# $\mathbb{E}$-RECURSIVELY ENUMERABLE DEGREES

by

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ABSTRACT

Normal Kleene recursion is considered in the general setting of a universe of sets, a formulation due to Normann and Moschovakis. In Chapter 0 Kleene's original definitions are reviewed and reformulated as an iteration of first order definability relative to a predicate. The theory of the $\kappa_r$-function for the normal Kleene theory is reviewed.

Chapter 1 defines $\mathbb{E}$-recursion and recasts it in terms of hierarchies and constructibility. The Moschovakis Phenomenon is defined and the $\kappa_r$-theory developed on initial segments of $L$.

In Chapter 2, using the parameters $\eta$ and $\rho$ first defined by G. Sacks, we prove

Theorem. $\rho$ RE-regular $\longrightarrow \exists$ minimal pairs of $\mathbb{E}$-RE degrees.

In Chapter 3, working in $\mathbb{E}(A)$-recursion on $\mathbb{E}$-closed $L(\kappa)$, we prove

Theorem: $A, B \subseteq \kappa$, RE with $A \leq_{\mathbb{E}} B$, but $B \not\leq_{\mathbb{E}} A$, then $\eta^A = \rho^A$ and $\rho^A$ $\mathbb{E}(A)$-RE regular $\longrightarrow \exists\, C \subseteq \kappa$ RE with $A <_{\mathbb{E}} C <_{\mathbb{E}} B$. In particular, this shows that the $\mathbb{E}$-RE degrees are dense, if the assumptions of the theorem are satisfied for every incomplete RE A.

Chapter 4 reviews absolute notion of degree for non-L-like $\mathbb{E}$-closed sets and indicates changes necessary in Chapters 2 and 3 to prove the analogous results for these RE degrees.

The Appendix contains the proofs of two selection theorems used in the proofs of Chapters 2 and 3.

Thesis Supervisor: Gerald E. Sacks

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# Chapter 0: Recursion in Higher Types

## §0. Normal Kleene Recursion

In preparation for our study of recursion in a universe of sets, we return to the original definition of Kleene [1959]. The partial recursive functions for recursion in a normal object of finite type are generated by induction via a list of nine schemata.

<u>Notation</u>:

| | |
|---|---|
| $a$ | number variable |
| $\alpha$ | function variable of type (1) |
| $q$ | natural number |
| $\psi, \chi$ | functions of indicated variables |
| $\phi$ | the function being defined. |

For the reader's convenience Kleene's schemata are given below with their "indices" and intended meaning on the right.

<u>Definition 0.0.0</u>:

S1 $\quad \phi(a,b) = a' = a+1$ $\qquad\qquad$ $\langle 1,\langle n_0,\ldots,n_r\rangle\rangle$
(successor function)

S2 $\quad \phi(b) = q$ $\qquad\qquad$ $\langle 2,\langle n_0,\ldots,n_r\rangle,q\rangle$
(constant function)

S3 $\phi(a,b) = a$ $\langle 3,\langle n_0,\ldots,n_r\rangle\rangle$ (projection)

S4 $\phi(b) = \psi(\chi(b),b)$ $\langle 4,\langle n_0,\ldots,n_r\rangle,g,h\rangle$ (composition)

S5 $\begin{cases} \phi(0,b) = \psi(b) \\ \phi(a',b) = \chi(a,\phi(a,b),b) \end{cases}$ $\langle 5,\langle n_0,\ldots,n_r\rangle,g,h\rangle$ (primitive recursion)

S6 $\phi(a) = \psi(a_1)$ $\langle 6,\langle n_0,\ldots,n_r\rangle,j,k,g\rangle$ (permutation of arguments)

S7 $\phi(\alpha,a,b) = \alpha(a)$ $\langle 7,\langle n_0,\ldots,n_r\rangle\rangle$ (application)

S8 $\phi(\alpha^j,b) = \alpha^j(\lambda\alpha^{j-2}\chi(\alpha^j,\alpha^{j-2},b))$ $\langle 8,\langle n_0,\ldots,n_r\rangle,j,h\rangle$ (application of type (j))

Kleene adds a scheme S9 in section 3.7 of Kleene [1959] which we shall consider in a moment.

Let $F$ be a type (2) variable. In computing $\phi(F)$ we allow ourselves to ask, at any step in computing, $F(\beta)$

if a procedure for computing $\beta(y)$ for any $y$ has arisen.

<u>Remark</u>: (i) $\phi(F)$ is computed via a preassigned procedure;

(ii) The only "non-mechanical" step is $\beta \longrightarrow F(\beta)$.

Here Kleene departs from classical recursion theory (CRT) in that the oracle for $F(\beta)$ (using computations of $\beta(y)$ for $y \in \omega$) <u>will</u> <u>not</u>, in general, be a finite object.

Returning to Kleene's schemata we note that we need, in giving an adequate definition of computability, elementary operations (primitive recursive functions) as well as the ability to "reflect" upon computation procedures (already set up) as objects. This is to say, we need a mechanism by which we are allowed to treat procedures already defined as parts of new computing procedures.

<u>Remark</u>: (i) This added feature of reflection will insure that there is no means of deciding in general whether a computation procedure terminates.

(ii) To make precise this notion of reflection, Kleene assigns numbers or indices (as above) to the recursive functions (used as Gödel numbers) following the conventions: For a primitive recursive function (a),

(a) $r$ = maximum type of $a$;

(b) $n_0,\ldots,n_r$ are the numbers of variables of each type;

(c) for S4, S5, S6 $g$ and $h$ are the indices already determined for $\psi$ and $\chi$ by descriptions of them as part of the description of $\phi$.

Finally, Kleene adds the scheme,

S9 $\phi(a,b,c) \simeq \{a\}(b)$ $\langle 9,\langle n_0,\ldots,n_r\rangle\rangle$ $\langle m_0,\ldots,m_s\rangle$ (reflection),

where $s$ = max type of $b$; $m_0,\ldots,m_s$ the numbers of variables in $b$ of types $0,\ldots,s$ respectively.

To obtain the partial recursive functions, use schemata S1-S9 using $\simeq$ as usual and interpreting,

$$\alpha^j(\lambda\alpha^{j-2}\chi(\alpha^j,\alpha^{j-2},b))$$

in S8 to be undefined when $\lambda\alpha^{j-2}\chi(\alpha^j,\alpha^{j-2},b)$ is incompletely defined. Since $\alpha^j$ is integer valued we may as well assume that it takes values in $\{0,1\}$ and hence is,

$$\alpha^j \subseteq tp(j-1)(\alpha^j \in tp(j)).$$

Nowhere in S8 do we have $\alpha^j$ in its entirety, we are only allowed to ask questions about its value at particular arguments. Notice that we are not even allowed this question at arbitrary arguments (i.e. elements of ${}^{tp(j-2)}tp(j-2)$), but only those computed in their entirety at some "previous" stage (the <u>height</u> of a computation is intuitively, the strict upper bound on the heights of its subcomputations--our ability to effectively compare the heights of computations will prove indispensable).

If one views Kleene recursion in a normal object of type $(j)$ as a hierarchy of objects of type $(j-1)$, then it is the scheme S8 which embodies the distinction between recursion relative to an element of (or "set" in) that hierarchy and recursion relative to a subset of (or predicate on) that hierarchy. For each predicate $\phi(x)$ on $tp(j-2)$, given by an integer code and a parameter already computed, we see what it has given us so far (those $x$ such that $\phi(x)$ has already been computed) and this is a typical element of $tp(j-1)$ available to us for application of $\alpha^j$.

The point of view that we are iterating first order definability relative to a predicate is instructive and was presented in detail first by Harrington [1973]. This answers, in part, the "conceptual" question raised by Kechris-Moschovakis [1977], namely the relation between higher types recursion and other work in foundations as well as its "proper place within definability theory".

## §1. Hierarchies and First Order Definability

In this section we endeavor to make explicit the models of Kleene's S1-S9 where $\alpha^j$ is of type $(j)$. We use symbols $\mathbb{F}$, G or $\mathbb{E}$ for functionals and add a superscript to indicate their type, for example, ${}^{n+2}\mathbb{F}$ is a functional of type $(n+2)$. This notation should serve to remind the reader that ${}^{n+2}\mathbb{F}$ takes elements of ${}^{\text{type}(n)}\text{type}(n) = \{f : f : \text{type}(n) \longrightarrow \text{type}(n)\}$ as arguments, i.e. $\text{dom}({}^{n+2}\mathbb{F}) \subseteq \text{type}(n+1)$. We fix, for $n \in \omega$, $x \in \text{type}(n+1)$, the functional

$$
{}^{n+2}\mathbb{E}(X) = \begin{cases} 0, & \text{if } X \neq \emptyset \\ \\ 1, & \text{if } X = \emptyset. \end{cases}
$$

<u>Definition 0.1.1</u>: ${}^{n+2}\mathbb{F}$ is <u>normal</u>, if for some $e$ an index generated by S1-S9,

$$\{e\}({}^{n+2}\mathbb{F}) = {}^{n+2}\mathbb{E}\,.$$

We shall be concerned only with normal higher type objects. Normality will be the source of our ability to compare heights of computations and of a "bounding principle" or limited replacement in models for S1-S9. The hierarchy defined below, following Harrington [1973], has precursors in hyperarithmetic theory introduced by Shoenfield [1968].

Fix an $n \geqslant 0$ and a normal ${}^{n+2}\mathbb{F}$. Let $I = \text{type}(n)$, the individuals. Given an $X \subseteq I$, let $\{e\}_p^X$ denote the $e^{\underline{th}}$ type $(n+1)$ functional primitive recursive in $X$ and ${}^{n+2}\mathbb{E}$. If $I$ is $R(\omega+n)$ = the collection of sets of rank $< \omega+n$, then consider the structure $\langle I,\in,X\rangle$. Fix a Gödel numbering of formulae and let $\{e\}_p^X$ be the $e^{\underline{th}}$ function from $I$ into $\omega$ which is first order definable over $\langle I,\in,X\rangle$. Let

$$W_e^X = \{a \in I \mid \{e\}_p^X(a) = 0\}.$$

<u>Definition 0.1.2</u>: Define $\mathcal{O}^{\mathbb{F}} \subseteq I$ and $|\cdot|^{\mathbb{F}} : \mathcal{O}^{\mathbb{F}} \longrightarrow$ ordinals and $\sigma \in \text{range } |\cdot|^{\mathbb{F}}$, $H_\sigma^{\mathbb{F}} \subseteq I$. By simultaneous induction for each $a \in I$, define $\mathcal{O}_a^{\mathbb{F}}$ and $|\cdot|_a^{\mathbb{F}} : \mathcal{O}_a^{\mathbb{F}} \longrightarrow$ ordinals:

$$\mathcal{O}^{\mathbb{F}} = \{\langle m,a\rangle \mid m \in \mathcal{O}_a^{\mathbb{F}} \wedge a \in I\}$$

$$|\langle m,a\rangle|^{\mathbb{F}} = |m|_a^{\mathbb{F}}; \quad H_0^{\mathbb{F}} = \emptyset$$

$$H_{\sigma+1}^{\mathbb{F}} = \{\langle e,a,0\rangle \mid a \in W_e^{H_\sigma^{\mathbb{F}}}\} \cup$$

$$\{\langle e,a,x+1\rangle \mid \mathbb{F}(\{e\}_p^{\langle H_\sigma^{\mathbb{F}},a\rangle}) = x\}, \quad \text{and}$$

$$H_\lambda^{\mathbb{F}} = \{\langle b,c\rangle \mid b \in \mathcal{O}^{\mathbb{F}} \wedge |b|^{\mathbb{F}} < \lambda \wedge c \in H^{\mathbb{F}}_{|b|^{\mathbb{F}}}\}$$

Remark: From now on we write $\mathbb{F}$ for ${}^{n+2}\mathbb{F}$ and omit superscripts where understood.

(i) $1 \in \mathcal{O}_a$, $|1|_a = 0$

(ii) $x \in \mathcal{O}_a \Rightarrow 2^x \in \mathcal{O}_a$ and $|2^x|_a = |x|_a + 1$

(iii) $m, e \in \omega$, if $m \in \mathcal{O}_a$, $|m|_a = \sigma$ and if $W_e^{\langle H_\sigma, a\rangle} \subseteq \mathcal{O}$, then $3^m \cdot 5^e \in \mathcal{O}_a$, $|3^m \cdot 5^e|_a$ = least limit ordinal $> \sigma$ and $> |b|$, for each $b \in W_e^{\langle H_\sigma, a\rangle}$.

From this definition we can give sense to the expression $\{e\}^{\mathbb{F}}(a) = x$ for $e \in \omega$, $a \in I$ i.e. let $e = \langle e_0, e_1\rangle$, $e_0 \in \mathcal{O}_a$ and $\{e_1\}^{H_{|e_0|_a}}(a) = x$. Then $\{e\}^{\mathbb{F}}(a)\downarrow$ (defined) simply means that for some $x$, $\{e\}^{\mathbb{F}}(a) = x$.

Definition 0.1.3: (i) $\phi$ : type$(n+2) \longrightarrow \omega$ is REC in $\mathbb{F}$, if $\exists\, e$ and $\forall\, G \in$ type$(n+2)$,

$$\phi(G) = \{e\}^{\langle \mathbb{F}, G\rangle}(0).$$

(ii) $A \subseteq$ type$(n+2)$ is RE in $\mathbb{F}$ (recursively enumerable), if $(\exists\, e)(\forall\, G \in \text{type}(n+2))$

$$G \in A \Leftrightarrow \{e\}^{\langle \mathbb{F}, G\rangle}(0)\downarrow .$$

There are some ordinal parameters associated with this hierarchy. That these definitions of recursion in ${}^{n+2}\mathbb{F}$ agree with those of Kleene can be verified as in Schoenfield [1968].

<u>Definition 0.1.4</u>: Fix normal ${}^{n+2}\mathbb{F}$ and let,

$$\kappa_0^{\mathbb{F}} = \sup\{|m|_0^{\mathbb{F}} \mid m \in \mathcal{O}_0^{\mathbb{F}}\}, \qquad \text{for} \quad i \leqslant n$$

$$\kappa_i^{\mathbb{F}} = \sup\{|m|_a^{\mathbb{F}} \mid a \in \text{type}(i) \wedge m \in \mathcal{O}_a^{\mathbb{F}}\}$$

$$= \sup\{\kappa_0^{<\mathbb{F},a>} \mid a \in \text{type}(i)\}$$

$$\kappa_n^{\mathbb{F}} = \kappa^{\mathbb{F}} = \mathcal{O}^{\mathbb{F}} \cap \text{ordinals}.$$

For $\sigma \in \mathcal{O}_n$, $\sigma$ <u>recursive in $\mathbb{F}$</u> iff $H_\sigma$ recursive in $\mathbb{F}$.

$\sigma$ <u>constructive in $\mathbb{F}$</u> iff $\sigma = |m|_0^{\mathbb{F}}$, for some $m \in \mathcal{O}_0^{\mathbb{F}}$.

<u>Remark</u>: (i) From now on omit $\mathbb{F}$ as superscript where understood and denote for $a \in I$, $i \leqslant n$ $\kappa_i^{<\mathbb{F},a>}$ by $\kappa_i^a$.

(ii) Definition 0.1.4 can easily be relativized to $a \in I$.

Recall the meaning given above to the expression $\{e\}_p^X$ as the $e^{\underline{th}}$ function from $I$ into $\omega$ which is first order definable over the structure $\langle I,\in,X\rangle$. Then the "sets" of the form,

$$W_e^X = \{a \in I \mid \{e\}_p^X(a) = 0\}$$

as $e$ ranges over a fixed Gödel numbering of formulae are nothing more than the collection of subsets of $I$ first order definable over $\langle I,\in,X\rangle$. With this in mind it's clear that the universe for Kleene recursion in normal ${}^{n+2}\mathbb{F}$ is nothing more than the result of iterating first order definability over a set of individuals, passing through limit stages only when we have defined an effective procedure which generates ordinals unbounded in that limit. What is called $\kappa^{\mathbb{F}}$ above is the first limit ordinal where we fail to generate such a procedure. The following precise expression of this is due to Harrington.

<u>Definition 0.1.5</u>: For $\sigma \in On$, define $L_\sigma[\mathbb{F}]$ as:

$L_0[\mathbb{F}] = I$

$L_{\sigma+1}[\mathbb{F}] = \{X \subseteq L_\sigma[\mathbb{F}] \mid X$ is first order definable with parameters over

$$\langle L_\sigma[\mathbb{F}],\in,\mathbb{F} \restriction (\text{type}(n+1) \cap L_\sigma[\mathbb{F}])\rangle\}$$

$L_\lambda[\mathbb{F} = \bigcup_{\delta<\lambda} L_\delta[\mathbb{F}]$. Let $M_\sigma[\mathbb{F}]$ denote the structure,

$$<L_\sigma[\mathbb{F}], \in, \mathbb{F} \restriction (\text{type}(n+1) \cap L_\sigma[\mathbb{F}])>$$

and $L$ be the first order language appropriate to the structures $M_\sigma[\mathbb{F}]$. $L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$ is then the structure for normal Kleene recursion in ${}^{n+2}\mathbb{F}$. The above is Gödel's constructible hierarchy relativized to a predicate $\mathbb{F}$, constructing over I. For details the reader is directed to Harrington [1973].

Remarks: (i) There is a one-to-one correspondence $e \longleftrightarrow \phi_e$ between integers and $\Sigma_1$ formulae of $L$, such that $\forall\ G \in \text{type}(n+2)$,

$$\{e\}^{<\mathbb{F},G>}(0)\downarrow \Leftrightarrow M_{\kappa_0^{<\mathbb{F},G>}}(<\mathbb{F},G>) \models \phi_e.$$

(ii) $A \subseteq I$ is RE in $\mathbb{F}$ iff there is a $\Sigma_1$ $\phi \in L$ such that $\forall\ a \in I$,

$$a \in A \Leftrightarrow M_{\kappa_0^a}[\mathbb{F}] \models \phi(a).$$

Here and in many similar situations we will have $\kappa_0^{<a,\mathbb{F}>} < \kappa^{\mathbb{F}}$. Thus, writing $\kappa_0^a$ for $\kappa_0^{<a,\mathbb{F}>}$, the study of RE subsets of I for $L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$ is the study of the function $a \longrightarrow \kappa_0^a$ for $a \in I$. For $n \geqslant 1$ the structures $L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$ will not be $\Sigma_1$-admissible and it is this theme of inadmissibility that gives this theory its peculiar flavor. There is, in fact, a $\Sigma_1$ formula $\phi(x) \in L$ such that $(\forall a \in I)$

$$M_{\kappa^{\mathbb{F}}}[\mathbb{F}] \models \phi(a)$$

however

$$M_{\kappa_0^a}[\mathbb{F}] \not\models \phi(a).$$

§2. $\Sigma_1$-Reflecting Ordinals and Selection

In general we say that a theory of recursive enumerability admits of full selection just in case for any RE set A defined by $\phi_e$ (for some Gödel number e), if $A \neq \emptyset$, then there is a uniform way of passing from e to a non-empty REC subset $Z \subseteq A$. In CRT this amounts to saying that given an index e for a non-empty, recursively enumerable set of natural numbers $W_e$, there is a recursive function f such that $f(e) \in W_e$. The procedure f will, of course, diverge if $W_e = \emptyset$ and can be taken to equal the least integer in $W_e$. Even for $L_{\kappa^{3\mathbb{E}}}[{}^3\mathbb{E}]$ this is too much to hope for since we cannot even make sense of the "least" $a \in 2^\omega$ in the absence of well-ordering of $2^\omega$. In this section we will review the limited selection theorems for $L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$ and develop the theory of $\Sigma_1$-reflecting ordinals that results from the failure of full selection. It is this reflection phenomenon which will form the basis for the priority method here and in more general settings.

<u>Definition 0.2.1</u>: (i) For $a \in I$ and $\alpha < \kappa^{\mathbb{F}}$ call $\alpha$ an <u>a-reflecting ordinal</u>, if for all $\Sigma_1$-formulae, $\phi(x) \in L$

$$M_\alpha[\mathbb{F}] \models \phi(a) \Rightarrow M_{\kappa_0^a}[\mathbb{F}] \models \phi(a);$$

(ii) the limit of a-reflecting ordinals is clearly a-reflecting so let $\underline{\kappa_r^a}$ = greatest a-reflecting ordinal (i.e. there is a $\Sigma_1$ formula $\theta_a(x)$ in $L$ such that $M_{\kappa_r^a}[\mathbb{F}] \not\models \theta_a(a)$, but $M_{\kappa_r^a+1}[\mathbb{F}] \models \theta_a(a)$).

The following proposition gives a hint of the connection between reflecting ordinals and selection principles. If $X \in L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$, we say that $a \in I$ <u>selects over X</u>, if whenever $A \subseteq X$, non-empty and RE in $\langle a,x \rangle$, then there is a $Z \subseteq A$, non-empty and REC in $\langle a,x \rangle$.

<u>Proposition 0.2.2</u>: Let $X \in L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$ and $a \in I$, then the following are equivalent:

(i) a selects over X.

(ii) $(\forall\, t \in X)[\kappa_r^{\langle a,x,t \rangle} = \kappa_r^{\langle a,x \rangle}]$.

<u>Proof</u>. (i) ⟶ (ii). Suppose (ii) fails, then

$$R = \{t \mid \kappa_r^{<a,x,t>} > \kappa_r^{<a,x>}\}$$

is RE in $<a,x>$, non-empty and clearly $a$ does not select on $R$.

(ii) $\longrightarrow$ (i). Let $R \subseteq X$ be RE in $<a,x>$, via $e$, then $(\exists\ t \in X)(\exists\ \gamma < \kappa_0^{<a,x,t>})$ $L_\gamma[\mathbb{F}]\ \exists\ \gamma'[|\{e\}(t,a,x)| < \gamma']$, but

$$\kappa_0^{<a,x,t>} < \kappa_r^{<a,x,t>} = \kappa_r^{<a,x>}, \quad \text{so}$$

$$L_{\kappa_r^{<a,x>}}[\mathbb{F}] \models (\exists\ t \in x)(\exists\ \gamma)[|\{e\}(t,a,x)| = \gamma],$$

so by reflection,

$$L_{\kappa_0^{<a,x>}}[\mathbb{F}] \models (\exists\ t \in x)(\exists\ \gamma)[|\{e\}(t,a,x)| = \gamma].$$

Then Gandy Selection computes such a $\gamma$. □

Thus the presence of an effective selection operator over a set $X$ is equivalent to a crude ordinal bound on computations in elements of $X$.

The following two results establish selection principles for particular $X \in L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$.

(Gandy Selection). There is a partial recursive operator $\phi$ such that for $A \subseteq \omega$ $A$ RE in $<a,\mathbb{F}>$ via $e$ for $a \in I$,

(i) $\phi(e,a)\downarrow \longleftrightarrow A \neq \emptyset$ and if $A \neq \emptyset$,

(ii) $\phi(e,a) \in A$.

This result is proved using stage comparison and the recursion theorem for normal Kleene recursion. One consequence of Gandy selection is that the union of RE sets is RE. It states, in effect, that the family of RE sets is closed under existential quantification over integers.

(Grilliot Selection). Let $n \geqslant 1$ and $\mathbb{F}$ of type $(n+2)$ normal and $J = \text{type}(n-1)$ (subindividuals). There is a partial recursive operator $\phi$ such that for $A \subseteq J$, $A$ RE in $\langle a,\mathbb{F}\rangle$ via $e$ for $a \in I$,

(i) $\phi(e,a)\downarrow \longleftrightarrow A \neq \emptyset$ and if $A \neq \emptyset$,

(ii) $Z = \{j \in J \mid |\langle e,aj\rangle|^{\mathbb{F}} \leqslant |\phi(e,a)|^{\mathbb{F}}\}$ is a non-empty REC in $\langle a,\phi,\mathbb{F}\rangle$ subset of $A$.

Grilliot's selection principle tells us that for any $a \in I$ and any $\Sigma_1$ formula $\phi(X)$ in $L$, if

$$M_{\kappa^a_{n-1}}[\mathbb{F}] \models \phi(a), \qquad \text{then}$$

$$M_{\kappa^a_0}[\mathbb{F}] \models \phi(a).$$

In other words, if $\phi(a)$ is true at some ordinal recursive in $a$, $\mathbb{F}$ and some subindividual $j \in I$, then $\phi(a)$ is true at some ordinal recursive in $a$, $\mathbb{F}$. As a consequence, the RE in $a$, $\mathbb{F}$ (for $a \in I$) subsets of $I$ are closed under the quantifier $\exists\, j \in J$. The proof of Gandy Selection can be found in Moldestad [1977] and that of Grilliot selection in Harrington-MacQueen [1976].

A natural question is then whether the family of RE in $\langle a, \mathbb{F}\rangle$ (for some $a \in I$) subsets of $I$ are closed under $\exists\, b \in I$. The answer, due to Moschovakis [1967], is a resounding "no!".

<u>Lemma 0.2.3</u> (Kechris [1973]): For $a \in I$, if $B$ is a nonempty subset of $I$, co-RE in $\langle \mathbb{F}, a\rangle$, then there is a $b \in B$ such that $\kappa_r^{\langle a,b\rangle} \leqslant \kappa_r^a$.

<u>Proof</u>. Suppose not and take $\Sigma_1$ formula $\phi(x)$ in $L$ such that

$$M_{\kappa_r^a+1}[\mathbb{F}] \models \phi(a), \qquad \text{but}$$

$$M_{\kappa_0^a}[\mathbb{F}] \not\models \phi(a).$$

For all $b \in B$, by assumption $\kappa_r^{\langle a,b\rangle} > \kappa_r^a$ and so

$$M_{\kappa_r^{\langle a,b\rangle}}[\mathbb{F}] \models \phi(a).$$

By reflection

$$M_{\kappa_0^{\langle a,b\rangle}}[\mathbb{F}] \models \phi(a).$$

Since $B$ is co-RE in $\langle \mathbb{F}, a\rangle$, there is a $\Sigma_1$ formula $\psi(x,y)$ of $L$ such that

$(\forall\ b \in I)[b \notin B$ <u>iff</u> $M_{\kappa_0^{<a,b>}}[\mathbb{F}] \models \psi(a,b)]$. Thus,

$$(\forall\ b \in I)[M_{\kappa_0^{<a,b>}}[\mathbb{F}] \models \psi(a,b) \vee \phi(a)]$$

and by bounding

$$(\forall\ b \in I)[M_{\kappa_0^a}[\mathbb{F}] \models \psi(a,b) \vee \phi(a)].$$

Since $B \neq \emptyset$, $\exists\ b \in I$ such that

$$M_{\kappa_0^a}[\mathbb{F}] \models \psi(a,b) \quad \text{and so}$$

$$M_{\kappa_0^a}[\mathbb{F}] \models \phi(a),$$

contradicting the choice of $\phi(x)$. □

Results of this sort are often called basis theorems, the point here is that if a co-RE set has an element, then it has an element whose $\Sigma_1$-reflecting ordinals are bounded by those of its defining parameter.

Intuitively, a computation is convergent iff its computation tree is well-founded. Through the computation tree of a divergent computation is an infinite descending path of subcomputations and divergence is a co-RE phenomenon. The next theorem states that, with sufficient coding and the basis result of the previous lemma, that <u>witness</u> to divergence will be in $L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$.

<u>Theorem 0.2.4</u> (Moschovakis [1967]): For any subset $A$ of $I$ RE in $\langle a,\mathbb{F}\rangle$ for $a \in I$, there is a relation $R(x,y)$ on $I$ RE in $\langle a,\mathbb{F}\rangle$ such that,

$$(\forall\ a \in I)[a \notin A \quad \text{iff} \quad (\exists\ b \in I)R(a,b)].$$

<u>Proof</u>. Consider $A = \mathcal{O}^{\mathbb{F}}$ the universal RE in $\mathbb{F}$ subset of $I$. Define an RE in $\mathbb{F}$ predicate $R'(x,y) \subseteq I \times I$ by: $R'(b,c) \Leftrightarrow$

(i) $b = \langle 2^m, a'\rangle$ for some $a' \in I$ and for some $m \in \omega$, then $c = \langle m,a'\rangle$;

(ii) $b = \langle 3^m \cdot 5^e, a'\rangle$ for some $a' \in I$ and some $m, e \in \omega$ and if $c \neq \langle m,a'\rangle$, then $\langle m,a'\rangle \in \mathcal{O}^{\mathbb{F}}$, $|m|^{\mathbb{F}}_{a'} = \sigma$ and $c \in W_e^{\langle H_\sigma, a'\rangle}$.

(iii) if $\forall\ a' \in I\ \forall\ m, e \in \omega[b \neq \langle 2^m,a'\rangle$ and $b \neq \langle 3^m \cdot 5^e, a'\rangle]$, then $c = 0$

(iv) $\forall\ a' \in I[b \neq \langle 1,a'\rangle]$.

