COBORDISM OF MANIFOLDS WITH w_1, w_2 AND w_4 VANISHING

by

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ABSTRACT

A cobordism theory is defined for manifolds whose first 4 Stiefel Whitney classes vanish. The classifying map of the stable normal bundle for such manifolds can be lifted to the 4-connected covering $BO\langle 4\rangle$ of BO. The cohomology of the Thom space $MO\langle 4\rangle$ of the canonical bundle is partially computed, and the results used to give information about the cobordism theory.

In analogy with the work of Brown and Peterson, an invariant ψ is defined on this cobordism theory in dimensions congruent to 6 mod 16, which reduces to the Kervaire-Arf Invariant Φ when the latter is defined. It is shown that Φ is zero on all stably-parallelizable manifolds of dimension 22 and 38, and some additonal results on the vanishing of Φ are obtained.

Thesis Supervisor: Franklin P. Peterson

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INTRODUCTION

In his thesis Thom [28] defined the relation of cobordism for closed, compact, C^∞ manifolds. Two such manifolds M and M' are cobordant if there is a compact C^∞ manifold W whose boundary is the disjoint union of M and M'. He also introduced the notion of the Thom space MV of a vector bundle V, and proved that the cobordism classes of manifolds form a graded ring $\mathfrak{N}_*$, which is isomorphic to the stable homotopy the Thom space MO of the canonical vector bundle EO over BO, the classifying space for stable vector bundles.

Since then cobordism theory has been generalized in many directions. All have in common an isomorphism into the stable homotopy of the Thom space of some bundle. Thom [28], Milnor [22], Dold [11] and Wall [31], determined the structure of the cobordism ring of oriented manifolds. Lashof [16] has shown that a cobordism relation can be defined with respect to any space X and map $f : X \to BO$. A manifold is considered only if the classifying map of its stable normal bundle $\nu : M \to BO$ can be factored through X. The cobordism relation must then of course preserve this factorization. The cobordism ring Ω_*^f obtained in this manner is isomorphic to the stable homotopy of the Thom space of $f^*(EO)$. This formulation gives as a special case all cobordism theories associated to a reduction of the structural group of the normal bundle of a manifold to a subgroup of the orthogonal group. For example oriented cobordism Ω_*, Spin cobordism, Ω_*^{spin} [3], [23], Unitary or complex cobordism, Ω_*^U [22], and special unitary cobordism, Ω_*^{SU} [2] [10].

The cobordism theory associated with manifolds whose normal bundle is trivial, commonly denoted by Ω_*^{framed} (which will be later denoted by Ω_*^∞), is especially interesting because Ω_*^{framed} is isomorphic to the stable homotopy of the sphere π^S. Kervaire and Milnor [11] study Ω_*^{framed} in detail. One of the questions they consider is: Given an n-dimensional framed manifold M, is M framed-cobordant to a homotopy sphere. They use the techniques of surgery, or spherical modifications to show that the answer is yes if n is odd. [20], [33]

In order to approach the problem for even dimensional manifolds, Kervaire [14] defined an invariant $\Phi(m) \varepsilon Z_2$ for 2k connected 4k+2 manifolds, and showed that $\Phi(M)=0$ iff M is framed-cobordant to a homotopy sphere. In [15] it is shown that Φ induces a homomorphism $\Phi : \Omega_{4k+2}^{framed} \to Z_2$. It is unknown whether this homomorphism is 0.

In [8] Brown and Peterson prove Φ is zero on 8k+2 dimensional manifolds. They define an invariant $\psi : \Omega^{SU} \to Z_2$ and show that the composition $\Omega_{8k+2}^{framed} \to \Omega_{8k+2}^{SU} \to Z_2$ is equal to Φ, and is 0.

In this work we attempt to adapt the work of [8] to show that Φ is zero on manifolds of dimension 16k+6. A cobordism theory $\Omega_*^{\langle 4\rangle}$ is defined, and a map ψ constructed such that the composition $\Omega_{16k+6}^{\text{framed}} \to \Omega_{16k+6}^{\langle 4\rangle} \to Z_2$ is equal to Φ, and some results on the vanishing of Φ are obtained.

CHAPTER I

STATEMENT OF RESULTS

Unless stated otherwise, the term "manifold" shall mean compact manifold, differentiable of class C^∞. M and N will denote closed manifolds, W a manifold with boundary. For an oriented manifold P, the same manifold with opposite orientation will be denoted by -P. Cohomology shall always mean cohomology with coefficients the field of integers modulo 2; $H^k(X) = H^k(X, Z_2)$.

In section 2, the cobordism ring $\Omega^{\langle 4\rangle}$ is defined, and using results of Lashof [16], its elementary properties are stated. One of the most useful is the following proposition.

<u>Prop</u>. Every cobordism class $\gamma \varepsilon \Omega^{\langle 4\rangle}$ has a representative M such that $H^q(M) = H^q(BO\langle 4\rangle)$ for $q < [n/2]$.

A quadratic operation ϕ, associated to the relation $Sq^{8k+4} = Sq^4 Sq^{8k} + Sq^2(Sq^4 Sq^{8k-2}) + Sq^1(Sq^2 Sq^4 Sq^{8k-3})$ is defined, and Adem's generalizations [1] of the Peterson-Stein formulae [25] are used to calculate ϕ. The bordism groups of a space X are introduced [9], and it is shown that ϕ induces a map $\overline{\phi} : \Omega^{\langle 4\rangle}_{16k+6}(K(Z_2, 8k+3)) \to Z_2$. Then ϕ is used to define a map $\psi : \Omega^{\langle 4\rangle}_{8k+2} \to Z_2$, and it is shown that $\psi = \Phi$ on 8k+2-connected manifolds of dimensions 16k+6.

Most of the theorems and proofs are modeled after [8].

Chapters 3 and 4 are devoted to technical details needed to study the structure of $\Omega^{<4>}$. In chapter 3 the cohomology of MO<4> as a module over the Steenrod algebra is partially determined. There is a monomorphism $A/A(Sq^4, Sq^2, Sq^1) \to H^*(MO\langle 4\rangle)$ which sends 1 into the Thom class U. It is shown that for small dimensions, $H^*(MO\langle 4\rangle)$ is the direct sum of cyclic modules over A. In chapter 4, $Ext_A(H^*(MO\langle 4\rangle)$ is partially determined. Most of the work is in determining $Ext_A(A/A(Sq^4,Sq^2,Sq^1),Z^2) = H^{**}(A_2)$, where A_2 is the sub-Hopf algebra of A generated by Sq^4, Sq^2, and Sq^1. This is done by using the spectral sequence of May [18], which converges to $H^{**}(A_2)$, and has E_2 term the cohomology of the associated graded algebra to A_2 with respect to the augmentation filtration. There are many purely technical details, and the proofs are referred to appendix I.

In chapter 5, some non-zero differentials in the Adams spectral sequence for $\pi^*(MO\langle 4\rangle) = \Omega_*^{<4>}$ are computed. In particular there are elements in the 12 and 15 stems on which d_2 is non-zero. A description is given of the k stems for k congruent to 6 mod 16.

In chapter 6, the action of ψ on products is computed. In particular we have the following theorems.

Theorem: Let $a \in \Omega^{<4>}_{16k+6}$, $b \in \Omega^{<4>}_{16j}$, and suppose a and b have representatives M and N such that $A^q H^{16k+6-q}(M) = 0$ for q odd, and that $A^p H^{16j-p}(N) = 0$ for p odd and less than $8j$. Then $\psi(ab) = \psi(a)\chi(b)$, where χ is the Euler characteristic reduced mod 2.

Theorem: Let $b \in \Omega^{<4>}_{16j}$ be such that b has a representative N with $\chi(N) = 0$. Let a be the class of $S^3 x S^3$. Then $\psi(ab) = 0$.

Using these theorems, and the results of chapter 5, it is shown that if $a \in \Omega^{<4>}_{16k+6}$ has a representative y in the E_2 term of the Adams spectral sequence, such that y lies in the image of the map $Ext_A(Z_2, Z_2) \rightarrow Ext_A(H^*(MO<4>), Z_2)$, then $\psi(y) = 0$, and that $\Phi(\Omega_n^{framed}) = 0$ if $n = 22$ or 38.

CHAPTER II

THE COBORDISM THEORY AND THE KERVAIRE INVARIANT: ELEMENTARY PROPERTIES

Let M be an n-dimensional manifold. Let BO_k be the classifying space for k-dimensional vector bundles. Given an embedding of M in some euclidean space R^{n+k}, let ν_k denote the normal bundle. ν_k can be regarded as a map $\nu_k : M \to BO_k$. Let $i : BO_k \to BO_{k+1}$ be the canonical map. The map $i\nu_k : M \to BO_{k+1}$ induces the bundle $\nu_k \oplus 1$, where $\oplus$ denotes Whitney sum, and 1 the trivial line bundle over M. If k is sufficiently large, the homotopy class of ν_k depends only on M, and the bundle ν_k is called the stable normal bundle. We drop the k and denote it by ν_M, or ν, when no ambiguity may arise. Similarly BO_k, for sufficiently large k will be denoted by BO. Define [34]

$$H^n(BO) = \lim_{n \to \infty} H^{n+k}(BO_k) \quad \text{and} \quad \pi_n(BO) = \lim_{k \to \infty} \pi_{n+k}(BO_k).$$

Let $BO\langle r\rangle_k$ denote the r^{th} connective covering of BO_k. $BO\langle r\rangle_k$ is the total space of a fibration $p_r : BO\langle r\rangle_k \to BO_k$ such that $BO\langle r\rangle_k$ is r-connected, and $p_{r*} : \pi_q(BO\langle r\rangle_k) \to \pi_q(BO_k)$ is an isomorphism for $q > r$. (For existence and other properties see [12]). The map $i : BO_k \to BO_{k+1}$

lifts to a map $i : BO\langle r\rangle_k \to BO\langle r\rangle_{k+1}$ and the diagram

$$\begin{array}{ccc} BO\langle r\rangle_k & \xrightarrow{i} & BO\langle r\rangle_{k+1} \\ \downarrow p_r & & \downarrow p_r \\ BO_k & \xrightarrow{i} & BO_{k+1} \end{array}$$

commutes. We have $BO\langle r\rangle$, $H^q(BO\langle r\rangle)$ and $\pi_q(BO\langle r\rangle)$ as for BO.

An r-structure on a manifold M is a lifting of the normal bundle ν_M to $\nu_M : M \to BO\langle r\rangle$. Let $\langle r\rangle^n$ be the set of all n dimensional manifolds with at least one r-structure, and $\langle r\rangle = \quad \langle r\rangle^n$.

There are several important examples of r-structures $BO_k(1) = BSO_k$, and a 1-structure is just an orientation. $BO(2)$ = B Spin and a 2-structure is a Spin structure.

This is a consequence of the following basic fact about connective coverings.

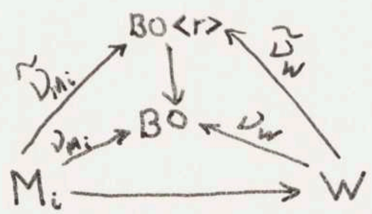
<u>Prop [12]</u>: A map $f : M \to BO\langle r\rangle_k$ has a lifting t. $BO\langle r+1\rangle_k$ iff $f^*(H^{r+1}(BO\langle r\rangle_k, \pi)) = 0$, where $\pi = \pi_{r+1}(BO\langle r\rangle_k) = \pi_{r+1}(BO_k)$. We have $\pi_1(BO) = \pi_2(BO) = Z_2$ $\pi_4(BO) = Z$, and hence a manifold has an orientation iff $w_1 = 0$, a Spin structure iff w_1 and $w_2 = 0$, and a 4-structure iff w_1 and $w_2 = 0$, and $w_4 = 0$ as an integral cohomology class. If $\nu : M \to BO$ can be lifted to $BO\langle r\rangle$ for every r, ν is homotopically trivial, and hence

the bundle ν_n is trivial. An $\langle\infty\rangle$ structure on a manifold is a trivialization of its normal bundle.

A cobordism relation can be defined on $\langle r\rangle$ as follows: Two n-dimensional manifolds M_1, $M_2 \varepsilon \langle r\rangle$ are r-cobordant if there is an (n+1) manifold W such that

1) $\partial W = M_1 + (-M_2)$

2) The diagram below commutes for $i = 1, 2$.



For $r = \infty$, the manifold W must be a framed manifold such that the framing on W restricts to that on M_1 and M_2.

Let EO_k be the canonical bundle over BO_k, and $p_r^*(EO_k) = EO\langle r\rangle_k$, the induced bundle over $BO\langle r\rangle_k$. Let $M(EO_k)$ be the Thom space of EO_k.

<u>Theorem 1</u>: The cobordism relation defined above is an equivalence relation. The set of equivalence classes $\Omega_n^{\langle r\rangle}$ of elements of $\langle r\rangle^n$ form a group under the operation induced by disjoint union, and $\Omega_*^{\langle r\rangle} = \sum_n \Omega_n^{\langle r\rangle}$ is a graded ring, the multiplication induced by cartesian product.

The map defined by Thom $\Omega_n^{<r>} \to \lim_k \pi_{n+k}(MEO_k)$ is an isomorphism. Set $MO<r>_k = MEO_k<r>$, and

$\lim_k \pi_{n+k}(MO<r>_k) = \pi_n(MO<r>)$.

<u>Proof</u>: The usual construction [28] gives a map $\Omega_n^{<r>} \to \pi_{n+k}(MEO_k)$ for large k. Embed M in an n+k dimensional sphere, k large. The normal bundle ν is induced by a map $M \to BO<r>_k$. This gives a map $M(\nu) \to MO<r>_k$. To get the map $S^{n+k} \to M(\nu)$ identify ν with a tubular neighborhood T of M in S^{n+k}. Then $M(\nu)$ = T/boundary of T = S^{n+k}/compliment of T. So there is a projection $S^{n+k} \to M(\nu)$, and by composing with the map above, we get an element of $\pi_{n+k}(MO<r>_k)$. The proof that this induces an isomorphism from $\Omega_n^{<r>} \to \pi_n(MO<r>)$ is given in [16]. It is essentially the same as Thom's.

Since the Thom space of a bundle over a point is a sphere, we have Ω_n^{∞} $(= \Omega_n^{\text{framed}}) \cong \pi_n^S = \lim_k \pi_{n+k}(S^k)$, the n-th stable stem of the homotopy of the sphere.

There is the obvious map $\rho_r : \Omega_n^{\infty} \to \Omega_n^{<r>}$ which sends a manifold in Ω_n^{∞} into its class in $\Omega_n^{<r>}$. The trivialization of ν_m determines a lifting of ν_M to $BO<r>$.

It will be very useful to be able to choose representatives of a given cobordism class such that they have partially known cohomology.

<u>Theorem 2</u>: Let $\omega \in \Omega_n^{\langle r \rangle}$. There is a manifold $M \in \omega$ such that $\nu^* : H^q(BO\langle r\rangle, Z_2) \to H^q(M, Z_2)$ is an isomorphism if $q < [^n/2]$.

<u>Proof</u>: This uses the technique of surgery, or spherical modifications. The theorem is proved in Lashof [16] so we give only definitions and a brief outline.

<u>Definition</u>: Let M^n be a C^∞ manifold without boundary but not necessarily compact. Let $f : S^p \to M$ be an embedding of S^p with trivial normal bundle, and let $F : S^p \times D^{q+1} \to M$ where $1 + p + q = n$ be a specific trivialization.

