to accentuate types of behavior observed in more general settings and so it is believed that the qualitative results we have obtained are of some general interest. Thus, the result of our analysis has yielded insight into the nature and causes of the behavior of the MMAC method. This understanding can be used to provide a basis for improving the design and for determining when it will work well. The main conclusions of this chapter are:

1) At best the MMAC system is neutrally stable about an equilibrium point in that the probability has no tendency to return to its initial value following a perturbation (see Section 4.1).

2) If $\underline{\tilde{A}}(P)$ is an unstable matrix for $P=1/2$, then for the structure of Section 3.1 small perturbations result in the probability oscillating which in turn results in behavior which is either stable or unstable. If $\underline{\tilde{A}}(P)$ is a stable matrix for $P=1/2$ either oscillatory or nonoscillatory behavior may occur depending on the size of perturbations and the stability of $\underline{\tilde{A}}(P)$ for $P=0$ and 1 (see Section 4.1). A procedure has been presented in Section 4.7 for determining a lower bound on the set of perturbations which yield oscillation-free responses. This procedure is valid for any two model structure with slight modification.

3) The rate of change of P is proportional to $||\underline{w}||^2$ while the rate of change of $||\underline{w}||$ is proportional to $||\underline{w}||$ to the first power (see Section 4.1). This results in P changing faster than $||\underline{w}||$ for $||\underline{w}||$ large and slower for $||\underline{w}||$ small. This partly causes the square-wave behavior often noted for P.

4) A necessary condition for $||w||$ to converge to zero shown to be that $\tilde{\underline{A}}(P)$ be a stable matrix for <u>some</u> value of P (see Section 4.1, 4.6).

5) A sufficient condition for exponential convergence has been shown to be that $\tilde{\underline{A}}(P)$ be a stable matrix for <u>all</u> values of P (see Section 4.3). This, of course, is very restrictive. Lacking this condition, the possibility of at least short-term oscillations must be recognized. Thus the MMAC method is probably a poor choice when such oscillations can not be tolerated.

6) If one model <u>matches</u> the true system <u>and</u> each model is diagonal, then $||w|| \to 0$ exponential (see Section 4.3). This says <u>nothing</u> about the behavior of $P(\cdot)$ however.

7) Whenever the MMAC method is used with two models, the equation for the probability can be divided into a scalar, static nonlinearity and a summation (i.e., the log likelihood ratio (see Section 4.4). Although not done here, this can be generalized in the N-model case to an N-1 variable static nonlinearity and N-1 log likelihood ratios. This approach is important as it emphasizes the switching behavior of the probability, allows relatively simple analysis to be done for the often seen

case when P is nearly piece-wise constant and provides for a slight numerical superiority in applications (see Section 5.3).

8) For two special cases specific conditions for the stability or instability of the oscillatory mode have been presented (see Sections 4.5 and 4.6). These results agree well with simulations and with each other. The basic result is that stability results when the stable and unstable eigenvalues are such that $ab < 1$ where a and b are appropriately defined as in Sections 4.5 and 4.6. The qualitative conclusion which follows is that for stability to occur the most unstable mode must be dominated by the stable modes.

9) The oscillatory response may be stable even if no value of P results in $\underline{\tilde{A}}(P)$ being a stable matrix (see Section 4.6). In this case the phase-plane plots resemble hyperbolas (see Figure 3.5). This emphasizes the nature of the controller in that it attempts to achieve stability by alternately controlling each mode of the true system.

10) For the special case of Section 3.1 with $\hat{g} = 0$, it has been shown that $v(\cdot)$ is decreasing for P=0 and 1 as long as the LQ design procedure is used with a nonzero state weighting penalty. Generalizations have evaded analysis but this is believed to be a result of the gain and phase margin properties of the LQ design [5].

Recall that in this special case the KF dynamics do not enter into the closed-loop behavior (since $\hat{g} = 0$). Thus, since the gain and phase margin properties can disappear when a KF is included in the feedback path [33], the extension to the general case ($\hat{g} \neq 0$) is in doubt.

11) In contrast to the previous conclusion, there are many values of the control gain which yield satisfactory matched behavior (i.e., stable) but which will result in unstable behavior in the MMAC system. Thus, care must be exercised in the choice of control gain.

Thus, two major types of behavior have been observed and analyzed; oscillatory and exponential (nonoscillatory). It has been seen that oscillatory behavior is very natural for the MMAC algorithm and that the conditions for excluding it are restrictive. Stability conditions for special cases have been presented and the qualitative implications for more general systems discussed. Thus, although based on special cases, the qualitative conclusions regarding the types of behavior and their underlying causes are believed to be of fairly general applicability.

CHAPTER 5

NUMERICAL ASPECTS OF MMAC

In the course of this study, various issues relating to the numerical sensitivity of the MMAC method when implemented on a digital computer have come to light. The purpose of the present chapter is to discuss these aspects and finally to propose an alternate formulation which appears to overcome some of the limitations.

In Chapter 4, while discussing the oscillatory behavior so often observed, it was noted that a lower limit on the probability tends to modify the behavior to some degree. Thus, while the analysis of Section 4.5 indicates that the peaks of the state trajectories should be decreasing for the simulation in Figure 3.2, in fact, the peaks are seen to be constant.

In Section 5.1 the oscillatory behavior is again analyzed, this time assuming that a lower limit on P is in use. This leads to the conclusion that if the limit is consistently achieved, then the peaks of the state trajectories will be equal when they would otherwise be decreasing. Furthermore, complete characterizations of both the period and peak amplitude of the states are given for both stable and unstable operation. The results are, except for the peak value, essentially the same as those in Section 4.5.

Section 5.2 contains a brief discussion on various forms of the probability equation. It is found that a few forms behave very poorly in the face of numerical roundoff caused by finite precision. Precision refers to the number of digits in the mantissa of the representation of a number in the computer. In addition, the need for extended range is also discussed. Range refers to the available size of the exponents in the computer. In particular, it is noted that due to large changes in the probability during oscillatory operation,(as seen in the simulations of Case 1 in Chapter 3) exponents of less than -75 often occur. Throughout this section the comments are made assuming that the algorithm is implemented using floating-point number representations.

In light of the results contained in Sections 5.1 and 5.2, it is clear that the MMAC method places large demands on the computer which calculates the control. However, as seen in Section 4.4, there is an alternate form to the probability equation. In Section 5.3, a discussion of the numerical properties of this form is given which concludes that using Equation (4.11) in order to calculate the probability results in reduced sensitivity to the problems discussed in Sections 5.1 and 5.2 compared to using Equation (2.22b). It is important to note that the proposed formulation does <u>not</u> eliminate the necessity of having a lower bound on the probability but merely allows the designer much greater latitude in the choice of this parameter.

One final comment is important. The remarks in Section 5.2 are <u>not</u> based on any <u>detailed analysis</u> of the numerical properties. The purpose of the thesis has been to examine the stability of the MMAC algorithm and <u>not</u> to prove anything about its numerical properties. However, in performing the simulations contained in Section 3.2, certain aspects of the numerical problem have become apparent.

The purpose of Section 5.2 is to point out these aspects. In keeping with this, no comment is made as to the numerical properties of the Kalman Filter, which is a significant problem in its own right. See, for example, the work of Sripad [30].

5.1 The Lower Limit Effect

The basic equations of the MMAC method have been presented in Chapter 2. As mentioned there, it is often necessary in practice to place an artificial lower limit on the probability of any model.

This is usually accomplished by simply setting the probability equal to the lower limit whenever the probability is less than the limit. Inclusion of a FORTRAN statement such as

```
IF(P(I) .LT. PLIM) P(I) = PLIM
```

is an example of the use of such a limit. The principal reason for such a limit is that without such a device, the probability of any model could be come exactly zero due to round-off errors. After this occurs, the probability remains zero as shown in Section 4.1. This is detrimental in cases in which the true model is changing and also in cases in which an oscillatory behavior occurs (such as in Section 3.2.2), for then roundoff may occur before the probability switches. In fact, the Case 1 simulations of Section 3.2 are just such a case. It should be recognized that although the lower limit is somewhat artificial, it is required in some form whenever a computer with finite precision is used for control calculations.

As mentioned in Section 4.5, the existance of a lower limit requires the modification of the analysis of the behavior of the method. The remainder of this section contains the modified analysis. As in Section 4.5, we study the canonical problem of Section 3.1. Further, we assume that the system is oscillating as in Figure 4.3 such that P is closely approximated by either 0 or 1. Note that this is consistent with Section 4.5.

Recall from Section 4.5 that the canonical problem (see Section 3.1) can be rewritten as

$$\begin{bmatrix} \underline{y}_1(k+1) \\ \underline{y}_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{\underline{A}}_1(P) & \underline{0} \\ \underline{0} & \tilde{\underline{A}}_2(P) \end{bmatrix} \begin{bmatrix} \underline{y}_1(k) \\ \underline{y}_2(k) \end{bmatrix} . \tag{4.8}$$

The log likelihood ratio, $\alpha(\cdot)$, can then be written as

$$\alpha(k) = \sum_{i=1}^{k} \underline{y}_2'(k)\underline{\phi}_2\underline{y}_2(k) - \underline{y}_1'(k)\underline{\phi}_1\underline{y}_1(k) \tag{5.1}$$

where $\underline{\phi}_i$ is defined in Section 4.5.

In Sections 4.4 and 4.5 it is argued that during oscillatory periods (i.e., when $P(\cdot)$ is alternately near zero and near one), then $\alpha(k)$ must behave as the solid line in Figure 5.1 and also that $\alpha(\cdot)$ can be approximated by a piecewise-exponential curve. It will now be argued that the affect of a lower limit on the probability is to change the trajectory of $\alpha(\cdot)$ to that of the dotted line in Figure 5.1.

In Section 4.4 a static relationship between $P(\cdot)$ and $\alpha(\cdot)$ has been derived as

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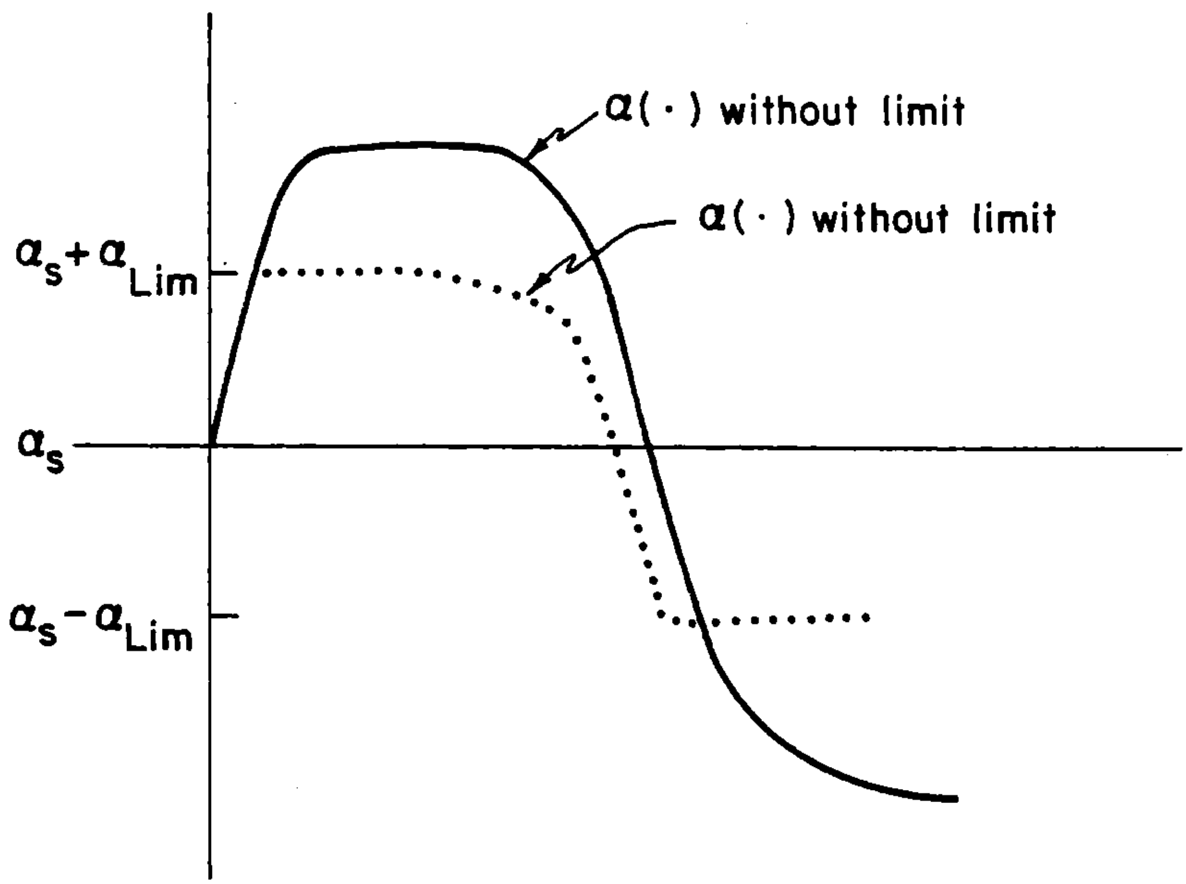