Then $R'(x,y)$ gives the relation of being an immediate subcomputation in $L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$. In the case that some $b \notin \mathcal{O}^{\mathbb{F}}$ we can then use $R'(x,y)$ to assert that fact by asserting that there is an encoding of path through the tree of subcomputations. $R'(x,y)$ fails to be REC because of clause (ii). More precisely,

(a) if $R'(b,c)$ and $b \in \mathcal{O}^{\mathbb{F}}$, then $c \in \mathcal{O}^{\mathbb{F}}$ and $|b|^{\mathbb{F}} > |c|^{\mathbb{F}}$; and

(b) if $b \notin \mathcal{O}^{\mathbb{F}}$, then $(\exists\, c \in I)[c \notin \mathcal{O}^{\mathbb{F}} \wedge R'(b,c)]$.

Define $R$ by:

$$R(a,b) \Leftrightarrow [b \in {}^{\omega}I \wedge (b)_0 = a \wedge$$

$$(\forall\, i \in \omega)[R'((b)_i,(b)_{i+1}]].$$

To see that $R(a,b)$ defines $a \notin \mathcal{O}^{\mathbb{F}}$: If $a \in \mathcal{O}^{\mathbb{F}}$ and $R(a,b)$, then by (a)

$$|a|^{\mathbb{F}} = |(b)_0|^{\mathbb{F}} > |(b)_1|^{\mathbb{F}} > \dots ,$$

an infinite descending chain of ordinals, which is absurd. If $a \notin \mathcal{O}^{\mathbb{F}}$, then one can find $(b_0,b_1,\dots$ such that $b_0 = a$ and $R'(b_i,b_{i+1})$, i.e. taking $b = \langle b_0,b_1,\dots\rangle$, then $R(a,b)$ holds. □

A path through the computation tree of a divergent $a \in I$ will be called a <u>Moschovakis Witness</u> to its divergence. As a consequence of Theorem 0.2.4, $L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$ is not $\Sigma_1$-admissible for ${}^{n+2}\mathbb{F}$ normal with $n \geq 1$. And the $\mathbb{F}$-RE subsets of $I$ are not closed under the quantifier $\exists\, b \in I$.

The next result, due to Harrington [1973], brings together the basis theorem of Kechris and the apparently negative result of Moschovakis to characterize the relation between $\Sigma_1$ reflecting ordinals and this failure of selection over I.

<u>Theorem 0.2.5</u> (Harrington): There is a formula (not $\Sigma_1$) in $L$, $\theta(x)$, such that for all $a \in I$,

$$M_{\kappa_r^a}[\mathbb{F}] \models \theta(a),$$

but for all $\sigma < \kappa_r^a$,

$$M_\sigma[\mathbb{F}] \not\models \theta(a).$$

<u>Remark</u>: We include Harrington's proof of this theorem to make $\theta(x)$ explicit. This result will allow us to turn a failure of selection into a powerful tool in preserving computations relative to an oracle in the setting of a priority argument. The import is that looking down from $\kappa_r^a$ is the same as looking down from $\kappa^{\mathbb{F}}$ when it comes to facts REC in $\langle a,\mathbb{F}\rangle$.

<u>Proof</u>. Since the predicate $R'(x,y)$ of Theorem 0.2.4 is RE in $\mathbb{F}$, there is a $\Sigma_1$ formula $\phi(x,y)$ in $L$ such that,

$(\forall\ b,\ c \in I)[R(b,c)$ __iff__ $M_{\kappa_0^{<b,c>}}[\mathbb{F}] \models \phi(b,c)$ __iff__ $M_{\kappa^{\mathbb{F}}}[\mathbb{F}] \models \phi(b,c)]$. Let $\psi(x)$ be a $\Sigma_1$ formula of $L$ such that, $(\forall\ a \in I)[a \in \mathcal{O}^{\mathbb{F}}$ __iff__ $M_{\kappa_0^a}[\mathbb{F}] \models \psi(a)]$. Let $\theta(x)$ be the following formula of $L$:

$$\theta(x) \Leftrightarrow (\forall\ m \in \omega)[\psi(<m,x>) \vee \exists\ b \in I$$

$$[b = <b_0,b_1,\ldots> \text{ and } b_0 = <m,x>$$

$$\text{and } (\forall\ i \in \omega)\phi(b_i,b_{i+1})]].$$

Thus for $\sigma \in On$ and $a \in I$, if

$$M_\sigma[\mathbb{F}] \models \theta(a), \quad \text{then}$$

$$(\forall\ m \in \omega)[m \in \mathcal{O}_a^{\mathbb{F}} \text{ iff } M_\sigma[\mathbb{F}] \models \psi(<m,a>)]$$

and therefore $\sigma \geq \kappa_0^a$ and $\sigma \geq \kappa_r^a$, for otherwise this fact would reflect below $\kappa_r^a$, a contradiction.

__Claim__: For any $a \in I$, $M_{\kappa_r^a}[\mathbb{F}] \models \theta(a)$.

__Proof of Claim__. Given $m \in \omega$ such that $<m,a> \notin \mathcal{O}^{\mathbb{F}}$, find $b = <b_0,b_1,\ldots> \in I$ such that $R(<m,a>,b)$ and $(\forall\ i \in \omega)[\kappa_r^{<b_i,b_{i+1}>} \leq \kappa_r^a]$. Construct $b$ as follows:

$b_0 = \langle m,a\rangle$ and given $b_i \notin \mathcal{O}^{\mathbb{F}}$ such that $\kappa_r^{b_i} \leq \kappa_r^a$, find $b_{i+1} \notin \mathcal{O}$ such that $R'(b_i,b_{i+1})$ and $\kappa_r^{\langle b_i,b_{i+1}\rangle} \leq \kappa_r^a$. As was already remarked, this presents no problem except in the case that $b_i = \langle 3^{m'} \cdot 5^e, a'\rangle$ for some $a' \in I$, $m'$ $e \in \omega$ and $\langle m',a'\rangle \in \mathcal{O}$, an "I-branching" in the computation tree. Let $\sigma = |m'|^{\mathbb{F}}_{a'}$ and $B = \{c \in I \mid c \notin \mathcal{O}^{\mathbb{F}} \wedge c \in W_e^{\langle H_\sigma,a'\rangle}\}$ which is co-RE in $\langle b_i,\mathbb{F}\rangle$. $B \neq \emptyset$, for otherwise $b_i \in \mathcal{O}^{\mathbb{F}}$, contradicting the choice of $b_i$. By Lemma 0.2.3 of Kechris, $\exists\, b_{i+1} \in B$ such that

$$\kappa_r^{\langle b_i,b_{i+1}\rangle} \leq \kappa_r^{b_i} \leq \kappa_r^a.$$

$b = \langle b_0,b_1,\ldots\rangle$ is the desired Moschovakis Witness (MW). □

The following facts are easily verified using Theorem 0.2.5.

<u>Facts</u>: Let $\kappa_r^{\emptyset} = \kappa_r$,

(i) $(\forall\, a \in I)$ $\kappa_r \leq \kappa_r^a$

(ii) $(\forall\, a \in I)[\kappa_r < \kappa_r^a$ $H_{\kappa_r}$ is REC in $\langle \mathbb{F},a\rangle]$

(iii) $B \subseteq I$, $B \neq \emptyset$ co-RE, then there is a non-empty subset of $B$ primitive recursive in $H_{\kappa_r}$ (so $H_{\kappa_r}$ can <u>select</u> from a non-empty co-RE set).

(iv) (Kechris) $A \subseteq I$, RE in $\mathbb{F}$ has non-empty <u>REC in $\mathbb{F}$</u> subset iff $(\exists\, a \in A)[\kappa_0^a \leq \kappa_r]$.

Remark: The reader can consult Harrington [1973] for the proofs of (i) - (iv).

Notice that fact (iv) gives the precise connection between selection and $\Sigma_1$-reflecting ordinals. Relativizing fact (ii) to $a \in I$ and using Theorem 0.2.5 $H_{\kappa_r^a}$ contains all information needed to decide the predicate "$m \in \mathcal{O}_a^{\mathbb{F}}$".

Proposition 0.2.6: Let $\mathbb{F} \in \text{type}(n+2)$ be normal with $n \geq 1$, then if $X \in L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$ and, $\sup_{t \in X} \kappa_r^t = \kappa^{\mathbb{F}}$, then

$$\sup_{t \in X} \kappa_0^t = \kappa^{\mathbb{F}}.$$

Proof. Notice first that Theorem 0.2.4 implies that, $(\forall\, a \in I)[\kappa_r^a < \kappa^{\mathbb{F}}]$. Suppose the proposition fails and fix $X \in L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$ such that,

$$\sup_{t \in X} \kappa_r^t = \kappa^{\mathbb{F}}, \quad \text{but}$$

$$\sup_{t \notin X} \kappa_0^t \leq \delta_0 < \kappa^{\mathbb{F}}.$$

We know that $\sup_{a \in I} \kappa_r^a = \kappa^{\mathbb{F}}$, so take $a_0 \in I$ such that $\kappa_r^{a_0} > \delta_0$. Then there exists a $\sigma < \kappa^{\mathbb{F}}$ such that,

$$M_\sigma[\mathbb{F}] \models \theta(a_0),$$

where $\theta(x)$ is the formula of $L$ constructed in Theorem 0.2.5. By reflection,

$$M_{\delta_0}[\mathbb{F}] \models \theta(a_0)$$

and so $\kappa_r^{a_0} \leqslant \delta_0$, which contradicts the choice of $a_0$. $\square$

This proposition will allow us to make use of the upper bound, given uniformly by Harrington's $\theta(x)$, in preserving a computation $\{e\}^A(a)$ where $A$ is an oracle for some predicate on $I$, $e \in \omega$ and $a \in I$. Through preserving the value of the characteristic function of $A$ we will not only be able to insure that $\{e\}^A(a)\downarrow$ for the final $A$, but, <u>in fact</u>, that $\{e\}^A(a)\uparrow$ for the final $A$, $\{e\}^{A\cap\kappa_r^{\langle e,a\rangle}}(a)\uparrow$. The fact that $\kappa_r^a < \kappa$ for $a \in I$ tells us that we need only preserve $A$'s values through $\kappa_r^{\langle e,a\rangle}$ to insure the behavior of $\{e\}^A(a)$! The details of this preservation strategy will be given in our chapter on RE degrees.

# Chapter 1: $\mathbb{E}$-Recursion

## §0. Background

This section is intended as an introduction to the theory of recursion on a universe of sets which was introduced independently by D. Normann [1978] and Y. N. Moschovakis [1976]. $\mathbb{E}$-recursion will be the theory of partial recursive set functions, that is, functions whose domain is contained in a universe of sets and which take sets as values.

The thought behind Normann's introduction of "Set Recursion" was that the companion for Kleene recursion in normal objects of finite type was easily identified with a natural universe for set recursion. Thus priority arguments can be carried out in this much more familiar set-theoretic setting and hence one can draw conclusions about the degrees of functionals. The success of that program is evidenced by Normann's results on Post's Problem and the existence of a minimal pair [1975] for Kleene recursion in a normal function $\mathbb{F}$ of type-$(n+2)$ for $n \geqslant 1$. Set recursion was the result of Normann's study of the companion of a normal functional of finite type.

The computation theory that resulted was far more general in nature than had been suggested by the intended application. In contrast to $\alpha$- and $\beta$-recrusion theories, one is given a priori search only over the integers. As a result it is a theory based on the notion of a computation and this notion is absolute between models of the theory, provided one insists that they be transitive. That is, given a domain $M$ which is $\mathbb{E}$-closed (closed with respect to $\mathbb{E}$-REC functions, a notion which will be made precise momentarily) and given $e \in \omega$ and $x \in M$, then the computation $\{e\}(x)$ is convergent in $M$ if and only if it is in $V$. Associated with such an $e$ and $x$ in $M$ is a computation tree, where branching is given by the relation of being a subcomputation and to say that $\{e\}(x)$ converges is simply to assert that the associated computation tree is well-founded. Thus, just as in classical recursion theory, a set $x$ will be put into an RE collection of sets just in case there is a well-founded computation tree on $x$ which is "computable" from $x$ and a definition of that RE collection. A typical step in such a computation will compute a set $W$ and then apply a preassigned procedure to each element $y$ of $W$. So, unlike CRT, the branching in a computation tree is potentially infinite depending on the "size" of $W$. As a result the very existence of solutions to many priority arguments will depend on the character of the power set operation and the existence of intimate ties between the sets of the recursion-theoretic universe and its ordinals.

## §1. $\mathbb{E}$-Recursive Functions

The domain for $\mathbb{E}$-Recrusion will be the universe of sets. If $R \subseteq V$, the universe, and $R \not\supseteq V$, then computation relative to $R$ means that we may ask if a given $x \in V$ is an element of $R$ in the course of computing. In other words, if we have generated a set $x$ in the process of computation we are allowed to compute $x \cap R$ and treat it as a set. By this we simply mean that we are allowed to treat $x \cap R$ as a finite entity, we may use all information about $x \cap R$ or information uniformly derived from elements of it at the same time. The notion of relative computability is reflected in scheme (vi). $\mathbb{E}$-recursion is nothing more than the rudimentary set functions of Jensen (cf. Devlin [1973]) augmented by a reflection (Kleene [1959]) or diagonalization (Normann [1978]) scheme.

Let $R \subseteq V$ be a relation. Define the functions partial recursive relative to $R$ with index $e$ by the following schemes:

<u>Definition 1.1.1</u>:

(i) $f(x_1,\ldots,x_n) = x_i$ $\quad e = \langle 1,n,i \rangle$ (projection)

(ii) $f(x_1,\ldots,x_n) = x_i \setminus x_j$ $\quad e = \langle 2,n,i,j \rangle$ (difference)

(iii) $f(x_1,\ldots,f_n) = \{x_i,x_j\}$ $\qquad$ $e = \langle 3,n,i,j\rangle$

(unordered pair)

(iv) $f(x_1,\ldots,x_n) = \bigcup_{y \in x_i} h(y,x_2,\ldots,x_n)$

$e = \langle 4,n,e'\rangle$, where $e'$ is an index for $h$.

(v) $f(x_1,\ldots,x_n) = h(g_1(x_1,\ldots,x_n),\ldots,g_m(x_1,\ldots,x_n))$

$e = \langle 5,n,m,e',e_1,\ldots,e_m\rangle$, where $e'$ is an index for $h$ and $e_1,\ldots,e_m$ are indices for $g_1,\ldots,g_m$, respectively.

(vi) $f(x_1,\ldots,x_n) \simeq x_i \cap R$ $\qquad$ $e = \langle 6,n,i\rangle$

(vii) $f(e_1,x_1,\ldots,x_n,y_1,\ldots,y_m) \simeq \{e\}^R(x_1,\ldots,x_n)$

$e = \langle 7,n,m\rangle$.

<u>Remark</u>: Scheme (iv) is defined just in case $h(y,x_2,\ldots,x_n)$ is defined for all $y \in x_1$.

The partial functions defined by these schemes are called <u>$\mathbb{E}$-recursive relative to R</u> and are denoted $\{e\}^R$. All functions that are rudimentary in R will be $\mathbb{E}$-recursive relative to R (<u>$\mathbb{E}(R)$-recursive</u>). Since the constant functions n, for $n \in \omega$ are rudimentary, they will be $\mathbb{E}$-recursive. Arguments may be commuted via schemata (i) and (v).

Because the $\mathbb{E}$-recursive ($\mathbb{E}$-REC) functions are generated inductively via these schemata, canonical notions of length of computation, subcomputation and computation tree are easily derived from Definition 1.1.1. Standard proofs yield the recursion theorems and the $s^n_m$-theorem. The following two results are due to Normann [1978] in $\mathbb{E}$- recursion and are two important aspects of the theory's flavor.

<u>Lemma 1.1.2</u> (Normality). There is in $\mathbb{E}$-recursion an index e such that for arbitrary R, x, e, $\vec{x}$:

$$\{e\}^R(x,e,x) \simeq \begin{cases} 0, & \text{if } (\forall y \in x)[\{e_1\}^R(y,\vec{x}) \simeq 0] \\ \\ 1, & \text{if } (\forall y \in x)[\{e_1\}^R(y,\vec{x})\downarrow] \text{ and} \\ & \quad (\exists y \in x)[\{e_1\}^R(y,\vec{x}) \neq 0] \end{cases}$$

where "$\downarrow$" indicates "is defined".

Proof. Take $\phi$ rudimentary such that

$$\phi(0) = 0 \quad \text{and}$$

$$\phi(x) = 1, \quad \text{for all} \quad x \neq \emptyset.$$

So assume that $\{e_1\}^R$ is a characteristic function taking values in $\{0,1\}$ and let

$$\{e\}^R(x,e_1,\vec{x}) = \bigcup_{y \in x} \{e_1\}^R(y,\vec{x})$$

using scheme (iv). □

This lemma justifies the use of the term "$\mathbb{E}$-Recursion" in analogy to the source of normality of a functional ${}^{n+2}\mathbb{F}$, namely ${}^{n+2}\mathbb{E}$ is recursive in ${}^{n+2}\mathbb{F}$ where,

$${}^{n+2}\mathbb{E}(\lambda\alpha\{e\}(\alpha^j,\alpha)) = \begin{cases} 0, & \text{if} \quad (\forall\ \alpha)[\{e\}(\alpha^j,\alpha) \simeq 0] \\ 1, & \text{if} \quad (\forall\ \alpha)[\{e\}(\alpha^j,\alpha)\ ] \ \text{ and } \\ & \qquad (\exists\ \alpha)[\{e\}(\alpha^j,\alpha) \neq 0] \end{cases}$$

The computability of the $\{e\}^R$ of Lemma 1.1.2 demonstrates sense in which arguments or sets are "finite" in the tradition of many generalized recursion theories. Information uniformly derived from a set can be used in its entirety.

Lemma 1.1.3 (Stage Comparison). There is an $\mathbb{E}$-REC function $p(x,y)$ such that $p(a,b)$ iff $a\downarrow$ or $b\downarrow$, where $a$ and $b$ are computations and

$$p(a,b) = \begin{cases} 0 & \text{if} \quad \|a\| \leqslant \|b\| \\ \\ 1 & \text{if} \quad \|a\| > \|b\|. \end{cases}$$

($\|\cdot\|$ refers to the canonical function giving the length of its argument, a computation.)

Proof. (Sketch) Apply recursion theorem to the definitions of $a$ and $b$ via the schemes. For the details of a similar argument see Moldestad [ 1977 ], where this result for recursion in a normal ${}^{n+2}\mathbb{F}$ is proven. $\square$

An immediate corollary of Lemma 1.1.3 is what we called Gandy Selection in the previous chapter.

Corollary 1.1.4 (Gandy Selection). There is in $\mathbb{E}$-recursion an index $e$ such that for any $R$, $e_1$, $\vec{x}$,

$$\{e\}^R(e_1,\vec{x})\downarrow \Leftrightarrow (\exists\, n \in \omega)[\{e_1\}^R(n,\vec{x})\downarrow],$$

and then

$$\{e_1\}^R(\{e\}^R(e_1,\vec{x}),\vec{x})\downarrow .$$

Again we refer the reader to Moldestad [1977] for proof.

There are two points we would like to emphasize before making explicit the relation between $\mathbb{E}$-recursion and the hierarchies of Chapter 0. First is that Scheme (iv) embodies what we will refer to as an effective bounding principle, a notion familiar to generalized recursion theory (for example, hyperarithmetic theory). Secondly, and as a consequence of this bounding principle, one can regard $\mathbb{E}$-recursion as a minimal formalization of the notion of Effective Transfinite Recursion (ETR). It is the presence of ETR, provable from schemata (i) - (vii), which gives a normed theory, i.e. an effective means of measuring the length of computation. For the moment let us say that a set $M$ is $\mathbb{E}$-closed iff $M$ is transitive, closed under pairing and if $e$ is an index for an $\mathbb{E}$-REC function on $M$, $X \in M$, then $\{e\}(x)\downarrow \longrightarrow \{e\}(x) \in M$ (recall the $\mathbb{E}$-REC functions are set-valued). We cannot yet give the precise definition of being $\mathbb{E}$-closed, but this intuition should motivate much of what follows.

## §2. Hierarchies of Computations

This section will present a definition of $\mathbb{E}$-recursion in terms of hierarchies of computations. This definition is a straightforward generalization of the hierarchies for recursion in higher types of the last chapter as pioneered by Schoenfield [1968], Grilliot [1969]. The equivalence of this definition for normal Kleene recursion to that via schemata can be found in Normann [1978]. The hierarchical definition serves to make explicit both the definition of computable set function and our choice of the domain for $\mathbb{E}$- recursion. The reader should keep in mind the hierarchy for Kleene recursion in a normal ${}^{n+2}\mathbb{F}$ as we proceed.

Preliminary to our final notion of computability is that of primitive recursions. Let $X \in V$ and fix a Gödel numbering of first order formulae. Let $\{e\}^X$ be the $e^{\underline{th}}$ function from the $TC(X)$ into $\omega$, which is first order definable over the structure $\langle TC(X), \in, X\rangle$ ($TC(X)$ denotes the transitive closure of $X$). Then $W_e^X = \{a \in TC(X) : \{e\}^X(a) = 0\}$ is the $e^{\underline{th}}$ subset of $TC(X)$ which is primitive recursive in $X$.

Define now by simultaneous effective transfinite recursion for each $\langle a,X\rangle$ with $a \in TC(X)$ a set of integers, ${}^{\langle a,X\rangle}$ and a map $|\cdot|^{\langle a,X\rangle} : {}^{\langle a,X\rangle} \longrightarrow On$, the ordinals. For each ordinal number $\sigma$ in the range of $|\cdot|^{\langle a,X\rangle}$, as $a$ ranges over $TC(X)$, define sets $H_\sigma^X$.

Let $\mathcal{O}^X = \{\langle n,a\rangle \mid n \in \mathcal{O}^{\langle a,X\rangle}\}$ and $|\langle e,a\rangle|^X = |e|^{\langle a,X\rangle}$. For A an $\mathbb{E}$-closed set let $\mathcal{O}^A = \bigcup_{X\in A} \mathcal{O}^X$, then the induction proceeds as follows,

Definition 1.2.0:

(i) $1 \in \mathcal{O}^{\langle a,X\rangle}$; $|1|^{\langle a,X\rangle} = 0$ and $H_0^X = \emptyset$;

(ii) If $n \in \mathcal{O}^{\langle a,X\rangle}$ and $|n|^{\langle a,X\rangle} = \sigma$, then $2^n \in \mathcal{O}^{\langle a,X\rangle}$; $|2^n|^{\langle a,X\rangle} = |n|^{\langle a,X\rangle} + 1 = \sigma + 1$ and

$$H_{\sigma+1}^X = \{\langle e,a,0\rangle : a \in W_e^{H_\sigma^X}\};$$

and finally,

(iii) Given $m, e \in \omega$, if $m \in \mathcal{O}^{\langle a,X\rangle}$, $|m|^{\langle a,X\rangle} = \sigma$ and $W_e^{H_\sigma^X,a,X} \subseteq \mathcal{O}^X$, then $3^m\cdot5^e \in \mathcal{O}^{\langle a,X\rangle}$. Let $|3^m\cdot5^e|^{\langle a,X\rangle}$ = first limit ordinal $\lambda$ $\lambda \geqslant \sup\{\sigma\} \cup \{|b|^X \mid b \in W_e^{H_\sigma^X,a,X}\}$. Let

$$H_\lambda^X = \{\langle b,c\rangle \mid b \in \mathcal{O}^X \wedge |b|^X < \lambda \wedge c \in H_{|b|^X}^X\},$$

the effective disjoint union of the $H_\sigma^X$'s for $\sigma < \lambda$.

From this hierarchy of primitive recursive set operations we define the usual recursion theoretic notions.

Definition 1.2.1:

(i) A set $Y \subseteq TC(X)$ is $\mathbb{E}$-recursive ($\mathbb{E}$-REC) in $a, X$ for some $a \in TC(X)$ if $Y = W_e^{H^{a,X}|\langle m,a\rangle|^X}$ for some $m \in \mathcal{O}^{\langle a,X\rangle}$ and some $e \in \omega$. This relation is denoted by $Y \leq_{\mathbb{E}} a, X$.

(ii) $Y \subseteq TC(X)$ is $\mathbb{E}$-recursively enumerable ($\mathbb{E}$-RE) in $a, X$ if $\exists\, e \in \omega \quad Y = \{b \in TC(X) : e \in \mathcal{O}^{\langle b,X\rangle}\}$.

We now turn to two notions of relative computability over a fixed $\mathbb{E}$-closed set $A$, where

Definition 1.2.2: A set $M$ is $\mathbb{E}$-closed, if $M$ is transitive, closed under pairing and given any $X \in M$ and $Y$ $\mathbb{E}$-REC in $X$, if $Y$ is an effective code for a set $Z$, then $Z \in M$.

By an effective code we mean that every element of $Y$ is an ordered pair and the partial order computed from $Y$ is the diagram for the $\in$-relation in $TC(Z)$. Thus 1.2.2 says simply that any $Z$ $\mathbb{E}$-recursively constructed from some $X \in A$ is an element of $A$. Call the least $\mathbb{E}$-closed set with $X$ as an element the $\mathbb{E}$-closure of $X$ ($\mathbb{E}$-cl(X)). In analogy with the previous chapter, there are some important ordinal parameters associated with $X$ and its $\mathbb{E}$-closure.

Definition 1.2.3:

(i) $\kappa_0^{<a,X>} = \sup\{|m|^{<a,X>} : m \in \mathcal{O}^{<a,X>}\}$; $\kappa_0^{<a,X>}$ is the supremum of the heights of prewellorderings of $TC(X)$ which are recursive in $a$, $X$.

(ii) $\kappa^X = \sup\{|<m,a>|^X : <m,a> \in \mathcal{O}^X\}$; then $\kappa^X$ is the height of the $\mathbb{E}$-closure of $X$.

Fix an $\mathbb{E}$-closed set $M = \mathbb{E}\text{-cl}(X)$ and $z \in M$ then,

Definition 1.2.4: A set $Y \subseteq TC(X)$ is $\mathbb{E}$-recursive in $a$, $X$, $Z$ for some $a \in TC(X)$, if $Y = W_e^{H^{a,X,Z}_{|<m,a>|^{X,Z}}}$ for some $m \in \mathcal{O}^{<a,X,Z>}$ and some $e \in \omega$. This relation is expressed by $Y \leq_{\mathbb{E}} X, a, Z$.

Proposition 1.2.5: Suppose $M = \mathbb{E}\text{-cl}(X)$ and $z \in M$, then $\mathcal{O}^{X,z}$ is recursively isomorphic to $\mathcal{O}^X$ above $|z|^X$.

Proof. (Sketch) We have trivially that $\kappa^X = \kappa^{X,z}$. Using a new definition of the hierarchy with $H_0^{X,z} = z$ and proceeding by ETR on $\kappa^X$ yields the result. Intuitively, there is a natural embedding of $\mathcal{O}^X$ into $\mathcal{O}^{X,z}$ and with the parameter $z \in \mathbb{E}$-closure of $X$ that embedding can be reversed. The details of the proof are omitted. □

This situation is altered radically if we consider $\mathcal{O}^{X,A}$ for $A$ some unbounded subclass of $\mathcal{O}^X$ or if $A = \mathbb{E}\text{-cl}(X)$. For such an $A$ we will have that $\kappa^{A,X} > \kappa^X$, since by the above this corresponds to the $\mathbb{E}$-closure

(A,X). Thus for purposes of relative $\mathbb{E}$-computability on $\mathcal{O}^X$ we give a definition based on 1.2.0 relativized to predicate on $\mathcal{O}^X$. This corresponds to the effect of scheme (vi). It may still be the case that $\kappa^{X,A} > \kappa^X$, however for an interesting recursion-theoretic choice of A we will have that $\mathbb{E}(A)$-cl(X) $\mathbb{E}$-cl(X) and $\kappa^X = \kappa^{X,A}$. These RE A's will be called hyperregular.

<u>Definition 1.2.6</u>: $A \subseteq TC(X)$, $a \in TC(X)$,

(i) $1 \in \mathcal{O}^{<a,X,A>}$; $|1|^{<a,X,A>} = 0$ and

$$H_0^{X,A} = \emptyset.$$

(ii) If $n \in \mathcal{O}^{<a,X,A>}$, then $2^n \in \mathcal{O}^{<a,X,A>}$ and if $|n|^{<a,X,A>} = \sigma$ then $|2^n|^{<a,X,A>} = |n|^{<a,X,A>} + 1 = \sigma + 1$; and

$$H_{\sigma+1}^{X,A} = \{<e,a,0> : a \in W_e^{H_\sigma^{X,A},A}\}.$$

Finally,

(iii) Given $m, e \in \omega$ if $m \in \mathcal{O}^{<a,X,A>}$, $|m|^{<a,X,A>} = \sigma$ and $W_e^{H_\sigma^{X,A},a,X,A} \subseteq \mathcal{O}^{X,A}$, then $3^m \cdot 5^e \in \mathcal{O}^{<a,X,A>}$. Let $|3^m \cdot 5^e|^{<a,X,A>}$ = first limit ordinal $\lambda$, $\lambda \geqslant \sup\{\sigma\} \cup \{|b|^{X,A} \mid b \in W_e^{H_\sigma^{X,A},a,X,A}\}$. Let

$$H_\lambda^{X,A} = \{\langle b,c\rangle : b \in \mathcal{O}^{X,A} \wedge |b|^{X,A} < \lambda \wedge c \in H^{X,A}_{|b|^{X,A}}\},$$

the effective disjoint union of the $H_\sigma^{X,A}$'s for $\sigma < \lambda$.