Let $M' = (M - (S^p \times D^{q+1})) \cup_F (D^{p+1} \times S^q)$ i.e., remove $S^p \times D^{q+1}$ and put back $D^{p+1} \times S^q$ identifying along $S^p \times S^q = \partial(S^p \times D^{q+1}) = \partial(D^{p+1} \times S^q)$. (The standard picture is putting a handle on S^2 by starting with an embedding of S^0). Then we will say M' is obtained from M by the modification. The manifold W given by $W = M \times I \cup_g D^{p+1} \times D^{q+1}$ where $g : S^p \times D^{q+1} \to M \times I$ is given by $g(x, y) = (f(x, y), 1)$, is a 0-cobordism between M and M' i.e., $\partial W = M \cup M_1$. This is clear from picture. If W is an r-modification and M, M' and W are in $\langle r\rangle$, then F will be called an r-modification.

In order to prove the theorem, it is sufficient to show that any homotopy class of $\pi_q(M)$ can be killed using r-modifications, unless $\nu_*(\omega x) \neq 0$ in $\pi_q(BO\langle r\rangle)$, where q is in appropriate dimensions. This is not too difficult. The entire proof is in Lashof.

Corollary: Let $\Omega_n^{\langle 4\rangle}$. $n > 16$. Then there is an $M \varepsilon \omega$ which is 7 connected.

Proof: Since $\pi_5(BO) = \pi_6(BO) = \pi_7(BO) = \pi_8(BO) = Z$. $BO\langle 4\rangle = BO\langle 7\rangle$. There there is a manifold M in ω such that $H^k(M) = 0$ for $k \leq 7$, $H^8(M) = Z$. Since H^8 has no torsion, $H_i(M) = 0$ for $i < 8 \Rightarrow \pi_i(M) = 0$ for $i < 8$.

Now we restrict our attention to $\langle 4\rangle$ structures. For the remainder of this chapter m will be an integer $\equiv 3 \bmod 8$. $Sq^{i,j}$ will denote Sq^iSq^j. Define a secondary cohomology operation ϕ on a subgroup of $H^m(X)$ with values in a quotient of $H^{2m}(X)$ for any space X as follows: [6].

Theorem 3: Since Sq^{m+1} is 0 on m-dimensional cohomology classes, the relation

$Sq^{m+1} = Sq^4Sq^{m-3} + Sq^2(Sq^{4,m-5}) + Sq^1(Sq^{2,4,m-6})$ in the Steenrod algebra gives rise to a secondary cohomology operation $\phi : H^m(X)$ Ker Sq^{m-3} Ker $Sq^{4,m-5}$ Ker $Sq^{2,4,m-6}$ $\rightarrow H^{2m}(X)/Xq^4\, H^{2m-4}(X) + Sq^2H^{2m-2}(X) + Sq^1H^{2m-1}(X)$.

Furthermore ϕ is quadratic, i.e., if $\phi(x)$ and $\phi(y)$ are defined, so is $\phi(x+y)$ and $\phi(x+y) = \phi(x) + \phi(y) + xy$.

<u>Proof</u>: This is proved in a more general case in [6]. Recall how ϕ is defined. Let

$K = K(Z_2,\ 2m-3) \quad K(Z_2,\ 2m-1) \quad K(Z_2,\ 2m)$ where

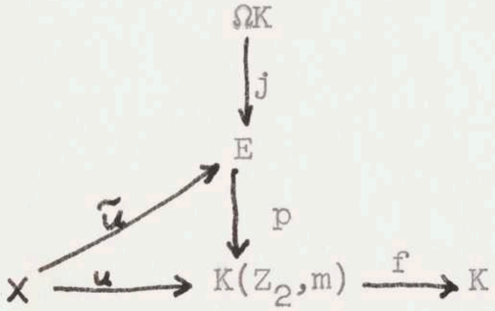
$m \equiv 3 \bmod 8$, and $m > 3$. Let $f : K(Z_2,\ m) \to K$

be such that $f^*(i_{2m-3}) = Sq^{m-3}i_m$; $f^*(i_{2m-1}) = Sq^{4,m-5}i_m$, and $f^*(i_{2m}) = Sq^{2,4,m-6}i_m$, where $i_j \in H^j(K(Z_2 z))$ is a generator. Let E be the fibre space over $K(Z_2,\ m)$ with fibre $\Omega K = K(Z_2,\ 2m-4) \times K(Z_2,\ 2m-2) \times K(Z_2, 2m-1)$.

Look at the cohomology spectral sequence for this fibre space. The element $x = Sq^4 i_{2m-4} + Sq^2 i_{2m-2} + Sq^1 i_{2m-1} \in H^{2m}(\Omega K)$ is transgressive, and transgresses to

$$Sq^4 Sq^{m-3} i_m + Sq^2 Sq^{4,m-5} i_m + Sq^1 Sq^{2,4,m-6} i_m = Sq^{m+1} i_m = 0.$$

Therefore there is an element $\phi \in H^{2m}(E)$, such that $j^*(\phi) = x$ where $j : \Omega K \to E$ is the inclusion. Consider the diagram



where X is any space, and $u \in H^m(X)$.

$Sq^{m-3}u = Sq^{4,m-5}u = Sq^{2,4,m-6}u = 0$ implies fu is homotopic to 0, and hence u lifts to a map $\tilde{u} : X \to E$. Define $\phi(u) = \tilde{u}(\phi)$. The indeterminacy of ϕ corresponds to the different choices of the lifting $\tilde{u}$.

The H-space structure on ΩK and $K(z_2, m)$ induces a multiplication $D : E \times E \to E$. Let

$K(Z_2, m) \times K(Z_2, m) \to K(Z_2, m)$ be the multiplication. By [6], Lemma 2.2, ϕ is not primitive, i.e.

$\nu^*(\phi) = \phi \otimes 1 + 1 \otimes \phi + Z$, where Z is non-zero. But $H^i(E) = 0$ for $m < i < 2m - 4$, so $H^{2m}(E) = H^{2m}(E) \otimes 1 \oplus H^m(E) \otimes H^m(E) \oplus 1 \otimes H^{2m}(E)$, and $H^m(E) = Z_2$, generated by $p^*(i_m)$. So $\nu^*(\phi) = \phi \otimes 1 + 1 \otimes \phi + p^*(i_m) \otimes p^*(i_m)$. Suppose $\phi(u)$ and $\phi(v)$ defined by liftings $\tilde{u}$ and $\tilde{v}$ respectively. Let $\Delta : X \to X \times X$ be the diagonal map. Then $\nu(\tilde{u} \times \tilde{v})\Delta$ is a lifting of $u + v$, since $p\nu(\tilde{U} \times \tilde{v})\Delta = \tau(u \times v)\Delta = u + v$, since that is how addition of maps is defined. It is a standard theorem that $(u + v)^* = u^* + v^*$. Then

$$\phi(u+v) = (\nu(\tilde{u} \times \tilde{v})\Delta)^*\phi = (\Delta(\tilde{u} \times \tilde{v}))^*\nu^*(\phi)$$

$$= (\Delta(\tilde{u} \times \tilde{v}))^*(\phi \otimes 1 + 1 \otimes \phi + p^*(i_m) \otimes p^*(i_m))$$

$$= \phi(u) + \phi(v) + uv \text{ modulo the indeterminacy of } \phi.$$

<u>Lemma 1</u>: Let $M \varepsilon \langle 4 \rangle^{2m}$, $\phi : H^m(M) \to H^{2m}(M)$. Then the indeterminacy of ϕ is zero.

<u>Proof</u>: Indeterminacy of $\phi = Sq^4H^{2m-4}(M) + Sq^2H^{2m-2}(M) + Sq^1H^{2m-1}(M)$ $= v_4H^{2m-4}(M) + v_2H^{2m-2}(M) + v_1H^{2m-1}(M)$. Here v_i denotes

the Wu class in dimension i. [21]. In general, if M is any manifold of dimension n, $v_i = v_i(M) \in H^i(M)$ is defined as that class such that

$$\langle v_i\, u, [M]\rangle = \langle Sq^i u, [M]\rangle, \quad \text{for all} \quad u \in H^{n-1}(M),$$

where $[M] \in H_n(M)$ is the fundamental class, and $\langle\ ,\ \rangle$ denotes evaluation. Two useful facts about these classes are

1) $w_i = \Sigma\, Sq^{i-j} v_j$, and $v_j = 0$ for $j > {}^n/2$.

In particular $w_1 = v_1$, $w_2 = v_2 + Sq^1 v_1$ etc. If $w_1 = 0$, then $w_2 = v_2$. Similarly if w_1 and $w_2 = 0$, $w_4 = v_4$. In the case above $w_1 = w_2 = w_4 = 0$, hence $v_1 = v_2 = v_4 = 0$ and the proposition is proved.

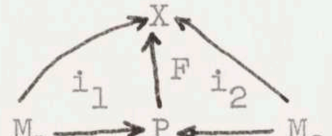
We now assume that all manifolds we are dealing with are connected. This is possible since one can change any manifold with a finite number of components by framed spherical modifications (∞-modifications) into a connected manifold. Under this modification disjoint union becomes connected sum [15]. The connected sum $M + M'$ of two n-manifolds M and N is obtained by embedding S^0 in their disjoint union $M \cup M'$ and replacing the normal bundle by $S^{n-1} \times I$, or equivalently, by removing a small n-disc from each manifold and identifying the boundary by a map of degree - 1.

From ϕ, we want to define a map from $\Omega_{2m}^{\langle 4\rangle} \to Z_2$. In order to do this, we need to know something about the way ϕ behaves with respect to cobordism, and have some way of calculating it. But ϕ of course depends on more than the space it is applied to. Being a cohomology operation, its value $\phi(u)$ depends on $u \in H^m(M)$. We want a cobordism theory to take this into account.

<u>Definition:1</u>: The 4-bordism groups of a space X, denoted by $\Omega_n^{\langle 4\rangle}(X)$ is the set of equivalence classes of pairs (M, f) where $M \in \langle 4\rangle^n$ and $f : M \to X$. The equivalence relation is given as follows: two pairs $(M_1 f_1)$, $(M_2 f_2)$ are bordant if there is a manifold $P \in \langle 4\rangle$ and a map $F : P \to X$ such that

1) $\partial P = M_1 \cup (-M_2)$ and P is a 4-cobordism

2) If $i_j : M_j \to P$ are the inclusion maps, the diagram



commutes.

Then $\Omega_n^{\langle 4\rangle}(X)$ is a group with operation of a disjoint union etc. For details see Conner-Floyd [9].

Now if we let $X = K(Z_2, n)$, a map $f : M \to X$ is just an n dimensional cohomology class of M. The next two lemmas show that ϕ defines a map

$\Omega_{2m}^{\langle 4 \rangle}(K(Z_2, m)) \to Z_2$, where $m = 8k + 3$, $k > 0$. (as above)

Lemma 2: Let $[M, u] \in \Omega_{2m}^{\langle 4 \rangle}(K(Z_2, m))$. Then there is a manifold M' and an element $u' \in H^m(M')$ such that $(M', u') \in [M, u]$, i.e. (M', u') is bordant to (M, u) and M' is 7 connected. Moreover there is a 4-cobordism N, v between (M, u) and (M', u') such that if

$i : M \to N$ and $j : M' \to N$ are the inclusion maps,

i_q^* is an isomorphism for $q > 8$ and

j_q^* is an isomorphism for $q < 2m - 8$.

Proof: M' is obtained from M by surgery, taking care to use only 4-modifications. N looks somewhat like $M \times I$, with cells of dimension 7 or less attached to kill off the homotopy of M dimensions ≤ 7. So N is $M \times I$ with some "handles" $D^i \times D^{2m-i+1}$ $0 < i < 8$ attached by maps $S^{i-1} \times D^{2m-i+1} \to M \times 0$. Hence, up to homotopy type N is $M \times I$ with i cells attached $(i < 8)$. Similarly N is M' with $2m-i$ cells $(i < 8)$ attached. The statement that the inclusion maps induce isomorphisms in the appropriate dimensions follows immediately from this description of N. Only necessary to check that the map $f : M \to K(Z_2, m)$ extends to a map $F : N \to K(Z_2, m)$. But f extends to $M \times I$ by $F(x, t) = f(x)$. To extend to N note that N is $M \times I$ with cells of dimension i attached, $i < 8 \Rightarrow i \neq m$.

Hence $F \mid (\partial$ of an attached cell) is an element of $\pi_1(K(Z_2, m)) = 0$ and so F extends over the cell. This completes proof of Lemma.

Lemma 3: The map $\phi : \Omega_{2m}^{\langle 4\rangle}(K(Z_2, m)) \to Z_2$ given by $\bar{\phi}([M, u]) = \phi(u)[M]$ is well defined, where $[M]$ denotes the fundamental cycle in $H_{2m}(M)$ (This lemma is just what we want. It says ϕ is some sort of bordism invariant.)

Proof: Lemma 2 says that any class ω in $\Omega_{2m}^{\langle 4\rangle} K(Z_2, m)$ contains a pair (M_1, u_1) where M_1 is 7 connected. Therefore $Sq^{m-3}H^m(M_1) \subset H^{2m-3}(M_1) = 0$ $Sq^{4,m-5}H^m(M_1) \subset H^{2m-1}(M_1)$ and $Sq^{2,4,m-6} : H^m(M_1) \to H^{2m}(M_1)$ is zero since $Sq^{2,4,m-6}H^m(M_1) \subset Sq^2H^{2m-2}(M_1) = w_2H^{2m-2}(M_1)$ where w_2 is second Stiefel Whitney class. But since $M \in \langle 4\rangle$ $w_2 = 0$. Hence if $u_1 \in H^m(M_1)$, $\phi(u_1)$ is defined. To show it is well defined, only need to know it is not dependent upon the representative of ω in $\Omega_{2m}^{\langle 4\rangle}K(Z_2, m)$, since the zero cobordism class is represented by $(M, 0)$ and $\phi(0) = 0$. Let $(M_2, u_2) \in \omega$ such that $\phi(u_2)$ is defined. Then (M_2, u_2) is bordant to $(M_1 u_1)$. Let N, v be a cobordism. If N is not 5-connected, we perform surgery on it, so we may assume N is 5 connected. We show $\phi(v)$ is defined, and $\phi(u_1)$ depends only on $\phi(v)$. Since $\phi(u_2)$ is defined, we have

$Sq^{m-3}u_2 = Sq^{4,m-5}u_2 = Sq^{2,4,m-6}u_2 = 0$. We wish to show corresponding relations for v. Let $j_i : M_i \to N$ be the inclusion. Then $j_i^*(v) = u_i$. The sequence $H^{2m-3}(N, M_2) \to H^{2m-3}(N) \xrightarrow{j_2^*} H^{2m-3}(M_2)$ is exact. By Poincaré duality $H^{2m-3}(N, M_2) \simeq H_4(N, M_1)$. But since N is 5 connected, and M_1 is 7 connected, $H_4(N, M_1)$ is 0. Therefore j_2^* is a monomorphism in dimension 2m-3. Then $0 = Sq^{2m-3}u_2 = Sq^{2m-3}j_2^*v = j_2^*Sq^{2m-3}v$. Hence $Sq^{2m-3}v = 0$. Exactly similar reasoning gives $Sq^{4,m-5}v = 0$ and $Sq^{2,4,m-6}v = 0$. Hence $\phi(v)$ is defined. Since $j_i^*(v) = u_i$ we have $j_i^*\phi(v) = \phi(u_i)$. But we have just seen that j_2^* is a monomorphism in dimension 2m (the proof of $Sq^{2,4,m-6}v = 0$ above) and j_1^* is an isomorphism in dimension 2m by lemma 2. Hence $\phi(u_1) = 0$ iff $\phi(u_2) = 0$, where = 0 means modulo the indeterminacy of ϕ. But M_1 and M_2 are <4> manifolds, and by Lemma 1 the indeterminacy of ϕ is 0. Hence the lemma is proved.

We now state a few lemmas which will help to calculate ϕ in certain cases.