Fig. 5.1 Effect of Limit on α(•)

$$P(k) = \frac{P(0)\tilde{\beta}e^{-\frac{1}{2}\alpha(k)}}{P(0)\tilde{\beta}e^{-\frac{1}{2}\alpha(k)} + (1 - P(0))} \quad . \tag{4.11}$$

Thus, if the probability of each model is constrained to be always greater than some limit P_{lim}, then it can be shown that this is entirely equivalent to constraining $\alpha(\cdot)$ such that

$$|\alpha(\cdot) - \alpha_s| < \alpha_{lim} \tag{5.2}$$

where α_s is the value of $\alpha(\cdot)$ when $P(\cdot) = 1/2$ (given by Equation (4.13)) and α_{lim} is given by

$$\alpha_{lim} = 2 \ln \left[\frac{1 - P_{lim}}{P_{lim}}\right] \quad . \tag{5.3}$$

Thus, placing a lower limit on the probability can be seen to modify the plot of $\alpha(\cdot)$ to the dotted line in Figure 5.1: Initially, until the limit is reached, the limited and non-limited behavior are identical, but when the limit is reached, further increases in the magnitude of $\alpha(\cdot)$ are ignored until $\alpha(\cdot)$ again decreases (in magnitude). Note that since the size of the term in Equation (5.1) which forces $\alpha(\cdot)$ to return to zero is independent of the existence of a limit, $\alpha(\cdot)$ must cross the axis earlier when a limit is in effect than otherwise. This of course will influence future periods. It is thus argued that during oscillatory behavior in which the lower limit on the probability is achieved, in analyzing the behavior of the MMAC algorithm the positive term of Equation (5.1) can be set equal to α_{lim} for $P \approx 0$ while the negative term can be set equal to $-\alpha_{lim}$ for $P \approx 1$ (assuming P_{lim} small).

Thus, we see that for the analysis of Section 4.5, the effect of a lower limit on the probability is to allow us to replace Equation (4.23), repeated here for convenience,

$$\sum_{i=1}^{T_1} a_2^i ||\underline{y}_2(0)||^2 \sigma_2 \approx \sum_{i=1}^{T_1} a_1^i ||\underline{y}_1(0)||^2 \sigma_1 \tag{4.23}$$

by the equation

$$\alpha_{1im} \approx \sum_{i=1}^{T_1} a_2^i ||\underline{y}_2(0)||^2 \sigma \tag{5.4}$$

where T_1 has been defined in Section 4.4 as the time of the probability transition from $P=1$ to $P=0$,

$$a_i = \max_P ||\tilde{\underline{A}}_i(P)||^2$$

and σ_i is the maximum eigenvalue of $\underline{\phi}_i$ (where, by symmetry, $\sigma_1 = \sigma_2 = \sigma$). Noting as in Chapter 4 that $||y_2(T_1)||^2 \approx a_2^{T_1} ||y_2(0)||^2$, Equation (5.4) can be rewritten as

$$\alpha_{1im} = \frac{\sigma a_2}{(a_2 - 1)} [||\underline{y}_2(T_1)||^2 - ||\underline{y}_2(0)||^2] \tag{5.5}$$

or, solving for $||y_2(T_1)||^2$

$$||\underline{y}_2(T_1)||^2 \approx \frac{\alpha_{1im}}{R_1} + ||\underline{y}_2(0)||^2 \tag{5.6}$$

where

$$R_1 = \frac{a_2 \sigma}{(a_2 - 1)} \quad . \tag{5.7}$$

In the common case (see the simulations of Case 1 in Section 3.2) in which

$$||\underline{y}_2(0)||^2 << ||\underline{y}_2(T_1)||^2 \tag{5.8}$$

then Equation (5.6) becomes

$$||\underline{y}_2(T_1)||^2 \approx \frac{\alpha_{1im}}{R_1} \quad . \tag{5.9}$$

For the Case 1 parameters, this indicates that the probability will change when $||\underline{y}_2(T_1)||^2 \approx 100$ which agrees well with the simulations when the integer nature of T_1 is considered. Note that Equation (5.9) implies that the probability will change when $||y_2(0)||^2$ equals a constant if the lower limit on the probability is achieved; this constant is dependent only on the system parameters.

In order to determine any change in period due to changes in the state, consider the situation in Figure 5.2 in which $P=1$ for $k=0$ to T_1 and $P=0$ for $k=T_1$ to T_2+T_1 where T_1 and T_1+T_2 are defined as the switching times of the probability. Thus, from Equation (4.20)

$$||y_1(k)||^2 \leq \begin{cases} a_k^k \; ||\underline{y}_1(0)||^2 & k \in [0,T_1) \\ a_2^{k-T_1} \; a_1^{T_1} \; ||\underline{y}_1(0)||^2 & k \in [T_1,T_1+T_2) \end{cases}$$

where,

$$a_1 = ||\tilde{\underline{A}}_1(1)||^2 < 1$$

$$a_2 = ||\tilde{\underline{A}}_1(0)||^2 > 1$$

by the construction of $\tilde{\underline{A}}_1(P)$. From Equation (5.9) it is clear that $||\underline{y}_1(0)||^2 \approx ||\underline{y}_1(T_1+T_2)||^2$ or

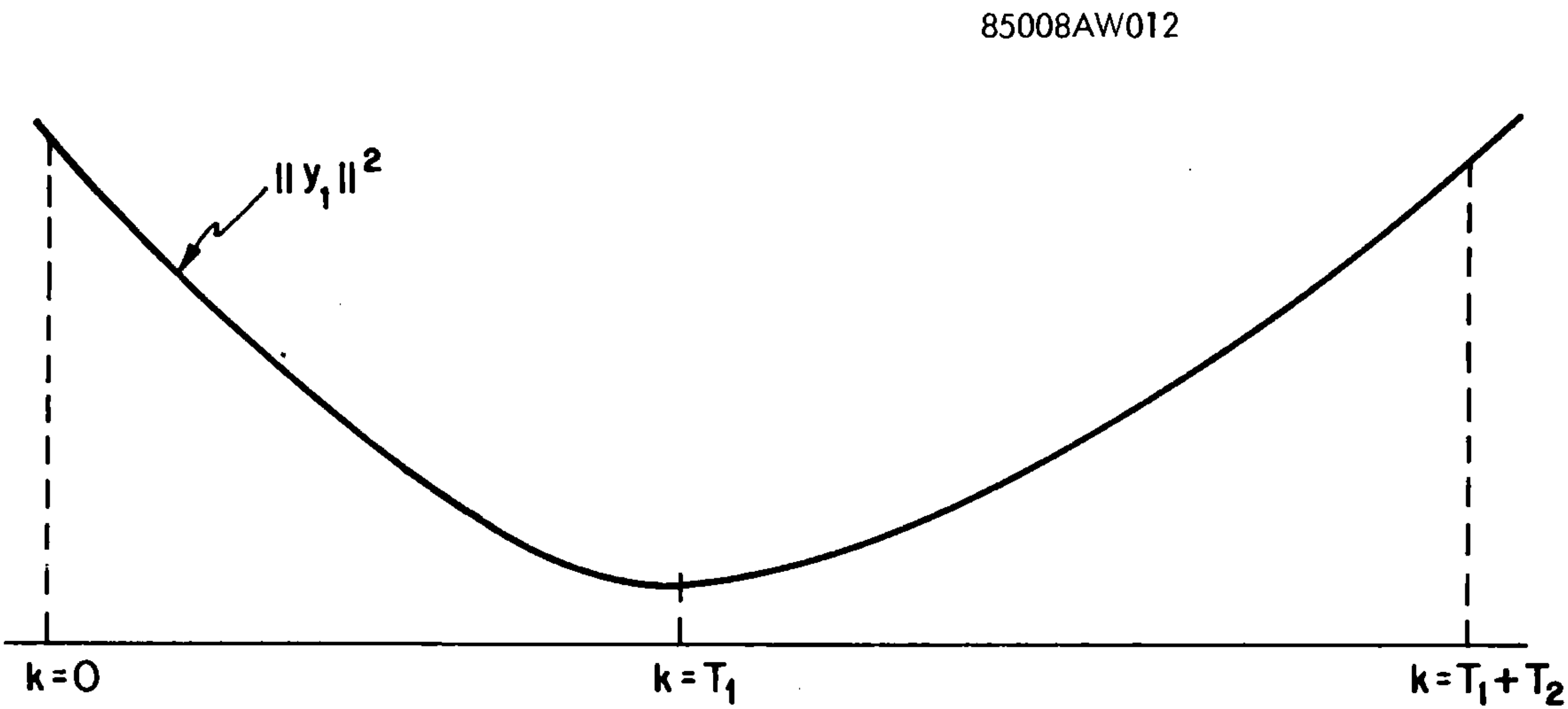

Fig. 5.2 Definition of T_1 and T_2

$$||\underline{y}_1(0)||^2 \le a_2^{T_2} a_1^{T_1} \, ||\underline{y}_1(0)||^2 \tag{5.10}$$

which, after some manipulation, yields

$$\frac{T_2}{T_1} \ge \frac{\ln a_1}{\ln a_2} \; . \tag{5.11}$$

Evaluation of this results in the following conclusion: If $a_1a_2 < 1$, then the peaks of the trajectories of $||\underline{y}_i(\cdot)||$ are constant and the period is increasing on each cycle of the probability. Furthermore, if $a_1a_2 = 1$, then the peaks are constant and the period approaches a constant. The case in which $a_1a_2 > 1$ is indeterminant from the above analysis although a plausible result, which is consistent with the simulations and the results of Section 4.6, is that the period would decrease until Equation (5.8) would no longer be a reasonable approximation. At this point, the peak amplitude of $||y_i(\cdot)||$ would grow. Basically in this case the states are destablized more than they are stablized and the overall system is unstable. Neither term (i.e., the positive or negative term) in the equation for $\alpha(\cdot)$ (5.1) need become small. Note that $a_1a_2 \le 1$ results in stable behavior which is in agreement with the results in Chapters 3 and 4. However, in this case, note that the peaks remain constant and so asymtotic stability does not result. In fact, the oscillations will continue forever with constant amplitude and increasing period. This is indeed an unusual type of behavior.

5.2 Forms of the Equations and Accuracy

In the course of doing the simulations contained in this thesis, a few points regarding the numerical aspects of the MMAC method have become apparent. These effects will not be dwelt upon as whole volumes could be written. However, they do form part of the basis for the form of implementation advocated in Section 5.3 and since such issues are often overlooked, a brief discussion is in order.

For the simulations in Chapter 3, the variable P makes large excursions, in some cases changing by 50 orders of magnitude in one or two time steps. This, of course, makes implementation very crucial. For example, throughout this thesis, the identity

$$P_2 = 1-P_1 \tag{5.12}$$

has been used to reduce the number of variables under consideration. However, using Equation (5.12) in a recursive equation such as

$$P_1(k+1) = \frac{P_1(k)p(\underline{r}_1)}{P_1(k)p(\underline{r}_1) + (1-P_1(k))p(\underline{r}_2)} \tag{2.22b}$$

can result in very poor accuracy due to roundoff. Thus, on a computer with 6-digit accuracy, this roundoff will occur for $P_2(\cdot) \approx 10^{-6}$ resulting in $P_2(\cdot)$ becoming exactly zero and remaining there unless some form of lower limit is placed on the probabilities as discussed in Section 5.1. Thus, using $P_2(\cdot) = (1-P_1(\cdot))$ places severe restrictions on the choice of a lower limit on the probability, which may be undesirable. The following three conclusions then are apparent.