From this hierarchy of primitive recursive set operations relative to $A$ we can now derive the relative notions of computability.

<u>Definition 1.2.7</u>: (i) A set $Y \subseteq TC(X)$ is <u>$\mathbb{E}(A)$-recursive</u> ($\mathbb{E}(A)$-REC) <u>in $a, X$</u> for some $a \in TC(X)$ if $Y = W_e^{H^{a,X,A}_{|\langle m,a\rangle|^{X,A}}}$ for some $m \in \mathcal{O}^{\langle a,X,A\rangle}$ and some $e \in \omega$. This relation is expressed by $Y \leqslant_{\mathbb{E}(A)} X, a$.

(ii) A set $Y \subseteq TC(X)$ is <u>$\mathbb{E}(A)$-recursively enumerable</u> ($\mathbb{E}(A)$-RE) <u>in $a, X$</u> if $\exists\, e \in \omega$

$$Y = \{b \in TC(X) : c \in \mathcal{O}^{\langle b,X,A\rangle}\}$$

This will be our choice of relative computability and before the end of this section we focus on the case of $\mathbb{E}$-closed initial segments of $L$, Gödel's constructible universe. Before that we note two facts about certain $\mathbb{E}$-RE subsets of $TC(X)$.

<u>Definition 1.2.8</u>: $K \subseteq TC(X)$ is <u>complete $\mathbb{E}$-RE</u> over $A = \mathbb{E}\text{-cl}(X)$, if for all $\mathbb{E}$-RE $Z \subseteq TC(X)$, $Z \leqslant_{\mathbb{E}} X, K$.

Remark: For this choice of $K$, $\kappa^{X,K} > \kappa^X$ since $\mathcal{O}^X \in \mathbb{E}(K)\text{-cl}(X)$. $K$ can be thought of as the canonical complete $\mathbb{E}$-RE subclass of $TC(X)$, i.e.

$$K = \{<e,a> : e \in \mathcal{O}^{X,a} \wedge a \in TC(X) \wedge e \in \omega\}.$$

Definition 1.2.9: (i) $A \subseteq TC(X)$ is *regular*, if for all $x \in \mathbb{E}\text{-cl}(X)$, $A \cap x \in \mathbb{E}\text{-cl}(X)$.

(ii) $A \subseteq TC(X)$ is *hyperregular*, if $\mathbb{E}(A)\text{-cl}(X) = \mathbb{E}\text{-cl}(X)$.

Proposition 1.2.10: Let $A \subseteq TC(X)$ be $\mathbb{E}$-RE and incomplete, then $A$ is hyperregular, in particular, $\kappa^{A,X} = \kappa^X$.

Proof. First observe that any incomplete RE $A$ is regular, for otherwise take $z_0 \in \mathbb{E}\text{-cl}(X) = M$ such that,

$$A \cap z_0 \notin M.$$

Now with $z_0$ as a parameter define $f : z_0 \longrightarrow On \cap M$ by,

$$f(a) = \begin{cases} |e|^{\mathcal{O}^{X,<b,a>}}, & \text{if } a \in A \cap z_0 \\ 0, & \text{if } a \notin A \cap z_0, \end{cases}$$

where $e \in \omega$ and $b$ $TC(X)$ and $<e,b>$ is an index for $A$ as an RE set. Then using $z_0 \in M$ and the fact that $f \leq_{\mathbb{E}(A)} z_0$ (viewing $f$ as a set), we have,

<u>Claim</u>: $A$ is complete.

<u>Proof of Claim</u>. Suffices to show that $\mathcal{O}^X \leq_{\mathbb{E}} A, z_0$ or $\mathcal{O}^X \leq_{\mathbb{E}} f, z_0$, but

$$b \in \mathcal{O}^X \longleftrightarrow (\exists\ \sigma < \sup(\mathrm{rng}(f)))[b \in \mathcal{O}^X_\sigma].$$

Suppose $A$ is non-hyperregular and so, in particular,

$$\mathrm{On} \cap \mathbb{E}(A)\text{-cl}(X) > \mathrm{On} \cap \mathbb{E}\text{-cl}(X).$$

Thus $\exists\ \delta < \kappa^X$ and $\mathbb{E}$-REC $f$ such that

(i) $f \upharpoonright \delta$ total, and

(ii) $\sup\limits_{\gamma<\delta} f(\gamma) = \kappa$.

But, as before

$$b \in \mathcal{O}^X \longleftrightarrow (\exists\ \sigma < \sup_{\gamma<\delta} f(\gamma))[b \in \mathcal{O}^X_\sigma],$$

contradicting $A$ incomplete.□

<u>Remark</u>: Suppose there is an $\mathbb{E}$-REC well-ordering of $TC(X)$. That there is a regular complete $\mathbb{E}$-RE set was first noticed by G. Sacks [1980].

<u>Proposition 1.2.11</u>: The RE predicates in $\mathbb{E}$-recursion are closed under the quantifier $\exists\ n \in \omega$.

<u>Proof</u>. Immediate from Gandy Selection. □

<u>Corollary 1.2.12</u>: If $\kappa_0^{<a,X>} = \phi^X$ for some $a \in TC(X)$, then $\mathbb{E}\text{-cl}(X)$ is $\Sigma_1$-admissible.

<u>Proof</u>. Since the integer notations relative to $<a,X>$ are unbounded in $\mathcal{O}^X$ an arbitrary quantifier $\exists\ t < \kappa^X$ can be converted into $\exists\ e \in \omega\ \exists\ t < |<e,a>|_{\mathcal{O}^X}$. Then apply Gandy Selection. □

<u>Proposition 1.2.13</u> (Moschovakis): There is a r.e. in $X$ relation $<_R \subseteq TC(X) \times TC(X)$ such that

$$\forall\ e \in \omega\ \forall\ b \in TC(X)[<e,b> \in \mathcal{O}^X \Longleftrightarrow <_R ,\ \text{restricted to}$$

$$\{b \in TC(X) : b <_R <e,b>\}\ \text{is well-founded}].$$

<u>Proof</u>. $<_R$ is defined inductively. Say that $y$ is an immediate subcomputation of $x$, if

(i) $x = <0,a>$ then $x$ has no immediate subcomputations.

(ii) $x = <2^n,a>$ and $y = <n,a>$.

(iii) $x = <3^m \cdot 5^e,a>$ and $y = <m,a>$ or $<m,a> \in \mathcal{O}^X$ and $y \in W_e^{H^{X,a}_{|<m,a>|^X}}$.

(iv) none of the above then $y = x$.

Let $z <_R y$ if there is a finite sequence $y_0, \ldots, y_n$ such that $y_{i+1}$ is an immediate subcomputation of $y_i$, $y_0 = y$ and $y_n = z$. $<_R$ is r.e. since $\mathcal{O}^X$ is. Given $y$, then $y \in \mathcal{O}^X$ iff $<_R$ is wellfounded below $y$.

($\Longrightarrow$) by ETR on $|y|^{\mathcal{O}^X}$ and

($\Longleftarrow$) by ETR on $|y|_{<_R}$ which is well defined if $<_R$ is wellfounded below $y$. □

<u>Definition 1.2.14</u>: An $\mathbb{E}$-closed set $A$ satisfies the <u>Moschovakis Phenomenon (MP)</u> if: Taking $<_R$ as in 1.2.13, if $<_R$ is not well-founded below a point $b$, then there is an infinite descending $<_R$-chain below $b$ in $A$. Such a sequence will be referred to as a <u>Moschovakis Witness</u> for $b$.

Among the examples of $\mathbb{E}$-closed sets satisfying MP are:

(i) The companion structure for Kleene recursion in ${}^{k+2}\mathbb{E}$ (Normann [1978] and Moschovakis [1976]);

(ii) an inadmissible $\mathbb{E}$-closed $L(\kappa)$, and

(iii) any $\mathbb{E}$-closed set $A$ modelling the CO-RE dependent choice scheme, i.e. $A \models DC_\omega(R)$, where $R$ is CO-RE, and $DC_\omega(R)$:

$$(\forall x)(\exists y)R(x,y) \longrightarrow (\exists z)(\forall i \in \omega)[R((z)_i,(z)_{i+1})].$$

As in Chapter 0 we will now recast $\mathcal{O}^X$ in the setting of constructibility. The reader familiar with Gödel's L will notice that this amounts to constructing over some set TC(X).

<u>Definition 1.2.15</u>: For an ordinal $\sigma$, define $L_\sigma(X)$ by,

$$L_0(X) = TC(X)$$

$$L_{\sigma+1}(X) = \{y \subseteq L_\sigma(X) : Y \text{ is first order definable over } \langle L_\sigma(X), \in, X\rangle \text{ with parameters from } L_\sigma(X)\}$$

$$L_\lambda(X) = \bigcup_{\sigma<\lambda} L_\sigma(X).$$

Let $M_\sigma(X) = \langle L_\sigma(X), \in, X\rangle$. $\kappa^X = \sup_{b\in TC(S)} \kappa_0^{X,b}$, then as before the relationship between $L_{\kappa^X}(X)$ and $\mathcal{O}^X$ is very close. There is a one-to-one correspondence, $e \longleftrightarrow \phi_e$, between integers $e$ and $\Sigma_1$ formulae $\phi_e$ such that,

$$\langle e,a\rangle \in \mathcal{O}^X \Longleftrightarrow M_{\kappa_0^{\langle a,X\rangle}}(X) \models \phi_e(a).$$

The definition of $\mathcal{O}^X$ at limit stages is analogous to the following bounding principle in $L_{\kappa^X}(X)$.

Proposition 1.2.16 (Bounding Principle): Suppose $\forall\ b \in TC(X)$

$$M_{\kappa_0^{<a,b,X>}}(X) \models \phi(a,b)$$

for some $\phi \in \Sigma_1(L_{\kappa^X}(X))$. Then there is a $\sigma < \kappa_0^{<a,X>}$ such that $\forall\ b \in TC(X)$,

$$M_\sigma(X) \models \phi(a,b).$$

If $L$ is the language appropriate to $M_{\kappa^X}(X)$, we have

Definition 1.2.17: For $a \in TC(X)$ and $\alpha < \kappa^X$ say that $\alpha$ is <u><a,X>-reflecting</u>, if for all $\Sigma_1$ formulae $\phi(y)$ in $L$:

$$M_\alpha(X) \models \phi(a) \quad \text{iff} \quad M_{\kappa_0^{<a,X>}}(X) \models \phi(a).$$

Let $\underline{\kappa_r^{<a,X>}}$ be the greatest $<a,X>$ reflecting ordinal $\leqslant \kappa^X$.

## §3. $\mathbb{E}$-closed Ordinals

We now focus our attention on a special class of $\mathbb{E}$-closed sets whose structure is closely tied to the ordinals. One can think of them either as the result of applying the $\mathbb{E}$-REC functions to ordinals or, alternatively, as domains which compute well-orderings of themselves.

Definition 1.3.0: Let $\kappa \in On$, then $\kappa$ is $\mathbb{E}$-closed if $L(\kappa)$ (Gödel's constructible universe through $\kappa$) is an $\mathbb{E}$-closed set.

The following results provide the basis for the priority method on $\mathbb{E}$-closed ordinals.

Proposition 1.3.1: Let $\kappa$ be $\mathbb{E}$-closed and suppose there is no greatest cardinal in $L(\kappa)$. Then $L(\kappa)$ is $\Sigma_1$-admissible.

Proof. By the standard proof in $\alpha$-recursion theory, the cardinals of an initial segment of $L$ are $\Sigma_1$-reflecting. The $\Sigma_1$-admissibility of $L(\kappa)$ follows immediately. $\square$

We now focus on inadmissible (not $\Sigma_1$-admissible), $\mathbb{E}$-closed $L(\kappa)$. Thus $L(\kappa)$ has a greatest cardinal by 1.3.1 which we denote by $gc(\kappa)$ = greatest cardinal of $L(\kappa)$. By 1.2.12, $\kappa_0^a < \kappa$ for $a \in L(\kappa)$. The next two results establish that every inadmissible, $\mathbb{E}$-closed ordinal satisfies MP.

<u>Proposition 1.3.2</u>: Suppose $\kappa$ is $\mathbb{E}$-closed and $X \in L(\kappa)$ is well-ordered in type $gc(\kappa)$. For each $e \notin \mathcal{O}^X$, a Moschovakis witness for $\langle e,x \rangle$ is constructed at $\kappa_r^X + 1$.

<u>Proof</u>. Suppose $e \notin \mathcal{O}^X$, then the following procedure generates the leftmost path through $<_R$ restricted to $\langle e,x \rangle$. By the well-ordering of $X$ all computations are indexed by ordinals less than $gc(\kappa)$. The computation tree for $\langle e,x \rangle$ is ill-founded and breaks up into two parts, namely, those subcomputations which are convergent and those which are divergent. At a branching in the tree we can imagine the subcomputations as well-ordered in type $cg(\kappa)$ since $x$ is. Thus we compute along that well-ordering. Since $\{e\}(x)\uparrow$ there is a least, in the sense of that well-ordering, point in the branching where we have divergence. By leftmost we then mean leftmost among the divergent subcomputations. Thus everything to the left of the resulting path will be convergent subcomputations and, hence, serve to define that path at a sufficiently large ordinal. We construct the $\omega$-sequence $f : \omega \longrightarrow R$ as follows:

(i) If $f(n) = \langle 2^m, \delta \rangle$, then $f(n+1) = \langle m, \delta \rangle$.

(ii) If $f(n) = \langle 3^m \cdot 5^e, \delta \rangle$, then either $f(n+1) = \langle m, \delta \rangle$ and $\langle m, \delta \rangle \notin \mathcal{O}^X$ or $\langle m, \delta \rangle \in \mathcal{O}^X$ and

$f(n+1) \in W_e^{H^{|\langle m,\delta\rangle|^X}}$ and $\forall\ \delta' < f(n+1)$

$\delta' \in W_e^{H^{X,|\langle m,\delta\rangle|^X}} \longrightarrow \delta' \in \mathcal{O}^X$.

If we let $WF(R)_{\langle e,x\rangle}$ be the well-founded portion of this tree such that in any branching at level $n$, $WF(R)_{\langle e,x\rangle}$ restricted to that branching contains only points less than $f(n)$. $WF(R)_{\langle e,x\rangle}$ is $\mathbb{E}$-RE and well-founded. Then $\alpha = \sup\{\text{height}(WF(R)_{\langle e,x\rangle}) \mid e \in \omega\} \geqslant \kappa_r^X$, for an $\alpha < \kappa_r^X$ and would reflect the complete subset of $\omega$ relative to $X$ to some $\sigma \leqslant_{\mathbb{E}} X$, which is absurd. Likewise $\alpha$ cannot be greater than $\kappa_r^X$ for otherwise an enumeration of an unbounded subset of $\alpha$ chopped off at $\kappa_r^X$ would witness $\kappa_r^X \leqslant_{\mathbb{E}} X$, contradicting $\kappa_0^X < \kappa_r^X$. Thus $\alpha = \kappa_r^X$ and the preceding gives a definition of $f$ at $\kappa_r^X + 1$. $\square$

<u>Corollary 1.3.3</u>: Under the assumptions of 1.3.2, there is a $\Sigma_1$ formula $\phi(z)$ for $a \in TC(X)$, such that

$$M_{\kappa_r^{a,X}+1}(X) \models \phi(a,X) \qquad \text{and}$$

$$M_{\kappa_r^{a,X}}(X) \not\models \phi(a,X) \qquad \text{and}$$

this formula is uniform in $a$, $X$.

Proof. $\phi(z)$ says that there is a Moschovakis witness for all divergent computations in $z, X$ for $z \in TC(X)$. $\square$

The analogy of this for Kleene recursion in normal ${}^{n+2}\mathbb{F}$ $(n \geq 1)$ was first obtained by Harrington [1973, where he considered $\kappa_r^{a,I}$, for $a \in I$.

Proposition 1.3.4: Suppose $\kappa$ is inadmissible and $\mathbb{E}$-closed, then for every $\langle a,X\rangle \in L(\kappa)$, $a \in TC(X)$ we have $\kappa_r^{\langle a,X\rangle} < \kappa$.

Proof. Take $a$, $X$ as above and suppose $\kappa_r^{\langle a,X\rangle} = \kappa$, then $(\forall\, b)[\kappa_r^{\langle a,b,X\rangle} = \kappa]$. We get that $\kappa$ is $\Sigma_1$-admissible since the witnesses for $\Sigma_1$ formulae with parameters in $X$ are contained in $L_{\kappa_0^X}$ by reflection. Since $\kappa_0^X < \kappa$ by 1.2.12 and $\kappa$ is inadmissible. But then $\kappa$ is $\Sigma_1$-admissible, a contradiction. $\square$

Corollary 1.3.5: If $\kappa$ is inadmissible and $\mathbb{E}$-closed, then $L(\kappa) \models MP$.

Proof. Immediate from 1.3.2 and 1.3.4. $\square$

As we will see later when we consider much more general $\mathbb{E}$-closed sets than $\mathbb{E}$-closed $L(\kappa)$, the Moschovakis phenomenon is a consequence of dependent choice along CO-RE set relations. The next two chapters contain results on the RE degrees of an $\mathbb{E}$-closed, inadmissible

$L(\kappa)$ which make heavy use of 1.3.5. In our final chapter we describe the changes necessary for extending these methods to Kleene recursion in a normal functional, ${}^{n+2}\mathbb{F}$, for $n \geqslant 1$.

# Chapter 2: $\mathbb{E}$-RE Degrees and the Priority Method

## §0. Preserving Computations

We now narrow our study to $\mathbb{E}$-closed $L(\kappa)$, $\kappa \in On$, which are not $\Sigma_1$-admissible. By 1.3.5, such an $\mathbb{E}$-closed set satisfies the Moschovakis Phenomenon. In fact if $\{e\}(x)\uparrow$ for $e \in \omega$ and $X \in L(\kappa)$, then as we saw in the last chapter a Moschovakis Witness to divergence is constructed at $\kappa_r^X + 1$ and $\kappa_r^X + 1 < \kappa$. Thus for $e \in \omega$ the question whether $\{e\}(x)\downarrow$ or $\{e\}(x)\uparrow$ is completely determined at $\kappa_r^X + 1$ and can therefore be used in the course of a construction in $\kappa$-many stages.

Now suppose we are constructing an $\mathbb{E}$-RE $A \subseteq L(\kappa)$ in the sense that at stage $T \subset \kappa$ we consider certain $X \in L(\kappa)$ and if $X$ satisfies a condition defined in $L(\kappa)$ we enumerate $X$ into $A_{T+1}$ (= amount of $A$ enumerated through stage $T + 1$). Further suppose that as we construct $A$ we consider the values of the indices for $\mathbb{E}$-recursion applied to $A$ as an oracle, i.e. $\{e\}^A$ using scheme ( vi ) of relativization. Thus at stage $T < \kappa$ $\{e\}_T^{A_T}(x)$ may appear to converge or diverge using positive and negative information about $A_T$ ($\gamma < T$ and $\gamma \in A_T$ or $\gamma \notin A_T$). Suppose that $A$ preserves the $\mathbb{E}$-closure of $L(\kappa)$, i.e. $L_\kappa[A]$ is $\mathbb{E}$-closed over $A$ (briefly $L(\kappa)$ is $\mathbb{E}(A)$-closed). Then

$\kappa_0^X[A] = \sup\{\gamma \leq \kappa \mid \gamma \leq_{\mathbb{E}(A)} X\} < \kappa$ and, as we shall see,

$\kappa_r^X[A] =$ (greatest X-reflecting ordinal over A) $< \kappa$.

Thus, as before in the unrelativized case, if $e \in \omega$ then either

(i) $L_{\kappa_r^X[A]}[A] \models \{e\}^A(x)\downarrow$ or

(ii) $L_{\kappa_r^X[A]}[A] \models \{e\}^A(x)\uparrow$

and in case (ii) a Moschovakis witness for $\{e\}^A(x)\uparrow$ is definable in $L_\kappa[A]$ at level $\kappa_r^X[A]$.

Therefore information about A used in $\{e\}^A(x)$ is contained in $A \upharpoonright \kappa_r^X[A]$ (actually $\chi_A \upharpoonright \kappa_r^X[A]$ where $\chi_A : \kappa \longrightarrow \{0,1\}$ given by

$$\chi_A(\gamma) = \begin{cases} 0, & \text{if } \gamma \in A \\ 1, & \text{if } \gamma \notin A \end{cases} .$$

However as we construct an $\mathbb{E}$-RE A, $A \upharpoonright \kappa_r^X[A]$ may well change by our enumerating some $\gamma < \kappa_r^X[A]$ into A. If $\{e\}_T^{A_T}(x)\downarrow$ we can insure that $\{e\}_\sigma^{A_\sigma}(x)\downarrow$ for $\sigma \geq T$ by freezing that portion of $\chi_A$ used in $\{e\}_T^{A_T}(x)$ (and all of its subcomputations, of course). If $\{e\}^{A_\tau}(x)\uparrow$ we can insure that $\{e\}_\sigma^{A_\sigma}(x)\uparrow$ for $\sigma \geq \tau$ by doing the same.

Note: 1. $\kappa_r^X[A] < \kappa$ (for this choice of $A$),

2. $\kappa_r^X[A]$ is an upper bound on restraint for the sake of preserving $\{e\}_\tau^{A_\tau}(x)$.

Unlike classical or ordinal recursion theory divergence is $\Sigma_1(L(\kappa))$ or in the above $\Sigma_1(L_\kappa[A])$. Thus given $f : \kappa \longrightarrow \{0,1\}$ the condition $R_e : \{e\}^A \neq f$ can be satisfied in one of two ways. Find $x$ such that

(i) $\{e\}^A(x)\downarrow$ and $f(x)\downarrow$ where

$$\{e\}^A(x) \neq f(x); \quad \text{or}$$

(ii) $\{e\}^A(x)\uparrow$ and $f(x)\downarrow$

and preserve $\chi_A$ through $\kappa_r^X[A]$ to extablish (i) or (ii) once and for all. Thus, we have established,

Lemma 2.0.0 (Persistence Lemma): Let $A \subseteq L(\kappa)$ and suppose $L(\kappa)$ is $\mathbb{E}(A)$-closed and not $\Sigma_1$-admissible. Then for $e \in \omega$,

$$L_\kappa[A] \models \{e\}^A(x)\downarrow \ \underline{\text{iff}}\ L_{\kappa_r^X[A]}[A] \models \{e\}^A(x)\downarrow.$$

§1. $\mathbb{E}$-RE Projecta

Conditions such as $R_e : \{e\}^A \neq f$ above are indexed by codes for the $\mathbb{E}$-recursive functions on $L(\kappa)$. As in ordinal recursion theory we use the shortest possible listing of these codes in a priority argument whose goal it is to satisfy $R_{<e,x>}$ for each $<e,x>$, $e \in \omega$ and $x < \kappa$ a code for a parameter in $L(\kappa)$.

Definition 2.1.0: Let

$$\text{(i)} \quad \rho^{L(\kappa)} = \mu\gamma \leq \kappa \; (\exists\, \gamma_0 < \kappa)[(\forall\, \delta < \kappa)(\exists\, \sigma < \gamma)$$

$$(\exists\, e \in \omega)[\delta = |\{e\}(\gamma_0,\sigma)|\,] \quad \text{and}$$

$$\text{(ii)} \quad \eta^{L(\kappa)} = \mu\gamma \leq \kappa \; (\exists\, \gamma_0 < \kappa)[R \subseteq \gamma \text{ is RE in } \gamma_0,$$

$$\text{but } R \not\leq_{\mathbb{E}} \delta \text{ for any } \delta < \kappa]$$

Remark: These parameters were first defined by G. Sacks [1980] for the special case of $\mathbb{E}\text{-cl}(2^\omega)$ where $\mathbb{E} = {}^3\mathbb{E}$ and there is a REC well-ordering of $2^\omega$. They correspond roughly to the two notions of $\Sigma_1$-projectum defined by Jensen [cf. Devlin [1973]]. Sacks called them the "greater" and "lesser" RE projectum, respectively. His intention is implicit in (i) of the following proposition.

<u>Proposition 2.1.1</u>: Let $\eta$ and $\rho$ be as in 2.1.0 (we omit the superscript from now on), then

(i) $\eta \leqslant \rho$; and

(ii) $\eta$ and $\rho$ are cardinals in $L(\kappa)$;

(iii) $\rho < \kappa$ implies $\rho = gc(\kappa)$.

<u>Proof</u>. (i) Define

$$N_\rho = \{\tau < \rho \mid \{(\tau)_0\}(\gamma_0,(\tau)_1)\downarrow \wedge (\tau)_0$$

$$\text{is an index in } \mathbb{E}\text{-recursion}\},$$

where $\gamma_0$ is the parameter in the definition of $\rho$. Then $N_\rho$ is simply the set of ordinals less than $\rho$ which are notations for ordinals less than $\kappa$, i.e. $\{(\tau)_0\}(\gamma_0,(\tau)_1)$ could be written $(\tau)_0 \in \mathcal{O}^{<\gamma_0,(\tau)_1>}$. Then we have

$$\sup\{|(\tau)_0|^{<\gamma_0,(\tau)_1>} \mid \tau \in N_\rho\} = \kappa,$$

by the definition of $\rho$.

Thus $N_\rho$ is RE\REC in $\gamma_0$ and $\eta \leqslant \rho$ .

(ii) If $\exists\, \delta < \eta\ \exists\, f : \delta \longleftrightarrow \eta$ such that $f \in L(\kappa)$ then if $R \subseteq \eta$ witnesses the definition of $\eta$, then

$$\{f^{-1}(\delta) \mid \delta < \eta \wedge \delta \in R\}$$

is RE\REC, contradicting $\eta$ least such.

If $\exists\, \delta < \rho\ \exists\, f : \delta \longleftrightarrow \rho$ such that $f \in L(\kappa)$, then

$$(\forall\, \sigma < \kappa)(\exists\, \gamma < \delta)(\exists\, e \in \omega)[\sigma = |\{e\}(\gamma_0, f(\gamma))|]$$

where $\gamma_0$ defines $\rho$, contradicting $\rho$ least such.

(iii) (ii) $\longrightarrow \rho \leqslant gc(\kappa)$. If $\rho < gc(\kappa)$, the function $f : \kappa \longrightarrow \rho$ given by

$$f(\sigma) = \mu\gamma < \rho(\exists\, e \in \omega)[|\{e\}(\gamma_0,\gamma)| = \sigma]$$

is REC and total. Thus $\{f(\sigma) \mid \sigma < gc(\kappa)\} \in L(\kappa)$ contradicting $gc(\kappa)$ a cardinal. □

We shall use $\rho$ to list the $\mathbb{E}$-recursive functions. The assumption $\eta = \rho$, sometimes termed "adequacy" in abstract recursion theories (cf. Fenstad [1980]), has played an important role in most generalizations of the priority method to abstract recursion theories.

<u>Lemma 2.1.2</u>: Suppose $L(\kappa)$ is $\mathbb{E}$-closed, not $\Sigma_1$-admissible and $\eta, \rho$ as above. Then if $\sigma_0 < \kappa$, then

$$\delta < \eta \Rightarrow [\sup_{\sigma \leqslant \delta} \kappa_r^{<\sigma,\sigma_0>} < \kappa].$$

<u>Proof</u>. Suppose not and fix $\sigma_0 < \kappa$ and $\delta_0 < \eta$ such that,

$$\sup_{\gamma<\delta_0} \kappa_r^{<\sigma_0,\gamma>} = \kappa .$$

Let

$$N(\delta_0) = \{\gamma < \delta_0 \mid \exists\, e \in \omega \{e\}(\sigma_0,\gamma)\downarrow\}$$

is RE in some $e_0 \in \omega$ by Gandy Selection. If $N(\delta_0)$ were REC in some $\tau < \kappa$, then the function $f : \delta_0 \longrightarrow \kappa$ given by

$$f(\gamma) = \begin{cases} |\{e_0\}(\sigma_0,\tau,\gamma)|, & \text{if } \gamma \in N(\delta_0) \\ 0 & , \text{otherwise} \end{cases}$$

would contradict $L(\kappa)$ $\mathbb{E}$-closed. But this contradicts $\eta$ least such. $\square$

## §2. Effective Cofinalities

Let $L(\kappa)$ be $\mathbb{E}$-closed and not $\Sigma_1$-admissible.

<u>Definition 2.2.0</u>: Let $\lambda \leqslant \kappa$ be a cardinal of $L(\kappa)$ (i.e. $L(\kappa) \models$ "$\lambda$ is a cardinal"), then

(i) <u>$\mathrm{REC\text{-}cf}(\lambda)$</u> $= \mu\gamma \leqslant \lambda[\exists\ \mathbb{E}\text{-REC}\ f : \gamma \longrightarrow \lambda$ with range $(f)$ unbounded in $\lambda]$; and

(ii) $\lambda$ is a <u>regular cardinal</u> of $L(\kappa)$ <u>iff</u> $\mathrm{REC\text{-}cf}(\lambda) = \lambda$.