We can express ϕ as a functional cohomology operation by factoring it as follows: Let $\Delta : H^q(X) \to H^q(X) \oplus H^q(X) + H^q(X)$ be the map $\Delta(u) = (u, u, u)$. Let

$\varepsilon : H^q(X) + H^q(X) + H^q(X) \to H^q(X)$ be the map $\varepsilon(u_1, u_2, u_3) = u_1 + u_2 + u_3$. These are defined for all spaces X and integers q. Let

$b : H^m(X) \oplus H^m(X) \oplus H^m(X) \to H^{2m-3}(X) \oplus H^{2m-1}(X) \oplus H^{2m}(X)$

be given by $b_m(u_1, u_2, u_3) = (Sq^{m-3}u_1, Sq^{4,m-5}u_2, Sq^{2,4,m-5}u_3)$

and $a : H^{2m-3}(X) \oplus H^{2m-1}(X) \oplus H^{2m}(X)$ by $a(v_1, v_2, v_3) = Sq^4v_1 + Sq^2v_2 + Sq^1v_3$. Let $\alpha = \varepsilon a$ and $\beta_m = b_m\Delta$. Then $\alpha\beta_m = \varepsilon ab_m\Delta = Sq^{m+1}$ (remember $m = 8k+3$ $k > 0$) and the relation $\alpha\beta_m = 0$ on classes in dimension m was what gave us ϕ.

Proposition: Let α and β_m be as above. Let $f : X \to Y$ and $u \in H^m(Y)$. Suppose that $f^*\beta_m(u) = 0$. We already know that $\alpha\beta_m(u) = 0$. Then the operations $\phi(u)$ and $\alpha_f\beta_m(u)$ are defined, and are equal modulo

$Sq^4H^{2m-4}(X) + Sq^2H^{2m-2}(X) + Sq^1H^{2m-1}(X) + f^*H^{2m}(X)$.

Proof: This is just theorem 5.2 of Adem.[1] It is basically the same as the formula of Peterson-Stein but in the case where α is an operation which takes several variables into one. For completeness, we define α_f. $f : X \to Y$ can be regarded as an inclusion by using the mapping cylinder. Then the exact cohomology sequence of the pair Y, X gives the following diagram.

$$\begin{array}{ccccccc}
 & & H^P(Y, X) & \xrightarrow{j^*} & H^P(Y) & \xrightarrow{f^*} & H^P(X) \\
 & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha \\
H^{k-1} & \xrightarrow{\delta} & H^k(Y, X) & \to & H^k(Y) & \to & H^k(X)
\end{array}$$

where $H^P = H^{k-4} \oplus H^{k-2} \oplus H^{k-1}$.

Given an element $(u_1, u_2, u_3) \in H^{k-4}(Y) \oplus H^{k-2}(Y) \oplus H^{k-1}(Y)$ such that $f^*(u_1, u_2, u_3) = 0$ and $\alpha(u_1, u_2, u_3) = 0$ define $\alpha_f(u_1, u_2, u_3)$ to be the set of all elements in $\delta^{-1}\alpha j^{*-1}(u_1, u_2, u_3)$, i.e. pull back along the dotted line. The vanishing of $\alpha(u_1, u_2, u_3)$ and $f^*(u_1, u_2, u_3)$ imply that this can be done.

<u>Lemma 4</u>: Let α, β_m be defined as above. Let $u \in H^m(X)$. Then $u : X \to K(Z_2, m)$. If $\phi(u)$ is defined, then so is $\alpha_u\beta(i_m)$ and $\alpha_u\beta(i_m) = \phi(u)$ modulo the indeterminacy of α_u.

<u>Proof</u>: The above proposition.

In the following we drop the m and write only β.

In general the indeterminacy of $\alpha_u\beta_m(i_m)$ is too large. We have already noted that the indeterminacy of ϕ is zero when it is applied to manifolds $M \in \langle 4 \rangle$. Denote by I_L the indeterminacy of $\alpha_u\beta_m(i_m)$. One of the methods we use will be to choose things so that $I_L = 0$ whenever possible. We need to know how ϕ behaves on products. Let α be as above.

Lemma 5. Let $f : X \to Y$ be a map of spaces. Let $u = (u_1, u_2, u_3) \in H^{p_1}(Y) \oplus H^{p_2}(Y) \oplus H^{p_3}(Y)$ $v \in H^q(Y)$ with $p_1 + 4 = p_2 + 2 * p_3 + 1$. Suppose $\alpha(u) = 0$, $Sq^1 v = Sq^2 v = Sq^4 v = 0$, and $f^*(u) = 0$. Then $\alpha_f(uv)$ is defined, and $\alpha_f(uv) = f^*(v)\alpha_f(u)$.

<u>Proof</u>: Let $H^p(Y) = H^{p_1}(Y) \oplus H^{p_2}(Y) \oplus H^{p_3}(Y)$

Then we have the commutative diagram

$$\begin{array}{ccccccc}
H^{p-1}(X) & \to & H^p(Y, X) & \to & H^p(Y) & \to & H^p(X) \\
\Big\downarrow f^*(v) & & \Big\downarrow v & & \Big\downarrow v & & \Big\downarrow f^*(v) \\
H^{p-1}(X) & \to & X^{p+q}(Y,X) & & H^{p+q}(Y) & & H^{p+q}(X)
\end{array}$$

where the horizontal lines are the cohomology sequence of the pair (Y, X) and vertical maps are multiplication by the element shown. Furthermore the operations α and multiplication by v commute, by Cartan formula since $Sq^1 v = Sq^2 v = Sq^4 v = 0$. Thus by applying α to above diagram, we get a three dimensional diagram, and chasing around it gives result. We will apply this in the case $Y = M \times N$, $X = K(Z_2, m) \times N$.

<u>Lemma 6</u>: Let $M \in \langle 4 \rangle^{16k+6}$ $N \in 4^{16j}$, $k > 0$, M, N 7 connected. Let $u \in H^{8k+3}(M)$, $v \in H^{8j}(N)$. Then $\phi(u \otimes v)$ is defined and $\phi(u \otimes v) = \phi(u) \otimes v^2$ modulo some indeterminacy which will come out in the proof.

<u>Proof</u>: Since M and N are 7 connected, so is $M \times N$, and

$$\phi(u \otimes v) = \alpha_{u \times v}\beta(i_{8(j+k)+3}) = \alpha_{u \times 1}\beta(i_{8k+3} \otimes v)$$

The first equality is of course Lemma 5, the second is naturality. Note an increase in indeterminacy at each step. Now apply Cartan formula and fact that N is 7 connected to get $\beta(i_{8k+3} \otimes v) = (\beta\, i_{8k+3}) \otimes v^2$.

So $\phi(u \otimes v) = \alpha_{u \times 1}(\beta\, i_{8k+3} \otimes v^2)$. But this is $\alpha_u\beta\, i_{8k+3} \otimes v^2 = \phi(u) \otimes v^2$, by lemma 5.

The indeterminacy is the indeterminacy of $\alpha_{u \times 1}$ which is $(u \times 1)^*H^{16k+6}(K(Z_2, 8k+3)) \otimes H^{16j}(N) = u^*H^{16k+6}(K(Z_2,(8k+3))) \subset A^{8k+3}H^{8k+3}(M)$ where A^{8k+3} denotes all the Steenrod operations of degree 8k+3, since $H^{16k+6}(K(Z_2, 8k+3)) = \{Sq^I i_{8k+3}$, and $I = (i_1, \ldots, i_k)$ with $\Sigma i_j = 8k+3$, with suitable restrictions on the $i_j\}$.

<u>Lemma 7</u>: Let N be as in Lemma 5, $M = S^3 \times S^3$, $M \in H^3(M)$ and $v \in H^{16j}(N)$ such that $v^2 = 0$. Then $\phi(u \otimes v)$ is defined, and $\phi(u \quad v) = 0$ modulo 0.

<u>Proof</u>: We have $Sq^{m-3}u \quad v = u \otimes v^2$ <u>$m = 8j + 3$</u>

$Sq^{4,m-5}u \otimes v = Sq^{2,4,m-6}u \otimes v = 0$. So $v^2 = 0$ implies $\phi(u \otimes v)$ is defined. Then by same argument as in lemma 6, $\phi(u \otimes v) = \phi(u) \otimes v^2 = 0$. The indeterminacy is 0 since

$A^3H^3(S^3 \times S^3) = 0$. In fact, all Steenrod operations on $S^3 \times S^3$ vanish.

We are now ready to define a map $\psi : \Omega_{2m}^{<4>} \to Z_2$, the Arf invariant. Let $\omega \in \Omega_{2m}^{<4>}$ and $M \in \omega$ (Recall $m = 8k+3$, $k > 0$). Since M is a manifold whose dimension is congruent to 2 mod 4, and M is orientable, the square of any element in $H^m(M)$ is 0, and $H^m(M)$ is even dimensional as a vector space over Z_2. Hence we can choose a basis $\{x_i, y_i \ i = 1, \ldots k\}$ for $H^m(M)$ with the following property: $x_i x_j = y_i y_j = 0$; $x_i y_j = 0$ iff $i \neq j$. Such a basis is called a symplectic basis for $H^m(M)$. We define

$\psi'(M)$ by $\psi'(M) = \sum_{i=1}^{k} \phi(x_i)[M]\cdot\phi(y_i)[M]$ where $\cdot$ indicates multiplication in Z_2, and $\psi(\omega)$ by $\psi'(M)$ where M is in ω. Since there is an m in ω which is 7-connected, we know there is an M such that $\psi'(M) = \psi(\omega)$ is defined. There are a few things to check to see that this definition makes sense. First that ψ' is independent of the choice of basis $\{x_i \ y_i\}$. This follows from the work of Arf, [4] since $\phi(x + y) = \phi(x) + \phi(y) + xy$, and the quadratic form $x, y \to xy$ is non singular (Poincaré Duality). So we need only check that ψ is independent of the representative chosen.

Proposition: ψ is well defined.

Proof: Suppose $\psi'(M_1)$ is defined. Then there is a 7 connected manifold M_2 which is 4-cobordant to M_1 obtained by surgery. By lemma 2 $j_i^* : H^m(N) \to H^m(M_i)$ is an isomorphism for each i. Hence a symplectic basis in $H^m(M_1)$ is carried by $j_2^* j_1^{*-1}$ into a basis for $H^m(M_2)$. Recall by proof of Lemma 3 j_2^* is a monomorphism in dimension 2m, and by Lemma 2, j_1^* is an isomorphism. So $j_2^* j_1^{*-1}$ takes a symplectic basis into a symplectic basis. ϕ is defined on all of $H^m(M_2)$ since M_2 is 7 connected, so $\psi'(M)$ is defined. By Lemma 3, $\psi'(M_1) = \psi'(M_2)$. So we may assume each representative 7-connected. Next claim ψ is additive with respect to addition in $\Omega_{2m}^{<4>}$.

Addition was originally defined by disjoint union, but since we are considering connected representatives, it is replaced by connected sum. It is clear that the connected sum is 4-cobordant to disjoint union, and so the group structure is the same. If we denote the connected sum of M_1 and M_2 by $M_1 + M_2$, we know that $H^m(M_1 + M_2) = H^m(M_1) \oplus H^m(M_2)$ and if $x \in H^m(M_1)$, $y \in H^m(M_2)$, then $xy = 0$ in $H^{2m}(M_1 + M_2)$. Then a symplectic basis for $M_1 + M_2$ can be given by $\{u_1, \ldots, u_i, u_{i+1} \ldots u_k; v_1, \ldots, v_i, v_{i+1}, \ldots v_k\}$ where $\{u_j, v_j\}$ $j \leq i$ are a symplectic basis for M_1 and $\{u_j, v_j\}$ $j > 1$ are a symplectic basis for M_2. Then we have, setting

$M = M_1 + M_2$

$$\psi'(M_1+M_2) = \sum_{j=1}^{k} \phi(u_j)[M]\phi(v_j)[M] = \Sigma\, \phi(u_j)[M_1+M_2]\phi(v_j)[M_1+M_2]$$

$$+ \sum_{j=i+1}^{k} \text{same thing.}$$ But $\phi(u_j)[M_1*M_2] = \phi(u_j)[M_1]$ if $j \leq i$

$\phi(u_j)[M_2]$ if $j > i$ and similarly for the v's. This says $\psi'(M_1+M_2) = \psi'(M_1) + \psi'(M_2)$. So to complete the proof we need only show that ψ is zero on the zero class. Let $M \in 0 \in \Omega_{2m}^{<4>}$, $N \in <4>$ such that $M = \partial N$, N is a 4-cobordism, N 1-connected. Sufficeth to show $\psi'(M) = 0$. This will be obvious if we choose a symplectic basis carefully. Let $u_1 \in H^m(M)$, $u_1 \neq 0$, $\delta^* : H^m(M) \to H^{m+1}(N, M)$, $j^* : H^m(N) \to H^m(M)$ the inclusion. If $\delta^*(u_1) = 0$, let x_1 be in $H^m(N)$ such that $j^*(x_1) = u_1$ and let $y_1 \in H^m(M)$ such that $y_1u_1 \neq 0$ (y_1 exists by Poincare duality). If $\delta^*(u_1) \neq 0$, then by Poincare duality there is an element $x_1 \in H^m(N)$ such that $x_1\delta^*(u_1) = 0$. But $x_1\delta^*(u_1) = \delta(j^*(x_1)\cdot u_1) \neq 0$. Hence $j^*(x_1)\cdot u_1 \neq 0$. Set $y_1 = u_1$. So we have elements $x_1 \in H^m(N)$, $y_1 \in H^m(M)$ with $j^*(x_1)\cdot y_1 \neq 0$. By using the same technique on the set $\{Z \in H^m(M) \mid Z\cdot y_1 = j^*(x_1)\cdot Z = 0\}$, we get elements x_2, y_2 with $j^*(x_1x_2) = y_1y_2 = 0$, $j^*(x_2)y_1 = j^*(x_1)y_2 = 0$ and

$j^*(x_2)y_2 \neq 0$. Proceeding similarly we get a symplectic basis $\{j^*(x_i), y_i\}$ for $H^m(M)$. The proof of Lemma 3 shows ϕ is defined. Then $\psi'(M) = \Sigma\, \phi(j^*(x_i))[M]\phi(y_i)[M]$. But $\phi(j^*(x_i)) = j^*\phi(x_i) = 0$ since $\phi(x_i) \in H^{2m}(N) = 0$. Therefore $\psi'(M) = 0$ and ψ is well defined. We drop the ' and denote both ψ' and ψ by ψ.

To complete the definition we define ψ on $\Omega_6^{<4>}$. $\Omega_6^{<4>} = \lim\limits_k \pi_{k+6}(MO<4>_k) = \lim\limits_k \pi_{k+6}(S^k)$, since $H_i(MO<4>_k) = 0$ $i < k;\ k < i < k + 8$ and $H_k(MO<4>_k) = Z$, and the Whitehead theorem. $\lim\limits_k \pi_{k+6}(S^k) = \pi_{14}(S^8) = Z_2$. It is not hard to see that $[S^3 \times S^3] \neq 0$ in $\Omega_6^{<4>}$. It follows from above that $\Omega_6^8 \to \Omega_6^{<4>} = \lim\limits_k \pi_{k+6}(S^k)$ is an isomorphism, and $[S^3 \times S^3]$ is not zero in Ω_6^∞. So $[S^3 \times S^3]$ generates $\Omega_6^{<4>}$. Define $\psi[S^3 \times S^3]$ to be 1.

Theorem 4: Let $M \in <4>^{16k+6}$ $m = 8k+3$, M m-1 connected, stably parallelizable. Then $\psi(M) = \Phi(M)$, where $\Phi(M)$ is the Kervaire invariant.

Proof: The proof is similar to that in [5].