1) Use of separate equations for $P_1(\cdot)$ and $P_2(\cdot)$ is desirable if a recursion on P is to be performed as in Equation (2.22b). Thus, in place of Equation (2.22b) it is preferred to use the pair of equations

$$P_1(k+1) = \frac{P_1(k)p(\underline{r}_1)}{P_1(k)p(\underline{r}_1) + P_2(k)p(\underline{r}_2)}$$

$$P_2(k+1) = \frac{P_2(k)p(\underline{r}_2)}{P_1(k)p(\underline{r}_1) + P_2(k)p(\underline{r}_2)}$$

which avoids the direct recursive calculation of $[1 - P_1(k)]$. Thus, when using a floating point representation with finite precision $P_2(k)$ in the above may be nonzero even when $[1 - P_1(k)]$ is zero. Note that this does not significantly increase the computational load since both the numerator and denominator of the second equation are contained in the first. An alternate approach to solving this problem is discussed in the next section.

2) Due to the fact that large variations in both the probability and the states estimates often occur, the maximum number of digits possible should be maintained. It should be noted that this is very important for the probabilities but less so for the other variables since the rounding off of the state does not adversely affect the overall system.

3) There is a direct relationship between the smallest possible value for P_{LIM} and the possible range of exponents in the floating point representation. This tradeoff between design freedom (i.e., choice of P_{LIM}) and computational complexity (i.e., needing large

magnitude exponents) must be recognized in any application.

As shown in the first section of this chapter, the achieving of a lower limit by the probability can result in some change in the behavior observed in a system. Also, a number of the precision aspects of the problem have been shown to most directly affect the probability variable. It thus is of some advantage to develop an implementation which minimizes the effects of such phenomenon. Such an implementation is discussed in the next section.

5.3 Proposed Implementation

As shown in the previous sections of this chapter, the form of the equations used to implement the MMAC algorithm can affect the achieved results. The major area in which problems occur has been seen to be in the calculation of the probability. In this section an alternate form of the probability equation is proposed which, although not eliminating the problems, does tend to minimize their effect.

The existance of a lower limit on the probability, although somewhat artificially imposed, is a necessary result of using a computer with finite precision. Thus, in <u>any</u> application of the MMAC method, a lower limit is present, in one form or another. The form of the probability equation about to be presented does <u>not</u> eliminate the lower bound problem but instead reduces the smallest possible value beyond that possible with the form given in Chapter 2. It thus allows increased freedom of choice for the designer.

From Section 5.2, one possible implementation of the probability calculation is

$$P_i(k+1) = \frac{P_i(k)\,p(\underline{r}_i)}{P_1(k)p(\underline{r}_1) + P_2(k)p(\underline{r}_2)} \qquad i = 1,2 \quad .$$

It is argued there that this form is to be preferred to the form given by Equation (2.22b) in that the subtraction $1 - P_1(\cdot)$ is avoided. In Section 4.4, it is shown that Equation (2.22b) is entirely equivalent to

$$P_1(k) = \frac{P_1(0)\tilde{\beta}e^{-\frac{1}{2}\alpha(k)}}{P_1(0)\tilde{\beta}e^{-\frac{1}{2}\alpha(k)} + P_2(0)} \tag{4.11}$$

with

$$\alpha(k) = \sum_{i=1}^{k} [\underline{r}_1'(i)\underline{\theta}_1^{-1}\underline{r}_1(i) - \underline{r}_2'(i)\underline{\theta}_2^{-1}\underline{r}_2(i)] \quad . \tag{5.13}$$

It is believed that Equation (4.11) is numerically superior to <u>either</u> of the other approaches. Note that, while a term such as $[1 - P_1(0)]$ is still required to compute $P_2(\cdot)$ for the calculation of the control, any errors in this computation do <u>not</u> get accumulated in the probability. Thus, the real advantage of using Equation (4.11) is that it is static, involving no recursive calculations. Thus, while roundoff obviously will occur, it will <u>not</u> influence future values as would happen with the other approaches. It should be pointed out that the recursive nature of the probability has, in effect been retained in the calculation of $\alpha(\cdot)$ and that now care must be taken to guarantee its numerical accuracy. However, by its nature, it tends to be a better-behaved function. For example, a change in P from 1 to 10^{-50} corresponds to a change in $\alpha(\cdot)$

from approximately +225 to -225. Thus, the recursion has effectively been moved to the exponent (i.e., the Log Likelihood Ratio) rather than in the probability itself. The importance of this can best be seen by considering the following problem. Assume one wishes to calculate the quantity $(e^a e^b)$. Once approach is to compute (e^a), then compute (e^b) and finally $(e^a)(e^b)$. This is effectively the kind of approach Equation (4.22b) uses. However, it is more accurate (for finite precision calculations) to compute $e^{(a+b)}$, which is analogous to what Equation (4.11) does.

A further advantage of using Equation (4.11) is that it can be approximated by any one of a number of functions without having to be overly concerned about the accuracy of the approximation. This is due to the fact that Equation (4.11) itself is not recursive and thus errors made in approximating P do not accumulate except through the true states. One example of such an approximation is a switch such that $P(\cdot) = 0$ for $\alpha(\cdot) > 0$ and $P = 1$ for $\alpha(\cdot) < 0$ while another has been presented in Section 4.5. Note that for the variations of the MMAC algorithm which use the control for the most probable model rather than the weighted control, a switch is an exact representation and not an approximation.

It should be pointed out that this approach can be generalized to an N model situation. In this case, one can define the N-1 log likelihood ratios based on, for example, model N as

$$\alpha_i(k) = \sum_{j=1}^{k} [\underline{r}_i'(j)\underline{\theta}_i^{-1}\underline{r}_i(j) - \underline{r}_N'(j)\underline{\theta}_N^{-1}\underline{r}_N(j)]$$

where all terms are defined analogously to their two-model-case counterparts. Repeating the derivation in Section 4.4 yields the N-1 nonlinear equations for the probabilities

$$P_i(k) = \frac{P_i(0)\tilde{\beta}_i e^{-\frac{1}{2}\alpha_i(k)}}{\sum_{j=1}^{N-1}\left[P_j(0)\tilde{\beta}_j e^{-\frac{1}{2}\alpha_j(k)}\right] + [1 - P_N(0)]} .$$

5.4 Summary

In this chapter a number of issues relating to the behavior of the MMAC algorithm when implemented on a digital computer have been examined. The most important of these is contained in Section 5.1 in which the effect of placing a lower bound on the probability is discussed. The principle conclusion of that section is that during oscillatory periods in which the lower limit is achieved, the conclusions of Chapter 4 are still valid except that the peaks of the trajectories of $||\underline{y}_i||$ are constant even during stable operation. Table 5.1 thus summarizes the conclusions about the stability whenever the lower limit is achieved. These results are seen to agree well with the simulation studies in Section 3.2.2.

It should be born in mind that, as discussed earlier in the chapter, there are philosophical reasons for choosing a relatively large value for the variable P_{LIM}. Note that such a choice results in a tendency to reduce the amplitude of oscillations as well as the period since the unstable state must now change less to cause $\alpha(\cdot)$ to change by α_{LIM}

a_1a_2	Peaks of $\|\|y_i\|\|$	Period	Stability
<1	constant	increasing	stable
=1	constant	constant	just stable
>1	constant changing to increasing	decreasing changing to indeterminant	unstable

TABLE 5.1

Stability Summary - P Limited

(i.e., cause P to reach P_{LIM}).

Section 5.2 has presented a brief discussion of the numerical properties of various forms of the probability equation. The principle conclusion is that the MMAC algorithm, if implemented using the straight forward equations such as Equation (2.22b), places large demands on a digital computer due to the recursive nature of the calculations, the use of a term such as $(1 - P(k))$ at each iteration, and a possibly large dynamic range of the probability. An alternate formulation, based on the analysis of Section 4.4 is presented in Section 5.3. It is shown that this formulation tends to be superior to the others presented in that the recursion is done on the exponent where accuracy can be better controlled.

CHAPTER 6

COMMENTS AND CONCLUSIONS

This thesis has presented an analysis of the behavior of the Multiple Model Adaptive Control algorithm through both analysis and simulation. Extensive use has been made of a canonical system (see Section 3.1) in which certain assumptions on the structure of the true model are made, chief of which is that it is diagonal. While somewhat extreme, this sample structure has been carefully chosen to display what we feel are the critical characteristics of the method as observed in more general applications and which, unlike the more general problem, also has been amenable to detailed analysis.

The purpose of this chapter is twofold. First of all, various ad hoc modifications have been proposed for the MMAC algorithm which are aimed at overcoming some of the undesirable behavior observed in applications. An example of such a modification is the introduction of a low pass filter to smooth out the probabilities. Section 6.1 contains a brief discussion of the most prominent modifications along with three methods for improving the response which are suggested by the results of the analysis of this thesis. The basic conclusion is that with <u>any</u> of these modifications great care needs to be taken to ensure that the response is not degraded.

The second purpose of this chapter is to provide a summary of the major conclusions of this thesis. This is done in Section 6.2 in which the specific conclusions of the analysis contained in this thesis are summarized. Also in this section are a list of qualitative conclusions regarding the MMAC system when applied to general systems. These result from extrapolating the specific conclusions to more general systems. The chapter concludes with a list of suggestions for future research in understanding the MMAC algorithm.

6.1 Modifications

Various ad hoc modifications have been proposed to overcome some of the undesirable properties of the MMAC algorithm which have been observed in applications. In this section two of the most prominent of these are discussed. First of all, in order to make the algorithm more sensitive to changes in the true model, a form of the MMAC algorithm which possesses a finite memory property can be used. In the first subsection, it is shown that most of the properties of the MMAC method apply in this case. The exception to this is that no convergence property can be given due to the finite memory.

The next subsection contains a brief discussion of the addition of a low pass filter in the probability calculation. This modification has been proposed [23] to smooth out the rapid probability transitions and oscillations which can occur and thereby attempt to get a smoother state response. It is argued in Section 6.1.2 that, especially when no model matches the true system, this can result in at best no change in performance and at worst in a destablilizing effect. This section then concludes with a discussion of three modifications which have been suggested by the analysis of this thesis.

6.1.1 Finite Memory MMAC

One of the many ad hoc modifications to the MMAC method which has been proposed has been termed Finite Memory MMAC (FM-MMAC). It can best be understood by considering the form of the probability equation given in Section 4.4:

$$P(k) = \frac{P(0)\tilde{\beta}e^{-\frac{1}{2}\alpha(k)}}{P(0)\tilde{\beta}e^{-\frac{1}{2}\alpha(k)} + (1-P(0))} \tag{4.11}$$

where

$$\alpha(k) = \sum_{i=1}^{k} \underline{r}_1'(i)\underline{\theta}_1^{-1}\underline{r}_1(i) - \underline{r}_2'(i)\underline{\theta}_2^{-1}\underline{r}_2(i) \quad . \tag{4.10}$$

This form of the equation makes it clear that <u>all</u> of the past data are <u>equally</u> weighted in determining the present probability. This is not necessarily what one would desire in an adaptive controller since data from the far past may <u>not</u> be relevant to the current operating conditions. There are, of course, many ways to modify the method so as to make it more responsive to the immediately past data than it is to the more remote data. For example, one may <u>exponentially</u> weight the past data:

$$\tilde{\alpha}(k) = \sum_{i=1}^{k} a^{(k-i)}[\underline{r}_1'(i)\underline{\theta}_1^{-1}\underline{r}_1(i) - \underline{r}_2'(i)\underline{\theta}_2^{-1}\underline{r}_2(i)] \tag{6.1}$$

for some $a < 1$. An alternative is to merely use the last M values of the residuals. Thus, one could use

$$\tilde{\alpha}(k) = \sum_{i=k-M+1}^{k} [\underline{r}_1'(i)\underline{\theta}_1^{-1}\underline{r}_1(i) - \underline{r}_2'(i)\underline{\theta}_2^{-1}\underline{r}_2(i)] \tag{6.2}$$

where M is a free design parameter which would be chosen partly on the basis of how fast the parameters of the true model are expected to change. This is referred to as the Finite Memory MMAC algorithm.