<u>Remark</u>: 2.2.0 (ii) has been called "recursive regularity" by Normann [1975].

For $\kappa$ $\mathbb{E}$-closed obviously there is no $\mathbb{E}$-REC $f$ with $\mathrm{dom}(f) = \lambda < \kappa$ and range $(f)$ unbounded in $\kappa$. However it can happen that $gc(\kappa)$, for example, is recursively singular, i.e. $\mathbb{E}\text{-cf}(gc(\kappa)) < gc(\kappa)$. Examples of this are $\kappa = \mathbb{E}\text{-cl}(\aleph_\omega)$ and $\kappa = \mathbb{E}\text{-cl}(\aleph_{\omega_1})$, where in $\mathbb{E}\text{-cl}(\aleph_\omega)$ $\mathbb{E}\text{-cf}(\aleph_\omega) = \omega$ and in $\mathbb{E}\text{-cl}(\aleph_{\omega_1})$ $\mathbb{E}\text{-cf}(\aleph_{\omega_1}) = \omega_1$.

<u>Remark</u>: $\mathbb{E}\text{-cl}(\aleph_\omega)$ is $\Sigma_1$-admissible (cf. Kirousis [1979]) and $\mathbb{E}\text{-cl}(\aleph_{\omega_1})$ is not $\Sigma_1$-admissible (cf. Moschovakis [1967]).

Since we demand that an $\mathbb{E}$-REC function $f : \mathrm{On} \longrightarrow \mathrm{On}$ satisfy

$$f(\delta) \leqslant_{\mathbb{E}} \delta,\ \rho,$$

where $p$ is a parameter encoding $f$, problems non-existent in admissibility theory crop-up. For example there can be RE sequences cofinal in $\kappa$ of order type less than $\kappa$.

<u>Definition 2.2.1</u>: Let $\lambda$ be a cardinal of $L(\kappa)$, then

(i) $\text{RE-cf}(\lambda) = \mu\gamma \leq \lambda$ [there is an RE $X \subseteq \lambda$ cofinal in $\lambda \wedge \text{o.t.}(X) = \gamma$]

(ii) $\lambda$ is RE-regular <u>iff</u> $\text{RE-cf}(\lambda) = \lambda$.

<u>Proposition 2.2.2</u>: $\text{RE-cf}(\kappa) \leq \eta$.

<u>Proof</u>. Recall that

$$\eta = \mu\gamma \leq \kappa (\exists R)_{R \subseteq \gamma} [(\exists \sigma < \kappa)[R \text{ is RE in } \sigma] \wedge (\forall \tau < \kappa)[R \text{ is not REC in } \tau]$$

and fix the parameter $\langle e,\sigma\rangle$ defining $\lambda$. Then

$$W = \{\langle\gamma,\tau\rangle \mid \gamma < \eta \wedge \tau < \kappa \wedge |\{e\}(\sigma,\gamma)| = \tau\}$$

is RE of order type $\leq \eta$ and $\cup W = \kappa$, by the definition of $\eta$. $\square$

There are cases where $\text{RE-cf}(\kappa) < \rho$.

Proposition 2.2.3: Let $\kappa = \mathbb{E}\text{-cl}(\aleph_{\omega_1})$ then $\text{RE-cf}(\kappa) = \omega_1 < \rho = \aleph_{\omega_1}$.

Proof. $\rho \leqslant \aleph_{\omega_1}$ by the choice of $\kappa$, however we cannot have $\rho < \aleph_{\omega_1}$ since $\aleph_{\omega_1}$ is an L-cardinal. In addition $\text{RE-cf}(\kappa) > \omega$ since $\aleph_{\omega_1}$ cannot be cofinalized with $\omega$. For $\alpha < \omega$, let

$$\beta_\alpha = \sup_{\gamma<\aleph_\alpha} \kappa_r^{<\gamma,p>},$$

where $p$ is the parameter in the definition of $\rho$. As before $\eta = \rho = \aleph_{\omega_1}$ implies that $(\forall\ \alpha)[\beta_\alpha < \kappa]$ and by the definition of

$$<<\alpha,\beta_\alpha> \mid \alpha < \omega_1>$$

is $\mathbb{E}$-REC (hence RE) and cofinal in $\kappa$. □

Since we plan to index the partial $\mathbb{E}$-REC functions by $\rho$, in the course of the argument for the existence of minimal pairs that follows, fortunately $\kappa$ and $\eta$ have the same RE-cofinality.

Lemma 2.2.4 (Sacks): $L(\kappa)$ $\mathbb{E}$-closed, not $\Sigma_1$-admissible and $\eta$ as above, then

$$\text{RE-cf}(\eta) = \text{RE-cf}(\kappa).$$

<u>Proof</u>. If $\eta = \kappa$, the lemma is trivial so assume $\eta < \kappa$. RE-cf$(\eta) \leq$ RE-cf$(\kappa)$ for let $R(\kappa) \subseteq \kappa$ witness RE-cf$(\kappa)$, i.e. $(\exists\, \sigma_0 < \kappa)(\exists\, e \in \omega)[R(\kappa) = \{\gamma < \kappa \mid \{e_0\}(\sigma_0,\gamma)\downarrow\}]$, with $\cup\, R(\kappa) = \kappa$. Let $R \subseteq \eta$ witness the definition of $\eta$, i.e. $(\exists\, \sigma_1 < \kappa)(\exists\, e_1 \in \omega)[R = \{\gamma < \eta \mid \{e_1\}(\sigma_1,\gamma)\downarrow\}]$ and $\bigcup_{\gamma\in R} [|\{e_1\}(\sigma_1,\gamma)|] = \kappa$ by $R$ non-REC. Then define an RE subset $S$ of $\eta$ as follows: If $\gamma_1$ is first element enumerated into $R(\kappa)$, then enumerate $\delta \in R$ into $S$ least such that, $|\{e_0\}(\sigma_0,\gamma_1)| \leq |\{e_1\}(\sigma_1,\delta)|$. Continue in this fashion insuring that the element enumerated into $S$ at a given stage is also larger than all previous elements enumerated into $S$. $S$ is then RE in parameters $\langle\sigma_0,\sigma_1\rangle$ and unbounded in $\eta$ by 2.1.2. Hence RE-cf$(\eta) \leq$ RE-cf$(\kappa)$. RE-cf$(\kappa) \leq$ RE-cf$(\eta)$: Let $R(\eta) \subseteq \eta$ witness the RE-cf$(\eta)$, i.e. $(\exists\, \sigma_0 < \kappa)(\exists\, e_0 \in \omega)$ $[R(\eta) = \{\gamma < \eta \mid \{e_0\}(\sigma_0,\gamma)\downarrow\}]$ and $\bigcup_{\gamma\in R(\eta)} [|\{e_0\}(\sigma_0,\gamma)|] = \kappa$ by $R(\eta)$ non-REC. Define for $\gamma \in R(\eta)$,

$$\beta_\gamma = \sup_{\delta<\gamma} \kappa_r^{\langle\sigma_0,\delta\rangle},$$

then $(\forall\, \gamma \in R(\eta)) \not\vdash [\beta_\gamma < \kappa]$ by 2.1.2. Then the sequence $S = \langle\beta_\gamma \mid \gamma \in R(\eta)\rangle$ is RE with parameter $\sigma_0$, has order type that of $R(\eta)$ and is unbounded in $\kappa$, by $\bigcup_{\gamma\in R(\eta)} [|\{e_0\}(\sigma_0,\gamma)|] = \kappa$. $\square$

In the course of the construction of the next section, we will proceed in $\kappa$-many stages to satisfy conditions listed in order type $\rho$. We will make use of the assumption $\text{RE-cf}(\rho) = \rho$ to insure that the construction succeeds.

The final Lemma of this section tells us in terms of Levy definability exactly what the $\text{RE-cf}(\kappa)$ is.

Definition 2.2.5: Fix $\beta \in \text{On}$ a limit ordinal and let (in $L(\beta)$),

$$\underline{\Sigma_1\text{-cf}(\beta)} = \mu\gamma \leqslant \beta[\exists\ f : \gamma \longrightarrow \beta \wedge f \in \Sigma_1(L(\beta))$$

$$\wedge \ \text{range}\ (f)\ \text{unbounded in}\ \kappa].$$

<u>Remark</u>: $\Sigma_1$-cf$(\beta) = \beta$ iff $\beta$ is $\Sigma_1$-admissible.

Since we are considering $\mathbb{E}$-closed $L(\kappa)$ which are not $\Sigma_1$-admissible, by the remark, $\Sigma_1$-cf$(\kappa) < \kappa$.

<u>Lemma 2.2.6</u>: $\Sigma_1$-c$(\kappa)$ = RE-cf$(\kappa)$.

<u>Proof</u>. That RE-cf$(\kappa) \leqslant \Sigma_1$-cf$(\kappa)$ is immediate from the fact that RE $\subseteq \Delta_1(L(\kappa))$. To see that $\Sigma_1$-cf$(\kappa) \leqslant$ RE-cf$(\kappa)$, choose $f : \Sigma_1\text{-cf}(\kappa) \longrightarrow \kappa$ $f \in \Sigma_1(L(\kappa))$ and consider

$$S = \langle\langle\gamma, f(\gamma), w_\gamma\rangle \mid \gamma < \Sigma_1\text{-cf}(\kappa)\rangle,$$

where $w_\gamma \in L(\kappa)$, $\forall\ \gamma$ and is the "witness" to the $\Sigma_1$ definition of $f$. (Here we have suppressed the possible parameter in the definition of $f$). Then $S$ is $\mathbb{E}$-REC (hence RE) and cofinal in $\kappa$ since range $(f)$ is. $\square$

<u>Definition 2.2.7</u>: Let $\kappa$ be $\mathbb{E}$-closed and not $\Sigma_1$-admissible, then let

(i) $\underline{\Sigma_1\text{-pr}(\kappa)} = \mu\gamma \leqslant \kappa[\exists\ f : \kappa \xrightarrow{1\text{-}1} \gamma \wedge f \in \Sigma_1(L(\kappa))]$;

then

(ii) $\kappa$ is <u>weakly inadmissible</u>, <u>iff</u>

$$\Sigma_1\text{-pr}(\kappa) \leqslant \Sigma_1\text{-cf}(\kappa)$$

and $\kappa$ is <u>strongly</u> <u>inadmissible</u>, <u>iff</u>

$$\Sigma_1\text{-cf}(\kappa) < \Sigma_1\text{-pr}(\kappa).$$

<u>Proposition 2.2.8</u>: $\Sigma_1\text{-pr}(\kappa) \leqslant \rho$.

<u>Proof</u>. Recall that,

$$\rho = \mu\gamma \leqslant \kappa \ \exists\ \gamma_0[(\exists\ \sigma < \kappa)(\exists\ \delta < \gamma)(\exists\ e \in \omega)[\sigma = |\{e\}(\gamma_0,\delta)|]]$$

and consider $f : \kappa \longrightarrow \rho$ given by, for $\gamma < \kappa$,

$$f(\gamma) = \mu\delta < \rho[(\exists\ e \in \omega)|\{e\}(\gamma_0,\delta)| = \gamma],$$

where $\gamma_0$ is the parameter defining $\rho$. Then $f$ is $\mathbb{E}$-REC via $\gamma_0$ and hence $\Sigma_1(L(\kappa))$. Thus

$$\Sigma_1\text{-pr}(\kappa) \leqslant \rho. \quad \square$$

Proposition 2.2.8 tells us that, in the case that $\kappa$ is strongly inadmissible, there are RE sequences of order type less than $\rho$ which are cofinal in $\rho$. This should motivate the following definition.

<u>Definition 2.2.9</u>: Let $R \subseteq \kappa$ be RE, then $R$ is <u>scattered</u>, if

(i) $R$ is not REC; and

(ii) o.t.(R) = "order type of $R$" $\rho$.

There are scattered sets which are complete (cf. T. Slaman [1981]). To see this consider the example of $\mathbb{E}\text{-cl}(\aleph_{\omega_1})$ and define the following sequence of ordinals, indexed by $\omega_1$: $\gamma < \omega_1$

$$\beta_\alpha = \sup_{\gamma<\aleph_\alpha} \kappa_r^\gamma;$$

$$a_\alpha = \text{index below } \rho \text{ for } \beta_\alpha.$$

At limits take supremums. Then obviously

$$(\forall \alpha < \omega_1)[\beta_\alpha < \kappa] \quad \text{and}$$

$$\sup_{\alpha<\omega_1} \beta_\alpha = \kappa.$$

Finally take $A$ to be $\{<\beta_\alpha, a_{\alpha+1}> : \alpha < \omega_1\}$, then a straightforward ETR shows that there is an $f \leq_{\mathbb{E}} A$ and $f : \omega_1 \longrightarrow \kappa$ unboundedly, where $f(\alpha) = \beta_\alpha$.

Stable sequences of this sort are familiar in admissibility theory and, in fact, give easy solutions to Post's Problem for some $\Sigma_1$-admissible $\alpha$ (cf. Simpson-Hrbacek [1979]).

§3. Minimal Pairs of RE Degrees

Given a notion of relative computability $\leq_R$, (classically for $A, R \subseteq \omega$ $A \leq_T R$ read "A is Turing reducible to B"), which is reflexive, antisymmetric and transitive, one takes the R-degree of a set $A$ to be its equivalence class under $\leq_R$. Intuitively, the degree of $A$ is a measure of the difficulty of 'computing' membership facts about $A$. The relation $\leq_{\mathbb{E}}$ satisfies these conditions as a relation between hyperregular subsets of $L(\kappa)$.

Definition 2.3.0: (Recall for $A, R \subseteq L(\kappa)$ $A \leq_{\mathbb{E}} R$ allows a parameter in $L(\kappa)$, then for $A$ RE and incomplete,

(i) $\mathbb{E}$-dg(A) $= \{B \subseteq L(\kappa) \mid B$ RE and $[B \leq_{\mathbb{E}} A \cap A \leq_{\mathbb{E}} B]\}$

(ii) Let $A, B \subseteq L(\kappa)$ RE such that

(a) $A \not\leq_{\mathbb{E}} B$ and $B \not\leq_{\mathbb{E}} A$; and

(b) [C RE such that $C \leq_{\mathbb{E}} A$ and $C \leq_{\mathbb{E}} B$] $\longrightarrow$ C REC. Then dg(A) and dg(B) (omitting $\mathbb{E}$ in $\mathbb{E}$-dg$(\cdot)$)) form a minimal pair of RE degrees.

Minimal pairs of Turing degrees were constructed independently by Yates [1966] and Lachlan [1966] using a strengthening of the finite-injury priority method, dubbed the "finite-injury-infinite preservation" method. If $\alpha$ is a $\Sigma_1$-admissible ordinal, then for many $\alpha$ there exist minimal pairs of $\alpha$-recursively enumerable degrees (cf. Lerman-Sacks [1972] and later Shore [1978]). Below we demonstrate there existence for $\mathbb{E}$-recursion on $\mathbb{E}$-closed, inadmissible $L(\kappa)$, assuming that $\rho$ is RE-regular (see 2.2.1 (ii)).

Theorem 2.3.1: Let $L(\kappa)$ be $\mathbb{E}$-closed, not $\Sigma_1$-admissible and suppose $\eta = \rho$ and $\rho$ is RE-regular. Then there are RE $A, B \subseteq L(\kappa)$ such that,

(i) $A, B$ are non-REC; and

(ii) $\forall C \subseteq L(\kappa)[(C \text{ RE} \wedge C \leq_{\mathbb{E}} A \wedge C \leq_{\mathbb{E}} B) \Rightarrow C \text{ REC}]$.

Proof. As the partial REC functions on $L(\kappa)$ can be indexed by $\rho$, so too will our conditions. Assume $\{W_e\}_{e<\rho}$ is a listing of the domains of partial REC functions.

Positive Conditions: For $e < \rho$, let

$$P_e^A: \quad W_e \text{ unbounded} \longrightarrow A \cap W_e \neq \emptyset \qquad \langle 1,e \rangle$$

$$P_e^B: \quad W_e \text{ unbounded} \longrightarrow B \cap W_e \neq \emptyset \qquad \langle 2,e \rangle.$$

These conditions are called positive since they entail our putting elements into $A$ and $B$. If $P_e^A$ and $P_e^B$ are satisfied, then obviously neither $A$ nor $B$ is REC.

Negative Conditions: For $e < \rho$, let

$$N_e : [\{(e)_0\}^A = \{(e)_1\}^B = f \text{ total}]$$

$$\Rightarrow f \text{ is } \mathbb{E}\text{-REC} \qquad \langle 3,e\rangle.$$

(Here $f$ is treated as the characteristic function of some subset of $L(\kappa)$ and free use is made of a REC pairing function).

These conditions are called "negative" since our strategy for satisfying them will be to preserve negative information about $A$ and $B$.

Ordering of Conditions:

For $e, f < \rho$ we let $R_e$ have higher priority than $R_f$ just in case $e < f$ as ordinals less than $\rho$. The partial REC functions have a standard indexing via $\gamma < \kappa$. Recall that

$$\rho = \mu\gamma \leqslant \kappa[(\forall\ \sigma < \kappa)(\exists\ \delta < \gamma)(\exists\ e \in \omega)[|\{e\}(p,\delta)| = \sigma]$$

where $p$ is the defining parameter. Using this parameter we have a partial REC $f : \rho \twoheadrightarrow \kappa$ (onto) which, in turn, can be used to interpret a partial REC function $\lambda x\{\sigma\}(x)$, indexed by $\sigma < \kappa$, as a function $\lambda x\{\sigma'\}(x)$, where

$\sigma' < \rho$ and $f(\sigma') = \sigma$. The tameness of this indexing is the content of the next lemma.

<u>Lemma 2.3.2</u>: Assume $\eta = \rho$ and let $\gamma < \rho$ and $f : \rho \twoheadrightarrow \kappa$ (partial REC, onto) be given by the definition of $\rho$. Then

$$\mathrm{dom}(f) \cap \gamma \in L(\kappa).$$

<u>Proof</u>. Suppose not and let $\gamma_0 < \rho$ witness the failure of the lemma. Then

$$\{\delta \mid \delta < \gamma \wedge f(\delta)\downarrow\} \subseteq \gamma$$

is RE and not REC. But this contradicts 2.1.0 (ii). □

Thus the indexing of the partial REC functions by $\rho$ has the property that we eventually (at an ordinal less than $\kappa$) are correct about an initial segment. As changes in our guess (i.e. $f_\sigma \restriction \delta = \{\gamma < \delta \mid L(\sigma) \models f(\gamma)\downarrow\}$ for $\gamma < \rho$, $\sigma < \kappa$) appear, we drop all preservations created on the basis of false information about a given initial segment, that is, we respect the new priority ordering. By Lemma 2.3.2 these errors made in preservation will eventually stop (before $\kappa$).

We will define length of agreement and restraint functions. The first will measure the degree to which some index appears at any stage to be computing the same set from $A$ and $B$. The restraint function will monitor the values of the length of agreement function and, under certain circumstances, preserve negative information about $A$ and $B$ needed in the associated computations giving agreements. If for some index $\emptyset = \langle e,f\rangle$ we have that

$$\{e\}^A = \{f\}^B$$

is total, then the length of agreement function will increase unboundedly in $\kappa$ for that index. If the negative information about $A$ and $B$ were to remain unchanged, we could then use $d$ to read off the values of $g = \{e\}^A = \{f\}^B$ in an $\mathbb{E}$-REC way. This won't happen, of course, since we will put elements into $A$ and $B$ in order to satisfy the positive conditions $P_{e'}^A$ and $P_{e'}^B$. $A$ and $B$ will be RE by the construction, since we will only put $x$ into $A$ or into $B$ at an ordinal REC in $\langle x,p\rangle$ for some $p \in L(\kappa)$. Neither $A$ nor $B$ will be REC provided $P_e^A$ and $P_e^B$ are satisfied for all $e < \rho$.

The basic conflict is between preserving negative information about $A$ and $B$ for the sake of some $N_e$ and enumerating elements into $A$ and $B$ for the sake of $P_e^A$ and $P_e^B$. Our restraint function will resolve this difficulty by ignoring all but the negative restraint required by conditions of higher priority.

This strategy would still be our undoing, since bounded initial segments of negative conditions could successfully thwart all our attempts to satisfy positive conditions of lower priority. Thus the restraint function will also allow us to enumerate elements into one of $A$ or $B$ at stage $\sigma$ for the sake of positive conditions of lower priority provided both sides of the computation

$$\{e\}^A = \{f\}^B$$

associated with $N_{<e,f>}$ are defined and equal on a longer initial segment of $\kappa$ that at any stage before $\sigma$. For an entire initial segment of negative conditions of higher priority than a given positive condition, they are brought to drop back restraint simultaneously be a simple hand-over-hand argument.

Remark: This is a departure from the proof of Lachlan [1966] and Yates [1966] in CRT, since there are no limit ordinals less than $\omega$.

In so doing, if we destroy a computation $\{e\}^A(x)$ for some argument less than the length of agreement, then we preserve $\{f\}^B(x)$ until $\{e\}^A(x)$ returns and the length of agreement advances still further (this will happen, if $\{e\}^A = \{f\}^B$ is total). We will then act to satisfy the positive condition of highest priority as yet unsatisfied. Our strategy for $N_e$ will nonetheless succeed for those $e$ such that $\{(e)_0\}^A = \{(e)_1\}^B =$ total, since at any given stage beyond the activity of $P^A_{e'}$ and $P^B_{e'}$ for $e' < e$, one or the other side of the computation is present and computes the correct value. The parameter that then gives $f$ as a total $\mathbb{E}$-REC function will be an encoding of that stage which bounds both the activity of $P^A_{e'}$ and $P^B_{e'}$ for $e' < e$ and bounds the stages at which computations, destroyed as a result of this activity, return if they ever will.

Definition 2.3.3: If $e < \rho$, we say that $N_e$ is an A-injured at stage $\sigma + 1$, if there is an active A-requirement for $e$ at stage $\sigma$, i.e. $z \leqslant \delta \leqslant \sigma$ such that $z \cap A_\sigma = \emptyset$ $z$ used negatively in $\{(e)_0\}^A(y)$ for some $y$, and for some $x \in z$, $x \in A_{\sigma+1} \setminus A_\sigma$ (similarly for B-injuries).

Auxilliary Functions:

Definition 2.3.4: For $d < \rho$, $d = \langle e,f\rangle$ and $\sigma < \kappa$, let

$$\ell(d,\sigma) = \begin{cases} \sup\limits_{\delta\leqslant\sigma} \alpha < \delta[\{e\}_\sigma^{A_\sigma}(\alpha) = \{f\}_\sigma^{B_\sigma}(\alpha)], \\ \quad \text{if there is such a } \delta. \\ \\ 0, \quad \text{otherwise;} \end{cases}$$

and

$$m(d,\sigma) = \sup_{\tau<\sigma} \ell(d,\tau).$$

<u>Note</u>: We write $A_\sigma$ and $B_\sigma$ to be the portion of $A$ and $B$ enumerated by stage $\sigma < \kappa$.

<u>Proposition 2.3.5</u>: If $e < \rho$, $\sigma < \kappa$, then $\ell(e,\sigma)$ and $m(e,\sigma)$ are REC functions of $e$ and $\sigma$.

<u>Proof</u>. By convention all information in $\{e\}_\sigma^{A_\sigma}(\alpha)$ is less than $\sigma$. The proposition then follows from stage comparison for $\mathbb{E}$-recursion. □

Thus if we have that $\ell(d,\sigma) \geqslant m(d,\sigma)$, it appears that $d$ is defining a total function on $A$ and $B$ giving the same set. If so we act to preserve the corresponding positive and negative facts about $A$ and $B$, tempered by our need to satisfy positive conditions.

Since $\kappa_r^X < \kappa$ for all $x \in L(\kappa)$, there will be two ways of achieving the inequality

$$\{e\}^A \neq \{f\}^B,$$

if it is to be achieved, namely,

(i) for some $\gamma < \kappa$,

$$\{e\}^A(\gamma) \neq \{f\}^B(\gamma); \quad \text{or}$$

(ii) for some $\gamma < \kappa$, either

$$\{e\}^A(\gamma)\uparrow \quad \text{or} \quad \{f\}^B(\gamma)\uparrow.$$

If either of (i) or (ii) obtains then we preserve $A$ (or $B$) through $\kappa_r^{<e,\ >}[A]$ $(\kappa_r^{<e,\ >}[B])$ and, in so doing, satisfy $N_{<e,f>}$.

<u>Definition 2.3.6</u>: For $e < \rho$, $\sigma < \kappa$, let

$$r(e,\sigma) = \begin{cases} \cup\ \{\kappa_r^{\ell(e',\sigma)} : e' < e \wedge \kappa_r^{\ell(e',\sigma)} < \sigma\}, \\ \qquad \text{if } \ell(e,\sigma) \geqslant m(e,\sigma) \\ \\ \cup\ \{\lambda < \sigma : \ell(e,\lambda) = m(e,\lambda)\}, \\ \qquad \text{otherwise} \end{cases}$$

and

$$R(e,\sigma) = \cup\ \{r(e',\sigma) \mid e' < e\}.$$

Remark: To insure that $\ell(e,\lambda)$ is defined and $\geq \delta$ for $\lambda$ a limit ordinal, if it is so unboundedly in $\lambda$, we restrict further injury to individual computations:

If $\{e\}_\sigma^{A_\sigma}(y)$ is destroyed by enumerating some $x$ into $A_{\sigma+1}$ for the sake of $R^A_{e'}$, then if $\{e\}_\tau^{A_\tau}(y)$ for $\tau > \sigma + 1$, we only allow $\{e\}^A(y)$ to be destroyed again at stage $\delta \geq \tau$ for the sake of $P^A_{e''}$, where $e'' < e$. (Similarly for $\{f\}^B(z)$).

With this constraint on the construction, if a fixed computation were injured infinitely often, we would thereby generate an infinite descending chain of ordinals below $e'$ (the source of the initial injury). Therefore injury to a fixed A- or B-computation is truly finite.

Remark: This will not stop our satisfying the positive conditions $P^A_e$ and $P^B_e$, since the amount of $A$ used in $\{e\}^A(y)$ is bounded by $\kappa_r^{<e,y>}[A]$.

Construction: Set

$$A_0 = \emptyset \text{ and } B_0 = \emptyset.$$

Definition 2.3.7: Let $e < \rho$ and $\sigma < \kappa$ and say that $e$ requires attention at stage $\sigma + 1$, if

(i) $e = <1,\tau>$ or $e = <2,\tau>$ and

$(\exists\, x < \sigma \mid [x > R(e,\sigma) \wedge \sigma \leq_{\mathbb{E}} x]$; or

(ii) $e = \langle 3,d,f \rangle$, if $\exists\, y < \sigma$ and there is no active A- or B-requirement at stage $\sigma$ for $N_d$ at argument $y$, where $y$ satisfies either,

$$\text{(a)} \quad [\{d\}_\sigma^{A_\sigma}(y) = 0 \wedge \{f\}_\sigma^{B_\sigma}(y) = 1] \vee$$

$$[\{d\}_\sigma^{A_\sigma}(y) = 1 \wedge \{f\}_\sigma^{B_\sigma}(y) = 0];$$

or

$$\text{(b)} \quad [\{d\}_\sigma^{A_\sigma}(y)\uparrow \wedge \{f\}_\sigma^{B_\sigma}(y)\uparrow].$$

Note: An A-requirement $z$ for $N_e$ is active at stage $\sigma$, if $A_\sigma \cap z = \emptyset$.

Stage $\sigma + 1$: Check to see whether there is an element of

$$\{\gamma < \sigma \mid L(\sigma) \models f(\gamma)\downarrow\}$$

which requires attention. If no, proceed to the next stage.

If yes, let $e$ be the least $\gamma < \sigma$ such that $L(\sigma) \models f(\gamma)\downarrow$ and (identifying $e$ with $f(e)$) proceed by cases depending on the form of $e$.

Case 1. $e$ is of the form $\langle 1,e_0 \rangle$ i.e. we consider condition $P_{e_0}^A$ and since it requires attention $\exists\, x < \sigma$

such that $x \in W_{e_0}$ $(x \in W_{e_0,\sigma})$. For some $x \in W_{e_0,\sigma}$ $x > R(e,\sigma)$ so check to see whether enumerating $x$ into $A_{\sigma+1}$ will destroy any previously destroyed computation for the sake of some $P^A_{e'}$. If there is no such computation, let $A_{\sigma+1} = A_\sigma \cup \{x_0\}$ and go to the next stage, where $x_0$ is least such. If there is, then if $e < e'$ let $A_{\sigma+1} = A_\sigma \cup \{x_0\}$ for lease such $x$ and go to the next stage. If $e' \leqslant e$, let $A_{\sigma+1} = A_\sigma$ and proceed to the next stage.