Recall the characterization of Φ in [15]. $\Phi = \Sigma\theta(x_i)\theta(y_i)$ where $\{x_i y_i\}$ is a symplectic basis for M_1 and θ is the secondary operation $\theta : H^m(M) \to H^{2m}(M)$ with the following property. Let $f : S^m \to M$ be an embedding, and ν the normal bundle. Let $v \in H^m(M)$ be the cohomology class dual to the embedded sphere. Then $\Theta(v)$ is zero iff ν is trivial. So it sufficeth to show that $\phi(v) = 0$ iff ν is trivial. Let $M(\nu)$ be the Thom space of ν, and $U \in H^m(M(\nu))$ the Thom class. Then the map $g^* : H^*(M(\nu)) \to H^*(M)$ induced by the projection $g : M \to M(\nu)$ takes U into v, i.e. $g^*(U) = v$, and g^* is an isomorphism in dimension 2m. So we need only show that $\phi(U) = 0$ iff ν is trivial. If ν is trivial $M(\nu) = S^m v\, S^{2m}$, and hence $\phi(U) = 0$. If ν is non trivial $M(\nu) = S^m \cup_{[i,i]} e^{2n}$, where $i \in \pi_m(S^m)$ is the generator, and $[i, i]$ is the Whitehead square [5].

The folding map $S^m v\, S^m \to S^m$ extends to a map $h : S^m \times S^m \to M(\nu)$, since the obstruction to the extension is just $[i, i]$, which is 0 in $M(\nu)$. Let $x \in H^m(S^m)$ be the generator. Then $h^*(\phi(U)) = \phi(h^*(U)) = \phi(x \otimes 1 + 1 \otimes x)$ $= \phi(x \otimes 1) + \phi(1 \otimes x) + x \otimes x = x \otimes x \neq 0$. Hence $\phi(U) \neq 0$, and the theorem is proved.

CHAPTER III

THE COHOMOLOGY OF BO<4> AND MO<4>

In[27], Stong determined the cohomology of BO<r>. He proved the following proposition.

Proposition: $H^*(BO\langle 4\rangle)$ is a polynomial algebra on those classes $w_i \varepsilon H^i(BO\langle 4\rangle)$ such that $i-1$ has at least three ones in its dyadic expansion, i.e., such that $i \neq 2^\alpha + 2^\beta + 1$ for any α, β non-negative integers or $-\infty$.

Corollary 1: $w_i = 0$ if $i < 8$, and if $i = 9, 10, 11, 13, 17, 18, 19, 21, 25$, or 33.

Proof: There are no multiples of the generators in those dimensions.

Corollary 2: If i is any of the integers in corollary 1 then $H^i(BO\langle 4\rangle) = 0$.

The operation of the Steenrod algebra A on $H^*(BO\langle 4\rangle)$ is given by the Cartan formula

$Sq^n w_i w_j = \Sigma\, Sq^{n-k} w_i Sq^k w_j$ and the Wu formulas,

$$Sq^i w_j = \sum_{k=0}^{i} \binom{j-i-1+k}{k} w_{i+j-k} w_k .$$

where $\binom{a}{b}$ is binomial coefficient $a!/b!(a-b)!$ reduced mod 2.

Corollary 3: As a graded group $H^*(MO\langle 4\rangle)$ is isomorphic to $H^*(BO\langle 4\rangle)$ via the Thom isomorphism $H^*(BO\langle 4\rangle) \rightarrow H^*(MO\langle 4\rangle)$

given by $w_i \to w_iU$, where $U \in H^o(MO\langle 4\rangle)$ is the Thom class.

The structure of $H^*(M)\langle 4\rangle)$, which we now call H for convenience, as a module over the Steenrod algebra is quite different from that of $H^*(BO\langle 4\rangle)$, since $Sq^iU = w_iU$. Using this, it is possible by brute force to determine the structure of H as a module over A in low dimensions.

If $x,y,z,\ldots$ are elements of A, we denote by $A/A(x,y,z,\ldots)$ the quotient of A by the left ideal generated by the elements $x,y,z,\ldots$. A_i will denote the subalgebra of A generated by Sq^o, $Sq^1,\ldots,Sq^{2i}$. Then the following is true.

<u>Theorem 5</u>: In dimensions less than 55, H is the direct sum of cyclic modules over A of six different types. The list below gives each type, together with the dimension in which generators for it appear.

Type	dimension
A/AA_2	0 16 32(2 copies) 48(3 copies)
$A/A(Sq^1,Sq^5,Sq^6,Sq^{13})$	20 36(2 copies) 52(2 copies)
$A/A(Sq^1,Sq^9)$	40
$A/A(Sq^1,Sq^5)$	44
$A/A(Sq^2,Sq^2Sq^1)$	46(2 copies)
A/AA_1	48

We will study a module of the second type briefly at the end of the next chapter. The remainder of this chapter and most of the next will be denoted to a study of A/AA_2.

<u>Lemma 8</u>: Let A^* be the dual of A, $(A/AA_2)^*$ the dual of A/AA_2. A^* is a polynomial algebra on generators ξ_i in degree 2^i-1. $(A/AA_2)^*$ is the subalgebra of A^* generated by $\xi_1^8, \xi_2^4, \xi_3^2, \xi_i \quad i \geq 4$.

<u>Proof</u>: We must show that the annihilator of A_2 in A^* is precisely the subalgebra described above. This is just all those elements in A^* which are taken into 0 by an element of A_2 acting on the right. This action can be described as follows: Let $\phi^*: A^* \to A^* \otimes A^*$ be the diagonal map. Let $\xi^R = \xi_1^{r_1} \ldots \xi_k^{r_k}$, and $\phi^*(\xi^R) = \Sigma \xi^S \otimes \xi^T$. Then $\xi^R \tau = \langle \xi^S, \tau \rangle \xi^T$, where $\tau \in A$, and $\langle\ ,\ \rangle$ is evaluation. Furthermore the diagonal map ϕ^* in A^* is given by $\phi^*(\xi_i) = \xi_{i-j}^{2^i} \otimes \xi_j$. So in order for $\xi^R \tau \neq 0$ we must have $\langle \xi^S, \tau \rangle \neq 0$. Since A_2 is generated by Sq^1, Sq^2, and Sq^4, it suffices to find which elements of A^* are non-zero on these. But they are exactly those which have a ξ_1^4, a ξ_1^2, or a ξ_1 as the first factor in some term of their diagonal expansion. But these are just $\xi_1^k, \xi_2^m, \xi_3^n$, where $k \not\equiv 0 \bmod 8$, $m \neq 0 \bmod 4$ and $n \not\equiv 0 \bmod 2$. Thus the lemma is proved.

The Whitney sum of vector bundles induces a map $BO\langle 4\rangle \times BO\langle 4\rangle \to BO\langle 4\rangle$ just as for BO, which gives $H^*(BO\langle 4\rangle)$ the structure of a coalgebra over Z_2. Since A_2 is a sub-Hopf algebra of A, the diagonal map on A induces a coalgebra structure on A/AA_2. Then we have the following lemma.

Lemma 9: There is a monomorphism $v : A/AA_2 \to H$ such that $v(1) = U$.

Proof: Let $\tau \in A/AA_2$, $a \in A$ which represents τ. Define $v(\tau)$ to be aU. Let $b \in AA_2$. Then $b = xSq^1 + ySq^2 + zSq^4$, where $x,y,z \in A$. But $Sq^1U = Sq^2U = Sq^4U = 0$, and so v is well defined. By naturality, v is a map of coalgebras. Hence by proposition 3.9 of [24] v is a monomorphism if and only if it is a monomorphism on the primitive elements of A/AA_2. If ϕ is the diagonal map on A/AA_2, an element $\tau \in A/AA_2$ is primitive if $\phi(\tau) = \tau \otimes 1 + 1 \otimes \tau$. The primitive elements in A/AA_2 are just the duals of the indecomposable elements in $(A/AA_2)^*$. Let Q_i be the dual of ξ_{i-1}. Then Q_i, $i \geq 3$, is primitive in A/AA_2. In fact it is even primitive in A. For $i < 3$ Q_i is zero in A/AA_2. The only other indecomposable elements in $(A/AA_2)^*$ are $\xi_1^8, \xi_2^4, \xi_3^2$. These have duals Sq^8, $Sq^{4,8}$, and $Sq^{2,4,8}$, respectively. So we need to show that none of these are mapped by v into zero. Since they are all in different dimensions, the images are clearly independent.

$v(Sq^8) = Sq^8U = w_8U$. $v(Sq^{4,8}) = w_{12}U$, $v(Sq^{2,4,8}) = w_{14}U$.

We know $Q_i = Q_{i-1}Sq^{2^i} + Sq^{2^i}Q_{i-1}$. Therefore $v(Q_3) =$ $Sq^{1,2,4,8}U = w_{15}U$. Therest of the proof is by induction.

We show (1) $v(Q_i) = w_{2^i-1}U$ + (decomposable elements of $H^*(BO\langle 4\rangle)$)U.

The Cartan and Wu formulas imply that any Steenrod operation on a decomposable element gives a decomposable element.

We have $v(Q_3) = w_{15}U$. Suppose (1) holds for i less than k. Then $v(Q_k) = Q_{k-1}Sq^{2^k}U + Sq^{2^k}Q_{k-1}U = Q_{k-1}w_{2^k}U + Sq^{2^k}w_{2^k-1}U +$ (decomposables)U. Show $Sq^{2^k}w_{2^k-1}U$ cannot possibly have a term $w_{2^{k+1}}$, so we have $v(Q_k) = Q_{k-1}w_{2^k}U$ + (decomposables)U.

Now $Q_{k-1}w_{2^k}U = Q_{k-2}Sq^{2^{k-1}}w_{2^k}U$ + (decomposables)U = $Sq^1Sq^2 \ldots Sq^{2^{k-1}}w_{2^k}U$ + (decomposables)U. But $Sq^1, \ldots Sq^{2^{k-1}}w_{2^k} =$

$$\binom{2^k + 2^{k-1} + 2 - 1}{1} \cdots \binom{2^k + 2^{k-1} - 1}{2^{k-2} - 1}\binom{2^k - 1}{2^{k-1}} w_{2^{k+1}-1} +$$

(decomposables)U by iterated application of the Wu formula. But all the binomial coefficients above are 1, and therefore the Lemma is proved.

CHAPTER IV

$\mathrm{Ext}_A(H^*(MO\langle 4\rangle; Z_2); Z_2)$

In this chapter we compute $\mathrm{Ext}_A(A/AA_2, Z_2)$ and compute $\mathrm{Ext}_A^{s,t}(A/A(Sq^1, Sq^5, Sq^6, Sq^{13})$ for $t-s \leq 20$. This will give the E_2 term of the Adams spectral sequence up to dimension 40, since H is the direct sum of modules of the above form in dimensions less than 40.

$\mathrm{Ext}_A(A/AA_2, Z_2)$.

Liulevicius [17] has shown that $\mathrm{Ext}_A(A/AA_2, Z_2)$ is isomorphic to $\mathrm{Ext}_{A_2}(Z_2, Z_2)$, which is commonly called the cohomology of A_2, and denoted by $H(A)$, $H^*(A)$, or $H^{**}(A)$, depending on how one writes bigraded objects. We will use the usual grading, $H^{s,t}(A_2) = \mathrm{Ext}_A^{s,t}(A/AA_2, Z_2)$, where the grading in the Steenrod algebra is t, and s is the homological, or resolution degree. By dimension, or stem we mean t-s.

We use the techniques of Peter May [18], [19] to compute $H(A_2)$. In outline it goes as follows: 1. Define a filtration on A_2 such that the associated graded algebra E^0A_2 is a primitively generated Hopf algebra. 2. Compute $H^*(E^0A_2)$ by using the theorem of Milnor and Moore that a primitively generated Hopf algebra is isomorphic to the universal enveloping algebra of its restricted Lie algebra of primitive elements. 3. Use a spectral sequence to get from $H^*(E^0A_2)$ to $H^*(A_2)$. The process is extremely technical, and most of the proofs are deferred to the appendix.

<u>Corollary</u> (To theorem 7 to follow) $H^*(A_2)$, as an algebra over Z_2 is a free module over the polynomial ring $Z_2[\omega, \omega_2]$ where $\omega \in H^{4,8}(A_2)$ and $\omega_2 \in H^{8,56}(A_2)$.

<u>Definition</u>: A graded, restricted Lie algebra over Z_2 is a graded Lie algebra L over Z_2, together with a map $\beta: L \to L$ such that $[\beta(x),y]=[x,[x,y]]$, and $\beta(x+y) = \beta(x) + \beta(y) + [x,y]$, x,y in L where $[\ ,\ ]$ denotes the multiplication in L. If G is an associative algebra over Z_2, it can be made into a restricted Lie algebra G_L by the definitions $[g,h] = gh-hg$ and $\beta(g) = g^2$ for all $g, h \in G$. The universal enveloping algebra $V(L)$ of the restricted Lie algebra L is defined by the following universal mapping property: There is an associative algebra with identity $V(L)$ and a homomorphism of restricted Lie algebras $i: L \to V(L)_L$ such that if G is an associative algebra, with identity, and $f: L \to G_L$ a homomorphism of restricted Lie algebras, then there is a unique homomorphism $g: V(L) \to G$ such that $f = gi$.

Theorem (Birkhoff, Witt, Poincare) Let L be a restricted Lie algebra over Z_2. Order the elements of L in some way. Then a basis for $V(L)$ is the set of all monomials $u_{i_1} \ldots u_{i_k}$, where u_{i_j} is less than $u_{i_{j+1}}$ for all j, i.e., all monomials in elements of L providing the elements are written in increasing order. [13]

There is a map $\phi : V(L) \to V(L) \otimes V(L)$ given by $\phi(u) = u \otimes 1 + 1 \otimes u$ for $u \in L$, and $\phi(uv) = \phi(u)\phi(v)$. This map makes $V(L)$ into a Hopf algebra.

Proposition: Let F_p be the increasing filtration of A_2 defined by:

$F_p(A_2) = A_2$ if p is greater than or equal to 0.

$F_{-1}(A_2) = I(A_2)$, the elements of positive degree in A_2

$F_{-n}(A_2) = I(A_2)F_{-n+1}(A_2)$

This is commonly called the augmentation filtration, $I(A_2)$ the augmentation ideal. Let $E^o_{p,q} = E^o_{p,q}(A_2) = (F_p(A_2)/F_{p-1}(A_2))_{p+q}$ where the last subscript indicates grading in A_2. Let $E^o_r = \sum_{p+q=r} E^o_{p,q}$, and $E^o = \sum E^o_r$. Then:

1. E^o is a primitively generated, graded connected Hopf Algebra.

2. $E^o = A_2$ as a vector space over Z_2.

3. $E^o = V(P(E^o))$, where $P(E^o)$ is the restricted Lie algebra of primitive elements in E^o.

Proof: 1 and 2 are obvious, where connected means $E^o_o = Z_2$. 3 is just the theorem of Milnor-Moore quoted above (Theorem 6.11).

For the remainder of this chapter we use the Milnor basis for A. If $R = (r_1,\ldots,r_k)$ is a finite sequence of non-negative integers, let $\xi^R \varepsilon A^*$ be the element $\xi_1^{r_1} \xi_k^{r_k}$, and let $Sq(R)$ be the element dual to it in A.

Proposition: A basis for A_2 as a vector space over Z_2 is given by $Sq(r_1,r_2,r_3)$ where $r_1 < 8$, $r_2 < 4$, and $r_3 < 2$.

Proof: This follows immediately from lemma 7 of last chapter.

Proposition: $P(E^o) = \{Sq(R) \mid R$ has only one non-zero entry and this is a power of 2$\}$ i.e., $\{Sq(1), Sq(2), Sq(4), Sq(0,1)\ Sq(0,2)$ and $Sq(0,0,1)\}$.