It should be noted that a significant amount of the analysis of systems of these types has already been done in Chapter 4. To illustrate this, a discussion of the second of the proposed schemes with M=1 is given. Thus, take

$$\hat{\alpha}(k) = \underline{r}_1'(k)\underline{\theta}_1^{-1}\underline{r}_1(k) - \underline{r}_2'(k)\underline{\theta}_2^{-1}\underline{r}_2(k) \quad . \tag{6.3}$$

First of all, from Section 4.1, the basic linearized analysis conclusions such as the neutral stability are unchanged as are the results of Sections 4.2 (Universal Stability). Note that the state convergence results of Section 4.3 do not carry over in this case. The technical reason is that $P_1(\cdot)$ is no longer non-decreasing. In fact, it is easy to show that

$$||\underline{w}(\cdot)|| \to 0 \Rightarrow P_i(\cdot) \to \frac{1}{N}$$

where N is the number of models. Thus, if the closed loop system is not stable for the case in which all models are equally likely (i.e., the N-model extension of the condition that $\tilde{\underline{A}}(P)$ is stable for P = 1/2) then $||\underline{w}||$ can not approach zero. This is a direct result of the finite memory assumption and occurs for any finite M. This is not, however, necessarily bad for an adaptive controller. It roughly corresponds to the notion that without information as to which model is correct, (i.e., $\underline{r}_i = 0$) one can assume that each model is equally likely.

The results of Section 4.4 on the analysis of the probability equation also hold since they do not depend on how $\alpha(\cdot)$ is computed. Thus, Figure 4.1 is again useful as are the approximations

$$\begin{aligned} \alpha(\cdot) >> \alpha_s \quad &\Rightarrow \quad P(\cdot) \approx 0 \\ \alpha(\cdot) << \alpha_s \quad &\Rightarrow \quad P(\cdot) \approx 1 \end{aligned} \tag{4.12}$$

and (4.14), where α_s is as in Equation (4.13). The qualitative behavior can best be appreciated by noting that combining Approximation (4.12) with Equation (6.3) yields

$$\begin{aligned} \|\underline{r}_1(k)\|_{\underline{\theta}_1^{-1}} >> \|\underline{r}_2(k)\|_{\underline{\theta}_2^{-1}} &\Rightarrow P(k) \approx 0 \\ \|\underline{r}_1(k)\|_{\underline{\theta}_1^{-1}} << \|\underline{r}_2(k)\|_{\underline{\theta}_2^{-1}} &\Rightarrow P(k) \approx 1 \end{aligned} \tag{6.4}$$

where, as in Chapter 4 we have assumed $\alpha_s = 0$. (Removing the assumption that $\alpha_s = 0$ merely results in α_s being added to the appropriate term in Approximation (6.4).) Thus one can expect that, compared to the infinite memory case, the period of oscillation for the M=1 case will be shorter due to a lack of the accumulation of the residuals over time. Effectively, by removing the summation of the residuals (the discrete time analogue of an integrator), a lag has been removed from the system. Thus, one would expect the peaks of the state curves to be smaller than when the summation is included. Note that it is possible, following the analysis of Chapter 4, to calculate approximations to the switching times T_1 and T_2. In fact, it becomes somewhat easier to compute since no summation is involved, Note, however, that care must be used in applying the results due to the assumption that P is always near zero or one.

Simulation results of this case are shown in Figure 6.1. The conditions of this simulation are identical to those of Figure 3.2 except that M=1 is used in the probability calculation. Note, however, that the character of the response has changed markedly. The probability (Figure 6.1a) changes much more rapidly and takes on intermediate values occasionally as it transitions between zero and one. This, of course, complicates any analysis such as the computation of the switching times T_1 and T_2. The state trajectories are shown in Figure 6.1b. Note that they are also much more oscillatory and in fact appear to limit-cycle. It is unknown at present if this is in fact a true limit cycle. Also, clearly shown is the reduction in the peaks of the state trajectories from about 17 for the unmodified situation to about 1.2 for the M=1 case.

The variable $\ln(x_1\, x_2)$, shown in Figure 6.1c, also is highly oscillatory and does not exhibit the negative slope which is evident in the unmodified case (Figure 3.2). This is further evidence of a limit cycle and indicates that the closed loop system is only neutrally hyperbolically stable. It should be noted that the analysis of Sections 4.5 and 4.6 on the conditions for hyperbolic stability required that the probability be always near zero or one. As long as this condition is met, the results of those sections still apply to the Finite Memory case. Note, however, that the assumption on the probability may not be as reasonable due to the presence of the high frequency oscillations.

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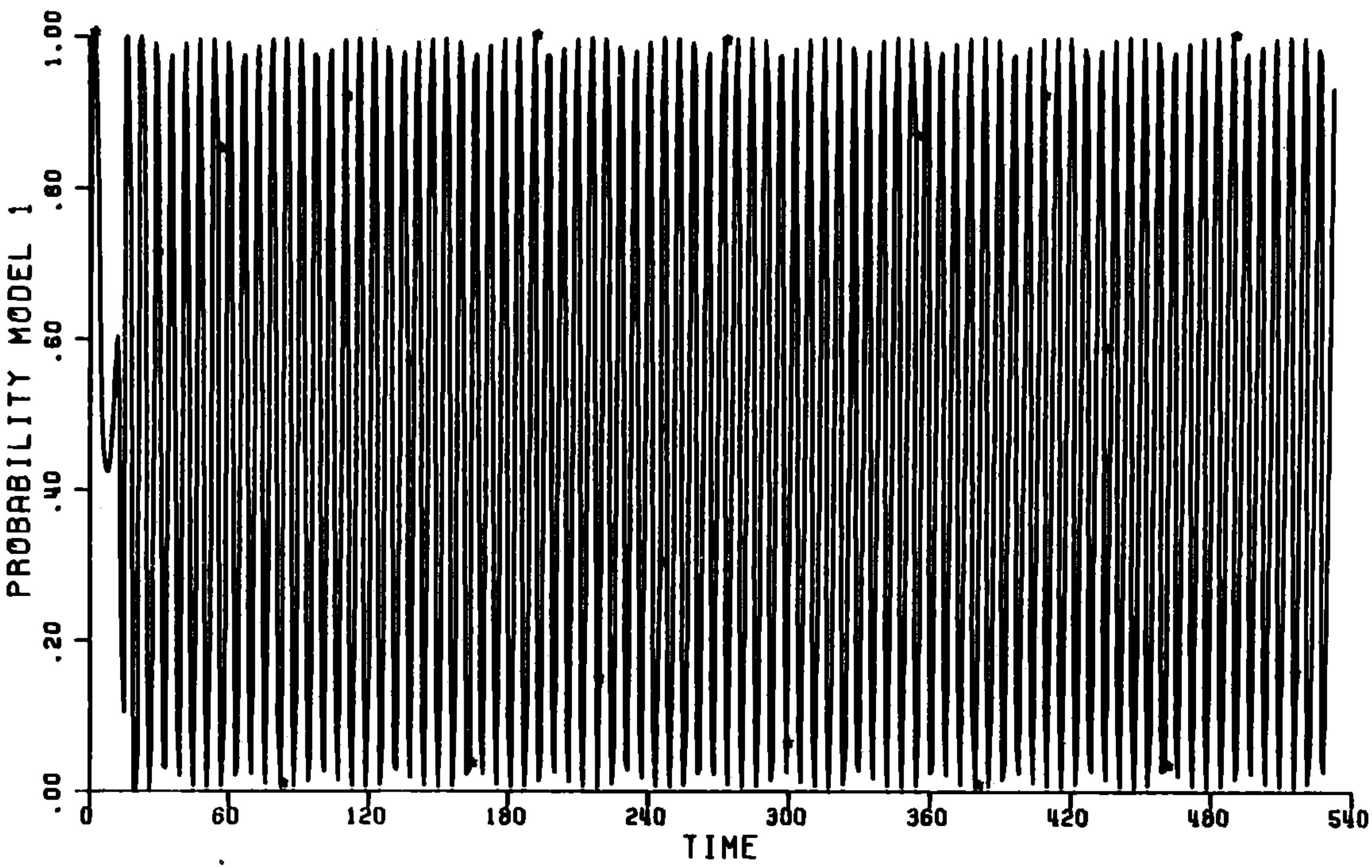

Fig. 6.1 Simulation of Finite Memory MMAC
(Case 1a with M=1)

a) Probability of Model 1

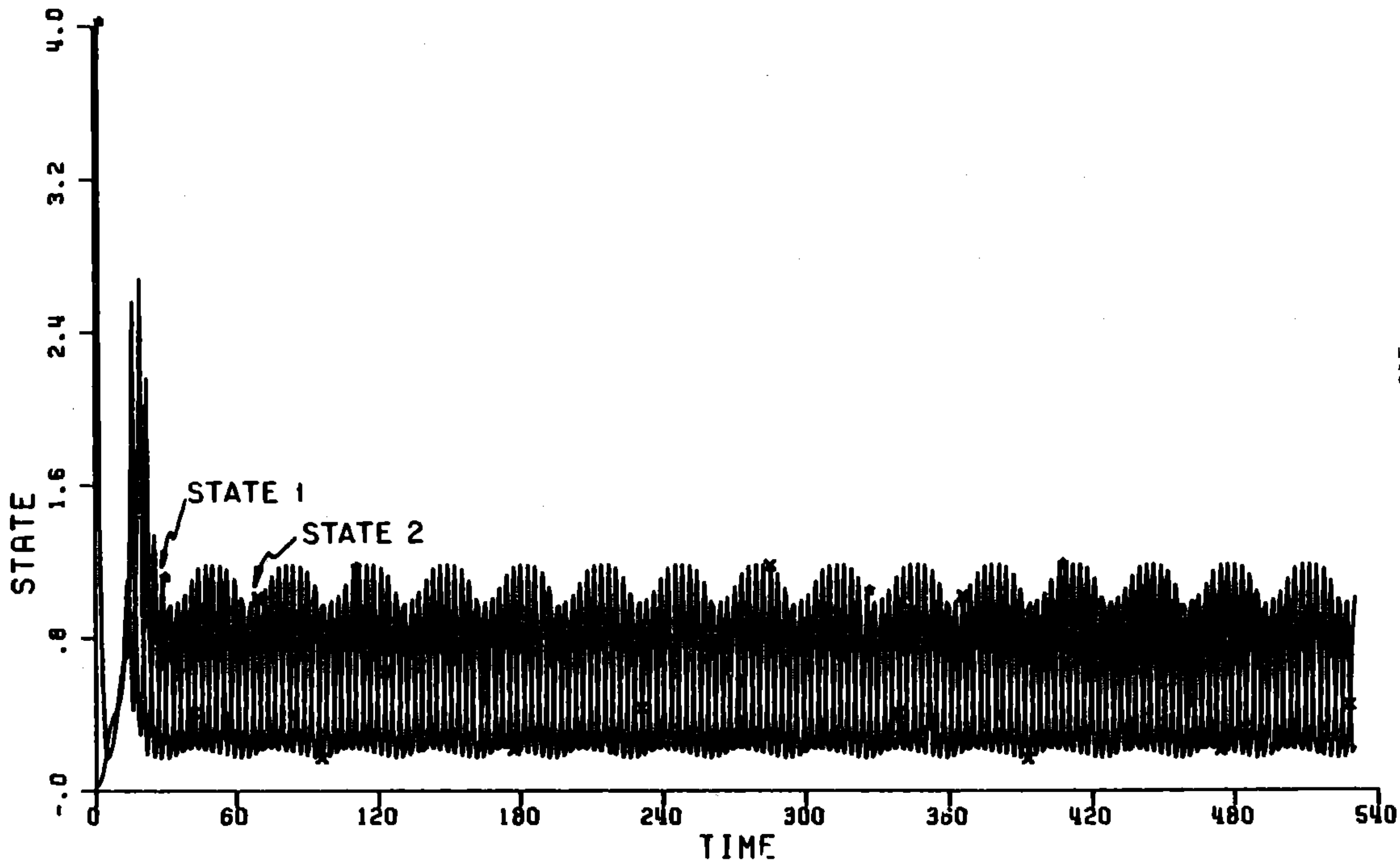

Fig. 6.1 Simulation of Finite Memory MMAC
(Case 1a with M=1)