Case 2. $e$ is of the form $\langle 2,e_0\rangle$, then proceed as in Case 1 with the roles of $A$ and $B$ interchanged.

Case 3. $e$ is of the form $\langle 3,f,g\rangle$. This case breaks up into two subcases depending on whether $e$ requires attention because of 2.3.7 (i) or 2.3.7 (ii).

Subcase 2.3.7 (i): So there is no active $A \cup B$-requirement for $e$ at argument $y$, where

$$[\{f\}_\sigma^{A_\sigma}(y) = 0 \wedge \{g\}_\sigma^{B_\sigma}(y) = 1]$$

or

$$[\{f\}_\sigma^{A_\sigma}(y) = 1 \wedge \{g\}_\sigma^{B_\sigma}(y) = 0].$$

Then let $\sigma \backslash A_\sigma \cup B_\sigma$ be an $A \cup B$-requirement for $e$ at argument $y_0$, the least such $y$.

Subcase 2.3.7 (ii): There is no active A-requirement (B-requirement) for $e$ at argument $y$, where

$$\{f\}_{\sigma}^{A_\sigma}(y)\uparrow \qquad (\{g\}_{\sigma}^{B_\sigma}(y)\uparrow)$$

(If both give preference to A-computation.) Suppose $\{f\}_{\sigma}^{A_\sigma}(y)\uparrow$ and let $\sigma\backslash A_\sigma$ be an A-requirement for $e$ at argument $y_0$, the least such $y$.

Stage $\lambda$ ($\lambda$ a limit ordinal): Let

$$A_\lambda = \bigcup_{\delta<\lambda} A_\delta \quad \text{and} \quad B_\lambda = \bigcup_{\delta<\lambda} B_\delta$$

and proceed to the next stage.

Finally let,

$$A = \bigcup_{\delta<\kappa} A_\delta \quad \text{and} \quad B = \bigcup_{\delta<\kappa} B_\delta .$$

End of Construction. □

The following sequence of lemmata establish that $A$ and $B$ as constructed satisfy the conclusion of Theorem 2.3.1. This first lemma states that $\ell(e,\sigma)$ behaves as desired at limit stages.

Lemma 2.3.8: Let $e < \rho$ and $\lambda, \delta < \kappa$, where $\lambda$ is a limit ordinal. If $\{\sigma < \lambda \mid \ell(e,\sigma) \geqslant \delta\}$ is unbounded in

$\lambda$, then $\ell(e,\lambda) \geqslant \delta$. Thus

$$\{\sigma \mid \ell(e,\sigma) = m(e,\sigma)\} \quad \text{and so}$$

$$\{\sigma \mid r(e,\sigma) = 0\}$$

are closed in the usual set theoretic sense.

Proof. (The proof follows R. Shore [ 1978 ].) Fix an argument $x < \delta$ and assume that $\lambda$, $\delta$, $e$ satisfy the hypotheses of the lemma. By the constraint put on cases 1 and 2 of the construction, there are finitely many stages $\tau$ such that a computation

$$\{f\}^A(x) \quad \text{or} \quad \{g\}^B(x)$$

is injured. So if $\{f\}^A(x)$ and $\{g\}^B(x)$ are defined unboundedly in $\lambda$, they must eventually be constant, i.e. the same computations are uninjured from some stage onward; thus it is a computation of

$$\{f\}_\lambda^{A_\lambda}(x) \quad \text{and} \quad \{g\}_\lambda^{B_\lambda}(x).$$

Since they agree unboundedly often at $x$, they agree at the limit, so

$$\{f\}_{\lambda}^{A_{\lambda}}(x) = \{g\}_{\lambda}^{A_{\lambda}}(x),$$

for all $x < \delta$. The second assertion is immediate for if $\ell(e,\sigma_{\beta}) = m(e,\sigma_{\beta})$ for increasing $\{\sigma_{\beta}\}_{\beta<\gamma}$ such that

$$\lambda = \sup_{\beta<\gamma} \sigma_{\beta} ,$$

then $\ell(e,\lambda) = m(e,\lambda) = \bigcup_{\beta<\gamma} m(e,\sigma_{\beta})$. $\square$

<u>Lemma 2.3.9</u>: $A$ and $B$ are not REC, i.e. $(\forall\ e < \rho)$ $[P_e^A$ and $P_e^B$ are satisfied].

<u>Proof</u>. Proceed by induction on $\rho$. Fix $e < \rho$ and suppose all $e' < e$ of the form $P_{e'}^A$, or $P_{e'}^B$, are satisfied by stage $\sigma_0 < \kappa$. Without loss of generality assume that $e$ is of the form $\langle 1,e_0\rangle$ (symmetric for the case $e = \langle 2,e_0\rangle$), thus we are concerned with putting an element of $W_{e_0}$ into $A$ to satisfy $P_{e_0}^A$.

By lemma 2.3.2 $\mathrm{dom}(f) \cap e_0 \in L(\kappa)$ where $f : \rho \longrightarrow$ onto (partial, onto) given by the definition of $\rho$. Thus take $\sigma_1 \geqslant \sigma_0$ such that $\sigma_1 \geqslant \mu\gamma[\mathrm{dom}(f) \cap e_0 \in L(\gamma)]$. After stage $\sigma_1$ the information we have concerning $\mathrm{dom}(f) \cap e_0$ is correct. Therefore we know conditions $R_{e'}$ for $e' < e_0$, in particular, $N_{e'}$ for $e' < e$. These $N_{e'}$ for $e' < e$ break into two cases:

<u>Case 1</u>. Those $e'$ such that for some $y < \sigma$ we have at

$\delta \leqslant \sigma$,

$$\{(e')_0\}_\sigma^{A_\sigma}(y)\downarrow \quad \text{and} \quad \{(e')_1\}_\delta^{B_\delta}(y)\downarrow$$

and

$$\{(e')_0\}_\delta^{A_\delta}(y) \neq \{(e')_1\}_\delta^{B_\delta}(y)$$

corresponding to subcase 2.3.7 (i) of the construction; or such that for some $y < \sigma$ we have at $\delta \leqslant \sigma$,

$$\{(e')_0\}_\delta^{A_\delta}(y)\uparrow \quad \text{or} \quad \{(e')_1\}_\delta^{B_\delta}(y)\uparrow$$

corresponding to subcase 2.3.7 (ii) of the construction; and

<u>Case 2</u>. Those $e' < e$ such that for all $y < \sigma$,

$$\{(e')_0\}_\sigma^{A_\sigma}(y)\downarrow, \quad \{(e')_1\}_\sigma^{A_\sigma}(y)\downarrow$$

and

$$\{(e')_0\}_\sigma^{A_\sigma}(y) = \{(e')_1\}_\sigma^{A_\sigma}(y).$$

For fixed $e' < e$ the restraint imposed is bounded above by $\kappa_r^{\langle e',y_0\rangle}$, where $y_0$ is least such. The set

$$X = \{\langle e', y\rangle \mid e' < e \wedge y < \sigma \text{ and Case 1 obtains}$$

$$\text{for } e' \text{ at } y \text{ in } L(\sigma)\}$$

satisfies $X \leq_{\mathbb{E}} \sigma, e$ and so, by $\eta = \rho$,

$$\sup_{\tau \in X} \kappa_r^{\langle \tau,\sigma,e\rangle} < \kappa.$$

Take $\sigma_2 \geq \sigma_1$ such that $\sigma_2 \geq \sup_{\tau\in X} \kappa_r^{\langle \tau,\sigma,e\rangle}$.

By $W_{e_0}$ unbounded there is an $X \in W_{e_0}$ with $X \geq \sigma_2$. Now a hand-over-hand argument on

$$\{e' \mid e' < e \wedge \text{Case 2 obtains for } e'\}$$

will yield an ordinal $\sigma_3$ such that

(i) $\sigma_3 \leq_{\mathbb{E}} X$ and

(ii) $X > R(e, \sigma_3)$.

At that stage $P_e^A$ has highest priority among unsatisfied positive conditions, requires attention and is therefore satisfied by our enumerating $X$ into $A_{\sigma_3+1}$. □

<u>Lemma 2.3.10</u>: Let $e < \rho$ of the form $e = \langle 3,f,g\rangle$ and suppose

$$\{f\}^A = \{g\}^B = h \qquad \text{is total.}$$

Then $h$ is REC.

<u>Proof</u>. $e < \rho = \eta$ so by lemma 2.3.2,

$$\mathrm{dom}(f) \cap e \in L(\kappa),$$

where $f : \rho \longrightarrow \kappa$ (partial, onto) $\mathbb{E}$-REC map given by the definition of $\rho$. Take $\sigma_0$ such that

$$\sigma_0 \quad \mu\sigma[\mathrm{dom}(f) \cap e \in L(\kappa)].$$

Thus beyond stage $\sigma_0$ we are correct about the ordering below $e$.

By lemma 2.3.9 each $P^A_{e'}$ and $P^B_{e'}$ is eventually satisfied. Consider the sequences

$$X^A = \{<e',x> \mid e' \text{ is of the form } P^A_{e'} \text{ and}$$

$$x \text{ is enumerated into } A \text{ to satisfy } P^A_{e'}\} \text{ and}$$

$$X^B = \{<e',x> \mid e' \text{ is of the form } P^B_{e'} \text{ and}$$

$$x \text{ is enumerated into } B \text{ to satisfy } P^B_{e'}\}.$$

They are clearly RE and so by Gandy Selection their union $X^A \cup X^B$ is RE. Since put at most one $x$ into $A$ or into $B$ to satisfy such a $P^A_{e'}$ or $P^B_{e'}$

$$\text{o.t.}(X^A \cup X^B) \leq e.$$

By RE-cf$(\kappa) = \rho$ and $e < \rho$, there exists an ordinal $\sigma_1 > \sigma_0$ bounding

$$\{\sigma < \kappa \mid \exists\ \langle e', x\rangle \in X^A \cup X^B \quad x \text{ is enumerated at stage } \sigma \text{ into } A \text{ or } B\}.$$

Now let,

$$K = \{\langle e', \{d\}^C(y), x, \sigma\rangle \mid e' < e \wedge (d = f \vee d = g) \wedge (C = A \vee C = B) \wedge \{d\}^C(q) \text{ is injured due to } e' \text{ by enumerating } x \text{ into } C \text{ and } \{d\}^C(y) \text{ returns at stage } \sigma\},$$

then by the assumption that

$$\{f\}^A = \{g\}^B = h \qquad \text{is total}$$

such $\sigma$'s exist and $K$ is RE. By the constraint on

Cases 1 and 2 of the construction and the fact that we enumerate at most one $x$ into $A$ or $B$ to satisfy $P^A_{e'}$, or $P^B_{e'}$. We have

$$\text{o.t.}(K) \leqslant e < \rho.$$

Thus by $\text{RE-cf}(\kappa) = \rho$, there exists $\sigma_2 \geqslant \sigma_1$ such that,

$$\sigma_2 \geqslant \sup\{\pi_4(\tau) \mid \tau \in K\}.$$

With the parameter $\langle e,\sigma_2\rangle = p$, we show that REC in $x, p$ we can compute

$$h = \{f\}^A = \{g\}^B \qquad \text{(total)}.$$

(We have 'translated' the computation of $h(x)$ by $\sigma_2$ representing interference posed by positive conditions of higher priority.)

<u>Claim</u>: $\ell(e,\sigma)$ is unbounded as $\sigma$ varies over $\kappa$.

<u>Proof of Claim</u>. Suppose not and let $\beta = \mu\gamma[(\exists\ \tau_0)(\forall\ \tau > \tau_0)(\ell(e,\tau) < \gamma)]$. Fix $\tau_0$ as in the definition of $\beta$.

$\beta = \delta + 1$ : since $A$ and $B$ are regular

$\exists\ \tau_1 > \tau_0$

$$(\forall\ \tau \geq \tau_1)[\{f\}_\tau^{A_\tau}(\delta) = \{g\}_\tau^{B_\tau}(\delta) =$$

$$\{f\}^A(\delta) = \{g\}^B(\delta)]$$

and since $\beta$ was minimal $\exists\ \tau > \tau_1$ such that,

$$\ell(e,\tau) = \delta;$$

but then by definition $\ell(e,\tau) \geq \delta + 1$, a contradiction. $\beta = \lambda$ a limit ordinal, then consider

$$\delta(e) = \{\sigma \mid \ell(e,\sigma) \geq \delta\}$$

for $\delta < \lambda$. $\delta(e)$ is $\mathbb{E}$-REC and unbounded, hence for $\delta < \lambda$ there is a common limit point $\tau > \tau_0$ and by lemma 2.3.8, $\ell(e,\tau) \geq \delta$, a contradiction. claim □

Sublemma 2.3.11: If $\rho$ is REC regular and $\delta < \rho$, then $\{\sigma \mid \sigma < \delta\}$ is $\delta$-reflecting, i.e. for $p < \kappa$, a parameter,

$$\sup_{\sigma<\delta} \kappa_0^{<p,\sigma,\delta>} \leq \kappa_r^{<p,\delta>}.$$

Proof. This is essentially due to Normann [1979], for normal Kleene recursion in ${}^{k+2}\mathbb{E}$, where he proves the analogous result for $\rho = |\prec|$, $\prec$ a recursive well-ordering of $I$ (individuals). Proceed as in Normann, but do an effective version of Grilliot Selection on $L(\kappa)$ (see Appendix for details). Sublemma $\square$

We now proceed by induction to show that for every $\tau > \sigma_2$ at least one of

$$\{f\}_{\tau}^{A_\tau}(x) \quad \text{or} \quad \{g\}_{\tau}^{B_\tau}(x)$$

is defined. Then we show by ETR on $\rho$ that there is such a $\tau \leqslant x, \sigma_2$.

If $\ell(e,\tau) = m(e,\tau) \geqslant x$, then both of $\{f\}_{\tau}^{A_\tau}(x)$ and $\{g\}_{\tau}^{B_\tau}(x)$ are correct by induction. Otherwise let $\gamma < \tau$ be the last stage greater than $\sigma_2$ such that

$$\ell(e,\gamma) = m(e,\gamma) \geqslant x.$$

By lemma 2.3.8 only one was injured at stage $\gamma$, say $\{f\}_{\gamma}^{A_\gamma}(x)$. By the definitions of $r(e,\sigma)$ and $R(e,\sigma)$ nothing has been allowed to injure the computation of

$$\{f\}_{\gamma}^{A_\gamma}(x)$$

and it is preserved through $\tau$ by the choice of $\gamma$.
Thus,

$$\{f\}_{\gamma}^{A_{\gamma}}(x) = \{f\}_{\tau}^{A_{\tau}}(x).$$

But $\{f\}_{\gamma}^{A_{\gamma}}(x) = \{g\}_{\gamma}^{B_{\gamma}}(x)$ by the choice of $\gamma$ and one is correct by induction.

To compute $h(x)$: suppose we have computed $h(y)$ for all $y < x$. Considered as pairs $\langle\sigma_1, y\rangle$, by sublemma 2.3.11,

$$\sup_{y<x} \kappa_0^{\langle\sigma_2, y\rangle} < \kappa_r^{\langle\sigma_2, x\rangle}.$$

Thus at some $\sigma < \kappa_r^{\langle\sigma_2, x\rangle}$ all $h(y)$ have been computed. At $\sigma$ we can ask $\{f\}_{\sigma}^{A_{\sigma}}(x)$ or $\{g\}_{\sigma}^{B_{\sigma}}(x)$ for a value and be correct since from $\sigma$ onward we protected with highest priority the computations $\{f\}^{A}(x)$ and $\{g\}^{B}(x)$ associated with $e$. By reflection $\exists\ \sigma < \kappa_0^{\langle\sigma_2, x\rangle}$ at which this is true and, hence, at any $\tau \geqslant \sigma$, $\tau \leqslant_{\mathbb{E}} \langle\sigma_2, x\rangle$ we compute $h(x)$ and get the correct value.

$\square$

This completes the proof of Theorem 2.3.1. $\square$

<u>Remark</u>: A closer look at the proof of Theorem 2.3.1 reveals that the following strengthening is possible.

<u>Theorem 2.3.12</u>: Let $C \subseteq \kappa$ be RE, non-REC and incomplete and RE-cf$(\rho) = \rho$. Then there exist $A, B \subseteq \kappa$, RE and non-REC such that,

(i) If $D \subseteq \kappa$ is RE, $D \leq_{\mathbb{E}} A$ and $D \leq_{\mathbb{E}} B$, then $D$ is REC; and

(ii) both $[A \not\leq_{\mathbb{E}} C \wedge C \leq_{\mathbb{E}} A]$ and
$[B \not\leq_{\mathbb{E}} C \wedge C \leq_{\mathbb{E}} B]$.

The proof of 2.3.12 proceeds as in Theorem 2.3.1 with additional conditions:

$$R_e^A: \qquad \{e_0\}^A \neq C \wedge \{e_0\}^C \neq A; \qquad e = \langle 4,e_0\rangle$$

$$R_e^B: \qquad \{e_0\}^B \neq C \wedge \{e_0\}^C \neq B; \qquad e = \langle 5,e_0\rangle.$$

For clause one of each $R_e^A$ and $R_e^B$ introduce preservations which will insure that if, say, $\{e\}^A = C$, then $C$ is REC, contradicting the choice of $C$. For clause two of each of $R_e^A$ and $R_e^B$ which are positive, introduce constraints on putting $x$ into $A$ or $B$ for $R_e^A$ or $R_e^B$ similar to those on $P_e^A$ and $P_e^B$ in the construction of 2.3.1. The proof of this will appear in E. Griffor [1980]. (At this writing the author has noticed that the assumption of RE-regularity on $\rho$ can be eliminated by an entirely different proof for another choice of the parameter $\rho$ (see E. Griffor [1980]).)

# Chapter 3: $\mathbb{E}(A)$-Recursion and the Density Theorem

## §0. $\mathbb{E}(A)$-Recursion

In Chapter 2 we made use of the $\kappa_r$-function in the setting of $\mathbb{E}$-closed, inadmissible $L(\kappa)$ to carry out a priority argument yielding a minimal pair of RE-degrees. As remarked, this minimal pair can be taken to be incomparable with an arbitrary incomplete RE $A \subseteq \kappa$. The $\kappa_r$-function has also been used by G. Sacks [1980] to give a positive solution to Post's Problem for every $\mathbb{E}$-closed, inadmissible $L(\kappa)$. His solution uses a 'wait-and see' argument without injuries. Thus what was a finite-injury priority argument in ORT and $\alpha$-recursion theory can be done without injury. We will see in this chapter that the density theorem for RE-degrees is, in many cases, a 'finite-injury' argument, i.e. injuries bounded below $\kappa$. Again $\Sigma_1$-reflecting ordinals will play the key role in our strategy, where here we are concerned with $\Sigma_1(A)$-reflecting ordinals for $A \subseteq \kappa$ RE and incomplete. For this choice of $A$, $L_\kappa[A] = L_\kappa$ and $L_\kappa$ is $\mathbb{E}(A)$-closed, i.e. $A$ does not violate the $\mathbb{E}$-closure of $L(\kappa)$. If $L(\kappa)$ is inadmissible, then so is $L_\kappa[A]$ and this fact combined with $L(\kappa)$ $\mathbb{E}(A)$-closed provide the basis for an argument using $\Sigma_1(A)$-reflecting ordinals.

<u>Proposition 3.0.0</u>: $A \subseteq \kappa$, then

$$\Sigma_1(A)\text{-cf}(\kappa) \leqslant \Sigma_1\text{-cf}(\kappa).$$

<u>Proof</u>. Immediate from the definition of $\Sigma_1(A)$-cf($\kappa$).

□

We will see that the RE(A)-cf($\kappa$) is $\Sigma_1(A)$-cf($\kappa$).

<u>Definition 3.0.1</u>: For $\delta$, $\sigma < \kappa$, $L(\kappa)$ $\mathbb{E}$-closed and $A \subseteq \kappa$, call $\sigma$ <u>$\delta$(A)-reflecting</u>, if for all $\Sigma_1(A)$-formulae, $\phi(x)$, of $\mathcal{L}(A)$

$$L_\delta[A] \models \phi(\delta) \Rightarrow L_{\kappa_0^\delta[A]}[A] \models \phi(\delta).$$

<u>Remark</u>: (i) $\mathcal{L}(A)$ is the language $\mathcal{L}$ of Chapter 1 with an additional predicate letter A to be interpreted as a chosen subset of $\kappa$.

(ii) Recall from Chapter 1 that $\kappa_0^\delta[A]$ = "supremum of those $r < \kappa$ which are REC(A) via some integer in $\delta$". Obviously $\kappa_0^\delta[A] \geqslant \kappa_0^\delta$.

As before, the limit of $\delta$(A)-reflecting ordinals is $\delta$(A)-reflecting. Let $\kappa_r^\delta[A]$ be the last $\delta$(A)-reflecting ordinal. For $A \subseteq \kappa$ RE and incomplete we will have $L_\kappa[A] = L(\kappa)$ $\mathbb{E}$(A)-closed and, hence, by 3.0.0 for each $\delta < \kappa$ there is a $\Sigma_1(A)$ formula $\theta_\delta(x)$ of $\mathcal{L}(A)$ such that, $L_{\kappa_1^\delta[A]}[A] \not\models \theta_\delta(\delta)$, but $L_{\kappa_r^\delta[A]+1}[A] \models \theta_\delta(\delta)$ i.e. $\kappa_r^\delta[A] < \kappa$. For such an $A$ we exhibit the same uniformity which Harrington [1968] demonstrated in the setting of higher types.

<u>Lemma 3.0.2</u>: For $\delta < \kappa$ and $A$ RE and incomplete, if $B \subseteq \kappa$ non-empty and co-RE(A) via $\delta$, then there is a $\gamma \in B$ such that $\kappa_r^{<\delta,\gamma>}[A] \leq \kappa_r^\delta[A]$.

<u>Proof</u>. Suppose not and pick $\Sigma_1(A)$ $\phi(x)$ in $L(A)$ such that $L_{\kappa_r^\delta[A]+1}[A] \models \phi(\delta)$ and $L_{\kappa_0^\delta[A]}[A] \not\models \phi(\delta)$.

This can be done, since by the choice of $A$ $\kappa_r^\sigma[A] < \kappa$ for $\sigma < \kappa$. For all $\gamma \in B$ $\kappa_r^{<\delta,\gamma>}[A] > \kappa_r^\delta[A]$ and so $L_{\kappa_r^{<\delta,\gamma>}[A]}[A] \models \phi(\delta)$. By reflection

$$L_{\kappa_0^{<\delta,\gamma>}[A]}[A] \models \phi(\delta).$$

Since $B$ is co-RE(A) via $\delta$, there is a $\Sigma_1(A)$ formula $\psi(x,y)$ of $L(A)$ such that

$$(\forall\, \gamma < \kappa)[\gamma \notin B \text{ iff } L_{\kappa_0^{<\delta,\gamma>}[A]}[A] \models \psi(\delta,\gamma)].$$

Thus $\forall\, \gamma < \kappa$,

$L_{\kappa_0^{<\delta,\gamma>}[A]}[A] \models \psi(\delta,\gamma) \vee \phi(\delta)$ and so $\forall\ \gamma < \kappa$,

$L_{\kappa_0^{\delta}[A]}[A] \models \psi(\delta,\gamma) \vee \phi(\delta)$. We have used the bounding principle relativized to $A$, which holds since $A$ does not violate the $\mathbb{E}$-closure of $L(\kappa)$.

Since $B \neq \emptyset$, $\exists\ \gamma < \kappa$ such that $L_{\kappa_0^{\delta}[A]}[A] \not\models \phi(\delta,\gamma)$ and so $L_{\kappa_0^{\delta}[A]}[A] \models \phi(\delta)$, contradicting the choice of $\phi$.

$\square$

<u>Theorem 3.0.3</u>: $L(\kappa)$ $\mathbb{E}$-closed, inadmissible and $A \subseteq \kappa$ RE and incomplete. For any $B \subseteq \kappa$ RE(A) there is a relation $<_{R(A)}$ on $\kappa$ which is RE(A), such that $(\forall\ \gamma < \kappa)$ $\gamma \notin B$ <u>iff</u> $(\exists\ \delta < \kappa)[\delta$ encodes a $<_{R(A)}$-infinite descending path through the computation tree resulting by applying the definition of $B$ (rel. to $A$) to $\gamma]$.

<u>Proof</u>. Suppose $B$ RE(A) via $e$, i.e.

$$\gamma \in B \longleftrightarrow \{e\}^{A}(\gamma)\downarrow$$

(suppressing parameters). Then if $\gamma \in B$, $\{e\}^{A}(\gamma)\downarrow$ and $|\{e\}^{A}(\gamma)|^{A} = \sigma < \kappa$ and the computation uses positive and negative information about $A \cap \sigma \in L(\kappa)$. Similarly, if $\gamma \notin B$, then $\{e\}^{A}(\gamma)\uparrow$, so repeat the argument of Chapter

1 relative to $A$ to get an infinite descending path through $<_{R(A)}$ below $\langle e,\gamma\rangle$ (the subcomputation relation relative to $A$). By the fact that $A$ preserves the $\mathbb{E}$-closure of $L(\kappa)$ there will be $\sigma < \kappa$, such that

$$L_\sigma[A] \models \text{"}\exists \text{ inf descending path through } \{e\}^A(\gamma)\text{"}. \qquad \square$$

We now have our uniformity,

<u>Theorem 3.0.4</u>: Take $A \subseteq \kappa$ RE and incomplete, then there is a formula $\theta^A(x)$ of $L(A)$ (not $\Sigma_1(A)$) such that $\forall\ \delta < \kappa$

$$L_{\kappa_r^\delta[A]}[A] \models \theta^A(\delta),$$

but for all $\sigma < \kappa_r^\delta[A]$, $L_\sigma[A] \not\models \theta^A(\delta)$.

<u>Proof</u>. Take $<_{R(A)}$ as in 3.0.3 RE(A), so there is a $\Sigma_1(A)$ formula $\phi(x,y)$ of $L(A)$ such that,

$$(\forall\ \sigma,\tau < \kappa)[\sigma <_{R(A)} \tau \text{ iff } L_{\kappa_0^{\langle\sigma,\tau\rangle}[A]}[A] \models \phi(\sigma,\tau)$$
$$\text{iff } L_\kappa[A] \models \phi(\sigma,\tau)].$$

By our choice of $A$, there is a $\Sigma_1(A)$ formula $\psi(x)$ in

$\mathcal{L}(A)$ such that

$$(\forall\, \sigma < \kappa)[\sigma\downarrow \ \text{ iff } \ L_{\kappa_0^{\sigma}[A]}[A] \models \psi(\sigma)],$$

viewing $\sigma$ as a code for an $\mathbb{E}(A)$-computation. Let $\theta^A(x)$ be the following formula of $\mathcal{L}(A)$.

$$\theta^A(x) \equiv (\forall\, m \in \omega)[\psi(\langle m,x\rangle) \vee \exists\, \delta < \kappa$$

$$(\delta \ \text{ encodes } \ f : \omega \longrightarrow \kappa \wedge f(0) = \langle m,x\rangle$$

$$\text{and } \ (\forall\, n \in \omega)\psi(f(n),f(n+1))].$$

Thus for any $\sigma$, $\delta < \kappa$: if

$$L_\sigma[A] \models \theta^A(\delta), \qquad \text{then}$$

$$(\forall\, m \in \omega)[\{m\}(\delta)\downarrow \ \ \underline{\text{iff}} \ \ L_\sigma[A] \models \psi(\langle m,\delta\rangle)]$$

so $\sigma \geqslant \kappa_0^{\delta}[A]$ and, in fact, $\sigma \geqslant \kappa_r^{\delta}[A]$, for otherwise this fact could be reflected to some $\tau < \kappa_0^{\delta}[A]$, which is absurd.

<u>Claim</u>: For $\delta < \kappa$ $\ L_{\kappa_r^{\delta}[A]}[A] \models \theta^A(\delta)$.

Proof of Claim. It suffices to find, given $m \in \omega$ wush that $\{m\}^A(\delta)\uparrow$, an $f : \omega \longrightarrow \kappa$ such that $f(0) = \langle m,\delta\rangle$ and $(\exists\, n \in \omega)\ [\phi(f(n),f(n+1))]$ and $(\exists\, n \in \omega)[\kappa_r^{\langle f(n),f(n+1)\rangle}[A] \leqslant \kappa_r^{f(0)}[A]]$. Let $f(0) = \langle m,\delta\rangle$ and given $f(n)\uparrow$ such that

$$\kappa_r^{f(n)}[A] \leqslant \kappa_r[A],$$

find $f(n+1)\uparrow$ with

1. $\phi(f(n),f(n+1)$, subcomputation

2. $\kappa_r^{\langle f(n),f(n+1)\rangle}[A] \leqslant \kappa_r^{\delta}[A]$.