Proof: The filtration on $E^o \otimes E^o$ is defined by $F_p(E^o \otimes E^o) = \sum_{i+j=p} F_i(E^o) \otimes F_j(E^o)$. It is clear that those elements above are primitive. That they are the only ones is not hard to check from the diagonal formula $\phi(Sq(R)) = \Sigma\, Sq(R_1) \otimes Sq(R_2)$ where the sum is over all sequences R_1, R_2 such that $R_1 + R_2 = R$(+ denotes componentwise addition).

Let $P(i, j) = Sq(R)$, where R has one non-zero entry 2^i in the jth component. Then the primitive elements of E^0 are just $P(i,j)$ for the pairs $(i,j) = (0,1), (1,1), (2,1), (0,2), (1,2)$, and $(0,3)$.

Proposition: As a restricted Lie algebra $P(E^0)$ has

1. Basis given above.
2. $[P(i,j), P(k,m)] = \delta_{i,k+m} P(k,j+m)$
3. $\beta(P(i,j)) = 0$.

Proof: Follows from the multiplication formulas in A_2.

Let L denote $P(E^0)$ as a graded, restricted Lie algebra over Z_2. The grading is given by $u \in L$ has degree $(0,t)$, where t is the degree of u in A_2.

Proposition A: Let $X = V(L) \otimes \Gamma(L)$, where $\Gamma(L)$ is the algebra of divided powers on L. Bigrade X by degree of $\gamma_r(u) = (r,rt)$, where $u \in L$ has degree $(0,t)$, and requiring that the degree of a product = sum of degrees of factors. Then there is an algebra structure on X, and a differential, such that X is a $V(L)$ free resolution of Z_2.

Proof: See Appendix.

Proposition B: There is a natural coalgebra structure on X, $D: X \to X \otimes X$ given by $D(ux) = \phi(u)D(x)$ if $u \in V(L)$ and $x \in \Gamma(L)$, where ϕ is the diagonal map in $V(L)$

and $D(\gamma_r(v)) = \Sigma\, \gamma_i(v) \otimes \gamma_{r-i}(v)$. The dual X^* of X is a $V(L)^*$-free resolution of Z_2, and $\overline{X}^* = Z_2 \otimes_{V(L)^*} X^*$ is a polynomial algebra on generators $R(i,j) = \gamma_1(P(i,j))^*$.

The differential in $\overline{X}^*$ is given by

$$\delta R(i,j) = \sum_{i=1}^{j-1} R(i+k,\ j-k)R(i,k)$$

Proof: See Appendix.

The elements $R(i,1)$ $i=0, 1, 2$; $R(j,2)$ $j=0, 1$; $R(0,3)$ and $R(0,2)R(1,2) + R(1,1)R(0,3)$ in $\overline{X}^*$ are cycles. Let h_i, α_j, β, γ denote their respective homology classes.

Theorem 6: The elements h_i, $i=0,1,2$, α_j, $j=1,2$; β, and γ generate $H(\overline{X}^*)$ and hence $H^{**}(E^0)$. There are 4 relations: $h_i h_{i+1} = 0$ $i=0, 1$, $h_2\alpha_0 = h_0\gamma$, $h_2\gamma = h_0\alpha_1$ and $\gamma^2 = \alpha_0\alpha_1 + h_1^2\beta$.

Proof: This is by inspection. It is clear that the above elements are cycles. For the relations note for example $\delta(R(i,2)) = h_i h_{i+1}$, $\delta(R(1,2)R(0,3)) = h_0\alpha_1 + h_2\gamma$, etc.

We are now ready to compute $H^{**}(A_2)$.

Proposition C: There is a spectral sequence whose E^2 term is $H^*(E^0 A_2)$ and which converges to $H^*(A_2)$. Furthermore, the differentials can be described as follows:

$\delta_2(h_1) = 0$, $\delta_2(\alpha_o) = h_1{}^3 + h_o{}^2h_2$, $\delta_2(\alpha_1) = h_2{}^3$;

$\delta_2(\beta) = h_1\alpha_1$, $\delta_2(\gamma) = h_oh_2{}^2$. $\delta_3 = 0$, and $\delta_4 = 0$ except

$\delta_4(\beta^2) = h_2\alpha_1{}^2$. $E^5 = E^\infty$.

Proof: See Appendix.

Using the above proposition $H^{**}(A_2)$ can be computed almost by inspection. E^2 has non-bounding cycles h_1, α_o^2, α_1^2, γ^2, $h_1\gamma$, β^2, $h_o\beta$, $h_2\beta$, $\gamma\alpha_1$; and these form a set of generators for the cycles, and for E^3. The only element above which is not obviously a cycle is $\alpha_1\gamma$, but $d_2(\alpha_1\gamma) = h_2^3\gamma + \alpha_1h_oh_2^2 = h_2^2h_o\alpha_1 + \alpha_1h_oh_2^2 = 0$. $E^3 = E^4$, and in passing from E^4 to E^5, β^2 is no longer a cycle, but β^4 and $h_1\beta^2$ are. Name the classes of the classes of these elements in $E^5 = E^\infty$ by

name		class of	grading	stem
h_i	=	$[h_i]$	$(1,2^i)$	2^i-1
c_o		$[h_1\gamma]$	$(3,11)$	8
ω		$[\alpha_o^2]$	$(4,12)$	8
d_o		$[\gamma^2]$	$(4,18)$	14
e_o		$[\alpha_1\gamma]$	$(4,21)$	17
g		$[\alpha_1^2]$	$(4,24)$	20
τ		$[h_o\beta]$	$(3,15)$	12
$\mathfrak{A}$		$[h_2\beta]$	$(3,18)$	15
$\mathfrak{A}_2$		$[h_1\beta^2]$	$(5,30)$	25
ω_2		$[\beta^4]$	$(8,56)$	48

Then we have the following theorem.

<u>Theorem 7</u>: $H^{**}(A_2) = \mathrm{Ext}_A(A/AA_2, Z_2)$ is generated (multiplicatively) by the elements above. The relations are generated by those below. The multiplicative structure in E^∞ is the same as that in $H^{**}(A_2)$ except for the relation $\mathfrak{A}\mathfrak{A}_2 = 0$ in E^∞, which becomes $\mathfrak{A}\mathfrak{A}_2 = g^2$ in $H^{**}(A_2)$. This is proved as Proposition D in the Appendix. The elements denoted by roman letters are in the image of the map $\mathrm{Ext}_A(Z_2, Z_2) \to \mathrm{Ext}_{A_2}(Z_2, Z_2)$, and multiplication by either ω or ω_2 is a monomorphism

Relations:

I Among the h's

$\underline{h_i\ h_{i+1}} = 0$; $h_1^{\ 3} = h_o^{\ 2}h_2$

$h_2^{\ 3} = 0 \qquad h_oh_2^{\ 2} = 0$

II Without h's

(a) $c_o \cdot x = 0$ where $x \neq h_1, \omega, \omega_2, \quad {}_2$.

(b) $d_o^{\ 2} = \omega g \qquad {}_2{}^2 = h_1^{\ 2}\omega_2$

$e_o^{\ 2} = d_og \qquad \tau^4 = h_o^{\ 4}\omega_2$

$g\tau = e_o \qquad e_o\tau = \quad d_o$

$\tau \ {}_2 = 0$

(c) $\tau^2 \quad + \quad {}_2d_o = 0$

$\mathcal{H}^3 = g \quad {}_2$

$\tau\mathcal{H}^2 = \mathcal{H}_2e_o$

III $h_2d_o = h_oe_o \qquad h_o^{\ 2}d_o = h_2^{\ 2}\omega$; $h_1d_o = h_o^{\ 2}$

$h_1e_o = h_oh_2$; $h_2e_o = h_og$

$h_1g = h_2^{\ 2}$, $h_2g = 0$

$h_1\tau = 0 \qquad h_2\tau = h_o$

$h_1\mathcal{H} = 0$

$h_o\mathcal{H}_2 = 0 \qquad h_2\mathcal{H}_2 = 0$, $h_1^{\ 2}\mathcal{H}_2 = h_o\tau\mathcal{H}$.

IV $\mathcal{H}\mathcal{H}_2 = g^2$

Table 1. shows the structure of $H^{s,t}(A_2)$ for $t-s$ less than 25.

It is possible to compute $\mathrm{Ext}_A(A/A(Sq^1, Sq^5, Sq^6, Sq^{13}), Z_2)$ in low dimensions by merely constructing a minimal resolution. It is a module over $H^{**}(A_2)$, and its structure is given for $t-s < 20$ by table 2.

<u>Corollary</u>: $\mathrm{Ext}_A^{s,\ s+18}(A/A(Sq^1, Sq^5, Sq^6, Sq^{13}), Z_2)$ is zero if $s \neq 3$, and Z_2 if $s = 3$. Furthermore the generator in $\mathrm{Ext}^{3,21}$ is in the image of h_2^2.

CHAPTER V

DIFFERENTIALS IN THE ADAMS SPECTRAL SEQUENCE FOR $\pi_*(MO\langle 4\rangle)$

In this chapter two non-zero differentials in the Adams' spectral sequence [0] for $\pi_*(MO\langle 4\rangle)$ are computed. Recall the Adams spectral sequence has $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(MO\langle 4\rangle), Z_2)$ and converges to ${}_2\pi_*(MO\langle 4\rangle)$, the quotient of $\pi_*(MO\langle 4\rangle)$ by its subgroup of elements of odd order. Serre [22] has shown that the standard theorems about homotopy are also true for ${}_2\pi$. In particular, the homotopy exact sequence of a fibre space is still exact, and the Whitehead theorem relating homology and homotopy still holds. In what follows π will always stand for ${}_2\pi$.

We compute $\pi_n(MO\langle 4\rangle)$ in low dimensions $(n<15)$ by using a postnikov system decomposition of $MO\langle 4\rangle$, and the known results on the stable homotopy of spheres in low dimensions [29]. Let k be large and set $S = S^k$, and $\pi_n(S) = \pi_n{}^S = \pi_{n+k}(S^k)$. Then we have the tower of fiber spaces

$$\begin{array}{ccc}
K(Z_2, 14) & \to & E^5 \\
 & & \downarrow \\
K(Z_2, 11) & \to & E^4 \\
 & & \downarrow \\
K(Z_2, 9) & \to & E^3 \\
 & & \downarrow \\
K(Z_2, 8) & \to & E^2 \\
 & & \downarrow \\
K(Z_2, 7) & \to & E^1 \\
 & & \downarrow \\
 & & E^0 = MO\langle 4\rangle
\end{array}$$

where E^i is a fibration over E^{i-1} with fibre an Eilenberg-MacLane space. We have $H^q(E^5) = 0$ if $q \neq 0$, $q < 16$, and $H^{16}(E^5, Z)$ has no 2 torsion, so by the Whitehead theorem $\pi_q(E^5) = \pi_q(S)$ if $q < 16$.

Theorem 8: In the Adams spectral sequence for $\pi_*(MO\langle 4\rangle)$, there are at least two non-zero differentials $d_2(\tau) = h_2\omega$ and $d_2(\aleph) = h_o d_o$, where the notation is that of chapter 4.

Proof: First we calculate $d_2(\aleph)$. Recall the structure of $E_2^{s,t}$ for $t-s = 13$, 14, and 15. For $t-s = 13$ it is zero, and there are three non-zero entries each for 14 and 15, given by $h_o^i d_o$ for 14, and $h_o^i \aleph$ for 15, $i = 0, 1, 2$. If there is a non-zero differential on $\aleph$, then it must be d_2 or d_3. If $d_2(\aleph) = h_o d_o$, then $E_3^{s,t+14}$ has one non-zero entry d_o, and hence $\pi_{14}(MO\langle 4\rangle) = Z_2$. If

$d_2(\varkappa) = 0$ and $d_3(\varkappa) \neq 0$ then we have d_o and $h_o d_o$ are not zero in E_3, and hence E_∞. Therefore we have $\pi_{14}(MO\langle 4\rangle) = Z_4$ since multiplication by h_o in E_∞ corresponds to multiplication by 2 in the homotopy.

Now look at the homotopy exact sequence for the fibre space $E^5 \to E^4$. We have

$$0 \to \pi_{15}(E^5) \to \pi_{15}(E^4) \to \pi_{14}(K(Z_2,14)) \to \pi_{14}(E_5) \to \pi_{14}(E_4) \to 0$$

But $\pi_{15}(E_5) = \pi_{15}(S) = Z_2 \oplus Z_{32}$, $\pi_{14}(E_5) = \pi_{14}(S) = Z_2 + Z_2$, and $\pi_{14}(E^4) = \pi_{14}(MO\langle 4\rangle)$ so we have

$$Z_2 \to Z_2 \oplus Z_2 \to \pi_{14}(MO\langle 4\rangle) \to 0$$

Therefore

$$\pi_{14}(MO\langle 4\rangle) = \text{either } Z_2 \text{ or } Z_2 \oplus Z_2.$$

By the above argument it must be Z_2, and therefore $d_2(\varkappa) = h_o d_o$. Now $d_2(h_o\varkappa) = h_o^2 d_o = h_2^2\omega$ and $h_o\varkappa = h_2\tau$, so $d_2(h_2\tau) = h_2^2\omega$, which implies $d_2(\tau) = h_2\omega$.

Corollary: The $E_3^{s,t}$ term of the Adams spectral sequence for $\pi_*(MO\langle 4\rangle)$, $t-s \equiv 6 \bmod 6$, can be described as follows: It is the module over the polynomial ring $Z_2[\omega, \omega_2, g^4]$ generated by the elements h_2^2, ωd_o, $\tau^2 d_o$, $d_o g^2$, $\varkappa^2 g^2$, and $\tau d_o g^3$, in dimensions 6, 22, 38, 54, 70 and 86 respectively.

Proposition

$\pi_i(MO\langle 4\rangle)$ is given as follows:

i	$\pi_i(MO\langle 4\rangle)$
0	Z
1	Z_2
2	Z_2
3	Z_8
4	0
5	0
6	Z_2
7	0
8	$Z \oplus Z_2$
9	$Z_2 \oplus Z_2$
10	Z_2
11	0
12	Z
13	0
14	Z_2 .

Proof: The homotopy sequences of the fibre spaces above and the Adams spectral sequence give the above groups without difficulty.

CHAPTER VI

PRODUCT FORMULAS FOR ψ

In this chapter we study the behavior of ψ on products, and use the results to prove the theorems on the vanishing of the Kervaire invariant. We need some facts about quadratic forms.

Let V be a vector space over Z_2, and Q a non-degenerate quadratic form on V. Q is anti-symmetric if $Q(x,x) = 0$ and $Q(x,y) = Q(y,x)$ for all x, y in V. A collection of subspaces $V_1, \ldots, V_n$ is mutually orthogonal if $x \in V_i$, $y \in V_j$ $i \neq j$ implies $Q(x,y) = 0$.

<u>Lemma</u>: Let V be a vector space over Z_2, Q a non-degenerate anti-symmetric quadratic form on V. Then V is even dimensional, and if $V_1, \ldots, V_n$ is a collection of mutually orthogonal subspaces which span V, a symplectic basis may be chosen for each V_i, such that the union of these bases forms a symplectic basis for V.

We will apply this to the case where V is $H^m(M)$, $m = 8k+3$, $M \in \langle 4 \rangle^{2m}$ and $Q(x,y) = xy$ (cup product). Since M is orientable, w_1 vanishes, and therefore the square of any m-dimensional class is zero. This and Poincaré duality imply that cup product is a non-degenerate quadratic form on $H^m(M)$.

The idea of the proof of the product formulae is simple. Given two manifolds M and N let $m = 1/2$ (dimension of $M \times N$). In all cases we consider m will be an integer. Decompose $H^m(M \times N)$ into mutually orthogonal subspaces. Suppose V' is such a subspace, and V' has a symplectic basis $\{x_i, y_i\}$ $i = 1,\ldots,k$, and $\phi(x_i) = 0$ for each i. Then $(M \times N)$ is completely determined by the orthogonal complement W of V' where $W = \{x \in V = H^m(M \times N) \mid xv = 0$ for all $v \in V'\}$.