b) True States

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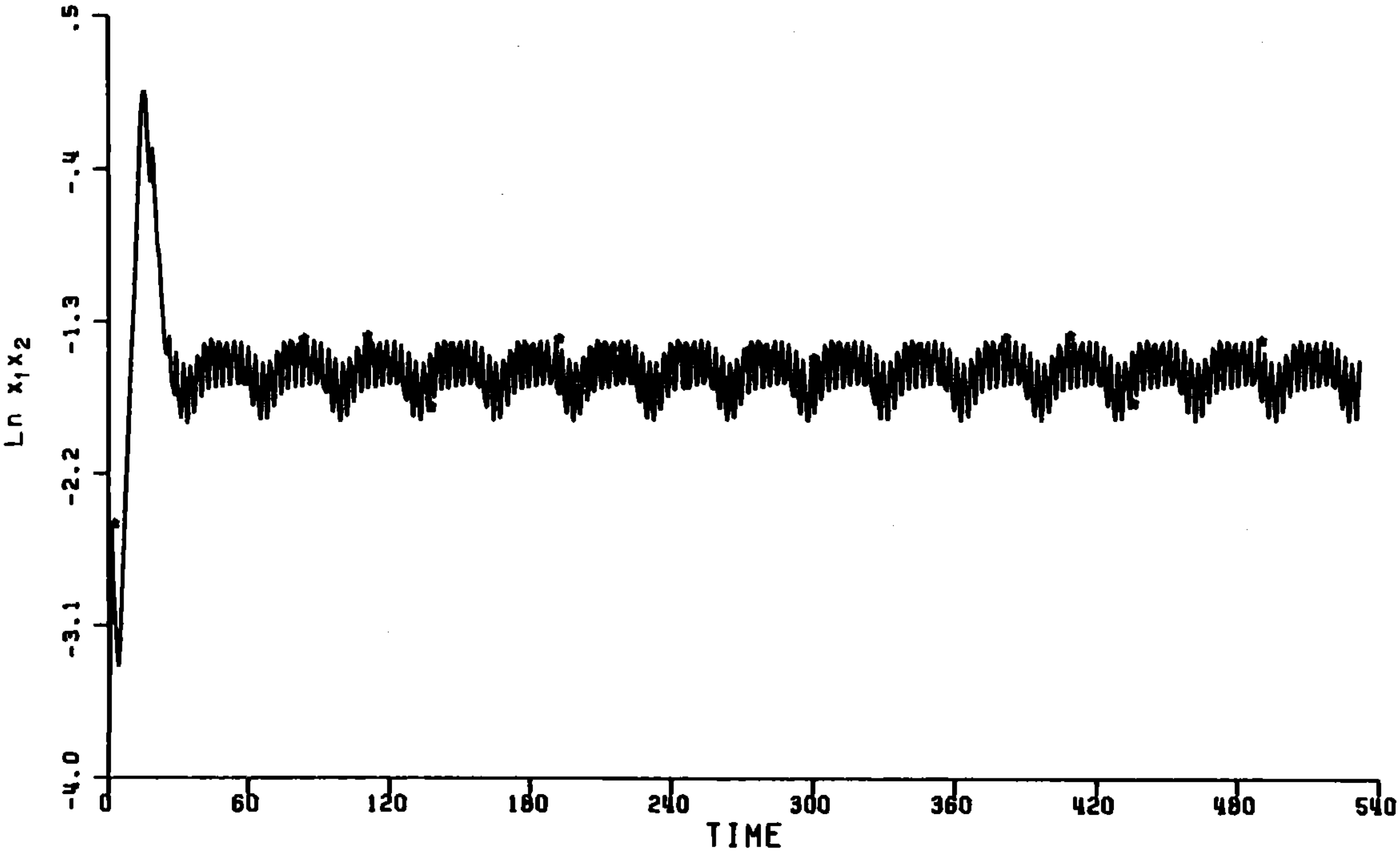

Fig. 6.1 Simulation of Finite Memory MMAC
(Case 1a with M=1)

c) Ln x_1x_2

6.1.2 Low Pass Filter

As seen in Section 3.2, the probability often exhibits a zero-one type behavior, rapidly changing between these values. In an attempt to smooth out these probability transitions, it has been proposed that a low pass filtered version of the probability be used for the control calculation. In this section a brief argument is presented which concludes that this may result in a degrading of the response.

As seen in Chapter 4, when $\tilde{A}(P)$ is an unstable matrix for all fixed P, the oscillatory behavior is necessary if hyperbolic stability is to be achieved. Thus, one can easily imagine cases in which low pass filtering the probability would result in a degradation in response. Although no detailed analysis has been done, the results of this thesis indicate that at least during the initial period, including a low pass filter on the probability would cause the probability to lag behind what it would be otherwise. This in turn would result in the unstable state growing to a larger value before the probability switches. Although the effects in later periods are unknown, it is reasonable to conclude that the net result is an increase in the values of the peaks of the state trajectories compared to the case in which no extra filtering is done.

6.1.3 Miscellaneous Modifications

In this section three relatively simple modifications to the MMAC algorithm which have been suggested by the analysis contained in Chapters 4 and 5 are introduced and discussed. The three modifications are: scale the residuals, increase the lower limit on the probability and take the square root of $\alpha(\cdot)$ in the probability equation.

In Chapter 4 it is shown that a discrepancy between the probability equation and the state equation exists in that for P near 1/2 the rate of change of the probability is proportional to the square of $||\underline{w}||$ while the rate of change of $||\underline{w}||$ is only proportional to $||\underline{w}||$ to the first power. This suggests that one way of smoothing the probabilities is to replace Equation (2.22b) by

$$P_1(k+1) = \frac{P_1(k)\tilde{p}(\underline{r}_1)}{P_1(k)\tilde{p}(\underline{r}_1) + P_2(k)\tilde{p}(\underline{r}_2)} \tag{6.5}$$

where

$$\tilde{p}(\underline{r}_i) = \beta_i e^{-\frac{1}{2}\sqrt{\underline{r}_i'\underline{\theta}_i^{-1}\underline{r}_i}} \quad . \tag{6.6}$$

Note that $\tilde{p}(\cdot)$ is no longer the Gaussian density function. This version will be referred to as the <u>square root modification</u> of the MMAC algorithm.

Simulations using this redefined function (shown in Figure 6.2) indicate that the probability still behaves in a zero-one fashion. Furthermore, note that the <u>peaks</u> of the state trajectories have <u>increased</u> significantly compared to the results using the normal probability function. (See Figure 3.2). This is believed to be due to the effective delay which results from $\alpha(\cdot)$ changing linearly rather than quadraticly. Thus, this modification does not accomplish the goal of smoothing the probability and results in degraded state responses. It is further concluded that the dominant nonlinearity in so far as the oscillatory behavior is concerned is that of Equation (4.11)

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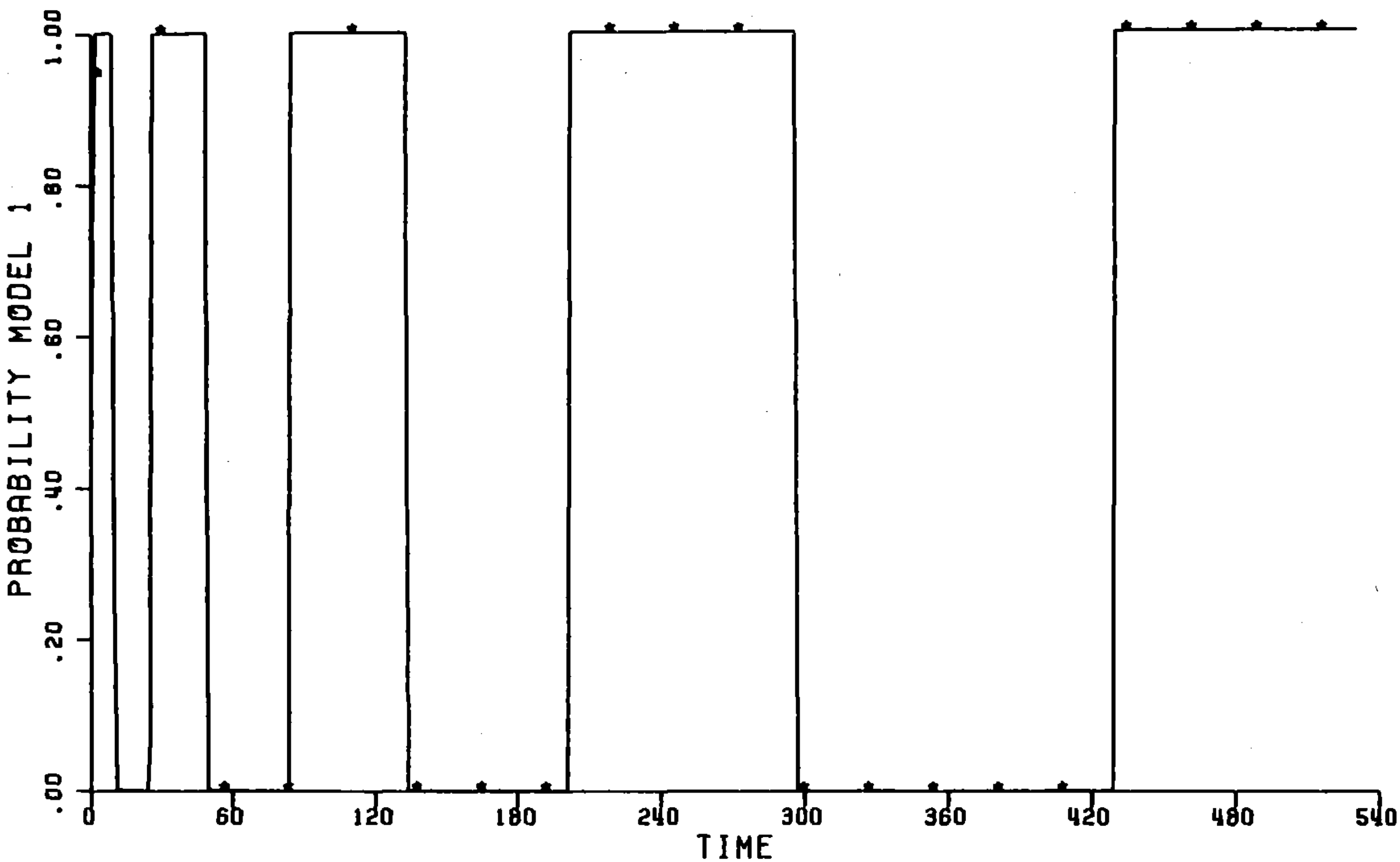

Fig. 6.2 Simulation of Square-Root Modification of MMAC (Case 1a)

a) Probability of Model 1

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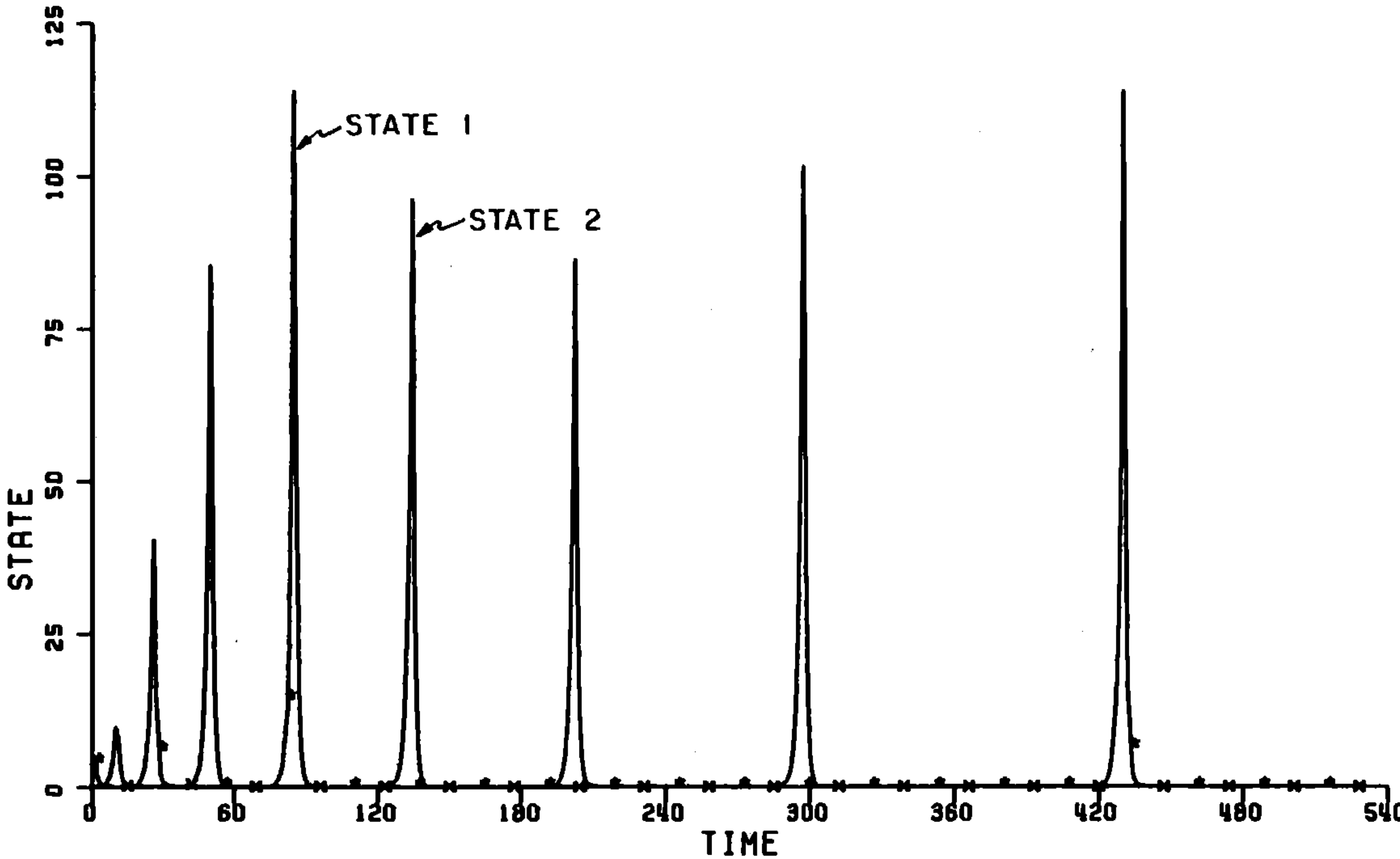

