PROTOCOLS FOR MULTI-ACCESS CHANNELS WITH CONTINUOUS ENTRY AND NOISY FEEDBACK

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ABSTRACT

Protocols for the accessing of a slotted multi-access channel have been presented in the literature. Recently, Hajek and Van Loon [2] have introduced a class of protocols which maintain system stability under a rather large class of restricted feedback. Modifications of their protocols are presented which maintain system stability under an even larger class of restricted feedback. Specifically, we consider the situation in which users observe the channel output through a discrete memoryless channel (DMC). We show as long as the DMC has non-zero capacity there exist algorithms maintaining stability.

In addition, we show that the throughput of any first come first served protocol is not above one-half, where throughput is the long term average rate of successful transmissions over the channel.

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Chapter One

Introduction

We are concerned with the problem of geographically separated sources (stations) sharing a common communication channel. In many situations it is impractical to share the channel capacity by classical methods such as frequency division multiple access (FDMA) where each source is allocated a fixed portion of the frequency spectrum in which it is allowed to transmit, and time division multiple access (TDMA) where each source is allocated a fixed subset of time during which it is allowed to transmit. For example, consider a satellite channel that is to be shared among a very large number of earth stations. Fixed allocation schemes such as TDMA and FDMA may be impractical for a variety of reasons. In many applications, each source in fact has no data to transmit most of the time, which results in inefficient utilization of channel capacity. Secondly, it may be that since the number of sources is so large, only a small amount of channel capacity can be allocated to each source. This can lead to unacceptable message delays when a source does in fact have data to transmit. Another, perhaps most important reason why these fixed allocation schemes may be inappropriate is that it may be incorrect to assume that the sources can agree a priori upon how the communication resources are to be divided among all the sources. For example, TDMA may be impossible because the sources have no way to agree on which subsets of time are to be allocated to which sources; FDMA could have an analogous problem. This type of situation arises when the number of sources is a random function of time. That is, it may be that new sources are continually entering the system, and that sources may cease to be functional. Thus there is a need for more suitable access schemes which will circumvent the problems mentioned above.

We will focus our attention on the following model for the communication channel to be shared and for the sources that share it. Time is divided into slots of equal length. All message lengths are such that it takes one slot of time to transmit a message. We will assume that there are an infinite number of users, each of which generates at most one message during its lifetime. In this way, we can focus on the more fundamental aspects of sharing the channel. We will assume that generation times of messages from all sources from a Poisson point process with intensity λ messages/slot. When two or more sources transmit a message in the same slot, the messages, "collide" and all messages involved in the transmission must be retransmitted at a later time. When exactly one source transmits a message in a given slot, we assume that the message is received at its destination error-free and hence need not be transmitted again.

A number of types of feedback have been considered in the literature for the above model. The term slotted "Aloha" channel has been used for the above channel when the only feedback available to the sources is acknowledgements of their own successful transmissions. A common access algorithm for this channel is known as the "Aloha" algorithm [3] which can be described simply as follows. When a source retransmits a message and does not receive an acknowledgement, that source retransmits that message in subsequent slots with a fixed probability f until the source transmits the message and hears an acknowledgement of his successful transmission. The Aloha algorithm is known to be unstable in the sense that eventually a large backlog of messages to be retransmitted will cause the probability of a successful transmission to approach zero. However, if messages are permanently rejected from the system after a finite number of retransmissions the system is stable and has maximum throughout $1/e \approx .368$, where throughout is defined as the expected long-term rate of successful transmission of messages over the channel [4].

A different feedback model has been considered in the literature which can be described as follows. In addition to the acknowledgement feedback available in the Aloha channel, each source can determine at the end of each slot whether that slot was empty (no transmissions, denoted by "0"), that slot had a successful transmission (denoted by "1"), or there was a collision in that slot (denoted by "e"). We will term this type of feedback as (0, 1, e) feedback. With this additional feedback there are known access algorithms that maintain stability. One particular access algorithm, the Gallager-Humblet algorithm [6], has throughout .488, which is the highest known throughput for this channel model. In [9]- [14] upper bounds are derived for the throughput of such systems. The lowest known upper bound is about .587.

For the same channel model Hajek and Van Loon [2] introduced multiaccess protocols similar to the simple Aloha protocol but different in that they maintain stability [1] by using the (0, 1, e) feedback to steer the traffic intensity toward an optimum level. Specifically, they allow the retransmission probability for sources to depend on past (0, 1, e) feedback.