$\psi(M \times N) = \Sigma\, \phi(w_i)[M]\phi(w_i')[M]$ where $\{w_i, w_i'\}$ is a symplectic basis for W. We call $\{w_i, w_i'\}$ an effective symplectic basis for $H^m(M \times N)$. A subspace W V is effective if it has an effective basis. Similarly the orthogonal complement of an effective subspace is an ineffective subspace. We find a small effective basis for $H^m(M \times N)$ with which we can compute.

<u>Theorem 9</u>: Let $N \in \langle 4\rangle^{16k}$, N 7-connected and $M = S^3 \times S^3$. Assume that $\chi(N)$ = Euler characteristic of N reduced mod 2 is 0. Then $\psi(M) = 0$.

<u>Proof</u>: Let $m = 8k+3$. Then $H^m(M) = H \oplus \hat{H}$ where

$H = H^3(S^3 \times S^3) \otimes H^{8k}(N)$ $\hat{H} = H^0(S^3 \times S^3) \otimes H^{8k+3}(N) \oplus H^6(S^3 \times S^3) \otimes H^{8k-3}(N)$.

First we show a symplectic basis for H is effective. It is clear that H and $\hat{H}$ are orthogonal, since $H^{16k-3}(N) = 0$.

Also the product of any two elements in the same summand of $\hat{H}$ is 0. Thus there is a symplectic basis $\{x_i, y_i\}$ for $\hat{H}$, where $x_i \in H^0(S^3 \times S^3) \otimes H^{8k+3}(N) = H^{8k+3}(N)$. We need to show $\phi(x_i)$ is defined, and equal to zero, with 0 indeterminacy. But $Sq^{8k}x_i \in H^{16k+3}(N) = 0$, $Sq^{4,8k-2}x_i \in H^{16k+5}(N) = 0$ etc. Moreover $\phi(x_i) \in H^{16k+6}(N) = 0$. The indeterminacy is also obviously zero.

Let $v_i \in H^i(N)$ be the Wu class. Then $v_{8k}u = u^2$ for all $u \in H^{8k}(N)$, and $v_i = 0$ if $i > 8k$. Let $x \in H^3(S^3)$ be a generator.

If $v_{8k} = 0$, then $u^2 = uv_{8k} = 0$ for all $u \in H^{8k}(N)$, and by Lemma 7, $\phi(1 \otimes x \otimes u) = \phi(x \otimes 1 \otimes u) = 0$, and therefore $\psi(M) = 0$. Suppose $v_{8k} \neq 0$, then $v_{8k}^2 = 0$. For $\Sigma\, Sq^i v_{16k-i} = Sq^{8k}v_{8k} = w_{16k}(N)$, the top dimensional Stiefel-Whitney class. But $w_{16k}(N) = \chi(N) = 0$ by hypothesis, and so $v_{8k}^2 = Sq^{8k}v_{8k} = 0$. By Poincare duality there is a class $v' \in H^{8k}(N)$ such that $v_{8k}v' \neq 0$. Let V = subspace of $H^{8k}(N)$ spanned by v_{8k} and v', and let W be its orthogonal complement. If $w \in W$, $w^2 = w \cdot v_{8k} = 0$ so $\phi \mid H^3(S^3 \times S^3) \otimes W$ is 0 by lemma 2. So an effective symplectic basis for M is

$\{1 \otimes x \otimes v_{8k},\ x \otimes 1 \otimes v_{8k};\ x \otimes 1 \otimes v',\ 1 \otimes x \otimes (v_{8k}+v')\}$.

But $\phi(1 \otimes x \otimes v_{8k}) = \phi(x \otimes 1 \otimes v_{8k}) = 0$ since $v_{8k}^2 = 0$. So $\psi(M) = 0$ and the theorem is proved.

<u>Theorem 10</u>: Let $M \varepsilon \langle 4\rangle^{16k+6}$, $N \varepsilon \langle 4\rangle^{16p}$, $k, p > 0$. M and N 7-connected. Let $n = 8k + 3$, $m = 8(k + p) + 3$. Suppose $A^q : H^{n-q}(M) \to H^N(M)$ is zero for q odd, and $A^q : H^{16p-q}(N) \to H^{16p}(N)$ is zero for q odd and $q > 8p$. (A^q denotes the elements of the Steenrod algebra of degree q.) Then $\psi(M \times N) = \psi(M)\chi(N)$. Compare [8] Theorem 1.6.

<u>Proof</u>: Let $\hat{H}_i = H^{n-i}(M) \otimes H^{8p+i}(N) \oplus H^{n+i}(M) \otimes H^{8p-i}(N)$ $i > 0$, and $H = H^n(M) \otimes H^{8p}(N)$. Then $H^m(M \times N) = H \oplus \Sigma H_i$, and all the summands are mutually orthogonal. First we show H is effective.

There are two cases to consider, i even and i odd. Consider i even. The hardest case is for i = 2. The proofs for larger i are analogous and easier. Since the product of any element in one summand of H_2 with another element in the same summand is 0, a symplectic basis $\{x_i, y_i\}$ for H_2 can be found such that

$x_i \varepsilon H^{n-2}(M) \otimes H^{8p+2}(N)$. We show ϕ is zero on that group. Let $u \varepsilon H^{n-2}(M)$, $v \varepsilon H^{8p+2}(N)$. Then $\phi(u \otimes v) =$

$$\alpha_{u x v}\beta(i_{8(k+p)+3}) = \alpha_{u x 1}\beta(i_{8k+1} \otimes v) =$$

$$\alpha_{u x 1}(Sq^{8(k+p)}, Sq^{4,8(k+p)-2}, Sq^{2,4,8(k+p)-3})(i_{8k+1} \otimes v)$$

$$= \alpha_{u x 1}(0, Sq^4 Sq^{8k} i_{8k+1} \otimes Sq^{8p-2} v; Sq^{2,4,8k-3} i_{8k+1} \otimes Sq^{8p-2} v)$$

by lemma 4, and fact that N is 7 connected and 16p dimensional.

But $Sq^{8p-2}v = Sq^2 Sq^{8(p-1)+4}v + Sq^1 Sq^{8(p-1)+4} Sq^1 v =$

$w_2\, Sq^{8(p-1)+4}v + w_1 Sq^1 Sq^{8(p-1)+4} Sq^1 v = 0$ since $N \in \langle 4 \rangle$.

So $\phi(u \otimes v) = 0$. Indeterminacy in above calculation is $(u \times 1)^* H^{2n}(K(Z_2,n-2)) \otimes H^{16p}(N) = u^* H^{2n}(K(Z_2,n-2)) \subset A^{n+2}H^{n-2}(M) = 0$, by hypothesis. So $\hat{H}_2$, and similarly $\hat{H}_{2i}$, is ineffective.

To prove H_i ineffective for i odd, it suffices to show for H_1, as proofs for other odd i are the same. Let $u \in H^{8k+4}(M)$, $v \in H^{8p-1}(N)$. By same argument as above, it is sufficient to show $\phi(u \otimes v) = 0$. Again apply lemma 4, to show that $\phi(u \otimes v) =$

$\alpha_{1xv}(Sq^{8k+2}u \otimes (Sq^{8p-2}i_{8p-1}, Sq^{4,8p-4}i_{8p-1}, Sq^{2,4,8p-5}i_{8p-1}))$.

But $Sq^{8k+2}u = Sq^2 Sq^{8k}u + Sq^1 Sq^{8k} Sq^1 u = 0$. So $\phi(u \otimes v) = 0$. Indeterminacy is $(1 \otimes v)^* H^{16k+6}(M) \otimes H^{16p}(K(Z_2 8p-1)) \subset A^{8p+1}H^{8p-1}(N) = 0$. So H is effective.

Let $v_{8p} \in H^{8p}(N)$ be the Wu class. If $v_{8p}^2 = 0$, then reasoning exactly as in proof of theorem 9, together with lemma 6 implies $\psi(M \times N) = 0 = \psi(M)\chi(N)$. So suppose $v_{8p}^2 \neq 0$. Let $U = \{x \in H^{8p}(N) \mid xv_{8p} = 0\}$. Then by Lemma 6, $H^n(M) \otimes U$ is ineffective. Let $\{x_i, y_i\}$ be a symplectic basis for $H^n(M)$. Then $\{x_i \otimes v_{8p}, y_i \otimes v_{8p}\}$ is an effective symplectic basis for $H^m(M \times N)$.

$$\psi(M \times N) = \Sigma\, \phi(x_i \otimes v_{8p})[M \times N]\phi(y_i \otimes v_{8p})[M \times N]$$

$$= \Sigma\, \phi(x_i)[M]v_{8p}^2[N]\phi(y_i)[M]v_{8p}^2[N]$$

$$(\Sigma\, \phi(x_i)[M]\, \phi(y_i)[M])v_{8p}^2[N] = \psi(M)\chi(N).$$

Corollary 1: Let $M \in \langle 4\rangle^{16k+6}$ $N \in \langle 4\rangle^{16p}$ $k,p > 0$. M stably parallelizable. M, N 7-connected. Then $\psi(M \times N) = \psi(M)\chi(N)$.

Proof: Same proof as theorem, except in the proof that $\hat{H}_i$ is ineffective for i odd. Instead of using $\phi(u \otimes v) = \alpha_{uxv}\beta(i) = \alpha_{1xv}\beta(u \otimes i)$ use $\phi(u \otimes v) = \alpha_{ux1}\beta(i \otimes v)$. Everything goes through, since M stably parallelizable implies all Steenrod operations into the top dimension vanish. Hence everything will be defined mod 0.

Corollary 2: Let $M \in \langle 4\rangle^{16k+6}$, $N \in \langle 4\rangle^{16p}$, N stably parallelizable. Then $\psi(M \times N) = \psi(M)\chi(N) = 0$.

Proof: N stably parallelizable implies all characteristic classes of M vanish, in particular $\chi(M) = 0$. The rest of the proof is as above.

In chapter 3 it was shown that there is a monomorphism $A/AA_2 \to H^*(MO\langle 4\rangle)$ given by $\alpha \to \alpha U$ for any $\alpha \in A/AA_2$, where $U \in H^0(MO\langle 4\rangle)$ is the Thom class, and that the image was an Amodule direct summand in dimensions less than 55. Now assume that it is an A module direct summand, so we have $H^*(MO\langle 4\rangle) = A/AA_2 \oplus K$ as A modules. Then

$Ext_A(H^*(MO\langle 4\rangle), Z_2) = Ext_A(A/AA_2, Z_2) \oplus Ext_A(K, Z_2)$.

The generator f of $\pi_o(MO\langle 4\rangle)$, corresponding to the cobordism class of a point induces a map $f^* : H^n(MO\langle 4\rangle_n) \to H^n(S^n)$ for n sufficiently large such that $f^*(U) = x$, where x is a generator of $H^n(S^n)$. The map f induces on homotopy is the map $\rho\Omega^{framed} \to \Omega^{\langle 4\rangle}$. Now f^* induces a map $Ext(f^*, 1) : Ext_A(Z_2, Z_2) \to Ext_A(H^*(MO\langle 4\rangle, Z_2)$ whose image lies in the summand $Ext_A(A/AA_2, Z_2)$. So every element in the E_2 term of the Adams spectral sequence for the homotopy of spheres is mapped into the first direct summand of the E_2 term of the Adams spectral sequence for the homotopy of $MO\langle 4\rangle$. It is not known whether every element in the image of ρ comes from this summand. But we can say much about those elements which do.

<u>Theorem 11</u>: As a module over the polynomial algebra $P = Z_2[\omega, g^4, \omega_2]$, $Ext_A^{s,t}(A/AA_2, Z_2)$ for $t-s \equiv 6 \bmod 16$ has 6 generators, h_2^2, ωd_o, $\tau^2 d_o$, $d_o g^2$, $\varkappa^2 g^2$, and $\tau d_o g^3$. See chapter 5 for details. Let G be the set of generators given above, and $P' = Z_2[\omega, g^4]$. Then

1. ψ is zero on all cobordism classes whose representatives in the E_2 term of the Adams spectral sequence for $\Omega_*^{\langle 4\rangle}$ lie in the P' module generated by G, except

the class of $S^3 \times S^3$, which is represented by h_2^2.

2. If $d_r(\omega_2) = 0$ for all $r > 5$, P' map be replaced by P.

The proof is many iterations of the proofs of the preceding two theorems. First we show ψ is zero on elements whose representatives lie in G, then apply Theorems 9 and 10 to give the result. The restriction in 2 is necessary in order to know that products in E_2 are products in $\Omega_*^{<4>}$. We need the following lemma.

Lemma 10: Suppose $M \varepsilon \omega \varepsilon \Omega^4$, and that ω is represented by an infinite cycle x in E_2 of the Adams spectral sequence, $x \varepsilon \omega \varepsilon \mathrm{Ext}_A^{s,t}(H^*(MO<4>), Z_2)$ with $s > 0$. Then $\chi(M) = 0$.

Proof: Let $r : \Omega_n^{<4>} \to \mathfrak{N}_n$ be the map induced by the covering $p_4 : BO<4> \to BO$. r is the map which takes the 4-cobordism class of a manifold into its ordinary unoriented cobordism class. The map $p_*: \mathrm{Ext}(H^*(MO<4>), Z_2) \to \mathrm{Ext}(H^*(MO), Z_2)$ carries x into 0, since $H^*(MO)$ is a free module over the Steenrod algebra and therefore $\mathrm{Ext}_A^{s,t}(H^*(MO), Z_2) = 0$ if $s > 0$. Furthermore, since everything in $\mathfrak{N}$ comes from something in filtration 0, and r cannot decrease filtration, we have $r(\omega) = 0$, i.e., M is unorientably cobordant to 0. Hence by [21] all Stiefel-Whitney numbers, and in particular

$w_n = \chi(M)$ are 0.

Corollary: Let $M \varepsilon \omega \varepsilon \Omega^{<4>}$, such that ω is represented in E_2 of the Adams spectral sequence by an element of the P module generated by G. Then $\chi(M) = 0$.

Proof of Theorem 11:

If $x \varepsilon Ext_A(H^*(MO<4>), Z_2)$ which is an infinite cycle in the Adams spectral sequence, let $[x] \varepsilon \Omega^{<4>}$ be the cobordism class it represents.

1. $\psi([\omega d_o]) = 0$. $[\omega d_o] = [\omega][d_o]$, since both are infinite cycles. $[\omega] \varepsilon \Omega_8^{<4>}$, $[d_o] \varepsilon \Omega_{14}^{<4>}$. By theorem 2 we can choose $M \varepsilon [d_o]$, $N \varepsilon [\omega]$ such that $H^q(M) \neq 0$ only if $q = 0, 7, 14$ and $H^q(N) \neq 0$ only if $q = 0, 4, 8$. Also all Steenrod operations in both N and M are zero. This is obvious from dimensional reasons and the fact that $Sq^4 : H^4(N) \to H^8(N)$ is multiplication by $v_4(N)$, which is 0 since $N \varepsilon <4>$. So we apply proof of theorem 2.

$H''(M \times N) = H^7(M) \quad H^4(M)$. Let $u \varepsilon H^7(M)$, $v \varepsilon H^4(N)$. Then $\phi(u \otimes v) = \alpha_{uxv}\beta(i_{11}) = \alpha_{1xv}\beta(u \otimes i_4) = \alpha_{1xv}(Sq^8, Sq^{4,6}, Sq^{2,4,5})u \otimes i_4 = \alpha_{1xv}(0) = 0$. Since the Steenrod operations are 0, the indeterminacy is 0 and $\psi([\omega d_o]) = 0$.