Fig. 6.2 Simulation of Square-Root Modification of MMAC
(Case 1a)

b) True State

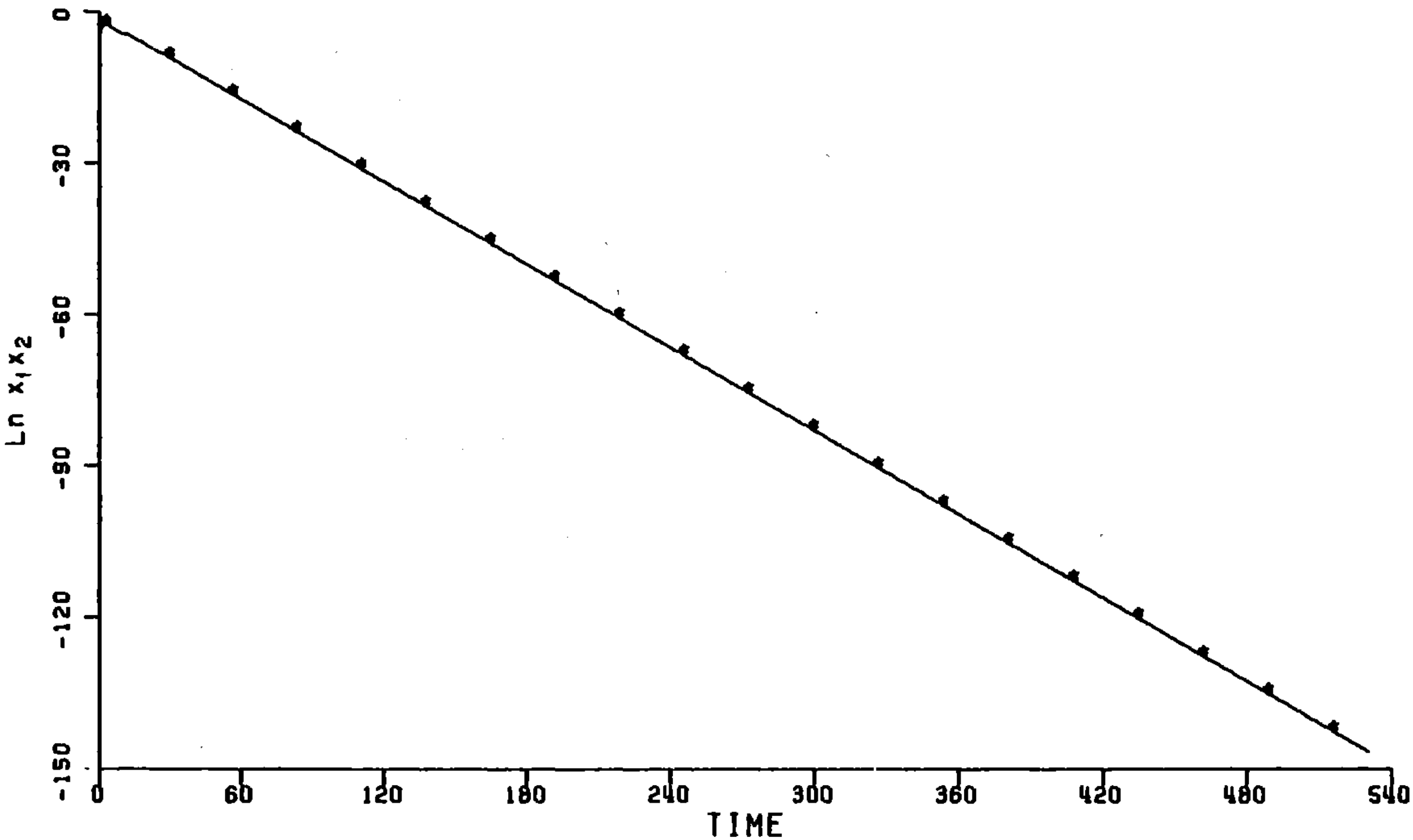

Fig. 6.2 Simulation of Square-Root Modification of MMAC
(Case 1a)

c) Ln x_1x_2

$$P(k) = \frac{P(0)\tilde{\beta}e^{-\frac{1}{2}\alpha(k)}}{P(0)\tilde{\beta}e^{-\frac{1}{2}\alpha(k)} + (1-P_0(0))} \tag{4.11}$$

and not those of the $\alpha(\cdot)$ term.

Finally, note that as predicted by the results of Chapter 4, the closed loop system is hyperbolically stable. This is best seen by considering the plot of $\ln(x_1x_2)$ given in Figure 6.2c in which this variable is seen to decrease linearly. This indicates that the states behave in such a fashion that

$$x_1x_2 = e^{bt}$$

where b is the slope of the line in Figure 6.2c.

The second possible modification is to change the lower limit on the probability, P_{lim}, discussed in Section 5.1. In that section it is shown that during periods when the lower limit is achieved, then the peaks of the state trajectories are given by α_{lim}/R_1 where

$$\alpha_{lim} = 2 \ln \frac{1-P_{lim}}{P_{lim}} \tag{5.3}$$

and R_1 is given by Equation (5.7). Thus it is possible to reduce the peaks merely by increasing the value of P_{lim} as long as the approximations

$$\underline{\tilde{A}}(1) \approx \underline{\tilde{A}}(1-P_{lim})$$

$$\underline{\tilde{A}}(0) \approx \underline{\tilde{A}}(P_{lim})$$

are valid. Even when these approximations are not valid, the approach is still valid if the effect on R_1 is accounted for by replacing $||\underline{\tilde{A}}_i(1)||$ with $||\underline{\tilde{A}}_i(1-P_{1im})||$ (see Section 5.1).

Simulations in which P_{1im} is changed have been done and they confirm this observation. However, two points need to be recognized in exploiting this fact. First of all, if P_{1im} is reduced too much, the assumption that α_{1im}/R_1 is greater than $||y_2(0)||^2$ (see Section 5.1) is violated resulting in the peaks not being as predicted by α_{1im}/R_1. Also, by Equation (5.3), the value of the peak varies as the natural logarithm of P_{1im} and so large changes in P_{1im} are necessary to make moderate changes in the peaks.

A third modification, suggested by the analysis of Section 4.4, is to scale the residuals. As seen in Section 4.4, the value of the probability is determined by $\alpha(\cdot)$. Thus, if the calculation of $\alpha(\cdot)$ is replaced by

$$\tilde{\alpha}(\cdot) = a\,\alpha(\cdot) \qquad (6.7)$$

with a>1 where $\alpha(\cdot)$ is given by Equation (4.10), then the changes in $\alpha(\cdot)$ are amplified. This, when combined with the existance of a lower limit tends to reduce the peak value of the state trajectoreis and the period (see Section 5.1). A simulation of this, shown in Figure 6.3, indicates that significant decreases in the peak amplitudes are possible. The simulation conditions are identical to those of Figure 3.2 except that the residuals are scaled by a factor of 10. This corresponds to letting a = 100 in Equation (6.7). As a comparison, the unmodified

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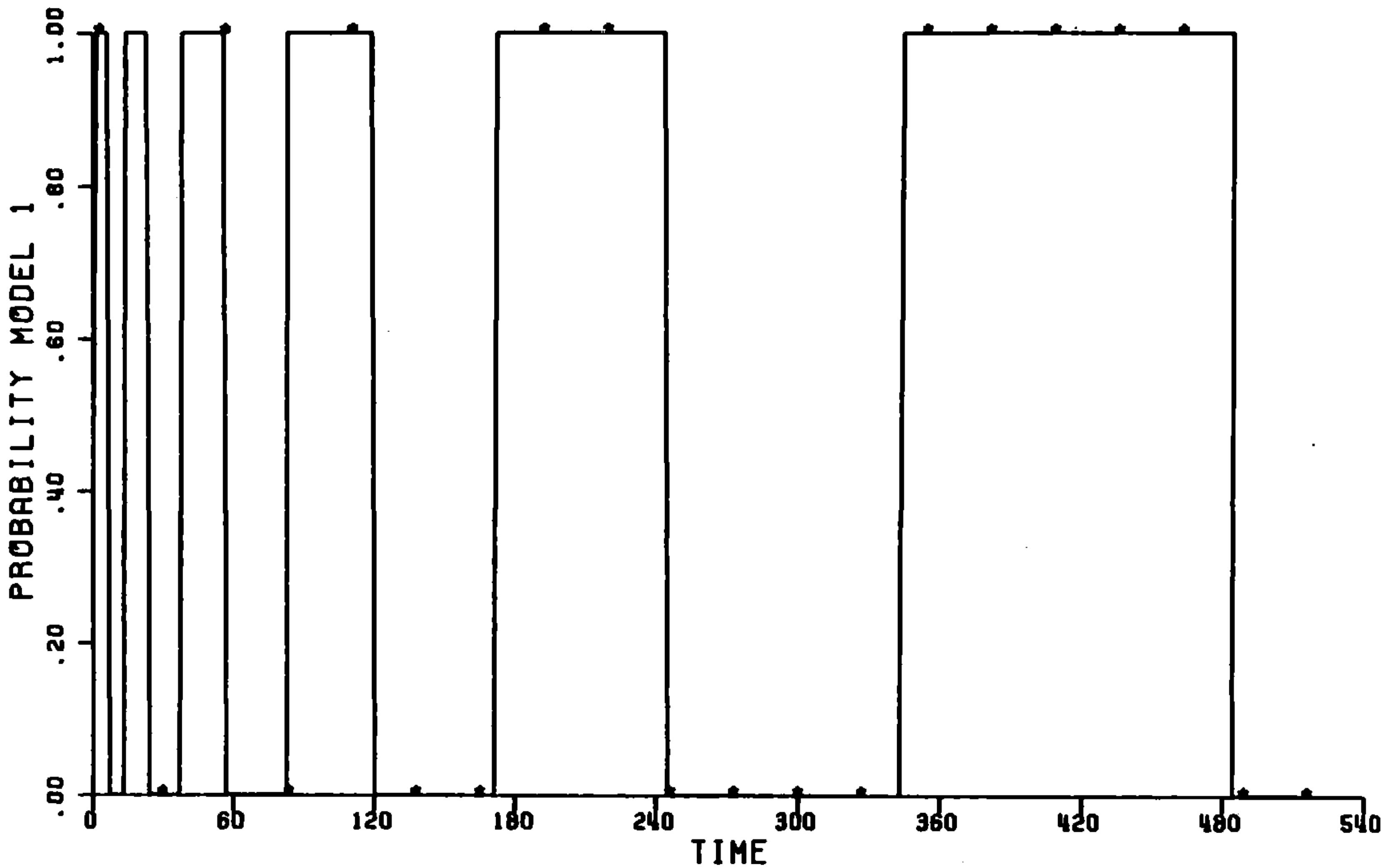