In chapters 2 and 3 we consider the following model. Number the slots

1,2,3..... Define $\tilde{\tilde{z}}_k = 0, 1, 2$ according to whether in slot k there were no attemtped transmissions, exactly one attempted transmission, or more than one attempted transmission respectively. At the end of slot k all stations observe the value of z_k where z_k is the output of a discrete memoryless channel (DMC) with input $\tilde{\tilde{z}}_k$. This type of model has been considered in [7] and [8]. For a specific form of the error probability matrix P describing the DMC, Ryter [7] devised and analyzed a modification of the algorithm in [6].

The type of algorithm introduced by Hajek and Van Loon [2] is particularly well suited to this type of noisy feedback. We show that there exist stable algorithms of this type when $\lambda < 1/e$ as long as the DMC has non-zero capacity. In chapter 2 we analyze the performance of the algorithm in [2] when P is of a certain form. In Chapter 3 we describe how the algorithm analyzed in Chapter 2 can be modified to maintain stability for any P that describes a DMC with non-zero capacity.

In Chapter 4 we consider the situation in which the stations do not have access to the same feedback. Specifically, each station observes the output of a DMC with input $\tilde{\tilde{z}}_k$ but the DMC's for any distinct stations are independent. In addition we assume that any station observes the output of the channel only after the arrival of a message to be transmitted (continuous entry). We conjecture that there are algorithms that maintain stability for this type of feedback, and we describe an algorithm which we believe is stable.

In Chapter 5 we consider algorithms that transmit messages in the order they were generated (we assume all stations are able to observe the $\tilde{\tilde{z}}_k$'s directly). For example the Gallager-Humblet algorithm [5], [6] is of this type and has a maximum throughput of about .488. We show the throughput of any algorithm of this type cannot be greater than ½.

CHAPTER TWO

A Stable Algorithm When P is of a Certain Form

2.1 Introduction

In this chapter, we assume that in addition to the acknowledgement feedback mentioned in Chapter 1 each station observes z_k at the end of slot k, where z_k is the output of a discrete memoryless channel (DMC) with input $\tilde{z}_k$. More specifically

$$\text{Prob}\,(z_k = j \mid \tilde{z}_k = i) = p_{ij} \qquad i,j = 0,1,2 \tag{1}$$

Define

$$P = \{p_{ij}\} \tag{2}$$

Define N_k to be the number of packets waiting to be retransmitted at the end of slot k. We will show that as long as $\lambda < e^{-1}$ and the capacity of the DMC described by P is non-zero, there exist algorithms that the stations can execute such that $E[N_k]$ remains bounded over k. We describe a method of analyzing a class of algorithms that maintain stability in the sense just mentioned. Specifically, for this class of algorithms we can compute numbers $\bar{N}$, K, and η such that

$$\limsup_k E[N_k] \leq \bar{N} \tag{3}$$

and

$$\limsup_k \text{Prob}(N_k \geq b) \leq K e^{-\eta b} \tag{4}$$

In section 2.2 we present an explicit algorithm that maintains stability when the channel error probability matrix P is of the form

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} \\ 0 & p_{11} & 1-p_{11} \\ 0 & p_{21} & 1-p_{21} \end{bmatrix} \tag{5}$$

with $p_{00} > 0$. In addition we describe a procedure for explicitly computing $\bar{N}, K, \eta$ satisfying (3) and (4) for the algorithm presented when P is of the form (5).

2.2 A Stable Algorithm When P Satisfies (5)

2.2.1 Description of Algorithm

The algorithm presented here is essentially identical to that presented by Hajek and Van Loon [2], the stability of which is established in Hajek [1].

We now describe the algorithm. When a packet is generated at a station, that station transmits the packet in the slot immediately following the slot in which it was generated. If the packet collides then the packet is retransmitted in subsequent slots, with proability f_k in slot k, until the packet is successfully transmitted. That is, if a station has a packet to be retransmitted it will decide whether or not to retransmit in slot k by flipping a biased coin that lands heads with proability f_k and tails with probability $1-f_k$. If the coin lands heads then the station retransmits the packet in slot k. If the packet collides, or if the coin lands tails, the station repeats the same procedure in slot k + 1 flipping heads with proability f_{k+1}. The f_k's are generated according to the recursive equation

$$f_{k+1} = \min\{\beta,\ f_k \cdot a^{\gamma}(Z_k)\} \tag{6}$$

where a is a function, $a:\{0,1,2\} \longrightarrow \mathbb{R}^+$

$$\gamma > 0$$

$$0 < \beta < 1$$

Intuitively the function a is chosen so that when the channel is experiencing a "high" rate of collisions the f_k's will decrease, and when there are many empty slots the f_k's will increase. γ controls the rate at which changes take place and should be small when λ is close to e^{-1}. When N_k is "large" or f_k is "small" the number of attempted transmissions in slot k is approximately Poisson with mean $\lambda + N_k f_k$ (see Lemma 8). Thus for N_k large or f_k small we have

$$\text{Prob}(\tilde{Z}_k = 1 \mid N_k, f_k) \approx (\lambda + N_k f_k)\exp(-\lambda - N_k f_k) \tag{7}$$

The expression on the right side of (7) is maximized for $N_k f_k = 1-\lambda$.