2. $\psi([\tau^2 d_o]) = 0$ $\tau^2 d_o$ is in dimension 38. The only possible non-zero differential on τ^2 is $d_3(\tau^2) = h_1 d_o \omega$.

But then $d_3(\tau^2 d_o) = h_1 d_o^2\omega = h_1\omega^2 g \neq 0$. So $[\tau^2 d_o] = [\tau^2][d_o]$. Let $M \varepsilon [d_o]$ as in 1). Let $N \varepsilon [\tau^2]$ with $H^q(N) = 0$ unless $q = 0, 8, 12, 16, 24$. Then $H^{19}(M \times N) = H^7(M) \otimes H^{12}(N)$. Let $u \varepsilon H^7(M)$, $v \varepsilon H^{12}(N)$. Then $\phi(u \otimes v) = \alpha_{uxv}\beta(i_{19}) = \alpha_{ux1}(\beta(i_7 \otimes v)) = \alpha_{ux1}((Sq^4 i_7, Sq^{4,2} i_7, Sq^{2,4,1} i_7) \otimes v^2 = \upsilon(u) \otimes v^2$, where υ is the secondary operation associated to the relation $Sq^4Sq^4 + Sq^2(Sq^4Sq^2) + Sq^1(Sq^2Sq^4Sq^1) = 0$. The indeterminacy is 0 since all Steenrod operations in M vanish. Then the same argument as in proof of theorem 10 shows that $\psi(M \times N) = \chi(N)(\Sigma\upsilon(x_i)[M]\upsilon(y_i)[N])$ where $\{x_i, y_i\}$ is a symplectic basis for M. But $\chi(N) = 0$, by lemma above, since $[\tau^2]$ is in filtration 6.

3. $\psi([d_o g^2]) = 0$. This is in $\Omega_{54}^{\langle 4 \rangle}$. Both d_o and g are infinite cycles, so $[d_o g^2] = [d_o][g][g]$. Let $M \varepsilon [d_o]$ as above, $N' \varepsilon [g]$ such that $H^q(N') = 0$ unless $q = 0, 8, 10, 12, 20$. Let $N = N' \times N'$. Then $M \times N \varepsilon [d_o g^2]$, $H^{27}(M \times N) = H^7(M) \otimes H^{20}(N)$ and by the same argument as case 2, $\psi([d_o g^2]) = 0$.

4. $\psi[\alpha^2 g^2]) = 0$. The only possibly non-zero differential on α^2 is $d_3(\alpha^2) = h_1\omega g$. If so, then

$d_3 \varkappa^2 g^2 = h_1 \omega g^3 \neq 0$. Choose $N \varepsilon [g^2]$ as above.
Choose $M \varepsilon [\varkappa^2]$ such that M has odd dimensional cohomology in dimension 15. This can be done by theorem 2, since $H^q(MO\langle 4\rangle) = 0$ for q odd and less than 15. Then $H^{35}(M \times N) = H^{15}(M) \otimes H^{20}(N)$. Let $u \varepsilon H^{15}(M)$, $v \varepsilon H^{20}(N)$, then $\phi(u \otimes v) = \alpha_{uxv}\beta(i_{35}) =$
$\alpha_{1xv}\beta(u \otimes i_{20}) = \alpha_{1xv}(u^2 \otimes Sq^{17} i_{20}, u^2 \otimes Sq^{4,15} i_{20}, u^2 \otimes Sq^{2,4,14} i_{20})$
$= \alpha_{1xv}(0)$ since $u \varepsilon H^{15}(M)$ implies $u^2 = 0$. Now $[g]$ is in image of ρ, hence stably parallelizable, and therefore all Steenrod operations into the top dimension are zero, hence indeterminacy is 0.

5. $\psi([\tau\, d_o g^3]) = 0$. $[\tau\, d_o g^3] \varepsilon \Omega_{86}^{\langle 4\rangle}$. d_o and g are infinite cycles, but $d_2(\tau) = h_2\omega$. Therefore $d_2(\tau g) = h_2 \omega g = 0$. The only other possible non-zero differential on τg is $d_6(\tau g) = \omega^2 h_1 d_o$. If that be so, then $d_6(\tau\, d_o g^3) = \omega^2 h\, d_o{}^2 g^2 = \omega^3 h_1 g^3 \neq 0$. So we have $[\tau\, d_o g^3] = [\tau g][d_o g^2]$. But $[\tau g]$ is in 32 stem $[d_o g^3]$ in the 54 stem, moreover by theorem 10, since $[d_o g^2]$ is in image of ρ, $\psi(\tau\, d_o g^3) = \psi[d_o g^2]\chi[\tau g]) = 0$. To complete the proof of the theorem we apply theorems 9 and 10. Theorem 1 shows that $\psi([h_2^2 p]) = 0$ where $p \varepsilon P$. Note

that any element of G has the property that a Steenrod operation from an odd dimension into the top dimension is 0. Representatives for $[\omega^2]^k$ and $[g^4]^k$ can be chosen with no odd dimensional cohomology, so Theorem 2 implies 1. To get 2 we need only to show $A^q H^{48-q}(N) = 0$ where $N \in [\omega^2]$ and q odd $q > 24$.

By Theorem 2, N can be chosen to have non-zero odd dimensional cohomology only in dimensions 15, 23, 24 and 33. Thus it suffices to show $A^{25}H^{23}(N) = A^{33}H^{15}(N) = 0$. But this is true since $H^{33}(BO\langle 4\rangle) = H^{25}(BO\langle 4\rangle) = 0$.

Theorem 12. Let $n = 22$ or 38. Then

$\Phi : \Omega_n^{\text{framed}} \to Z_2$ is 0.

Proof: We show that $\psi : \Omega_n^{\langle 4\rangle} \to Z_2$ is 0 on manifolds in the image of $\Omega_n^{\infty} \to \Omega_n^{\langle 4\rangle}$.

1. $n = 22$. There are 2 elements in $\Omega_{22}^{\langle 4\rangle}$, represented by ωd_o and $h_2^2 x$, where x is the element in $H^{0,0}(A_2)$ corresponding to the summand of $H^*(MO\langle 4\rangle)$ which is isomorphic to A/AA_2 and begins in dimension 16. We have already shown that $\psi[\omega d_o] = 0$. It is sufficient to show that $[h_2^2 x]$ cannot contain a stably parallelizable manifold. But the map $\Omega^{\infty} \to \Omega^{\langle 4\rangle}$ cannot decrease filtration, and the element $[h_2^2 x]$ is in filtration 2, and there are no elements in Ω_{22}^{∞} in filtration less than 4.

2. $n = 38$.

Let x be as above, y and y' be the elements corresponding to generators of the summands of $H^*(MO\langle 4\rangle)$ which start in dimension 32, and z the generator of the summand which starts in dimension 20. Then there are 6 elements in $\Omega_{38}^{\langle 4\rangle}$ to consider.

$[\tau^2 d_0]$, $[\omega^3 d_0]$, $[x\omega d_0]$ $[h_2^2 y]$; $[h_2^2 y']$ and $[h_2^2 \gamma_0 z]$.

By theorem 11, ψ is zero on the first two. By Theorem 9 $\psi(x\omega d_0] = \psi(\omega d_0)\chi(x) = 0$, and by Theorem 10, $\psi[h_2^2 \gamma_0 z] = \chi(\gamma_0 z)$ which is zero by Lemma 10. So we have only $[h_2^2 y]$ and $[h_2^2 y']$ to consider. These are both in filtration 2, and hence would have to be in the image of something in filtration 2 or less. There is one element in Ω_{38}^{∞} in filtration 2, $[h_3 h_5]$. But the map $\mathrm{Ext}_A(Z_2 Z_2) \to \mathrm{Ext}_{A_2}(Z_2 Z_2)$ on the E_2 term of the Adams spectral sequence sends h_3 into 0, hence $h_3 h_5$ goes into 0. Therefore $[h_2^2 y]$ and $[h_2^2 y']$ are not in the image of Ω^{∞}, and the theorem is proved.

APPENDIX

In this chapter the four propositions of chapter 4 are proved. The first three are contained in May's work [18], [19], and are included only for completeness. The last is a consequence of the work of Liulevicius [17], and gives the multiplicative structure in $H^{**}(A_2)$.

Let G be a Z_2-module. Recall the definition of the algebra of divided powers $\Gamma(G)$ on G. $\Gamma(G)$ has generators $\gamma_t(x)$ for each x in G and each non-negative integer t, subject to the relations

$$\gamma_0(x) = 1 \quad \text{for all } x.$$

$$\gamma_r(x)\gamma_x = \binom{r+s}{x}\gamma_{r+s}$$

$$\gamma_t(x+y) = \sum_{r+s=t} \gamma_r(x)\gamma_s(y) \qquad [30].$$

<u>Proposition A</u>: Let $L = P(E^0A_2)$, the graded restricted Lie algebra of primitive elements in E^0A_2, and $V(L)$ its associated enveloping algebra. Let $\Gamma(L)$ be the algebra of divided powers on L, and $X = V(L) \otimes \Gamma(L)$. With a bigrading, algebra and coalgebra structure, and differential as defined as below, X is a free $V(L)$ resolution of Z_2.

<u>Grading</u>: For any $u \in L$, assign degree $(0, t)$, where t is the degree of u in A_2. This induces a grading on $V(L)$, by setting grading uv = sum of gradings of u and v,

since the Birkoff-Witt-Poincare theorem says that monomials in the elements of L are a Z_2 basis for V(L). Let the grading of $\gamma_r(u)$ be (r,rt), where t is degree of $u \in A_2$, and again require that the grading of a product be the sum of the gradings.

Multiplication: Give V(L) and $\Gamma(L)$ their natural algebra structures, and subject the tensor product to only the following relations:

$$\gamma_1(u)v = v\gamma_1(u) + \gamma_1([v,u])$$

$$\gamma_{2n}(u)v = v\gamma_{2n}(u) + \gamma_1(u)\gamma_1([v,u])\gamma_{2(n-1)}(u)$$

$$\gamma_0(u) = 1$$

for all $u, v \in L$.

Diagonal map: Define $D : X \to X \otimes X$ by

$D(b,x) = \phi(b)D(x)$ where $b \in V(L)$, $x \in \Gamma(L)$ and ϕ is the diagonal map on V(L).

$D(\gamma_r(u)) = \gamma_i(u) \otimes \gamma_{r-i}(u)$ if $u \in L$ and D is a homomorphism on $\Gamma(L)$.

Differential: Define $d : X \to X$ by

$d(bx) = bd(x)$ for $b \in V(L)$, $x \in L$.

$d(\gamma_1(u)) = u$ for any $u \in L$,

$d(\gamma_{2n}(u)) = u\gamma_1(u)\gamma_{2(n-1)}(u)$, and

d is a derivation on (L).

Proof: The map $\varepsilon : X \to Z_2$ given by $\varepsilon(1) = 1$ and $\varepsilon(x) = 0$ for any $x \in X$, $x \neq 1$ is clearly an augmentation.

So we need only show $dd = 0$ and X is acyclic. To show $d^2 = 0$, it is enough to show it is zero on generators. $d^2(u) = 0$, $d^2\gamma_1(x) = d(x) = 0$ clearly.

$d^2\gamma_{2n}(x) = d(x\gamma_1(x)\gamma_{2(n-1)}(x)) = x\cdot x\gamma_{2(n-1)}(x)$

$+ x\gamma_1(x)x\gamma_1(x)\gamma_{2(n-2)}(x) = 0 + xx\gamma_1(x)\gamma_1(s)\gamma_{2(n-2)}(x)$

$+ x\gamma_1([x,x])\gamma_1(x)\gamma_{2(n-2)}(x) = 0$, since $[x,x] = 0$ in L, and xx is 0 in $v(L)$.

To complete the proof we need only show X is acyclic. Consider the filtration F_p defined on X by

1. $F_p(X) = \sum_{r+s=p} F_r(v(L)) \otimes F_s(\Gamma(L))$

2. F_p on $v(L)$ given by

$F_1(v(L)) = Z_2 \cup L$ (i.e. identity and elements of L)

$F_p(v(L)) = (F_1(v(L))^p \qquad p > 1$

and on $\Gamma(sL)$ by $\gamma_{r_1}(x_1) \ldots \gamma_{r_m}(x_m) \in F^q$ if $r_1+\ldots+r_m \leq q$. Clearly $d(F^q) \subset F^q$, so d induces a differential d_o on X_o the associated graded algebra. But X_o is just $V(L^a) \otimes \Gamma(L^a)$ where L^a is the abelian restricted lie algebra on the vector space L, i.e. $[u,v] = u$, and $\beta(u) = 0$ for all $u,v \in L^a$. This follows from definitions above and from the theorem of Milnor-Moore [24] which states that the associated graded algebra to $v(L)$ with the above filtration is $V(L^a)$. If X_o is acyclic, so is X. So we need only show X_o is acyclic. To show this construct a contracting homotopy, i.e., a map $x_o : X_o \to X_o$ such that $s_o d_o + d_o s_o = I + \varepsilon_o$, where $\varepsilon_o : X_o \to Z_2$ is the augmentation. Since L^a is abelian $L^a = L_1 \oplus \dots \oplus L_k$, where L_i are one dimensional restricted lie algebras and $X_o = X_1 \otimes \dots \otimes X_k$. Since X_i is isomorphic to X_j for all i and j, it is sufficient to show 1) there is a contracting homotopy on X_1, and 2) given a contracting homotopy on $X_2 \otimes \dots \otimes X_k$, we can extend it to X_o. Let $Y = X_2 \otimes \dots \otimes X_k$, $d_1 : X_1 \to X_1$ the differential, $s_1 : X_1 \to X_1$ the contracting homotopy $\varepsilon_1 : X_1 \to Z_2$ the augmentation.

d_1 and s_1 are defined by

$$d_1(1) = 0 \qquad s_1(1) = 0$$

$$d_1(u) = 0 \qquad s_1(u) = \gamma_1(u)$$

$$d_1(\gamma_1(u)) = u \qquad s_1(\gamma_1(u)) = 0$$

$$d_1(u\gamma_1(u)) = 0 \qquad s_1(u\gamma_1(u)) = \gamma_2(u)$$

$$d_1(\gamma_{2n}(u)) = u\gamma_1(u)\gamma_{2(n-1)}(u) \qquad s_1(u\gamma_1(u)\gamma_{2n}(u) = 0$$

for $u \in L_1$, the generator. Note d_1 is the same as $d_o | X_1 \otimes Z_2 \otimes \dots \otimes Z_2$. Clearly $d_1 s_1 + s_1 d_1 = 1 + \varepsilon_1$. Now suppose $\varepsilon_2 : Y \to Z_2$ is the augmentation d_2 the differential induced by $d_o : X_o \to X_o$, and s_2 the contracting homotopy for Y. Define

$$s_o = s_1 \otimes 1 + \varepsilon_1 \otimes s_2.$$

Note $d_o = d_1 \otimes 1 + 1 \otimes d_2$.

Then $d_o s_o = d_1 s_1 \otimes 1 + d_1 \varepsilon_1 \otimes s_2 + s_1 \otimes d_2 + \varepsilon_1 \otimes d_2 s_2$

$s_o d_o = s_1 d_1 \otimes 1 + \varepsilon_1 d_1 \otimes s_2 + s_1 \otimes d_2 + \varepsilon_1 \otimes s_2 d_2$

adding, and noting that $\varepsilon_1 d_1 = d_1 \varepsilon_1 = 0$ we have

$d_o s_o + s_o d_o = (d_1 s_1 + s_1 d_1) \otimes 1 + \varepsilon_1 \otimes (d_2 s_2 + s_2 d_2)$

$= (1+\varepsilon_1) \otimes 1 + \varepsilon_1 \otimes (1+\varepsilon_2) = 1 \otimes 1 + \varepsilon_1 \otimes \varepsilon_2$

$= 1 + \varepsilon_o$.