Fig. 6.3 Simulation of MMAC with Scaled Residuals
(Case 1a)

a) Probability of Model 1

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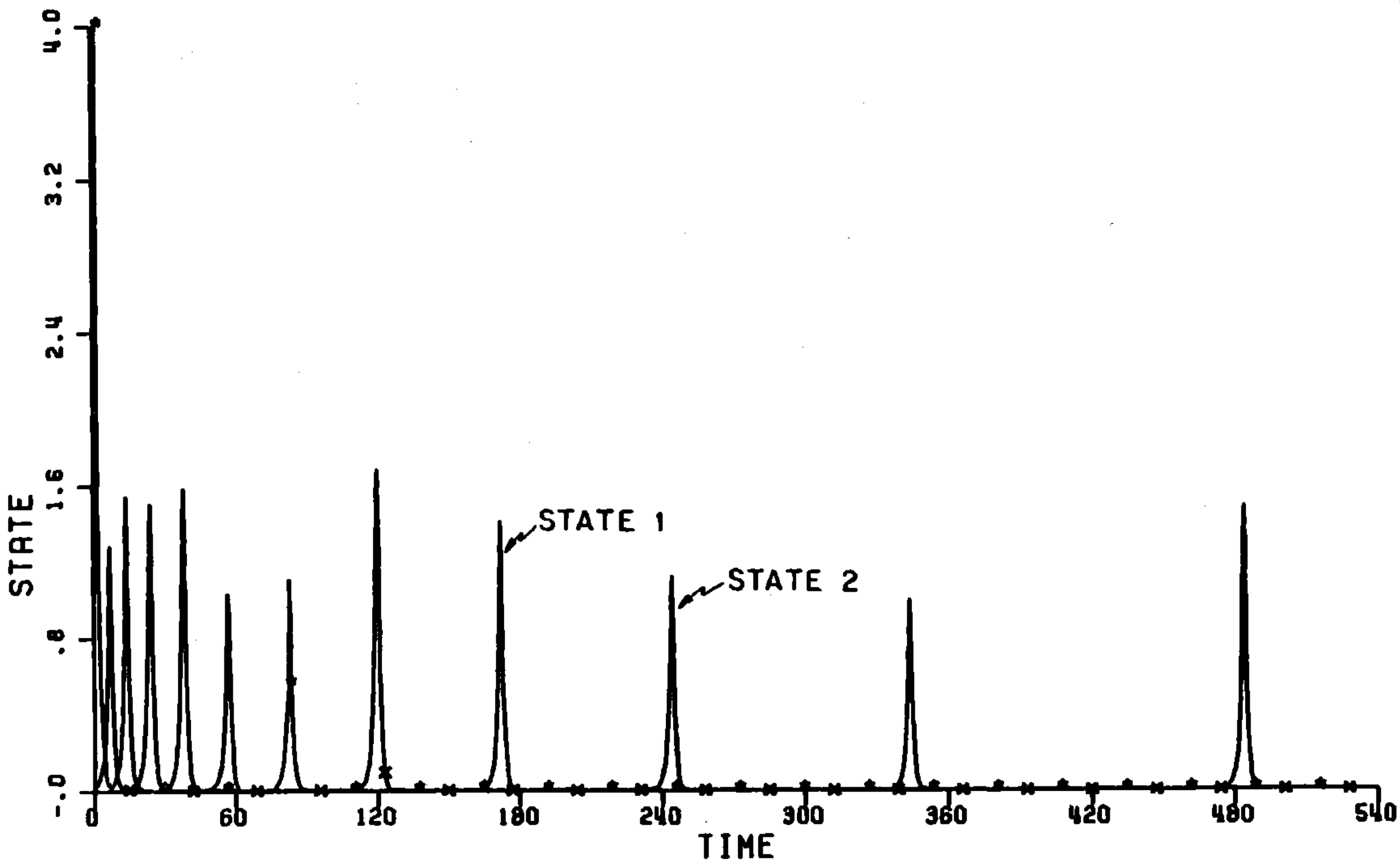

Fig. 6.3 Simulation of MMAC with Scaled Residuals
(Case 1a)

b) True State

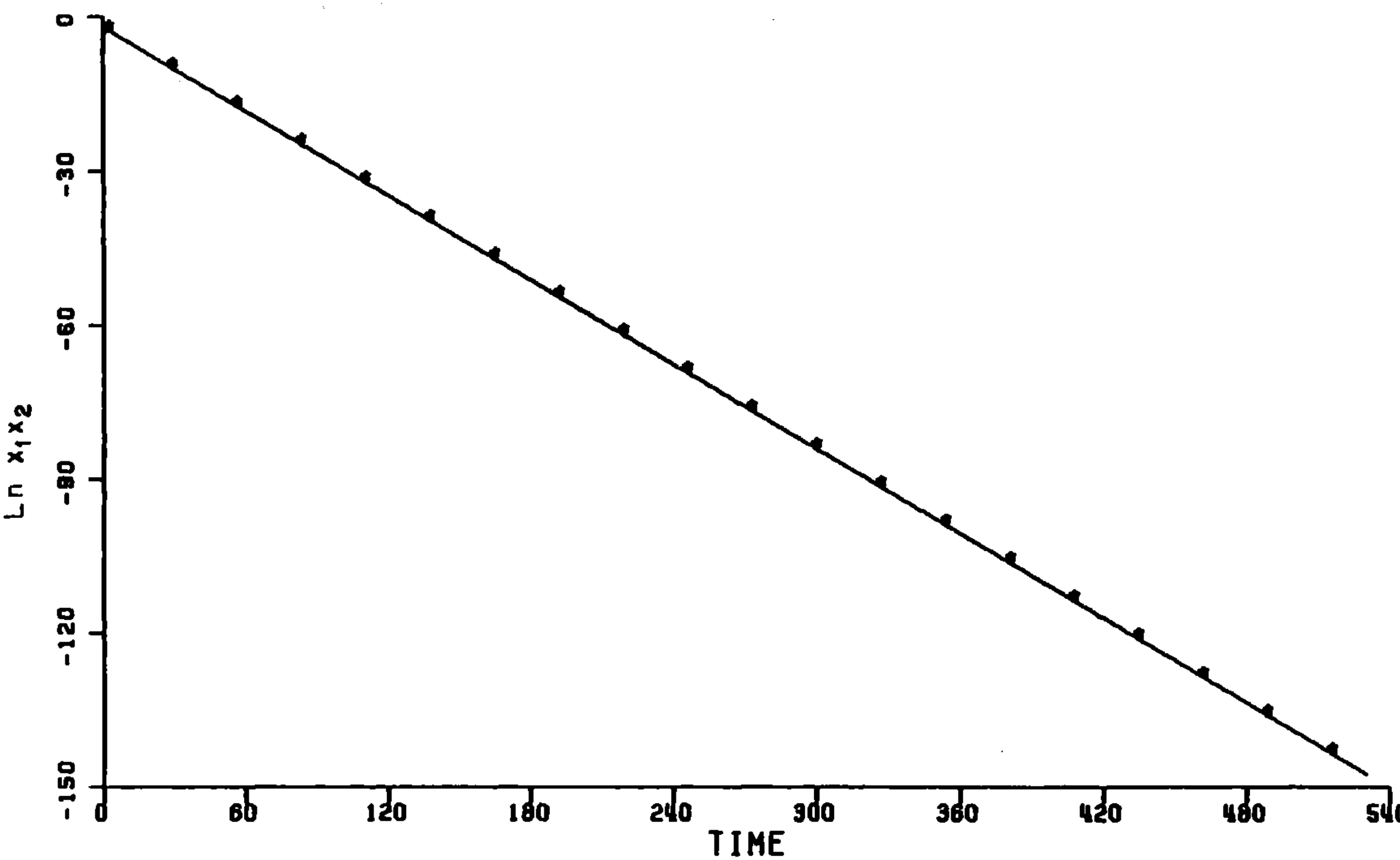

Fig. 6.3 Simulation of MMAC with Scaled Residuals
(Case 1a)

c) Ln x_1x_2

MMAC algorithm had a peak of about 17. (see Figure 3.2b), the FM-MMAC algorithm had a peak of about 1.4 (see Figure 6.1b) but oscillated rapidly, while scaling with a value a = 100 results in peaks of about 1.6 (see Figure 6.3b). Note also that the quantity $\ln x_1 x_2$ for the case when $\alpha(\cdot)$ is scaled (Figure 6.3c) behaves exactly as in the unmodified case while the FM-MMAC algorithm results in neutral hyperbolic stability (i.e., in $\ln x_1 x_2$ being constant).

This modification only reduces the peaks of the state trajectories when the lower limit on the probability is achieved. In fact, it follows directly from the anslysis of Section 5.1 and thus all of the analysis of that section can be repeated. In particular, the results on the hyperbolic stability and the calculation of the switching times follows analogously with the use of Equation (6.7). One drawback of this approach is that since $\alpha(\cdot)$ is scaled without modifying the associated KF's, $\alpha(\cdot)$ will tend to be affected more by noisy inputs than before. This in turn could significantly alter the response to noise in the observations (see Section 6.3).

6.2 Summary of Results

The major results of this thesis are of two basic types. The first type is the specific conclusions which have been derived for the special cases discussed in Chapter 3 and 4. These result in specific criteria for each type of behavior but, strictly speaking, apply only to the associated special case. Of possibly greater importance are the qualitative results which, while not proved in this thesis are reasonable extrapolations of the specific conclusions. Both types of results are summarized below. For the details of the specific conclusions, the reader is referred to the appropriate section of the body of the thesis.

The major specific conclusions of this thesis are:

1. At best the MMAC system is neutrally stable about an equilibrium point in that the probability has no tendency to return to its initial value following a perturbation. (See Section 4.1).

2. If $\tilde{\underline{A}}(P)$ is an unstable matrix for $P=1/2$, then for the structure of Section 3.1 small perturbations result in the probability oscillating which in turn results in behavior which is either bounded or unstable. If $\tilde{\underline{A}}(P)$ is a stable matrix for $P=1/2$ either oscillatory or nonoscillatory behavior may occur depending on the size of the perturbations and the stability of $\tilde{\underline{A}}(P)$ for $P=0$ and 1 (see Section 4.1). A procedure has been presented in Section 4.7 for determining a lower bound on the set of perturbations which yield oscillation-free responses. This procedure is valid for any two model structure with slight modification.

3. The rate of change of P is proportional to $||\underline{w}||^2$ while the rate of change of $||\underline{w}||$ is proportional to $||\underline{w}||$ to the first power (see Section 4.1). This results in P changing faster than $||\underline{w}||$ for $||\underline{w}||$ large and slower for $||\underline{w}||$ small. This partly causes the square-wave behavior often noted for P.

4. A necessary condition for $||\underline{w}|| \to 0$ has been shown to be that $\tilde{\underline{A}}(P)$ be a stable matrix for some value of P (see Section 4.1, 4.6).

5. A sufficient condition for the exponential convergence of the state has been shown to be that $\tilde{\underline{A}}(P)$ be a stable matrix for all values

of P. (see Section 4.3). This, of course is very restrictive. Lacking this condition, the possibility of at least short term oscillations in the probability which cause the states to increase must be recognized. Thus, the MMAC method is probably a poor choice when such oscillations can not be tolerated.

6. If one model matches the true system and each model is diagonal, then $||\underline{w}|| \to 0$ (see Section 4.3). This says nothing about the behavior of $P(\cdot)$ however.

7. Whenever the MMAC method is used with two models, the equation for the probability can be divided into a scalar, static nonlinearity and a summation (i.e., the log likelihood ratio) (see Section 4.4). Although not done here, this can be generalized in the N-model case to an N-1 variable static nonlinearity and N-1 log likelihood ratios. This approach is important as it emphasizes the switching behavior of the probability, allows relatively simple analysis to be done for the often seen case when P is nearly piece-wise constant and provides a slight numerical superiority in applications (see Section 5.3).

8. For two special cases specific conditions for the stability or instability of the oscillatory mode have been presented (see Sections 4.5 and 4.6). These results agree well with simulations and with each other. The basic result is that stability results when the slowest decaying stable eigenvalue and the most rapidly growing eigenvalue are such that their product is less than unity.

9. The oscillatory mode of response may be hyperbolicly stable in that the quantity $\ln(x_1 x_2)$ is decreasing even if $\underline{\tilde{A}}(P)$ is an unstable

matrix for all constant values of P (see Section 4.6). This is seen to correspond to the observation that the states are alternately controlled in a manner such that

$$x_1 x_2 = e^{bt}$$

where b is a constant which is negative for hyperbolicly stable systems. In this case the phase-plane plots resemble hyperbolas (see Figure 3.5). This illustrates the nature of the controller in that it attempts to achieve stability by alternately controlling each mode of the true system.