The hard case is a branching i.e. $f(n) = \langle 3^{m'}\cdot 5^{e},\tau\rangle$ for some $\tau < \kappa$, $m', e \in \omega$ and $\{m'\}(\tau)\downarrow$. Let $\sigma = |m'|_{\tau}^{A}$, then

$$B = \{\gamma < \kappa \mid \gamma\downarrow \wedge \gamma \in W_e^{\langle H_\sigma^A,\tau\rangle}\}$$

is co-RE(A) via $f(n)$ and so by 3.0.2 there is $f(n+1) \in B$ such that

$$\kappa_r^{\langle f(n),f(n+1)\rangle}[A] \leqslant \kappa_r^{f(n)}[A] \leqslant \kappa_r^{\delta}[A]$$

take $f(n+2)$ to be least such. $\square$

Remark: In the case that the universe is effectively well-ordered (as in $L$) we can actually piece together the infinite descending path. In general, as we will see in Chapter 4, we can only designate several paths.

## §1. Relativized RE Projecta

In order to show that the RE degrees are dense for $L(\kappa)$ we will need to work in $\mathbb{E}(A)$-recursion, that is, in the structure $L_\kappa[A]$, where $A$ is RE and incomplete. As we saw in the case of Theorem 2.3.1 (Minimal Pair) of the last chapter parameters such as $\eta^{L(\kappa)}$, $\rho^{L(\kappa)}$ and RE-cf$(\kappa)$ played important roles. As we will work in $L_\kappa[A]$ we will instead be concerned with $\eta^{L_\kappa[A]}$, $\rho^{L_\kappa[A]}$ and RE(A)-cf$(\kappa)$ among others.

<u>Definition 3.1.0</u>: Recall that the $\mathbb{E}(A)$-REC functions are denoted $\{e\}^A$ and let

(i) $\rho^{L_\kappa[A]} = \mu\gamma \leqslant \kappa\ \exists\ \sigma_0[(\forall\ \sigma < \kappa)(\forall\ \delta < \gamma)(\exists\ e \in \omega)$

$$[\sigma = |\{e\}^A(\sigma_0,\delta)|^A]$$

a "notation map relative to A"; and

(ii) $\eta^{L_\kappa[A]} = \mu\gamma \leqslant \kappa\ \exists\ \sigma_0[R \subseteq \gamma$ is RE(A) in $\sigma_0$

but $\forall\ \sigma < \kappa$ $R$ is not REC(A) in $\sigma$].

Then as before we have,

<u>Lemma 3.1.1</u>: (Abbreviate $\rho^{L_\kappa[A]}$ as $\rho^A$ and $\eta^{L_\kappa[A]}$ as $\eta^A$).

(i) $\eta^A \leqslant \rho^A$

(ii) $\eta^A$ and $\rho^A$ are cardinals in $L_\kappa[A]$.

<u>Proof</u>. (i) By the definition of $\rho^A$ there is a $\sigma_0 < \kappa$ such that

$$(\forall \sigma < \kappa)(\exists \delta < \rho^A)(\exists e \in \omega)[\sigma = |\{e\}^A(\sigma_0,\delta)|^A]. \qquad (*)$$

Let

$$\mathcal{O}^A = \{\gamma < \rho^A \mid \{(\gamma)_0\}^A(\sigma_0(\gamma)_1)\downarrow\},$$

then $\mathcal{O}^A$ is clearly $RE(A)\setminus REC(A)$.

(ii) Suppose $\exists f \in L_\kappa[A]$ such that $f : \eta^A \longleftrightarrow \gamma$ for some $\gamma < \eta^A$ and let $R \subseteq \eta^A$ witness the definition of $\eta^A$. Then

$$\{f(\delta) \mid \delta \in R\} \subseteq \gamma$$

is $RE(A)\setminus REC(A)$ by the choice of $R$, contradicting the choice of $\eta^A$ as least.

Similarly suppose $\exists f \in L_\kappa[A]$ such that $f : \rho^A \longleftrightarrow \gamma$ for some $\gamma < \rho^A$. By the definition of $\rho^A$, $\exists \sigma_0 < \kappa$ such that,

$$(\forall\ \sigma < \kappa)(\exists\ \delta < \rho^A)(\exists\ e \in \omega)[\sigma = |\{e\}^A(\sigma_0,\delta)|^A].$$

Then clearly,

$$(\forall\ \sigma < \kappa)(\exists\ \delta < \gamma)(\exists\ e \in \omega)[\sigma = |\{e\}^A(\sigma_0,f^{-1}(\delta)|^A]$$

namely fix $\sigma < \kappa$ and take $\delta < \rho^A$ and $e \in \omega$ with $\sigma = |\{e\}^A(\sigma_0,f(\delta))|^A$ and then $\sigma = |\{e\}^A(\sigma_0,f(\delta))|^A$. But this contradicts the choice of $\rho^A$ as least. □

<u>Definition 3.1.2</u>: Let $A \subseteq \kappa$, then

(i) $A$ is <u>regular</u> <u>iff</u> $(\forall\ \sigma < \kappa)[A \cap \sigma \in L(\kappa)]$

(ii) $A$ is <u>hyperregular</u> <u>iff</u> $L_\kappa[A]$ is $\mathbb{E}$-closed.

<u>Proposition 3.1.3</u>: Let $A \subseteq \kappa$ RE, then

(i) $A$ hyperregular $\longrightarrow A$ regular,

(ii) $A$ is complete RE $\longleftrightarrow A$ is not hyperregular

(recall that an RE $A$ is complete just in case $B \leqslant_{\mathbb{E}} A$ for all RE $B \subseteq \kappa$).

<u>Proof</u>. (i) Suppose $A$ is not regular and let $\sigma_0 < \kappa$ witness the non-regularity of $A$, i.e. $A \cap \sigma_0 \notin L(\kappa)$. Define $f : \sigma_0 \longrightarrow \kappa$, by

$$f(\gamma) = \begin{cases} |\{e\}(\gamma)|, & \text{if } \gamma \in A \cap \sigma_0 \\ 0 & , \text{otherwise.} \end{cases}$$

Then $f \leqslant_{\mathbb{E}} A$ and range(f) is unbounded in $\kappa$, for otherwise $\exists\, \sigma < \kappa$ such that range(f) $\subseteq \sigma$. But then $A \cap \sigma_0 \leqslant_{\mathbb{E}} \langle\sigma_0, e, \sigma\rangle$ and hence $A \cap \sigma_0 \in L(\kappa)$ contradicting the choice of $\sigma_0$. But now f witnesses $L_\kappa[A]$ not $\mathbb{E}$-closed, contradicting A hyperregular.

(ii) Case 1: $\rho < \kappa$, then let,

$$\mathcal{O} = \{\gamma < \rho \mid \{(\gamma)_0\}(\sigma_0, (\gamma)_1)\},$$

where $\sigma_0$ is the parameter in the definition of $\rho$. Recall that $\{|\gamma|^{\mathcal{O}} \mid \gamma \in \mathcal{O}\}$ are unbounded in $\kappa$ and let $f : \rho \longrightarrow \kappa$ be defined by,

$$f(\gamma) = \begin{cases} |\gamma|^{\mathcal{O}}, & \text{if } \gamma \in \mathcal{O} \\ 0, & \text{otherwise.} \end{cases}$$

Then since $\mathcal{O}$ is RE, $\mathcal{O} \leqslant_{\mathbb{E}} A$ and hence $f \leqslant_{\mathbb{E}} A$. Then f witnesses A not hyperregular (note that by (i) $L_\kappa[A] = L_\kappa$).

Case 2: Assume that if $\kappa_\delta < \kappa$ is $\mathbb{E}$-closed, then the function

$$\kappa_\delta \xmapsto{G} \rho^{L(\kappa_\delta)}, \quad \text{where}$$

$$\kappa = \bigcup_{\delta<\lambda} \kappa_\delta, \quad \text{is constant.}$$

Define for $\rho = G(\kappa_\delta)$, $\forall\ \delta$ by ETR $f : \rho \longrightarrow \kappa$

$$f(\delta) = \sup_{\tau<\rho} g_\delta(\tau),$$

where in $L(\kappa_\delta)$ we let

$$\mathcal{O}_{\kappa_\delta} = \{\gamma < \rho \mid \{(\gamma)_0\}(p_{\kappa_\delta},(\gamma)_1)\}$$

and $p_{\kappa_\delta}$ defines $\rho^{L(\kappa_\delta)}$. Also $g_\delta : \rho \longrightarrow \kappa_\delta$, by

$$g_\delta(\tau) = \begin{cases} (\tau)^{\mathcal{O}_{\kappa_\delta}}, & \text{if } \tau \in \mathcal{O}_{\kappa_\delta} \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \leqslant_{\mathbb{E}} A$ and witness $L_\kappa[A]$ not $\mathbb{E}$-closed. Hence $A$ is not hyperregular.

Now suppose $A$ is not hyperregular, i.e. there is $\delta < \kappa$ and $f : \delta \longrightarrow \kappa$ unbounded with $f \leqslant_{\mathbb{E}} A$, then

$$\{e\}(\gamma) \longleftrightarrow$$

$$(\exists\ \sigma < \sup_{\gamma<\delta} f(\gamma))[|\langle e,\gamma\rangle|^{\mathcal{O}} \leqslant \sigma].$$

Hence $A$ is complete RE. $\square$

Remark: Recall (Sacks [1980]) there is a regular complete RE subset of $\kappa$. We will use this and the fact that every incomplete RE subset of $\kappa$ is regular (by 3.1.3) in our argument for density of the RE degrees.

Lemma 3.1.4: Suppose $A \subseteq \kappa$ is RE and incomplete, then

(i) $L_\kappa[A] = L_\kappa$, and

(ii) $L_\kappa[A]$ is $\mathbb{E}$-closed.

Proof. (i) By 3.1.3 $A$ is regular and hence $L_\kappa[A] = L_\kappa$. (ii) by 3.0.3 $A$ incomplete iff $A$ hyperregular iff $L_\kappa[A]$ is $\mathbb{E}$-closed. $\square$

Remark: For incomplete RE $A \subseteq \kappa$ we will often write $L(\kappa)$ for $L_\kappa[A]$, noting that we are still concerned with those functions $\mathbb{E}(A)$-REC on $L(\kappa)$.

Lemma 3.1.5: Let $\eta^A$ and $\rho^A$ be as in 3.0.0, where $A \subseteq \kappa$ is RE and incomplete, then

(i) $\rho = \rho^A$; and

(ii) $\exists\ \kappa$ $\mathbb{E}$-closed such that

$$\eta^A < \rho^A.$$

Proof. (i) $\rho^A \leqslant \rho$ by definition so assume $\rho^A < \rho$ then the map

$$\sigma \xmapsto{f} \mu\gamma < \rho^A[\sigma = |\{e\}^A(\sigma_0,\gamma)|],$$

where $\sigma_0$ is the parameter in the definition of $\rho^A$ is $\mathbb{E}(A)$-REC and total on $\kappa$. By the definition of this notation map for fixed $\sigma < \kappa$

$$(*) \qquad \gamma < \rho^A \mid (\exists\, e \in \omega)[\sigma = |\{e\}(\sigma_0,\gamma)|^A]\}$$

$\in L_\kappa[A] = L(\kappa)$ and so consider

$$f \restriction \rho : \rho \longrightarrow \rho^A.$$

By (*) $f \restriction \rho$ uses at most $A \cap \rho \in L(\kappa)$, by regularity of $A$ and, therefore, $f \restriction \rho \in L(\kappa)$. But this contradicts $\rho$ a cardinal in $L(\kappa)$. (i) $\square$

(ii) By part (i) $\rho = \rho^A$ so suffices to give $\mathbb{E}$-closed $\kappa$ where $\eta < \rho$: Let $L(\kappa) = \mathbb{E}\text{-closure}(\aleph_{\omega_1})$ and in $L(\kappa)$ form

$$\mathbb{E}\text{-cl}(\omega_1 \cup \{\aleph_{\omega_1}\}) = M$$

and let $\overline{M}$ be the Mostowski collapse of $M$. Now $\rho^{L(\kappa)} = \aleph_{\omega_1}$ and hence $\rho^{\overline{M}} = \aleph_{\omega_1}^{\overline{M}}$, however clearly

$$[\sup_{\gamma<\omega_1} \kappa_0^\gamma]^{\overline{M}} = \mathrm{On} \cap \overline{M}$$

and hence $\eta^{\overline{M}} < \rho^{\overline{M}}$. Hence

$$\eta^A < \rho^A = \rho^{\overline{M}}$$

for any incomplete RE $A \subseteq On \cap \overline{M}$. (ii) $\square$

<u>Remark</u>: $\overline{M}$ is $\mathbb{E}$-closed since $\overline{M}$ is closed under $X \longrightarrow \kappa_r^X$. To see this notice that any infinite descending path through a divergent computation in $L(\kappa)$ can be taken to be a countable sequence of countable ordinals, hence REC encodable as a countable ordinal. At a branching at level $n$, simply take the least $\alpha < \omega_1$ such that for some $\gamma \leqslant \aleph_\alpha$ the computation procedure diverges on $\gamma$. This example was first noticed by T. Slaman.

## §2. Relativized RE Cofinalities

As in Chapter 2 an obstacle to the priority method is the existence of RE sequences of short order type cofinal in $\kappa$, so-called scattered sets. Such sequences arise naturally in the argument for the existence of a minimal pair of RE-degrees. Associated with a fixed negative condition are computations which are injured for the sake of positive conditions of higher priority. The set of such computations which eventually return form an RE sequence which may or may not be REC. Essential to the 'recovery' argument there was our ability to bound these sequences below $\kappa$. Working in $L_\kappa[A]$ we are concerned with sequences $RE(A)\setminus REC(A)$ of order type less than $\eta^A$ (and hence less than $\rho^A$ since we will assume that $\eta^A = \rho^A$).

<u>Definition 3.2.0</u>: Let $A \subseteq \kappa$ and $\lambda$ a cardinal of $L_\kappa[A]$, then

(i) <u>$RE(A)$-$cf(\lambda)$</u> $= \mu\gamma[\gamma$ is the order type of some $X \subseteq \lambda$ $RE(A)$ and unbounded in $\lambda]$

(ii) $\lambda$ is <u>$RE(A)$-regular</u> <u>iff</u> $RE(A)$-$cf(\lambda) = \lambda$.

<u>Proposition 3.2.1</u>: Let $A \subseteq \kappa$ be RE and incomplete, then

(i) $RE(A)$-$cf(\kappa) \leqslant RE$-$cf(\kappa)$; and

(ii) $RE(A)$-$cf(\kappa) = \Sigma_1(A)$-$cf(\kappa)$.

<u>Proof</u>. (i) Immediate since any $R \subseteq \kappa$ which is RE is RE(A). (i) $\Box$

(ii) $\Sigma_1(A)$-cf$(\kappa) \leq$ RE(A)-cf$(\kappa)$ since any $R \subseteq \kappa$ RE(A) is $\Sigma_1(A)$, where the witness in the $\Sigma_1(A)$ definition is the computation $\{e\}^A(\gamma) = W$, $e, \gamma < \kappa$. $W \in L_\kappa[A]$ by the fact that A preserves the $\mathbb{E}$-closure of $L(\kappa)$, i.e. if $\{e\}^A(\gamma)\downarrow$, then $\exists\ \sigma < \kappa$ such that

$$|\langle e,\gamma\rangle|^A = \sigma$$

and the information about A used in the computation, say $A \cap \sigma$, is an element of $L(\kappa)$ (since A regular). Thus,

$$\{e\}^A(\delta) \longleftrightarrow [\text{there is an ordinal } \sigma < \kappa \text{ and a neighborhood } A \cap \gamma \text{ for some } \gamma < \sigma \text{ such that } \{e\}_\sigma^{A\cap\gamma}(\delta)\downarrow].$$

The right hand side is $\Sigma_1(A)$, i.e. $\Sigma_1$ treating A as a $\Delta_0$ predicate with some parameter less than $\kappa$.

RE(A)-cf$(\kappa) \leq \Sigma_1(A)$-cf$(\kappa)$: let $f : \gamma \longrightarrow \kappa$ witness $\Sigma_1(A)$-cf$(\kappa)$ and for each $\delta < \gamma$, let $W_\delta$ be the witness to $f(\delta) = \sigma < \kappa$. Then the sequence

$$<<\delta, f(\delta), W_\delta> \mid \delta < \gamma>$$

is RE(A) and has order type $\gamma$. Thus

$$\mathrm{RE}(A)\text{-cf}(\kappa) = \Sigma_1(A)\text{-cf}(\kappa),$$

as desired. (ii) $\square$

If we use $\rho^A$ ($\rho^A = \rho$ for our choice of A by 3.1.5 (i)) to index the conditions of our argument, then RE(A)-cf($\kappa$) is somewhat well behaved.

Lemma 3.2.2: Let $\eta^A$ be as in 3.1.0 (ii), then for $\sigma < \kappa$ and $\gamma < \eta^A$

$$\sup_{\delta<\gamma} \kappa_r^{<\sigma,\delta>}[A] < \kappa.$$

Proof. Suppose not and let $\gamma_0$ and $\sigma_0$ witness,

$$\sup_{\delta<\gamma} \kappa_r^{<\sigma,\delta>}[A] = \kappa.$$

Sublemma 3.2.3: $\sup_{\delta<\gamma} \kappa_r^{<\sigma,\delta>}[A] = \kappa \longrightarrow \sup_{\delta<\gamma} \kappa_0^{<\sigma,\delta>}[A] = \kappa$ .

Proof of Sublemma. Suppose that $\sup_{\delta<\gamma} \kappa_r^{<\sigma,\delta>}[A] = \kappa$, but $\sup_{\delta<\gamma} \kappa_0^{<\sigma,\delta>} < \kappa$ and take $\lambda_0 < \kappa$ such that for all $\delta < \gamma$

$$\kappa_0^{<\sigma,\delta>}[A] \leq \lambda_0.$$

By assumption there is a $\delta_0 < \gamma$ such that

$$\lambda_0 < \kappa_r^{<\sigma,\delta_0>}[A] < \kappa.$$

Without loss of generality $\gamma$ is closed under pairing, so take $\delta_1 < \gamma$ such that

$$\kappa_r^{<\sigma,\delta_0>}[A] \leq \kappa_r^{<\sigma,\delta_1>}[A] < \kappa$$

and let $\delta = <\delta_0,\delta_1>$. Then $\delta < \gamma$ and

$$\kappa_r^{<\sigma,\delta_0>}[A] \leq \kappa_r^{<\sigma,\delta>}[A] < \kappa,$$

so $L_{\kappa_r^{<\sigma,\delta>}[A]}[A] \models$ "$\exists\sigma\ L_\sigma[A] \models \theta^A(\delta_0)$", where $\theta^A(x)$ is the formula of Theorem 3.0.4. By reflection,

$$L_{\kappa_0^{<\sigma,\delta>}[A]}[A] \models \text{"}\exists\sigma\ L_\sigma[A] \models \theta^A(\delta_0)\text{"},$$

but then

$$\kappa_r^{<\sigma,\delta_0>}[A] < \kappa_0^{<\sigma,\delta>}[A] \leq \lambda_0,$$

contradicting the choice of $\delta_0$. $\square$

Returning to 3.2.2, by the choice of $\gamma_0$ and $\sigma_0$ and 3.2.3

$$\sup_{\delta<\gamma_0} \kappa_0^{<\sigma_0,\delta>}[A] = \kappa, \quad \text{but}$$

$$R_{\gamma_0} = \{\delta < \gamma_0 \mid (\exists\, e \in \omega)\{e\}^A(\sigma_0,\gamma)\downarrow\}$$

is by Gandy Selection an $RE(A)\setminus REC(A)$ subset of $\gamma_0$, contradicting the definition of $\eta^A$ and $\gamma_0 < \eta^A$. Thus $\forall\, \sigma < \kappa,\ \ \forall\, \gamma < \eta^A$

$$\sup_{\delta<\gamma} \kappa_r^{<\sigma,\delta>}[A] < \kappa.$$

$\square$

With this lemma we can now show that $\kappa$ and $\eta^A$ have the same RE(A)-cofinality.

<u>Theorem 3.2.4</u>: $A \subseteq \kappa$ RE and incomplete, then

$$RE(A)\text{-}cf(\kappa) = RE(A)\text{-}cf(\eta^A).$$

<u>Proof</u>. $RE(A)\text{-}cf(\kappa) \leqslant RE(A)\text{-}cf(\eta^A)$: Let $R(\eta^A) \subseteq \eta^A$ witness the $RE(A)\text{-}cf(\eta^A)$, i.e. for some $e \in \omega$ and $\sigma_0 < \kappa$,

$$R(\eta^A) = \{\gamma < \eta^A \mid \{e\}^A(\sigma_0,\gamma)\downarrow\}$$

is RE(A) and $\cup\{\gamma \mid \gamma \in R(\eta^A)\} = \eta^A$. For $\gamma \in R(\eta^A)$, let

$$\beta_\gamma = \sup_{\delta<\gamma} \kappa_r^{<<\sigma_0,\sigma_1>_1\delta>}[A],$$

where $\sigma_1$ is the parameter in the definition of $\eta^A$. Then by lemma 3.2.2,

$$(\forall\, \gamma \in R(\eta^A))[\beta_\gamma < \kappa]$$

and the sequence

$$S = <\beta_\gamma \mid \gamma \in R(\eta^A)>$$

is RE(A) via $<\sigma_0,\sigma_1>$. By the definition of $\eta^A$ $S$ is also unbounded in $\kappa$ and also has order type that of $R(\eta^A)$. Thus RE(A)-cf$(\kappa) \leqslant$ RE(A)-cf$(\eta^A)$.

To show RE(A)-cf$(\eta^A) \leqslant$ RE(A)-cf$(\kappa)$, let $R(\kappa) \subseteq \kappa$ witness the RE(A)-cf$(\kappa)$, i.e. $(\exists\, e \in \omega)(\exists\, \sigma_0 < \kappa)$,

$$R(\kappa) = \{\gamma < \kappa \mid \{e\}^A(\sigma_0,\gamma)\downarrow\}$$

is RE(A) and $\cup\, R(\kappa) = \kappa$ and $R(\kappa)$ has order type RE-cf$(\kappa)$.

Fix a witness to the definition of $\eta^A$, i.e. $R \subseteq \eta^A$ RE(A) via $e' \in \omega$ and $\sigma_1 < \kappa$. Define $S \subseteq \eta^A$

as follows: Run algorithms for computing $R$ and $R(\kappa)$, if $x$ is enumerated into $R(\kappa)$ i.e. $\{e\}^A(\sigma_0,x)\downarrow$, then enumerate into $\zeta$ the first element $y \in R$ satisfying

$$|\{e\}^A(\sigma_0,x)| \leq |\{e'\}^A(\sigma_1,y)|.$$

Continue this process each time enumerating $y$ into $\zeta$ just in case $y$ is greater than all $y'$ previously enumerated into $\zeta$. Then $\zeta$ is $RE(A)$ and of order type that of $R(\kappa)$ by construction. $\cup\,\zeta = \eta^A$, for otherwise $\exists\ \gamma < \eta^A$

$$\sup_{\delta<\gamma} \kappa_0^{<\sigma,\delta>}[A] = \kappa$$

for some $\sigma < \kappa$, contradicting Lemma 3.2.2. $\square$

## §3. The Density Theorem

To carry out the preservation-recovery strategy in Chapter 2, we made use of what is called in the Appendix Normann Reflection, i.e. $\rho$ REC-regular <u>iff</u> $(\forall\ \sigma < \kappa)$ $(\forall\ \delta < \rho)$

$$\sup_{\gamma<\delta} \kappa_0^{<\sigma,\delta,\gamma>} \leqslant \kappa_r^{<\sigma,\delta>}.$$

An analogous result holds for $L_\kappa[A]$ for $A \subseteq \kappa$ RE and incomplete with minor changes in the proof. We state the result here with a sketch of the proof.

<u>Theorem 3.3.0</u>: Let $A \subseteq \kappa$ be RE and incomplete, then $\rho^A$ is $\mathbb{E}(A)$-REC regular <u>iff</u> $\forall\ \sigma < \kappa\ \ \forall\ \gamma < \rho^A$

$$\sup_{\delta<\gamma} \kappa_0^{<\sigma,\delta,\gamma>}[A] \leqslant \kappa_r^{<\sigma,\gamma>}[A].$$

<u>Proof</u> (Sketch): The proof of 3.3.0 proceeds as in the Appendix with some changes. Instead of the collapsing argument used to produce the parameter

$$B_\delta = \{f : \delta \longrightarrow \rho \mid f \in L(\kappa)\},$$

first notice that for $f : \delta \longrightarrow \rho^A$, $f \in L_\kappa[A]$ $\cup\, \mathrm{rng}(f) < \rho^A$ by the $\mathbb{E}(A)$-REC regularity of $\rho^A$, say $\cup\, \mathrm{rng}(f) = \sigma$. Then $f$ uses at most $A \cap \sigma$ in its computation and $A \cap \sigma \in L(\kappa)$ by regularity of $A$. By the lemma of the Appendix and $\rho = \rho^A$,

$$\{A \cap \sigma \mid \sigma < \rho^A\} \in L(\rho+1) \subseteq L(\kappa).$$

Thus, using $A$,

$$B_\delta[A] = \{f : \delta \longrightarrow \rho^A \mid f \in L_\kappa[A]\} \in L_{\rho+1}[A] \subseteq L(\kappa).$$

Then use $B_\delta[A]$ to carry out the selection argument to get $\Gamma^A(x)$ partial $\mathrm{REC}(A)$.

As before, suppose $\exists\ \sigma_0 < \kappa\ \ \exists\ \delta_0 < \rho^A$ such that,

$$\sup_{\delta<\delta_0} \kappa_0^{<\sigma_0,\delta,\delta_0>}[A] > \kappa_r^{<\sigma_0,\delta_0>}[A].$$

Now let

$$R = \{\delta < \delta_0 \mid (\exists\ e \in \omega)\{e\}^A(\delta_0,\sigma_0,\delta)\downarrow\},$$

then $R$ is $\mathrm{RE}(A)$ subset of $\delta_0$ via $<\delta_0,\sigma_0>$. Let

$$R(> \kappa_r^{<\sigma_0,\delta_0>}[A]) = \{\delta < \delta_0 \mid (\exists\ e \in \omega)\{e\}^A(\delta_0,\sigma_0,\delta)\downarrow$$

$$\wedge\ \ \kappa_r^{<\sigma_0,\delta_0>}[A] < |\{e\}^A(\delta_0,\sigma_0,\delta)|^A\},$$

then $R(> \kappa_r^{<\sigma_0,\delta_0>}[A])$ is $\mathrm{RE}(A)$ subset of $\delta_0$ via

$<\delta_0,\sigma_0>$, say via the integer $e_0$. Then $\Gamma^A_{<\delta_0,\sigma_0>}(e_0)\downarrow$ and

$$\min_{z\in R(> \kappa_r^{<\sigma_0,\delta_0>}[A])} |z| \leq |\Gamma_{<\delta_0,\sigma_0>}(e_0)|.$$

But by the definition of $\Gamma^A$,

$$\left|\Gamma^A_{<\delta_0,\sigma_0>}(e_0)\right| \leq \kappa_0^{<\delta_0,\sigma_0>}[A] < \kappa_r^{<\delta_0,\sigma_0>}[A],$$

contradicting the definition of $R(> \kappa_r^{<\delta_0,\sigma_0>}[A])$.

□

Theorem 3.3.1: Let $L(\kappa)$ be $\mathbb{E}$-closed and not $\Sigma_1$-admissible. Let $A, C \subseteq \kappa$ RE and $A$ be incomplete with $A \leq_{\mathbb{E}} C$ but $C \not\leq_{\mathbb{E}} A$ and further suppose that $\eta^A = \rho^A$ and $\rho^A$ is $\mathbb{E}(A)$-RE-regular. Then there exists $B \subseteq \kappa$ such that

(i) $B$ is RE

(ii) $[A \leq_{\mathbb{E}} B \wedge B \not\leq_{\mathbb{E}} A]$ and $[B \leq_{\mathbb{E}} C \wedge C \not\leq_{\mathbb{E}} B]$.

Proof. Strategy.

In proving 3.3.1 we employ the basic idea first used by G. Sacks [ 1964 ] to prove that the Turing degrees of recursively enumerable subsets of the integers are dense. To insure that $A \leq_{\mathbb{E}} B$ we plant with highest priority $A$ into $B$ as $A$ is enumerated on $L(\kappa)$. $B \leq_{\mathbb{E}} C$ by the construction and the fact that $A \leq_{\mathbb{E}} C$. To insure that $B \not\leq_{\mathbb{E}} A$ we will monitor, for each index $e$, the length of agreement between $\{e\}^A$ and $B$ and if this length appears to be increasing unboundedly in $\kappa$, then we will plant information about $C$ as it appears into $B$ in an $\mathbb{E}(A)$-REC way. If we then had $B = \{e\}^A$ finally, then we would have $C \leq_{\mathbb{E}} A$ contradicting the choice of $A$ and $C$. To insure that $C \not\leq_{\mathbb{E}} B$ we monitor, again, for each index $e$, the length of agreement between $\{e\}^B$ and $C$. If this length appears to be increasing unboundedly in $\kappa$, then we will preserve information about $B$ used in those computations. If we then had $\{e\}^B = C$ finally, then again we would have $C \leq_{\mathbb{E}} A$ contradicting the choice of $A$ and $C$. Without loss of generality we can assume that both $A$ and $C$ are regular. The fact that $A$ is incomplete insures that $L_\kappa[A] = L_\kappa$ and that $L_\kappa$ is $\mathbb{E}(A)$-closed.