Hence X_o is acyclic, and the proposition is proved.

Proposition B: Let X^* be the dual of X, $X^* = V(L)^* \otimes \Gamma(L)^*$, and $\overline{X}^* = Z_2 \otimes_{V(L)^*} X^*$. X^* is a free $V(L)^*$ resolution of Z_2, and $\overline{X}^*$ is a polynomial algebra on generators $R(i,j) = (\gamma_1(P(i,j)))^*$. The differential in $\overline{X}^*$ is given by $\delta(R(i,j)) = \sum_{k=1}^{j-1} R(i+k,j-k)R(i,k)$.

Proof: Everything except the last statement follows from Proposition A and the fact that the dual of an algebra of divided powers with the natural coalgebra structure (i.e., that which it has) is a polynomial algebra.

Grading X^* the same as X, we have that $\delta(R(i,j))$ must have grading $(2,t)$ for some t. So the only possible things it could be non-zero on are $\gamma_2(u)$ or $\gamma_1(v)\gamma_1(v)$.

$\delta(R(i,j))(\gamma_2(u)) = R(i,j)(u\gamma_1(u)) = 0.$

$\delta R(I,j)(\gamma_1(u)\gamma_1(v) = R(i,j)d(\gamma_1(u)\gamma_1(v)) = R(i,j)(u\gamma_1(v) + \gamma_1(u)v)$

$= R(i,j)(\gamma_1([u,v])$. This is non-zero iff $[u,v] = P(i,j)$.

Set $u = P(k,\chi)$ $v = P(m,n)$. Then $[u,v] = \delta_{k,m+n}P(M,\chi+n)$.

Therefore $k=m+n$, $m=i$, and $\chi+n = j$. Solving these we get the formula above.

Proposition C: The proof of proposition C is divided into two parts, the first setting up the spectral sequence, the second calculating differentials.

Proposition CI: There is a filtration $\overline{F}$ of the reduced Bar construction on A_2, $\overline{B}(A_2)$ such that $\overline{\mathbf{F}}$ gives rise to a spectral sequence $\{E^r\}$ such that

1. $\{E^r\}$ converges to $H_{**}(A_2)$, the homology of A_2

2. $E^1 \doteq \overline{B}(E^0A_2)$ as a differential graded Z_2 module

and hence

3. $E^2 = H_{**}(E^0A_2)$, the homology of E^0A_2.

4. The dual spectral sequence $\{E_r\}$ is obtained by dualizing everything above and

5. $\{E_r\}$ converges to $H^{**}(A_2)$ the desired algebra.

6. E_2 is isomorphic to $H^{**}(E^0A_2)$, which we have already computed.

We will use the homology spectral sequence only to calculate differentials.

Proof: First we define the filtration on $\overline{B}(A)$. Let F_p be the augmentation filtration on A_2 defined above. An element $[a_1|a_2|\ldots|a_n]$ is in $\overline{F}_p$ if $a_i \in F_{p_i}$, $p_i \leq -1$, and $\sum_{i=1}^{n} p_i + n = p$. This filtration gives rise in the usual way to a spectral sequence $\{E^r\}$. Furthermore the filtration is finite in each degree, and hence the spectral sequence converges to $H_{**}(A_2)$. Let $\overline{E}^o = \overline{F}_p/\overline{F}_{p-1}$, to distinguish it from E^o. The differential in the bar construction is given by:

$$d[a_1|\ldots|a_n] = a_1[a_2|\ldots|z_n] + \sum_{i=1}^{n} [a_1|\ldots|a_i a_{i+1}|\ldots a_n]$$

Therefore $d(F_p) \subset F_p$, hence $d_o = 0$ and $\overline{E}^o = E^1$.

To prove 3, note that tensor product over Z_2 is an exact functor. The sequences $0 \to F_p(A_2) \to A_2 \to A_2F_p(A_2) \to 0$ are split exact as Z_2 modules, and the filtration on $\overline{B}(A_2)$ is the augmentation filtration on each factor. Hence $E^1 \cong \overline{B}(E^oA_2)$ as graded Z_2 modules, and comparing the differentials one sees that they are the same. Hence 3 is proved. The rest follows by dualizing.

To calculate the differentials in the cohomology spectral sequence, we dualize to the homology spectral sequence, embed X in $\overline{B}(A_2)$ and compute the differentials. Then dualize back to the cohomology spectral sequence.

In order to compute we need an embedding of X in $\overline{B}$. Define the shuffle product in $\overline{B}$ as follows:

$$[a_1|\ldots|a_m]*[a_{m+1}|\ldots|a_{m+n}] = \sum_{\pi} [a_{\pi(1)}|\ldots|a_{\pi(m+n)}]$$

where the sum is taken over all permutations π of the integers $1, \ldots, n+m$ such that if $1 \leq i < j \leq m$ or $m+k \leq i < j \leq m+n$, then $\pi(i) < \pi(y)$. This is called an (m,n) shuffle.

Proposition. There is a monomorphism of differential algebras $\sigma : \overline{X} \to \overline{B}(A_2)$ where $\overline{X} = \Gamma(P(E^0A_2)) = Z_2 \otimes_{V(L)} X$ such that

1. $\sigma(\gamma_r(u)) = [u|\ldots|u]$, r factors

2. $\sigma(xy) = \sigma(x)*\sigma(y)$ where $u \in P(E^0A_2)$, $x,y \in \overline{X}$

3. With the natural coalgebra structure on $\overline{B}$, σ is a map of coalgebras.

This is theorem 18 of [19].

Lemma: We can trigrade E^1 by $E^1_{p,q,t}$ where an element $[a,|\ldots|a_n]$ has degree $[p,q,t]$ if

1. $\sum_{i=1}^{n}$ (degree of $a_i \in A_2$) $= t$

2. $\sum_{i=1}^{n}$ (filtration degree of $a_i \in E^0A_2$) $= q$.

$p+q=n$.

Note that $p \leq 0$ always, since the filtration degree is ≤ -1 for each element $a \in I^oA_2$ except 1. Furthermore, the generators of $E_2 = H^{**}(E^oA_2)$ are in the following trigradings:

	p	q	t	
h_i	0	1	2^i	i=0,1,2
α_i	-2	4	$2(2^{i+1}-1)$	i=1,2
β	-4	6	15	
γ	-2	4	9	

Proof: Just look at the definitions of the elements.

$\delta_r : E_r^{p,q,t} \to E_r^{p+r,\ q+1-r,t}$, so the following corollary is immediate.

Corollary: $\delta_r(h_i) = 0$ for all i, $\delta_{2i+1} = 0$ for $i > 0$.

Proposition CII:

$$\delta_2(\alpha_o) = h_1^3 + h_o^2h_2$$

$$\delta_2(\alpha_1) = h_2^3$$

$$\delta_2(\beta) = h_1\alpha_1$$

$$\delta_4(\beta^2) = h_2\alpha_1^2$$

and all other differentials are 0.

Proof: The computations are long and messy, and are all in May's thesis, so we give a sample computation.

(a) $\delta_2(\alpha_o) = h_1{}^3 + h_o{}^2 h_2$,

$\alpha_o \; \varepsilon \; E_2^{-2,4,6}$; $h_1{}^3, h_o{}^2 h_2 \; \varepsilon \; E^{0,3,6}$; $E^{-2,4,6} \to E^{0,3,6}$

so everything is in the right dimension.

$\alpha_o{}^*$ is represented by $\gamma_2(p(0,2))$; $h_1{}^*$ by $\gamma_1(p(1,1))$

$h_o{}^*$ by $\gamma_1(p(0,1))$ and $h_2{}^*$ by $\gamma_1(\rho(2,0))$.

Imbedding in $\overline{B}(A_2)$ we have

$\sigma(\gamma_2(p(0,2))) = [p(0,2)|p(0,2)]$

$\sigma(\gamma_1(R_{i,j})) = [p(i,j)]$.

So $(h_o{}^2 h_2)^*$ is represented by

$x = [p(2,1)]*[p(0,1)|p(0,1)]$ and $(h_1{}^3)^*$ by

$y = [p(1,1)|p(1,1)|p(1,1)]$.

$dx = [p(0,2)p(1,1)|p_1{}^o] + [p(0,1)|p(0,2)p(1,1)]$ and

$dy = [p(1,1)p(1,1)|p(1,1)]$.

Now consider the chain $u \; \varepsilon \; \overline{B}(A_2)$, $u = [p(2,1)]*[p(0,1)|p(0,1)]$

Then $du = [p(0,2)|p(0,2)] + [p(0,1)|p(1,1)p(0,2)] +$ [(
$[p(1,1)p(0,2)|p(0,1)]$. So in $\overline{B}(E^oA)$ we have

$[p(0,2)|p(0,2)] = d[p(2,1)]*[p(0,1)|p(0,1)]$ and therefore

$d_2(h_o{}^2 h_2)^* = \alpha_o{}^*$. Similarly we get $d_2(h_1{}^3)^* = \alpha_o{}^*$, and the other boundaries in E^2, and this gives the δ_2 above.

Since $\delta_3 = 0$, we calculate δ_4. For dimensional reasons, $\delta_4(Z) = 0$ for $Z \; \underline{\varepsilon} \; E_4$ except β^2. For example $\alpha_o{}^2 \; \varepsilon \; E_4^{-4,8,12}$ and $\delta_4 \alpha_o{}^2 \; \varepsilon \; E_4^{0,5,12} = 0$.

$$\beta^2 \ \varepsilon \ E_4^{-8,12,30} \quad \text{and} \quad \delta_4\beta^2 \ \varepsilon \ E_4^{-4,9,30}$$

$h_2\alpha_1{}^2 \ \varepsilon \ E_4^{-4,9,30}$, and it is the only element. The same computation as above shows $\delta_4(\beta^2) = h_2\alpha_1{}^2$. Once again dimensional arguments give $\delta_{2i} = 0$ if $i > 2$.

<u>Proposition D</u>: In $H^{**}(A_2)$, $\mathcal{H}\mathcal{H}_2 = g^2$.

<u>Proof</u>: Note first that this is possible. In the E_∞ term of May's spectral sequence, the above elements are represented by $\mathcal{H} \ \varepsilon \ E_\infty^{-4,7,18}$, $\mathcal{H}_2 \ \varepsilon \ E_\infty^{-8,13,30}$, and $g^2 \ \varepsilon \ E_\infty^{-16,24,48}$. Now $\mathcal{H}\mathcal{H}_2 = 0$ in E_∞, but it is 0 in filtration 12, and hence it is possible that in H^{**} $\mathcal{H}\mathcal{H}_2 = g^2$. Since the only elements in $H^{8,48}(A_2)$ are g^2 and 0, it follows that it is sufficient to show $\mathcal{H}\mathcal{H}_2 \neq 0$.

Let B be the two sided ideal in A_2 generated by Sq^i for $i = 1, 2$. We construct a map $\hat{j}: H^{**}(A_2) \to H^{**}(B)$, and show that $j(\mathcal{H})j(\mathcal{H}_2) = j(g^2) \neq 0$. B is a Hopf sub-algebra of A_2, and the quotient A_2/B is the module A_0''. As a vector space this quotient is isomorphic to $Z_2 \oplus Z_2$, with generators 1 and Sq^4, and it has the obvious structure as an A_2 module. We have the exact sequence of A_2 modules $0 \to Z_2 \xrightarrow{i} A_0'' \xrightarrow{j} Z_2 \to 0$, where i is multiplication by Sq^4, and j is the augmentation. Applying the functor $\mathrm{Ext}_{A_2}(\ ,Z_2)$ to this exact sequence, we get a long exact sequence

$$\xrightarrow{\partial} \mathrm{Ext}_{A_2}(Z_2,Z_2) \xrightarrow{i} \mathrm{Ext}_{A_2}(A_o'', Z_2) \xrightarrow{j} \mathrm{Ext}_{A_2}(Z_2, Z_2) \xrightarrow{\partial}.$$

Liulivicius [17] has shown that $\mathrm{Ext}_{A_2}(A_o'') \simeq \mathrm{Ext}_B(Z_2,Z_2) = H^{**}(B)$, and in the same paper he computed $H^{**}(B)$. The map $\hat{j}:H^{**}(A_2) \to H^{**}(B)$ given by the composition

$$H^{**}(A_2) = \mathrm{Ext}_{A_2}(Z_2,Z_2) \xrightarrow{j^*} \mathrm{Ext}_{A_2}(A_o'',Z_2) \xrightarrow{\approx} \mathrm{Ext}_B(Z_2,Z_2) = H^{**}(B)$$

is the same as that induced by the inclusion of B into A_2. Hence it is a ring homomorphism.

In [17] it is shown that there is an element k in $H^{1,6}(B)$ such that $k^r \neq 0$ for any positive integer r. By constructing minimal resolutions for over A_2 for Z_2 and A_o'', and lifting the map $\hat{j}$, one finds that $\hat{j}(\alpha) = k^3$, $\hat{j}(\alpha_2) = k^5$, and $\hat{j}(g^2) = k^4$. Hence the result follows.

TABLE 1

$Ext_{A_2}^{s,t}(Z_2, Z_2)$ for $t-s<25$

s \ t-s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
12																									↑
11																				$h_0^2h_2\omega^2$					ω^2c_0
10																			$h_1^2\omega^2$	$h_0h_2\omega^2$	↑		$h_2^2\omega^2$		
9																		$h_1\omega^2$		$h_2\omega^2$			$h_0\omega d_0$	$h_0^2\omega\partial\ell$	↑
8																	ω^2	$h_1\omega c_0$			$h_0\omega\tau$		ωd_0	$h_0\omega\partial\ell$	
7									↑			$h_1^3\omega$	↑				ωc_0	$h_0^2h_2d_0$			$\omega\tau$			$\omega\partial\ell$	$h_0\tau_2$
6											$h_1^2\omega$	$h_0h_2\omega$			$h_2^2\omega$			$h_0h_2d_0$			h_0g^2				τ^2
5									$h_0\omega$	$h_1\omega$	[redacted]	[redacted]$h_2\omega$	$h_0^2\tau$		h_0d_0	$h_0^2\partial\ell$		h_2d_0	$h_0h_2\partial\ell$		h_0g	$h_2^2\partial\ell$			
4									ω	h_1c_0		[redacted]	$h_0\tau$		d_0	$h_0\partial\ell$		e_0	$h_2\partial\ell$		g				
3	↑			$h_0^2h_2$					c_0				τ			$\partial\ell$									
2			h_1^2	h_0h_2			h_2^2																		
1	h_0	h_1		h_2																					
0	1																								

Table 2

$Ext_A^{s,t}(A/A(Sq^1, Sq^5, Sq^6, Sq^{13}), Z_2)$ for $t-s < 20$.

s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
8	↑																↑ α_2 ↑			
7	h_0^7														$h_1^2\alpha_0\alpha_1$		β_2			
6	h_0^6								↑				↑	$h_1\alpha_0\alpha_1$	$h_1^2\alpha_0\beta_1$		↑			
5	h_0^5								↑ $h_0\alpha_1$				↑ $\alpha_0\alpha_1$	$h_1\alpha_0\beta_1$			$h_0\gamma_1$			
4	h_0^4				↑				α_1 $h_0\beta_1$				$\alpha_0\beta_1$				γ_1			
3	h_0^3				$h_0^2\alpha_0$		$h_1^2\alpha_0$		β_1				↑		$h_1^2\gamma_0$	$h_0h_2\gamma_0$		$h_2\varphi_0$	$h_2^2\gamma_0$	
2	h_0^2				$h_0\alpha_0$	$h_1\alpha_0$	$h_1\beta_0$						$h_0\gamma_0$	$h_1\gamma_0$	φ_0	$h_2\gamma_0$				
1	h_0				α_0	β_0							γ_0							
0	Z																			
s / t−s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

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BIOGRAPHY

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