Define

$$\varphi_k = \ln((N_k \vee 1) f_k) \tag{8}$$

$$\varphi^* = \ln(1-\lambda) \tag{9}$$

$$m_k = \varphi_k - \varphi^* \tag{10}$$

where $(a \vee b) = \max(a,b)$.

We now motivate how $a(\cdot)$ is chosen so that $N_k f_k$ drifts toward $1-\lambda$, or equivalently so that m_k drifts toward zero. Note that

$$E[m_{k+1} - m_k | N_k, f_k] = E[\ln \frac{N_{k+1} \vee 1}{N_k \vee 1} | N_k, f_k] + E[\ln \frac{f_{k+1}}{f_k} | N_k, f_k] \tag{11}$$

If N_k is large enough it can be easily seen that the first term on the right side of (11) can be made to be negligible with respect to the second term. If f_k is small enough the min in (6) will not be taken by β. Hence for N_k large enough, f_k small enough we have

$$E[m_{k+1} - m_k | N_k, f_k] \approx \gamma m(\varphi_k) \tag{12}$$

where

$$m(\varphi_k) = (e^{-G}, Ge^{-G}, 1-(1+G)e^{-G}) \cdot (P\underline{c})^T \tag{13}$$

$$G = \lambda + e^{\varphi_k} \tag{14}$$

$$\underline{c} = \begin{pmatrix} \ln a(0) \\ \ln a(1) \\ \ln a(2) \end{pmatrix} \tag{15}$$

T denotes transpose.

Based on (7), (12) we choose the function $a(\cdot)$ or equivalently $\underline{c}$ so that $m(\varphi_k) = 0$ for $\varphi_k = \varphi^*$, $m(\varphi_k) < 0$ for $\varphi_k > \varphi^*$, and $m(\varphi_k) > 0$ for $\varphi_k < \varphi^*$. Note that this holds if

$$P\underline{c} = \begin{pmatrix} e-1 \\ -1 \\ -1 \end{pmatrix} \tag{16}$$

A sufficent condition (neccessary if P^{-1} exists) is

$$\underline{c} = \begin{pmatrix} \frac{e}{p_{00}} - 1 \\ -1 \\ -1 \end{pmatrix} \tag{17}$$

This yields

$$a(i) = \begin{cases} e^{(\frac{e}{p_{00}} - 1)} & ; i = 0 \\ e^{-1} & ; i = 1 \\ e^{-1} & ; i = 2 \end{cases} \tag{18}$$

We also make the choice

$$\beta = 1/2 \tag{19}$$

Throughout section 2.2 we assume that $a(\cdot), \beta$ in (6) are given by (18) and (19). The choice of $P_{\underline{c}}$ in (16) was made somewhat arbitrarily to simplify expressions used in the proof. Other choices for $P_{\underline{c}}$ (or equivalently other choices for $a(\cdot)$) might lead to better performance of the algorithm.

We will show that the intuitive reasoning that lead us to choose the values of $a(\cdot)$ above was correct, and that for all $\lambda < e^{-1}$, γ sufficiently small insures stability of the algorithm. This could be shown by arguments in Hajek [1] which show the stability of the algorithm presented in Hajek and Van Loon [2]. We choose an alternative approach, suggested also by Hajek in [1]. We commence our approach in section 2.2.3. In section 2.2.2 we repeat for reference results from Hajek [1] which we will use in our proof.

2.2.2 Results from Hajek [1]

Let $(Y_K)_{K\geq 0}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$ adapted to an increasing sequence $(\mathcal{F}_K)_{K\geq 0}$ of sub-σ-fields of $\mathcal{F}$ -- thus Y_K is $\mathcal{F}_K$ measurable for each k.

If Y and Z are random variables, then Z is said to stochastically dominate Y, written $Y \prec Z$, if $P(Y > c) \leq P(Z > c)$ for $-\infty < c < \infty$. If $\mathcal{D}$ is a sub-σ-field of $\mathcal{F}$ we write $(Y|\mathcal{D}) \prec Z$ if $P(Y > c|\mathcal{D}) \leq P(Z > c)$ for $-\infty < c < \infty$.