<u>Conditions and the Ordering</u>

The conditions to be satisfied will be indexed by $\rho^A = \rho$. By the definition of $\rho^A$ there is a partial $\mathbb{E}(A)$-REC function from $\rho^A$ onto $L(\kappa)$ and a total $\mathbb{E}(A)$-REC map of $\kappa$ into $\rho^A$. This latter map is given by, for $\sigma < \kappa$,

$$\sigma \longrightarrow (\mu\gamma < \rho^A)(\exists\, e \in \omega)$$

$$[\sigma = |\{e\}^A(\sigma_0,\gamma)|],$$

where $\sigma_0$ is the parameter in the definition of $\rho^A$. For $e < \rho^A$,

$$P^A\text{:}\quad A \leqslant_{\mathbb{E}} B$$

$$P_e^{A,B}\text{:}\quad \{e\}^B \neq A \qquad\qquad e = \langle 1,e'\rangle$$

We write $P^A$ and $P_e^{A,B}$ to indicate that these conditions are positive, i.e. they entail our putting elements into $B$ to satisfy them. For $e < \rho^A$,

$$N_e^{B,C}\text{:}\quad \{e'\}^B \neq C \qquad\qquad e = \langle 2,e'\rangle$$

We write $N_e^{B,C}$ to indicate that these conditions are negative, i.e. they entail our restraining certain elements from $B$ in order to preserve $\{e\}^B$ on longer and longer initial segments of $\kappa$. The <u>priority ordering</u> will be given by the ordering as ordinals of indices less than $\rho^A$:

1. $P^A$ has highest priority

2. For $e, e' < \rho^A$ if $R_e$, $R_{e'}$ range over positive and negative conditions then $R_e$ has <u>higher priority than</u>

$R_{e'}$ iff $e < e'$ (provided $e$ and $e'$ are indices for conditions).

Proposition 3.3.2: Under the assumptions of the theorem, if $\gamma < \rho^A$, then

(i) $\rho^A$ indexes the $\mathbb{E}(A)$-REC functions;

(ii) $\{f \mid f \ \mathbb{E}(A)\text{-REC with index } \delta \leq \gamma\} = \mathbb{P}(\gamma)$ is an element of $L_{\kappa_r^{\langle\gamma,\sigma_0\rangle}[A]}$.

Proof. (i) The $\mathbb{E}(A)$-REC function have a natural indexing via ordinals $\sigma < \kappa$. Using closure of $\rho^A$ under pairing and the definition of $\rho^A$ there is a $\delta < \rho^A$ such that for some $e \in \omega$,

$$\sigma = |\{e\}^A(\sigma_0,\delta)|,$$

($\sigma_0$ the parameter defining $\rho^A$) i.e. $\sigma$ is constructive in $\delta$ via $\langle e,\sigma_0\rangle$. (i) □

(ii) That $\mathbb{P}(\gamma) \in L(\kappa)$ follows from $\eta^A = \rho^A$ and the fact that $\mathbb{P}(\gamma)$ is an $RE(A)$ subset of $\gamma < \rho^A$. However Theorem 3.3.0 gives that

$$\sup_{\delta<\gamma} \kappa_0^{\langle\sigma_0,\delta,\gamma\rangle}[A] \leq \kappa_r^{\langle\sigma_0,\gamma\rangle}[A]$$

and hence $\mathbb{P}(\gamma) \in L_{\kappa_r^{\langle\sigma_0,\gamma\rangle}[A]}$. (ii) □ □

The proposition asserts the strong 'tameness' of or reindexing of the $\mathbb{E}(A)$-RE functions via $\rho^A$.

Auxilliary Functions

Definition 3.3.3: For $\sigma < \kappa$ and $e < \rho^A$ if $e = <1,e'>$, then let

$$\ell^{A,B}(\sigma,e) = \max_{\gamma \leqslant \sigma} (\forall\ \alpha < \gamma)[\{e'\}_\sigma^{B_\sigma}(\alpha) = A_\sigma(\alpha)],$$

and if $e = <2,e'>$, then let

$$\ell^{B,C}(\sigma,e) = \max \gamma \leqslant \sigma(\forall\ \alpha < \gamma)[\{e'\}_\sigma^{B_\sigma}(\alpha) = C_\gamma(\alpha)].$$

Remark: (a) $D_\sigma = \{\gamma < \kappa \mid \{e\}(\gamma)\downarrow \wedge |\{e\}(\gamma)| < \gamma\}$ where $e$ defines $D \subseteq \kappa$ RE.

(b) Subscript $\sigma$ on $\{e\}_\sigma(\gamma)$ indicates that it is evaluated in $L(\sigma)$ and, by convention, all information is coded by ordinals less than $\sigma$.

Definition 3.3.4: If $\sigma < \kappa$ and $e < \rho^A$, then if $e = <1,e'>$, let

$$m^{A,B}(e,\sigma) = \bigcup_{\tau<\sigma} \ell^{A,B}(e,\tau);$$

and if $e = <2,e'>$, let

$$m^{B,C}(e,\sigma) = \bigcup_{\tau<\sigma} \ell^{B,C}(e,\tau)$$

"longest initial segment of agreement before stage $\sigma$".

Suppose that,

$$p(e,\sigma) = \mu\gamma[\gamma \text{ bounds negative facts about } B_\gamma \text{ used in evaluating } \ell^{B,C}(e,\sigma)].$$

A bound on $p(e,\sigma)$ will arise in one of two ways:

1. $(\exists\ \alpha < \kappa)\{e'\}^B(\alpha)$ , in which case $p(e) = \lim_{\sigma\to\kappa} p(e,\sigma) \leqslant \kappa_r^{<e',\alpha>}[B]$; where $k^{B,C}(e) = \lim_{\sigma\to\kappa} \ell^{B,C}(e,\sigma) = \alpha$; or
2. $(\exists\ \alpha < \kappa)[C(\alpha) \neq \{e'\}^B(\alpha)]$ in which case $p(e) = \lim_{\sigma\to\kappa} p(e,\sigma) < \tau < \kappa_0^{<e',\alpha>}[B]$, where $\ell^{B,C}(e) = \lim_{\sigma\to\kappa} \ell^{B,C}(e,\sigma) = \alpha$.

Roughly, we will be able to conclude that 1. or 2. obtains after the fact by an ETR on $\gamma < e$.

<u>Definition 3.3.5</u>: For $e < \rho^A$ and $\sigma < \kappa$ say that $e$ <u>requires attention at stage $\sigma$</u> if $e$ is least such that either,

(i) $e = <1,e'>$ and $\ell^{A,B}(e,\sigma) > m^{A,B}(e,\sigma)$; or

(ii) $e = <2,e'>$ and $\ell^{B,C}(e,\sigma) > m^{B,C}(e,\sigma)$.

<u>Construction</u>

<u>Stage 0</u>: Set $B = \emptyset$

<u>Stage $\lambda$</u>: $\lambda \in LOR$, then let

$$B_\lambda = \bigcup_{\gamma<\lambda} B_\gamma$$

<u>Stage $\sigma$</u>: $\sigma \in SOR$. Let $e$ be the least index less than $\sigma$ which requires attention, then:

<u>Substage 1</u>. Check to see whether $A_\sigma \setminus A_\tau \neq \emptyset$, where $\sigma = \tau+1$. If so for each $\gamma \in A_\sigma \setminus A_\tau$ , enumerate $<0,\gamma>$ into $B$ and proceed to substage 2.

<u>Substage 2</u>.

<u>Case (a)</u>. $e = <1,e'>$ $(P_e^{A,B})$ and suppose that $\ell^{A,B}(e,\sigma) = z$. Consider the truth value of the sentence:

$$\phi^{A,B}(z,e,\sigma) \equiv (\forall\ x \leqslant \kappa)[A_\sigma(x) = \{e'\}_\sigma^{B_\sigma}(x)]$$

in $L(\kappa)$ (i.e. 0 or 1) and let $c^{A,B}(\gamma,e,\sigma)$ be that truth value. Let $p(z,e,\sigma) =$ "the number of changes in $c(z,e,\tau)$ for $\tau \leqslant \sigma$"; say that $p(z,e,\sigma) = \delta$. Form the pair $<z,\delta>$, then $\exists\ \alpha < \rho$ such that $\{(\alpha)_0\}((\alpha)_1)\downarrow$ and $|\{(\alpha)_0\}((\alpha)_1)| = \sigma$. Form $<\alpha,<z,\delta>>$ and check to see whether $<\alpha,<z,\delta>> > R(e,\sigma)$, where

$$R(e,\sigma) = \cup\{\kappa_r^{\ell^{B,C}}(d,\sigma) \mid d < e \wedge (\exists\ d')[d = \langle 2,d'\rangle]\}.$$

If $\langle\alpha,\langle z,\delta\rangle\rangle \leq R(e,\sigma)$, then go to the next stage. If $\langle\alpha,\langle z,\delta\rangle\rangle > R(e,\sigma)$, then let

$$[\langle\alpha,\langle z,\delta\rangle\rangle \in B \text{ iff } z \in C_\sigma]$$

and go to the next stage.

Case (b). $e = \langle 2,e'\rangle$ $(N_e^{B,C})$, then proceed to the next stage.

Finally, let $B = \bigcup_{\gamma<\kappa} B_\gamma$.

End of Construction.

We now show that the constructed $B$ satisfies the statement of the theorem.

Lemma 3.3.6: (i) $B$ is RE

(ii) $A \leq_{\mathbb{E}} B$ and $B \leq_{\mathbb{E}} C$.

Proof. (i) By the construction the elements of $A$ were enumerated into $B$ as they were enumerated into $A$, say $A$ defined via parameter $a < \kappa$, i.e.

$$\gamma \in A \longleftrightarrow \langle 0,\gamma\rangle \in B.$$

In addition, in Substage 2, Case (a) we had elements of $B$ of the form $\langle\alpha,\langle z,\delta\rangle\rangle$ and $\alpha$ was an index less than $\rho$

for the stage at which we enumerated $<\alpha,<z,\delta>>$ into B, if we did it at all. (i) □

(ii) $A \leqslant_{\mathbb{E}} B$ by construction and $B \leqslant_{\mathbb{E}} C$ by construction and the fact that $A \leqslant_{\mathbb{E}} C$. ii □ □

<u>Lemma 3.3.7</u>: $B \not\leqslant_{\mathbb{E}} A$ and $C \not\leqslant_{\mathbb{E}} B$ (allowing parameters in $L(\kappa)$).

<u>Proof</u>. We proceed by induction on $e < \rho^A$ and the form e (i.e. $e = <1,e'>$ or $e = <2,e>$ corresponding to $P_e^{A,B}$ and $N_e^{B,C}$). Fix e and assume that for $d < e$ we have that

(i) $B \neq \{d\}^A$, if $d = <1,d'>$ and

(ii) $C \neq \{d'\}^B$, if $d = <2,d'>$.

<u>Case 1</u>. $e = <1,e'>(P_e^{A,B})$. $\mathbb{P}(e) \in L(\kappa)$ so let $\sigma_0 = \mu\sigma < \kappa[\,\mathbb{P}(e) \in L(\sigma)]$. For $d < e$ corresponding to (i) and (ii) we have: (i') $d = <1,d'>(\exists\ \alpha < \kappa)$ such that $[\{d'\}^A(\alpha) \neq B(\alpha)]$ or $[\{d'\}^A(\alpha)\uparrow]$; or (ii') $d = <2,d'>$ $(\exists\ \alpha < \kappa)$ such that $[\{d'\}^B(\alpha) \neq C(\alpha)]$ or $[\{d'\}^B(\alpha)\uparrow]$. In either case let $\alpha(d)$ be the least such. By $\mathbb{E}(A)$-closure of $L(\kappa)$ this disagreement or divergence is established by $\sigma < \kappa_0^{<d',\alpha(d)>}[A]$ or $\kappa_r^{<d',\alpha(d)>}[B]$. Let for $d < e$:

$$
\beta(d) = \begin{cases}
\sigma[|\{d'\}^{A}(\alpha(d))| = \sigma], & \text{if } \{d'\}^{A}(\alpha(d)) \neq B(\alpha(d)) \land d = \langle 1,d' \rangle \\[2ex]
\kappa_{r}^{\langle d',\alpha(d) \rangle}[A], & \text{if } \{d'\}^{A}(\alpha(d))\uparrow \land d = \langle 1,d' \rangle \\[2ex]
\sigma[|\{d'\}^{B}(\alpha(d))| = \sigma], & \text{if } \{d'\}^{B}(\alpha(d)) \neq C(\alpha(d)) \land d = \langle 2,d' \rangle \\[2ex]
\kappa_{r}^{\langle d',\alpha(d) \rangle}[B], & \text{if } \{d'\}^{B}(\alpha(d))\uparrow \land d = \langle 2,d' \rangle
\end{cases}
$$

Then the sequence,

$$<\beta(\lambda) \mid d < e>$$

is RE(A) and of order $\leqslant e$, hence bounded by some $\sigma_1 \geqslant \sigma_0$ ($\mathbb{E}(A)$-RE-cf$(\kappa) = \rho^A$). For $d < e$ $d = <1,d'>$ we planted falsely in $B$, consider $F(e) = \{<z,d> \mid d < e \wedge z$ planted for the sake of $P_d^{A,B}\}$ $\mathbb{E}(A)$-RE of order type $\leqslant e$, hence bounded by some $\sigma_2 \geqslant \sigma_1$. At $\sigma_2$ we can compute that false planting activity and $\forall\ \tau \geqslant \sigma_2$, assuming $\{e'\}^A = B$, we plant for the sake of $e$ with highest priority in an $\mathbb{E}(A)$-REC way. Suppose we have computed $C(\gamma)$ $\forall$ $\gamma < z$ $\mathbb{E}(A)$-REC'ly via $f_A$ and show that $f_A(z) = C(z)$ can be computed. Now $e$ enumerated in $C$ then $|$"$e \in C$"$| < \kappa_0^{<p,z>}$, $p$ defining $C$, and

$$\sup_{\gamma<z} |f_A(\gamma)| \leqslant \kappa_r^{<\sigma,\gamma>}[A] \qquad \text{(by 3.3.0).}$$

Thus $\exists\ \sigma < \kappa_0^{<\sigma_2,\gamma>}[A]$, by which time we have $\{f_A(\gamma) \mid \gamma < z\}$ and $\tau = |\{p\}(z)|$ (assuming $\sigma_2 = <\sigma',p>$) and at stage $z$ was $\mathbb{E}(A)$-REC coded into $B$, thus $f_A(z)\downarrow$ and correct. But with $f_A$, $C \leqslant_{\mathbb{E}} A$, contradiction. $\square$

<u>Case 2</u>. $e = <2,e'>(N_e^{B,C})$ : $P(e) \in L(\kappa)$, let $\sigma_0 = \mu\sigma[\mathbb{P}(e) \in L(\kappa)]$. Take $<\alpha(d) \mid d < e>$ as in Case 1 and also $\beta(d)$ and the bound $\sigma_2$ and consider,

$$I(e) = \{\langle z,w\rangle \mid z \text{ planted for the sake of } P_d^{A,B} \wedge d < e \text{ injuring } w \text{ associated with } e \text{ and some argument } \alpha\}$$

which is $\mathbb{E}(A)$-RE in $\sigma_2$ of order type $\leqslant e$ and hence

$$\langle \sigma\langle z,w\rangle \mid \langle z,w\rangle \in I(e)\rangle$$

where $\sigma(z,w)$ = stage at which $w$ returns is also $\mathbb{E}(A)$-RE and, hence, bounded by some $\sigma_3 \geqslant \sigma_2$ (assuming $C = \{e'\}^B$ for a contradiction). We show by ETR that $C \leqslant_{\mathbb{E}} A$ with parameter $\sigma_3$. Suppose we have defined $f_A(\gamma)$, $\forall\ \gamma < z$. By construction we preserved $\{e'\}^B = C$ with highest priority after $\sigma_3$, hence

$$\sup_{\gamma<z} |f_A(\gamma)| \leqslant \kappa_r^{\langle\sigma_3,z\rangle}[A]$$

using $\mathbb{E}(A)$-closure of $L(\kappa)$ and 3.3.0 hence by reflection $\exists\ \sigma < \kappa_0^{\langle\sigma_3,z\rangle}[A]$ with $|\{p\}(z)| \leqslant \sigma$, where $p$ defines $C$. For all $\tau \geqslant \sigma$ we preserved $\{e'\}^B(z)$ with highest priority and $f_A(z) = \{e'\}_\sigma^{B_\sigma}(z)$ is correct. Thus, $C \leqslant_{\mathbb{E}} A$, a contradiction. □

Thus B satisfies the statement of the theorem.

□

Corollary 3.3.8: (Density). Let $L(\kappa)$ be $\mathbb{E}$-closed, not $\Sigma_1$-admissible and assume $\eta^A = \rho^A$ and $\rho^A$ $\mathbb{E}(A)$-RE regular for each $A, C \subseteq \kappa$ RE with $A \leq_{\mathbb{E}} C$, but $C \not\leq_{\mathbb{E}} A$. Then the $\mathbb{E}$-RE degrees on $L(\kappa)$ are dense.

Proof. Apply Theorem 3.3.1 to each pair A, C and take $\underset{\sim}{a} = \mathbb{E}$-degree (A), $\underset{\sim}{b} = \mathbb{E}$-degree (B) and $\underset{\sim}{c} = \mathbb{E}$-degree (C), where for $D \subseteq \kappa$

$$\mathbb{E}\text{-degree } (D) = \{B \subseteq \kappa \mid D \leq_{\mathbb{E}} B \wedge B \leq_{\mathbb{E}} D\} = \underset{\sim}{d}.$$

Then $\underset{\sim}{a} < \underset{\sim}{b} < \underset{\sim}{c}$. □

Remark: At this writing the author has found a means of eliminating the assumption of $\rho^A$ $\mathbb{E}(A)$-RE regular by an entirely different proof using a shorter 'tame' listing of the $\mathbb{E}(A)$-REC functions. Independently, T. Slaman has given a proof of density using his techniques for 'splitting' $\mathbb{E}$-RE sets (Griffor-Slaman [1980]).

# Chapter 4: Absolute Degree Theory: Normal Kleene-Recursion Revisited

## §0. Absolute Degrees

This last chapter will consider another particular case of an $\mathbb{E}$-closed set, namely the universe for normal Kleene recursion in ${}^{k+2}\mathbb{F} : Tp(k+1) \longrightarrow Tp(k+1)$, $k \geqslant 1$, reviewed in Chapter 1. There we described a hierarchy for ${}^{k+2}\mathbb{F}$ which amounted to iterated constructibility relative to a predicate contained in $Tp(k+1)$, $L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$, where $\kappa^{\mathbb{F}}$ was the least ordinal not recursive in $\langle\mathbb{F},a\rangle$ for some $a \in I = Tp(k)$. As Normann [1978] has shown, this theory can be shown equivalent to $\mathbb{E}$-Recursion on $\mathbb{E}$-cl$(Tp(k))$ with an additional predicate or relation, i.e. $\mathbb{E}(R)$-Recursion $R \subseteq Tp(k+1)$. In this case $R = {}^{k+2}\mathbb{F} \subseteq Tp(k+1)$. There may or may not be a well-ordering of $Tp(k)$ in what we will call $\mathbb{F}$-cl$(Tp(k))$, read the '$\mathbb{F}$-closure of $Tp(k)$'. However the ordinals less than $\kappa^{\mathbb{F}}$ form the spine of this universe and since $\mathbb{F}$-cl$(Tp(k))$ is indexed by elements of $Tp(k)$, it makes sense to focus our attention on those elements of $Tp(k)$ which have the same degree in the sense of $\mathbb{F}$ as an ordinal $< \kappa^{\mathbb{F}}$ modulo some integer. If we narrow our attention then to those elements of $\mathbb{F}$-cl$(Tp(k))$ which have the same degree $(\mathbb{F})$ as an ordinal modulo an integer, then the resulting domain will prove sufficiently 'L-like' to allow the arguments of the previous chapters.

<u>Definition 4.0.0</u>: For $k \geqslant 1$ and ${}^{k+2}\mathbb{F}$ normal consider $\mathbb{F}\text{-cl}(Tp(k))$ and let $\kappa^{\mathbb{F}} = On \cap \mathbb{E}\text{-cl}(Tp(k))$, then

$$OA(\mathbb{F}\text{-cl}(Tp(k))) = \{X \in \mathbb{F}\text{-cl}(Tp(k)) \mid (\exists\, e \in \omega)(\exists\, \gamma < \kappa^{\mathbb{F}})$$

"ordinal absolute part" $[X \leqslant_{\mathbb{F}} e, \alpha \wedge \alpha \leqslant_{\mathbb{F}} e, X]\} = OA_{\mathbb{F}}$.

<u>Remark</u>: We write $\leqslant_{\mathbb{F}}$ rather than $\leqslant_{\mathbb{E}(R)}$, $R = {}^{k+2}\mathbb{F} \subseteq Tp(k+1)$ for convenience.

Now $Tp(k)^{OA_{\mathbb{F}}} = OA(\mathbb{F}\text{-cl}(Tp(k))) \cap Tp(k) \subseteq Tp(k)$ and, in general, may not be all of $Tp(k)$, however, the least order type of a well-ordering of $Tp(k)^{OA_{\mathbb{F}}}$ will be a cardinal in $OA_{\mathbb{F}}$. If we then restrict our attention to the functions of $L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$ <u>$\mathbb{F}$-REC on $OA_{\mathbb{F}}$</u>, then something can be said about the $\mathbb{F}$-RE degrees on $OA_{\mathbb{F}}$.

<u>Definition 4.0.1</u>: (i) $f : OA_{\mathbb{F}} \longrightarrow OA_{\mathbb{F}}$ is <u>$\mathbb{F}$-REC on $OA_{\mathbb{F}}$</u>, if $\exists$ e an index in $\mathbb{E}(R)$-recursion for $R = \mathbb{F}$ and some $z \in OA_{\mathbb{F}}$, $w \in OA_{\mathbb{F}}$

$$f(w) \simeq \{e\}^{\mathbb{F}}(z,w);$$

(ii) $A \subseteq OA_{\mathbb{F}}$ is <u>$\mathbb{F}$-RE on $OA_{\mathbb{F}}$</u> in $a \in OA_{\mathbb{F}}$, if $e \in \omega$, $z \in OA$

$$z \in A \longleftrightarrow \{e\}^{\mathbb{F}}(a,z)\downarrow; \qquad \text{and}$$

(iii) $A \subseteq OA_{\mathbb{F}}$ is <u>$\mathbb{F}$-REC on $OA_{\mathbb{F}}$</u>, if $A$ and $OA_{\mathbb{F}} \setminus A$ are $\mathbb{F}$-RE on $OA_{\mathbb{F}}$ (suppressing parameters in $OA_{\mathbb{F}}$).

It remains only to make sense of relative $\mathbb{F}$-recursion and, hence, $\mathbb{F}$-degree (A) for $A \subseteq OA_{\mathbb{F}}$.

<u>Definition 4.0.2</u>: (i) $A, B \subseteq OA_{\mathbb{F}}$, then <u>$A \leq_{\mathbb{F}} B$ on $OA_{\mathbb{F}}$</u> ('A is $\mathbb{F}$-REC in B') if $(\exists\ e \in \omega)(\exists\ a \in OA_{\mathbb{F}})$:

$$(\forall\ z \in OA_{\mathbb{F}})\ [\{e\}^{B,\mathbb{E}}(a,z) = A(z)],$$

viewing $A$ as its characteristic function;

(ii) <u>$\mathbb{F}$-degree (A)</u> $= \{B \subseteq OA_{\mathbb{F}} \mid A \leq_{\mathbb{F}} B$ on $OA_{\mathbb{F}}$ and $B \leq_{\mathbb{F}} A$ on $OA_{\mathbb{F}}\}$.

<u>Remark</u>: We will omit the phase on $OA_{\mathbb{F}}$ and assume that parameters in $OA_{\mathbb{F}}$ are allowed unless stated otherwise.

## §1. The $\kappa_r$-Function

In this section we review the results necessary for the use of the $\kappa_r$-function in this setting.

<u>Lemma 4.1.0</u> (Kechris): If $a \in Tp(k)$ and $B \subseteq Tp(k)$ CO-$\mathbb{F}$-RE in $a$, then there is a $b \in B$ such that

$$\kappa_r^{\langle a,b\rangle} \leqslant \kappa_r^a \, .$$

<u>Theorem 4.1.1</u> (Moschovakis): For any subset $A$ of $Tp(k)$ $\mathbb{F}$-RE, there is a relation $R(x,y)$ on $Tp(k)$ which is $\mathbb{F}$-RE such that $\forall\, a \in Tp(k)[a \notin A$ <u>iff</u> $\exists$ $b \in Tp(k)R(a,b)]$.

<u>Remark</u>: The reader may consult Harrington [1973] for the proofs of 4.1.0 and 4.1.1. 4.1.0 is used in the proof of 4.1.1, which actually asserts that $L_{\kappa^{\mathbb{F}}}[\mathbb{F}]$ is not $\Sigma_1$-admissible.

Finally Harrington's characterization of $\kappa_r$.

<u>Theorem 4.1.2</u> (Harrington): There is a formula (not $\Sigma_1(L_{\kappa^{\mathbb{F}}}[\mathbb{F}])$) $\theta(x)$ in $L$ such that for all $a \in Tp(k)$ $M_{\kappa_r^a}[\mathbb{F}] \models \theta(a)$, but for all $\sigma < \kappa_r^a$, $M_\sigma[\mathbb{F}] \not\models \theta(a)$

($M_\sigma[\mathbb{F}]$ denotes the structure

$$\langle L_\sigma[\mathbb{F}], \in, \mathbb{F} \upharpoonright (Tp(k+1) \cap L_\sigma[\mathbb{F}])\rangle$$

and $L$ is the first order language appropriate to the structures $M_\sigma[\mathbb{F}]$).

Finally we state the result, due to Normann [1978], which combines a Skolem Hull argument with a result on selection to exhibit a regularity of the $\kappa_r$-function.

Definition 4.1.3: A subset $A \subseteq Tp(k)$ is reflecting, if for all $b \in Tp(k)$ such that $A \leq_{\mathbb{F}} b$ : $\sup_{a \in A} \kappa_0^{\langle a,b\rangle} \leq \kappa_r^b$ (throughout $\mathbb{F} = \langle {}^{k+2}\mathbb{E}, A\rangle$ where $A \subseteq OA_{\mathbb{F}}$ $\mathbb{F}$-RE on $OA_{\mathbb{F}}$ and incomplete, as in the next section).

Let $\prec$ be a wellordering of $Tp(k)$ and assume that $\prec \in \mathbb{F}\text{-cl}(Tp(k))$.

Definition 4.1.4: $\prec$ is $\mathbb{F}$-REC regular if there is no function $\mathbb{F}$-REC in some $a \in Tp(k)$ mapping an initial segment of $\prec$ onto a $\prec$-cofinal subset of $Tp(k)$.

Theorem 4.1.5 (Normann): The following are equivalent:

(i) $\prec$ is $\mathbb{F}$-REC regular

(ii) All proper initial segments of $\prec$ are reflecting.

Proof. [See Normann [1980]]. □

The results of this section provide the tools for proving the results of the next section.

## §2. Absolute RE Degrees

Fix ${}^{k+2}\mathbb{F}$ and work in $OA_{\mathbb{F}}$.

Definition 4.2.0: Recall that $\kappa_{\mathbb{F}}$ is the ordinal of $OA_{\mathbb{F}}$ and let

$$\eta = \mu\gamma \leqslant \kappa_{\mathbb{F}} (\exists\ R)_{R \subseteq \gamma} [R \text{ is } \mathbb{F}\text{-RE}/\mathbb{F}\text{-REC on } OA_{\mathbb{F}}]$$

$$\rho = \mu\gamma \leqslant \kappa_{\mathbb{F}} (\exists\ f \mid [f : \gamma \xrightarrow{\text{onto}} \kappa_{\mathbb{F}}] \quad .$$
$$\text{f partial } \mathbb{F}\text{-REC on } OA_{\mathbb{F}}$$

Remark: In this setting $\rho = |Tp(k)^{OA_{\mathbb{F}}}|$, where we simply mean the well-ordering inherited from the ordinals. Thus there will exist a total inverse of $f$ taking any $\tau < \kappa_{\mathbb{F}}$ to its least index in $|Tp(k)^{OA_{\mathbb{F}}}|$ = shortest order type of the wellorderings of $TP(k)^{OA_{\mathbb{F}}}$ in $OA_{\mathbb{F}}$. Obviously $\eta \leqslant \rho$.