10. For the special case of Section 3.1 with $\hat{g} = 0$, it has been shown that $v(\cdot)$ is decreasing for P=0 and 1 as long as the LQ design procedure is used with a nonzero state weighting penalty.

11. In contrast to the previous conclusion, there are many values of the control gain which yield satisfactory matched behavior but which will result in unstable behavior in the MMAC system. Thus, care must be exercised in the choice of control gain.

12. The existence of a lower limit on the probability slightly affects the results. When the limit is consistently achieved, the peaks of the state trajectories tend to a precomputable constant (see Section 5.1).

13. Various forms of the equation for the probability have been examined as to the effects of numerical roundoff (see Section 5.2) resulting in the proposal in Section 5.3 of a form which is believed to be superior to others in that it is less sensitive to roundoff effects.

14. Five ad hoc modifications to MMAC algorithms have been examined (see Section 6.1). The Finite Memory MMAC resulted in a significant reduction in the amplitude of the peaks of the state trajectories but also resulted in a significant increase in the frequency of the oscillation. Adding a low pass filter was seen to be possibily detrimental to the simulation results as was using the square root of $\alpha(\cdot)$ in the probability calculation. Two approaches which appear to improve the response have been given. These are to increase the smallest value that the probability is allowed to assume (i.e., increase P_{lim}) and also to scale the residuals by a scalar which is greater than one.

The qualitative results are:

1. If one of the models matches the true system, good behavior (as defined by the design cost function) probably results. Some doubt must remain due to the fact that convex combinations of stabilizing controls do not necessarily yield stabilizing controls.

2. If none of the models match the true system but if at least one model results in a stabilizing control (that is, if $\underline{A}(\underline{P})$ is the extension of $\tilde{\underline{A}}(P)$ to the N model case, then this results when there exists some i such that $P_i = 1 \Rightarrow \underline{A}(\underline{P})$ is a stable matrix) then stability probably results. However, it is clear that the quality of the response may still be poor in that it may be signifiantly different from the design performance specifications as measured by the response of the control system under perfectly matched conditions. Furthermore,

it is possible to get oscillations in this situation. Consider, for example, the case with two models where one model stabilizes the closed loop system but has a slow filter while the other destabilizes the system but has a fast filter. One could then expect a cycle to develop in which the system alternates between the stable and unstable control laws.

3. Even if no constant value for the probability results in a stabilizing control, the overall system may still be hyperbolically stable (where hyperbolic stability would have to be redefined for the general case). In this case, the probabilities would tend to rapidly change alternately controlling the various modes of the true system.

4. Even when a constant value of P results in a stable control, oscillations in the probability are likely to occur. In fact, the oscillations are a natural and necessary attribute of using the MMAC algorithm when perfect model matching does not occur.

5. The MMAC algorithm can be expected to provide good response in conditions in which the set of possible models is closely approximated by a finite set of models. An example of this would be the control of a system subject to discrete failures in various components.

6.3 Suggestions for Future Research

Many aspects of the behavior of the MMAC algorithm remain to be studied. A partial list follows.

1. Ljung et al [24, 25] have considered closed loop identification for some specific model structures and identification methods. Application of their results to the MMAC method remains to be done. This may result in the extension of Baram's results [15] to the closed loop case.

2. In Section 4.5, the analysis produced the conclusion that the peaks of the state trajectories decrease and the period increases if a_1a_2 is less than unity. However, this eventually results in the violation of the assumption that the probability is either zero or one. Furthermore, if the closed loop system with P constant is unstable for all values of P, then the state can not approach zero since, as seen in Chapter 4, this would imply that P approaches zero even faster than the state. Simulation results to date have been of little help in that in all simulations in which long term behavior was explored an underflow to zero occurred for at least one of the components of the state. Thus, the asymptotic properties of the MMAC method remain to be determined.

3. All of the analysis and simulation results of this thesis are based on a noise-free case. The effect of noise needs to be determined. Note that this is a non-trivial task since even if all the $\underline{r}_i(\cdot)$ are assumed zero mean, $\alpha(\cdot)$ is then Chi-Squared resulting in a very complex distribution for the probability.

4. Many of the qualitative properties discussed in Section 6.2, while substantially based on the specific conclusions of this thesis, remain to be proved in general. An example of this is the proof of the general convergence of the probabilities when one of the hypothesized models matches the true system but when the assumption that the true system is diagonal is not satisfied.

5. Much of the analysis contained in this thesis has been based on the canonical problem structure of Section 3.1 in which the true system is assumed to be diagonal with two states. Extensions to non-diagonal

systems and/or systems with more than two states are needed.

6. Much of the analysis has included the assumption that there were two hypothesized models. Extensions to the N-model case are needed.

7. In the analysis of Section 4.6 on hyperbolic stability it is assumed that the control gain corresponding to the mismatched state ($\hat{g}$) is zero. Extensions to the more general case have been attempted without success (see Section 4.6). However, such results would yield useful insight into the general MMAC system.

8. In Section 4.6 it is noted that for the special case under consideration, the LQ methodology is guaranteed to produce a control which meets one of the criteria for hyperbolic stability. It remains to determine if this is a general property or merely a result of the conditions of the special case.

9. Recall from Section 4.6 that hyperbolic stability is guaranteed when Conditions (4.41) and (4.42) are met. The first of these guarantees hyperbolic stability for $P=0$ and 1 while the latter does so for $P=1/2$. If it can be shown that in cases in which Condition (4.41) is met but (4.42) is not that P oscillates in such a fashion that it never takes on intermediate values, then Condition (4.41) would itself be sufficient to guarantee hyperbolic stability.

10. In Section 6.1 it is seen that using the finite memory MMAC algorithm with $M=1$ results in what appears to be a limit cycle. Simulation results indicate that the characteristics of the limit cycle are dependent on the system parameters and <u>not</u> on the initial conditions.

For example, when initialized with extremely small initial conditions (i.e., true states of 1×10^{-10} and 1×10^{-20} respectively) behavior very much like that in Figure 6.1 has been observed in that oscillations of the same amplitude and period start after about 200 time steps. It would thus be useful to examine the characteristics of this case to determine the amplitude, bias (see Figure 6.1) and frequency of this oscillation.

11. Throughout this thesis, extensive use has been made of the matrix norm $||\underline{A}||^2 = \text{Max}\ \lambda(\underline{A}\underline{A}')$. This results in conservative estimates, for example, for the domain of attraction in Section 4.7. Calculations in which $\text{Max}|\lambda(\underline{A})|$ is used in place of the norm have been done and estimates which agree better with simulation results thus obtained. However, this quantity is not, in general, a norm for the matrix $\underline{A}$. Research into using $\text{Max}|\lambda(\underline{A})|$ in place of the norm may result in the improvement in the estimates used throughout this thesis.

12. In this thesis, the concept of hyperbolic stability is introduced for the two-state diagonal system. Extensions of this useful concept to more general system structures needs to be done.

13. A few design modifications have been proposed and analyzed. It is believed that the results of this thesis can be used to develop additional design modifications which lead to improvements of the response.

14. The results of this thesis strongly indicate the types of behavior which a general MMAC algorithm can cause. Although no consistent general methodology has been developed, the qualitative results

of this thesis provide a basis for such methodology. Specifically, important design ground rules are needed for selecting an appropriate hypothesized model set given the expected range of parameters of the true model and for the selection of the control gains for each of the hypothesized models. Our analysis in Chapter 4 provides insights into the constraints which must be imposed upon such choices and thus should help form the basis for such a design methodology.

APPENDIX A

DEVELOPMENT OF THE ALGORITHM FOR THE RANGE OF $||w_0||$ FOR EXPONENTIAL BEHAVIOR

In Section 4.7 an algorithm has been presented which allows one to compute a bound on $||\underline{w}(0)||$ such that exponential (non-oscillatory) behavior results. This appendix contains a more detailed development of that algorithm.

Consider Equation (2.22)

$$\underline{w}(k) = \tilde{\underline{A}}(P)\ \underline{w}(k)$$

$$P(k+1) = \frac{P(k)\,p(\underline{r}_1)}{P(k)p(\underline{r}_1) + (1-P(k))p(\underline{r}_2)}\quad ,$$

We make the following assumptions as in Section 4.7:

1) $P(0) = 1/2$ and $\tilde{\beta} = 1$ (i.e., $\alpha_s = 0$)

2) $\tilde{\underline{A}}(P)$ is such that there exists an $\varepsilon < 1/2$ such that

$$||\tilde{A}(P)|| < 1 \qquad \forall\ P \in (1/2 - \varepsilon,\ 1/2 + \varepsilon)$$

$$||\tilde{A}(P)|| \geq 1 \qquad \text{otherwise.}$$

Define δ to be the value of $\alpha(\cdot)$ corresponding to $P = 1/2 - \varepsilon$; that is, define δ as the solution of

$$.5 - \varepsilon = \frac{\hat{\beta}e^{-\frac{1}{2}\delta}}{\beta e^{-\frac{1}{2}\delta}+1}\quad . \tag{A.1}$$

Thus, by the definition of ε and α it can be shown (see Section 4.4) that

$$P(\cdot) \in (1/2 - \varepsilon, 1/2 + \varepsilon) \Longleftrightarrow \alpha(\cdot) \in (\alpha_s + \delta, \alpha_s - \sigma) \quad . \tag{A.2}$$

Combining Equations (4.48) and (A.2) with the definition of $\alpha(\cdot)$ from Equation (4.10), it can be shown that

$$||\tilde{A}(P(k))|| < 1 \qquad \forall k \geq 0 \tag{A.3}$$

$$\Longleftrightarrow$$

$$||\alpha(\infty)|| = \left| \sum_{i=1}^{\infty} [\underline{r}_1'(i)\underline{\theta}_1^{-1}\underline{r}_1(i) - \underline{r}_2'(i)\underline{\theta}_2^{-1}\underline{r}_2(i)] \right| < \delta \, .$$

This condition merely states that for $\underline{\tilde{A}}(P(\cdot))$ to remain a stable matrix $P(\cdot)$, and therefore also $\alpha(\cdot)$, must remain close to their initial values.

It is now necessary to examine $\alpha(\cdot)$. Equation (4.10) can be rewritten as

$$\alpha(k) = \sum_{i=1}^{k} \underline{w}'(0) \left[\prod_{j=1}^{i} \underline{\tilde{A}}(P(i))\right]' \underline{\tilde{\Sigma}} \left[\prod_{j=1}^{i} \underline{A}(P(i))\right] \underline{w}(0) \tag{A.4}$$

where

$$\underline{\tilde{\Sigma}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \underline{\theta}_1^{-1} & 0 \\ 0 & 0 & -\underline{\theta}_2^{-1} \end{bmatrix} .$$

Define a such that

$$||\underline{\tilde{A}}(P(i))||^2 \leq a \qquad \forall i = 1,2,\ldots,k \quad .$$

Taking norms in Equation (A.4) then yields

$$|\alpha(k)| \leq \sum_{i=1}^{k} \frac{a^i}{\sigma} \, ||\underline{w}_0||^2 \qquad \text{(A.5)}$$

where $\sigma = \text{MIN}\{\lambda(\underline{\theta}_1), \lambda(\underline{\theta}_2)\}$ with $\lambda(\underline{\theta})$ representing the set of all eigenvalues of $\underline{\theta}$. Evluating the summation in Equation (A.5) then yields

$$|\alpha(k)| \leq \frac{a\,||\underline{w}_0||^2}{(1-a)\sigma} (a^k - 1) \quad . \qquad \text{(A.6)}$$

Thus, the approach in the remainder of this section is to bound changes in $\alpha(\cdot)$ so that $\tilde{\underline{A}}(P(\cdot))$ is always a stable matrix. If $\tilde{\underline{A}}(P(\cdot))$ is always stable, then a is less than one and Inequality (A.7) can be replaced by

$$|\alpha(k)| \leq \frac{a\,||\underline{w}_0||^2}{(1-a)\sigma} \quad \forall k \qquad \text{(A.7)}$$

since the right hand side of Inequality (A.7) is an increasing function of k. This leads to the procedure of Section 4.7 for finding $||\underline{w}_0||^2$ such that $|\alpha(\cdot)| < \delta \;\; \forall k$.

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