Definition 4.2.1: Let $\sigma$ be a cardinal of $OA_{\mathbb{F}}$ and let

$$\mathbb{F}\text{-RE-cf}(\sigma) = \mu\gamma \leqslant \sigma[\gamma \text{ is the order type of}$$
$$\text{an } \mathbb{F}\text{-RE } X \subseteq \sigma \wedge \cup X = \sigma].$$

Theorem 4.2.2 (Minimal Pair): Working in $OA_{\mathbb{F}}$ for normal $^{k+2}\mathbb{F}$. Assume that $\eta = \rho$ and $\rho$ is $\mathbb{F}$-RE regular (i.e. $\mathbb{F}$-RE-cf$(\rho) = \rho$). Then there exists a minimal pair of $\mathbb{F}$-RE degrees, i.e. $\exists\ A, B \subseteq \kappa_{\mathbb{F}}$ $\mathbb{F}$-RE/$\mathbb{E}$-REC such that $\forall\ C \subseteq \kappa_{\mathbb{F}}$ $\mathbb{F}$-RE $[(C \leq_{\mathbb{F}} A \wedge C \leq_{\mathbb{F}} B) \longrightarrow C$ is $\mathbb{F}$-REC$]$.

Proof. This proof proceeds just as in Chapter 2 on $L(\kappa)$ using the tools of 4.1. We need that $\mathbb{F}$-RE-cf$(\kappa_{\mathbb{F}})$ = $\mathbb{F}$-RE-cf$(\eta)$, the proof of which proceeds as those of the analogous fact in Chapter 2. 4.2.2 □

For the density theorem we work in $\mathbb{E}(\mathbb{F},A)$-recursion (or simply $\mathbb{F}(A)$-recursion) on $OA_{\mathbb{F}}$ as before.

Definition 4.2.3: $A \subseteq \kappa_{\mathbb{F}}$ and let

(i) $\underline{\eta^A} = \mu\gamma \leq \kappa_{\mathbb{F}}\ (\exists\ R)_{R \subseteq \gamma}[R$ is $\mathbb{F}(A)$-RE$\setminus\mathbb{F}(A)$-REC on $OA_{\mathbb{F}(A)}]$

(ii) $\underline{\rho^A} = \mu\gamma \leq \kappa_{\mathbb{F}}\ (\exists\ f \mid [f : \gamma \xrightarrow{\text{onto}} \kappa_{\mathbb{F}}]$.
f partial $\mathbb{F}(A)$-REC
on $OA_{\mathbb{F}(A)}$

Remark: Take $A \subseteq \kappa_{\mathbb{F}}$ $\mathbb{F}$-RE and incomplete, then as before $OA_{\mathbb{F}} = OA_{\mathbb{F}(A)}$ and $\rho = \rho^A$. Then $\eta^A \leq \eta \leq \rho = \rho^A$ for this A.

Remark: Take $A \subseteq \kappa_{\mathbb{F}}$ $\mathbb{F}$-RE and incomplete, then as before $OA_{\mathbb{F}} = OA_{\mathbb{F}(A)}$ and $\rho = \rho^A$. Then $\eta^A \leqslant \eta \leqslant \rho = \rho^A$ for this A.

Definition 4.2.4: Let $\sigma$ be a cardinal of $OA_{\mathbb{F}(A)}$ and let

$$\mathbb{F}(A)\text{-RE-cf}(\sigma) = \mu\gamma \leqslant \sigma[\gamma \text{ is the order type of an } \mathbb{F}(A)\text{-RE } X \subseteq \sigma \wedge \cup X = \sigma].$$

Theorem 4.2.5: Work in $OA_{\mathbb{F}}$ for normal ${}^{k+2}\mathbb{F}$. Take $A, C \subseteq \kappa_{\mathbb{F}}$ $\mathbb{F}$-RE such that $A \leqslant_{\mathbb{F}} C$, but $C \not\leqslant_{\mathbb{F}} A$; and assume that $\eta^A = \rho^A$ and $\rho^A$ is $\mathbb{F}(A)$-RE regular (i.e. $\mathbb{F}(A)\text{-RE-cf}(\rho^A) = \rho^A$). Then there exist $B \subseteq \kappa_{\mathbb{F}}$ $\mathbb{F}$-RE such that:

(i) $A \leqslant_{\mathbb{F}} B$, but $B \not\leqslant_{\mathbb{F}} A$; and

(ii) $B \leqslant_{\mathbb{F}} C$, but $C \not\leqslant_{\mathbb{F}} B$.

Proof. The proof proceeds as in the analogous result of Chapter 3 using the tools of 4.1. By the above Remark, $OA_{\mathbb{F}(A)} = OA_{\mathbb{F}}$ and $OA_{\mathbb{F}}$ is $\mathbb{F}(A)$-closed. The proof that $\mathbb{F}(A)\text{-RE-cf}(\kappa_{\mathbb{F}}) = \mathbb{F}(A)\text{-RE-cf}(\eta^A)$ proceeds as in Chapter 3. □

Corollary 4.2.6: Work in $OA_{\mathbb{F}}$ for normal ${}^{k+2}\mathbb{F}$. Assume that $\eta^A = \rho^A$ and $\mathbb{F}(A)\text{-RE-cf}(\rho^A) = \rho^A$ for all A

$\mathbb{F}$-RE and incomplete. Then the $\mathbb{F}$-RE degrees on $OA_{\mathbb{F}}$ are dense.

<u>Proof</u>. Apply the theorem to each pair $A, C \subseteq \kappa_{\mathbb{F}}$ $\mathbb{F}$-RE with $A \leq_{\mathbb{F}} C$, but $C \not\leq_{\mathbb{F}} A$ to construct $B$. Then the required degrees are $\underset{\sim}{a} =$ $\mathbb{F}$-degree $(A) < \underset{\sim}{b} =$ $\mathbb{F}$-degree $(B) <$ $\mathbb{F}$-degree $(C)$. $\square$

<u>Remark</u>: Analogs to 4.2.2 and 4.2.5 can be proved for an absolute notion of degree on arbitrary $\mathbb{E}$-closed sets $M$, Griffor [1980], which are not $\Sigma_1$-admissible and satisfy $DC_{\omega}(RE)$, the scheme for dependent choices of length $\omega$ along relations $R$ which are $\mathbb{E}$-RE on $M$.

Appendix: Selection Theorems in $\mathbb{E}$-Recursion

We include here the proofs of two selection theorems for $\mathbb{E}$-recursion on $\mathbb{E}$-closed $L(\kappa)$, Gandy Selection and an effective version of Grilliot Selection for $L(\kappa)$ where $\rho < \kappa$ and REC-regular. This second result was used in Chapters 2 and 3 in the proofs of Theorems 2.3.1 and 3.3.1. The proof of Gandy Selection is based on the proof given by Moschovakis [1967] in the setting of normal Kleene recursion in a functional of higher type. It should function as a 'warm-up' for the second result which we call Normann Selection [1980].

Theorem A-0 (Gandy Selection): Let $L(\kappa)$ be $\mathbb{E}$-closed. Then there exists a partial recursive function $\psi(e)$, $e \in \omega$ such that:

$$(\exists\, n \in \omega)[\{e\}(n)\ ] \longrightarrow$$

$$(\forall\, n \in \omega)[|\{e\}(\psi(e))| \leqslant |\{e\}(n)|].$$

Remark: Parameters are suppressed for clarify of exposition. If we are considering $e \in \omega$, $\tau < \kappa$ such that,

$$(\exists\, n \in \omega)[\{e\}(\tau,n)\downarrow]$$

then the value of $\psi$ will also depend on $\tau$.

Proof. Let $\mathcal{O}$ be the complete RE subset of $\omega$ in $L(\kappa)$ i.e.

$$\mathcal{O} = \{\langle e,n\rangle \mid \{e\}(n)\downarrow\}$$

and make use of $q : \omega \times \omega \longrightarrow \mathcal{O}$ such that

$$\{e\}(n)\downarrow \longleftrightarrow q(e,n) \in \mathcal{O}.$$

$|\cdot|^{\mathcal{O}}$ is simply height of computation and we omit the superscript. Using this norm $|\cdot|$ on computation one can define $\leqslant : \omega \times \omega \longrightarrow \{0,1\}$ partial REC

$$\leqslant (x,y)\downarrow \ \underline{\text{iff}}\ \ x\downarrow \ \text{ or } \ y\downarrow$$

(i.e. viewing $x$ as a pair $\{(x)_0\}((x)_1)\downarrow)$ and if either is defined, then

$$\leqslant (x,y) = \begin{cases} 0, & \text{if } |x| \leqslant |y| \\ 1, & \text{if } |y| < |x| \end{cases}$$

Remark: For $x\uparrow$, $|x| = \infty$, $\infty$ larger than any OR.

We shall define $\psi(e)$ by induction on $\min(e)$, where

Definition A-1: $\min(e) = \mu n(\forall m)[|\{e\}(n)| \leqslant |\{e\}(m)]$ ("the least integer which gives rise to the least height of computation"). Note that $\min(e)$ will be defined, just in case $(\exists n)[\{e\}(n)\downarrow]$.

Let $e'$ be primitive recursive function of $e$ such that,

$$\{e'\}(t) \simeq \{e\}(t+1).$$

Proposition A-2: $\min(e) > 0$, then

$$\min(e') = \min(e) - 1.$$

Proof. Since $\min(e)\downarrow$, let

$$\min(e) = s_0 \underset{df}{=} \mu s\ \forall\ t[|\{e\}(s)| \leqslant |\{e\}(t)|],$$

then $s_0-1 = \min(e')$, for: fix $t$, then

$$|\{e'\}(s_0-1)| = |\{e\}(s_0)| \leqslant |\{e\}(t+1)| \leqslant |\{e'\}(t)|$$

so $$t[|\{e'\}(s_0-1)| \leqslant |\{e'\}(t)|].$$

If $s_0-1$ were not least such, then $n < s_0-1$

$$\min(e') = n$$

and $\min(e) = n+1 \neq s_0$ , contradiction. A-2 $\square$

<u>Remark</u>: This method can't succeed for RE $W \subseteq \gamma$; $\omega < \gamma$ since for limit ordinals $\lambda < \gamma$, "$\lambda$-1" is meaningless. The idea is, however, still sound. One actually builds up to $\min(e) \in ON$ using the set of $e'$ for which $\min(e')$ has already been defined by effective transfinite recursion (cf. Harrington-MacQueen [1976]).

Keep in mind that $\psi(e) \cong \min(e)$, which is defined if $(\exists\, n \in \omega)[\{e\}(n)\downarrow]$. Also

$$\psi(e)\downarrow \ \ \underline{\text{iff}}\ \ q(\overline{\psi},e) \in \mathcal{O},$$

where $\overline{\psi}$ is an index for $\psi$ obtained by the recursion theorem.

To compute $\psi(e)$ from $\overline{\psi}$:

Try to compute $\leqslant (\{e\}(0),q(\overline{\psi},e'))$:

<u>Case 1</u>: If $\leqslant (\{e\}(0),q(\overline{\psi},e')) \simeq 0$, then $\{e\}(0)\downarrow$ so $\leqslant (\{e\},(0),z)\downarrow$ for any $z$, so ask whether

$$\forall\, t[\leqslant (\{e\}(0),\{e\}(t)],$$

i.e. if 0 gives computation of minimal height. If yes, output 0. If no, output $\psi(e') + 1$.

Case 2: $\leqslant(\{e\}(0),q(\overline{\psi},e')) \simeq 1$, then compute

$$\leqslant(\{e\}(0),\{e\}(\psi(e')+1))$$

(this is defined since $q(\overline{\psi},e')\downarrow \Rightarrow \psi(e')\downarrow \Rightarrow \{e'\}(\psi(e'))\downarrow$ $\longrightarrow \{e\}(\psi(e')+1)$ .

If $\leqslant(\{e\}(0),\{e\}(\psi(e')+1)) \simeq 0$, then as above ask whether $\forall\, t[\leqslant(\{e\}(0),\{e\}(t))]$. If yes, output 0. If no, output $\psi(e') + 1$.

If $\leqslant(\{e\}(0),\{e\}(\psi(e')+1)) \simeq 1$, output $\psi(e') + 1$.

Claim: $(\exists\, n)[\{e\}(n)\downarrow]$, then

$$\psi(e) = \min(e).$$

Proof of Claim. By induction on $\min(e) \in \omega$. $\min(e) = 0$ i.e.

$$(\forall\, t)[|\{e\}(0)| \leqslant |\{e\}(t)|]$$

and since $\{e\}(0)\downarrow$, $\leqslant(\{e\}(0),q(\overline{\psi},e'))\downarrow$: if $\leqslant(\{e\}(0),q(\overline{\psi},e')) \cong 0$, then by assumption,

$$(\forall\, t)[\leqslant(\{e\}(0),\{e\{(t)) = 0],$$

so we gave output, 0 as desired.

If $\leqslant(\{e\}(0),q(\overline{\psi},e')) = 1$, then

$$|q(\overline{\psi},e')| < |\{e\}(0)|,$$

and so $q(\overline{\psi},e')\downarrow$ so $\psi(e')\downarrow$ and it must be that

$$\leqslant(\{e\}(0),\{e\}(\psi(e')+1)) = 0$$

and hence $0$ is output, the $\min(e)$, since

$$|\{e\}(0)| \leqslant |\{e\}(\psi(e')+1)|.$$

$\min(e) > 0$; by induction hypothesis

$$\psi(e') = \min(e')$$

since $\min(e') < \min(e)$, by the proposition. Thus we show that,

$$\psi(e) = \psi(e') + 1 :$$

$$\psi(e')\downarrow \Rightarrow q(\overline{\psi},e')$$

$$\Rightarrow \leqslant(\{e\}(0),q(\overline{\psi},e'))\downarrow$$

if $\leqslant(\{e\}(0),q(\overline{\psi},e')) = 0$, by assumption

$$\neg(\forall\ t)[\leqslant(\{e\}(0),\{e\}(t))], \qquad \text{so}$$

$$\psi(e) = \psi(e') + 1, \qquad \text{as desired.}$$

If $\qquad \leqslant(\{e\}(0),q(\overline{\psi},e')) = 1,$ then if

$$\leqslant(\{e\}(0),\{e\}(\overline{\psi}(e')+1)) = 0$$

and we output $\psi(e') + 1$, as desired; or

$$\leqslant(\{e\}(0),\{e\}(\overline{\psi}(e')+1)) = 1 \quad \text{and again}$$

$$\psi(e) = \psi(e') + 1, \quad \text{as desired.}$$

claim □

Thus $\psi$ is our desired selection operator. □

In the following version of Grilliot Selection we assume that $\rho < \kappa$. In the case that $\rho = \kappa$ and we have RE $X \subseteq \delta < \rho$, let

$$M = \mathbb{E}\text{-cl}(gc(\kappa) \cup \delta \cup p+1)$$

where $p$ is the parameter defining $X$. Then $M$ is transitive and hence $M = L(\kappa')$ for some $\kappa' < \kappa$

(ow. $L(\kappa) = \mathbb{E}\text{-cl}(\gamma)$ for some $\gamma < \kappa$ and, hence, $\rho < \kappa$) and so carry out this argument inside $L(\kappa')$.

<u>Theorem A-3</u>: $L(\kappa)$ $\mathbb{E}$-closed not $\Sigma_1$-admissible, $\rho < \kappa$ and $\rho$ REC-regular iff

$$\forall \quad \delta[\delta < \rho \Rightarrow \sup_{\gamma<\delta} \kappa_0^{<\gamma,\delta>} \leqslant \kappa_r^{\delta}].$$

<u>Remark</u>: Parameters are suppressed for the sake of clarity. One actually proves, for $\gamma_0 < \kappa$

$$\delta < \rho \Rightarrow \sup_{\gamma<\delta} \kappa_0^{<<\gamma_0,\gamma>\delta>} \leqslant \kappa_r^{<\gamma_0,\delta>}.$$

<u>Proof</u>. ($\Longleftarrow$) Suppose $\rho$ is REC-singular and let $\delta < \rho$, $f : \delta \longrightarrow \rho$ witness this singularity with $f$ REC and $rg(f)$ unbounded in $\rho$.

Note that by Theorem 1.2.13 due to Moschovakis, the RE subsets of $\kappa$ are not closed under the quantifier $\exists\ \tau < \rho$. But, using $f$, we can write $\exists\ \tau < \rho$ as

$$(\exists\ \gamma < \delta)(\exists\ \sigma < f(\gamma)).$$

Thus RE sets cannot be closed under less than $\rho$ bounded existential quantification, so it must be that for some $\delta < \rho$ and $\gamma_0 < \kappa$

$$\sup_{\gamma<\delta} \kappa_0^{<\gamma,\gamma_0,\delta>} > \kappa_r^{<\gamma_0,\delta>} .$$

To show $\rho$ REC-regular implies

$$(\forall\ \delta)[(\delta < \rho) \Rightarrow \sup_{\gamma<\delta} \kappa_0^{<\gamma,\delta>} \leqslant \kappa_r^{\delta}].$$

<u>Lemma A-4</u>: If $\rho$ is REC-regular and $\delta < \rho$ then there is a partial REC $\Gamma$ (suppressing parameters) such that if

$$(\exists\ \gamma < \delta)[\{e\}(\gamma)\downarrow] \longleftrightarrow \Gamma(e)$$

and then if $\Gamma(e)\downarrow$,

$$\min_{\gamma<\delta}[|\{e\}(\gamma)|] \leqslant |\Gamma(e)|.$$

We can now complete the proof of the theorem. Suppose $\exists\ \delta < \rho$

$$\sup_{\gamma<\delta} \kappa_0^{<\gamma,\delta>} > \kappa_r^{\delta}$$

then let

$$R = \{<e,\alpha> \mid e \in \omega \wedge \gamma < \delta \wedge \{e\}(\gamma)\downarrow\}.$$

Then

$$\sup_{z \in R} |\{(z)_0\}((z)_1)| = \sup_{\gamma<\delta} \kappa_0^{<\gamma,\delta>} > \kappa_r^\delta.$$

Now define

$$R(> \kappa_r^\delta) = \{<e,\gamma> \mid \{e,\gamma> \in R \wedge |\{e\}(\gamma)| > \kappa_r^\delta\}.$$

Certainly $R(> \kappa_r^\delta)$ is RE in $\delta$ say via $e_0$ $\omega$ and contained in $\delta$. By assumption $R(> \kappa_r^\delta) \neq \emptyset$, so

$$\Gamma(e_0,\delta)\downarrow \quad \text{and}$$

$$\min_{\gamma<\delta}[|\{e_0\}(\gamma)|] \leqslant |\Gamma(e_0,\delta)|.$$

But $|\Gamma(e_0,\delta)| < \kappa_0^\delta$ and by the definition of $R(> \kappa_r^\delta)$

$$(\forall\ \gamma < \delta)[\kappa_r^\delta < |\{e_0\}(\gamma)|],$$

a contradiction. □

It remains only to prove the lemma.

<u>Proof of Lemma</u>:

Proposition A-5: Suppose $\rho$ is REC-regular, $\delta < \rho$ and $A \subseteq \delta$ and $A \in L(\kappa)$. Then $A \in L(\rho)$.

Proof of Proposition: Fix $\delta < \rho$ and $A \subseteq \delta$, $A \in L(\kappa)$ then $\exists\ \sigma \in \kappa$ $A \in L(\sigma)$. Then in $L(\sigma)$ let

$$M = \mathbb{E}\text{-cl}(\delta \cup \{\gamma\})$$

where $\gamma < \rho$ such that $\sigma \leqslant_{\mathbb{E}} \gamma$. Let $\overline{M}$ be the Mostowski collapse of $M$. $M \leqslant_{\mathbb{E}} \langle\delta,\gamma\rangle$ and is indexed by ordinals in $\omega \times \delta$. By $\rho$ REC-regular $\overline{M} \cap \rho$ is bounded. Since $A \subseteq \delta$ and the collapse is the identity on $\delta$, $A$ is mapped to $A$, so $A \in \overline{M}$.

Let $\gamma_0 = \text{On}\ \ \overline{M}$, then $\gamma_0 < \rho$ and

$$\overline{M} = L(\gamma_0)$$

and so $A \in L(\gamma_0) \subseteq L(\rho)$. □

By the proposition, if $\delta < \rho$,

$$B_\delta = \{f : \delta \longrightarrow \rho \mid f \in L(\kappa)\}L(\rho+1) \subseteq L(\kappa).$$

We will use this fact in defining $\Gamma(e)$.

Note that by the definition of $\rho$ we can view the $\mathbb{E}$-REC functions as being indexed by ordinals less than $\rho$.

If $a < \rho$ is a code for a computation, then $q(a) < \rho$ is an immediate subcomputation of $a$. $R_a = \{$immed. subcomputations of $a\}$ is, in general, RE\REC in $a$.

Let $X \subseteq \delta$ be RE via $e$ with $X \neq \emptyset$. By closure under pairing we can imagine $\mathcal{O} \subseteq \rho$

$$\mathcal{O} = \{<e,\delta> \mid \{e\}(\delta)\downarrow\},$$

then there is REC $h : \delta \longrightarrow \rho$ such that $b < \delta$

$$b \in X \longleftrightarrow h(b) \in \mathcal{O}.$$

These are the functions we'll consider in our proof. Our goal is to product a non-empty REC subset of $X$, $W$. Naturally enough

$$W = \{b \in X \mid |\{e\}(b)| \text{ is minimal}\}.$$

Keeping in mind the proof of Gandy Selection, for $f : \delta \longrightarrow \rho$, let

$$\min f = \min \{|f(b)| \cdot |b < \delta\}$$

where $f(b)$ is an instruction for computation.

We define $\psi(f,b) : {}^{\delta}\rho \times \rho \longrightarrow \{0,1\}$, by

$$\phi(f,b) = \begin{cases} 0, & \text{if} \quad |f(b)| = \min f < \infty \\ 1, & \text{if} \quad |f(b)| > \min f \\ \uparrow, & \text{if} \quad \min f = \infty. \end{cases}$$

So given $f : \delta \longrightarrow \rho$ with $\min f < \infty$,

$$\{b < \delta \mid |f(b)| = \min f\} \leqslant_{\mathbb{E}} f.$$

$\phi$ will be defined by ETR on $\min f$. Say $f$ given and $\phi$ is correctly defined for all $g$, $b$, $b < \delta$ such that,

$$\min g < \min f,$$

then we will define $\phi(f,b)$ for all $b < \delta$. Assume

$$0 < \min f < \infty.$$

Define $\psi : \rho \times \delta \longrightarrow \{0,1\}$ an approximation to

$$\{b < \delta \mid |q(f(b))| < \min f\},$$

where, recall, $q(a)$ is an immed. subcomputation of $a$, which must be evaluated before the remaining subcomputations can be determined. This corresponds to an application of scheme (iv).

$\psi(\sigma,b) = 0$ iff we know that

$$q(f(b))\downarrow,$$

hence the set of immed subcomputations is known and we can proceed.

$\psi(\sigma,b)$ is defined by ETR on $\sigma$ simultaneously for all $b$,

Stage 0: $\psi(o,b) = 1$ for all $b < \delta$.

Stage $\sigma > 0$: $\psi(\sigma,b)$ is defined as follows, if $\psi(\tau,b)$ has been defined $b < \delta$, let

$$M_\tau = \{g : \delta \longrightarrow \rho : g \in L(\tau) \wedge g(b) = \begin{cases} q(f(b)) \text{ if} \\ \psi(\tau,b) = 1 \\ \\ \text{some element of} \\ R_{f(b)} \text{ if} \\ \psi(\tau,b) = 0 \end{cases} \}$$

where $R_a = \{\text{immed subcomputations of } a\} \in RE(a)$.

Intuitively these functions $q : \delta \longrightarrow \rho$ "lie under" $f$ at stage $\tau$ constructible by $\tau$. Clearly

$$M_\tau \leq_{\mathbb{E}} f, \tau \qquad \text{since}$$

$$\psi(\tau,b) = 0 \Rightarrow q(f(b)) \; [\Leftrightarrow R_{f(b)} \leq_{\mathbb{E}} b].$$

Notice: $g \in M_\tau \Rightarrow \min g < \min f$ and so $\phi \restriction M_\tau \times \delta$ is available i.e. the min's of $g \in M_\tau$. Using $\phi$ define

$$N_\tau = \{g(b) : g \in M_\tau \wedge b < \delta \wedge \psi(g,b) = 0\}.$$

Finally assume $\psi(\tau,b)\downarrow$ defined $\forall\, \tau < \sigma$ and all $b < \delta$, then

$\psi(\sigma,b) = 0$ iff

$$(\exists\, \tau < \sigma)(\exists\, c \in N_\tau)[|q(f(b))| \leq |c|]$$

in particular $q(f(b))\downarrow$.

Facts: 1. $\tau < \sigma \Rightarrow \psi(\tau,b) \geq \psi(\sigma,b), \quad \forall\, b < \delta$.

2. $\tau < \sigma \Rightarrow M_\tau \subseteq M_\sigma$ and $N_\tau \subseteq N_\sigma$.

3. $g \in M_\tau \Rightarrow \min g < \min f$ so

$$c \in N_\tau \Rightarrow |c| = \min g \text{ for some } g \in M_\tau.$$

4. $\psi(\sigma,b) = 0 \Rightarrow |q(f(b))| < \min f$.

Intuitively, $\psi(\sigma,b) = 0$ tells us that in our effort to effectively generate the ordinal $\min f$, $q(f(b))$ gives a "false minimum".

There exists $\sigma < \rho$ such that $\forall\, b < \delta$

$$\psi(\sigma,b) = \psi(\sigma+1,b),$$

for otherwise using $\delta$ and $\rho$ and $B_\delta$ as parameters and assuming such a $\sigma$, say $\sigma(\gamma)$ has been determined for $\gamma < \delta$, the map

$$\gamma \longmapsto \sigma(\gamma) \qquad \text{is} \quad \mathbb{E}\text{-REC}.$$

$\sigma(\gamma)$ is increasing as a function of $\gamma$ so if there were no $\sigma(\delta) < \rho$ the map $h : \delta \longrightarrow \rho$ given by

$$h(\gamma) = \sigma(\gamma)$$

would give an $\mathbb{E}$-REC singularity in $\rho$, contradicting $\rho$ REC-regular. Thus $\sigma(\delta) < \rho$ and if $\sigma_0$ is the least such, $\psi(\sigma_0,b)$ is $\mathbb{E}$-REC(b) and hence $M_{\sigma_0}$ and $N_{\sigma_0}$ are $\mathbb{E}$-REC.

For $b < \delta$, define $\phi(f,b)$, by

$$\phi(f,b) = 0 \Leftrightarrow \psi(\sigma_0,b) = 0 \wedge (\forall\, d \in R_{f(b)})$$

$$(\exists\, c \in N_{\sigma_0})[|d| \leqslant |c|].$$

To show that $\phi(f,b)$ is correctly defined, we show

<u>Claim</u>: $\min f = \sup\{|c| + 1 \mid c \in N_{\sigma_0}\}$.

<u>Proof of Claim</u>. Assume not. We know

$$c \in N_{\sigma_0} \longrightarrow |c| < \min f$$

so, we assume

$$\min f > \sup\{|c| + 1 \mid c \in N_{\sigma_0}\}.$$

Let $Y = \{b < \delta \mid \psi(\sigma_0,b) = 0\}$, then $b \not\in Y$ implies $\psi(\sigma_0+1,b) = \psi(\sigma_0,b) = 1$, so by the definition of $\psi$,

$$|q(f(b))| \not\leqslant |c| \qquad \text{for any} \quad c \in N_{\sigma_0}.$$

If $b \in Y$, then $|f(b)| \geqslant \min f > \sup\{|c| + 1 \mid c \in N_{\sigma_0}\}$, so $\exists\, d \in R_{f(b)}$ such that $|d| \not\leqslant |c|$ for any $c \in N_{\sigma_0}$.

Define $g : \delta \longrightarrow \rho$ by

$$g(b) = \begin{cases} d \in R_{f(b)} \text{ such that } |d| \not\leq |c| \text{ for} \\ \quad \text{any } c \in N_{\sigma_0}, \quad \text{if } b \in Y \\ \\ g(f(b)), \quad \text{if } b \notin Y. \end{cases}$$

Then $g \in M_{\sigma_0}$, so for some $b < \delta$, $g(b) \in N_{\sigma_0}$, but by the definition of $g$,

$$|g(b)| \not\leq |c| \text{ for any } c \in N_{\sigma_0},$$

a contradiction. □

This shows that $\phi(f,b)$ is correctly defined. Now if $\min f < \infty$, then $\phi(f,b)$ is defined for all $b < \delta$ and if we let $e_f$ be an index for $f$,

$$\{b \mid \phi(f,b) = 0\} \leqslant_{\mathbb{E}} f \qquad \text{so let}$$

$$\Gamma(e_f,\delta) = \cup \{b < \delta \mid \phi(f,b) = 0\}, \quad \text{then}$$

$$|\Gamma(e_f,\delta)| = \cup \{|\phi(f,b)| + 1 : b \subset \delta \wedge \phi(f,b) = 0\}$$

and, then

$$\min_{\gamma<\delta}[|\{e_f\}(\gamma)|] \leqslant |\Gamma(e_f,\delta)|,$$

as desired. $\square$

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