Suppose $(Y_K, \mathcal{F}_K)_{K\geq 0}$ is such that

$$E[Y_{k+1} - Y_k + \varepsilon_0 \,;\, Y_k > \alpha \mid \mathcal{F}_k] \leq 0 \qquad \forall k \geq 0 \tag{20}$$

for some α and $\varepsilon_0 > 0$. In addition suppose there is a random variable Z and positive constants q, D such that

$$(|Y_{k+1} - Y_k| \mid \mathcal{F}_k) \prec Z \tag{21}$$

and

$$E[e^{qZ}] \leq D < \infty \tag{22}$$

Then (20), (21), and (22) imply that

$$E[e^{\eta Y_k} \mid \mathcal{F}_0] \leq \rho^k e^{\eta Y_0} + \frac{1-\rho^k}{1-\rho} D e^{\eta\alpha} \tag{23}$$

and

$$P(Y_k \geq b \mid \mathcal{F}_0) \leq \rho^k e^{\eta(Y_0 - b)} + \frac{1-\rho^k}{1-\rho} D e^{\eta(\alpha - b)} \tag{24}$$

where

$$\eta = \min(q, \tfrac{\varepsilon_0}{2c}) \tag{25}$$

$$\rho = 1 - \frac{\varepsilon_0 \eta}{2} \tag{26}$$

$$C = \frac{E[e^{qZ}] - 1 - qE[Z]}{q^2} = \sum_{K=2}^{\infty} \frac{q^{K-2}}{K!} E[Z^K] \qquad (27)$$

We now repeat the proof given in Hajek [1] of (23), (24).

<u>Lemma 1</u> Suppose X and Z are random variables such that $|X| < Z$ and $E[e^{qZ}] < +\infty$ for some $q > 0$. Then for $0 \leq \varepsilon \leq q$,

$$E[e^{\varepsilon X}] \leq 1 + \varepsilon E[X] + \varepsilon^2 C \qquad (28)$$

where C is given by (27). Hence if $E[X] \leq -\varepsilon_0 < 0$ and η, ρ satisfy (25), (26) then

$$E[e^{\eta X}] \leq 1 - \eta\varepsilon_0 + \eta^2 C \leq \rho < 1 \qquad (29)$$

To prove the lemma note that since $|X| < Z$, $E[|X|^K] \leq E[Z^K]$ so that if $0 \leq \varepsilon \leq q$, then $E[e^{\varepsilon X}]$ has an absolutely convergent series expansion and

$$\begin{aligned} E[e^{\varepsilon X}] &= 1 + \varepsilon E[X] + \sum_{K=2}^{\infty} \frac{\varepsilon^K}{K!} E[X^K] \\ &\leq 1 + \varepsilon E[X] + \varepsilon^2 \sum_{K=2}^{\infty} \frac{\varepsilon^{K-2}}{K!} E[Z^K] \\ &\leq 1 + \varepsilon E[X] + \varepsilon^2 C \quad . \end{aligned}$$

(23) clearly holds for $K = 0$. For $K \geq 0$,

$$E[e^{\eta Y_{k+1}} | \mathcal{F}_0] = E[E[e^{\eta(Y_{k+1}-Y_k)} | \mathcal{F}_k] e^{\eta Y_k} | \mathcal{F}_0] \tag{30}$$

Now

$$\begin{aligned} E[e^{\eta(Y_{k+1}-Y_k)} | \mathcal{F}_k] = {} & E[e^{\eta(Y_{k+1}-Y_k)} ; Y_k > \alpha | \mathcal{F}_k] \\ & + E[e^{\eta(Y_{k+1}-Y_k)} ; Y_k \le \alpha | \mathcal{F}_k] \end{aligned} \tag{31}$$

By the lemma,

$$E[e^{\eta(Y_{k+1}-Y_k)} ; Y_k > \alpha | \mathcal{F}_k] \le \rho I_{\{Y_k > \alpha\}} \le \rho \tag{32}$$

and by the hypothesis (21), (22)

$$E[e^{\eta(Y_{k+1}-Y_k)} ; Y_k \le \alpha | \mathcal{F}_k] \le D I_{\{Y_k \le \alpha\}} \tag{33}$$

Combining (30) - (33) and using the trivial inequality $I_{\{Y_k \le \alpha\}} e^{\eta Y_k} \le e^{\eta \alpha}$ yields that

$$E[e^{\eta Y_{k+1}} | \mathcal{F}_0] \le \rho E[e^{\eta Y_k} | \mathcal{F}_0] + D e^{\eta \alpha} \tag{34}$$

(23) now follows for all k from (34) by induction. (24) follows from (23) by Chebyshev's inequality.

2.2.3 Proof that algorithm is stable

2.2.3.1 Description of approach and definitions

Our approach to proving stability of the algoritm will be as follows. We consider the random sequence defined by

$$V_K = N_K + r g_\Delta(m_K) \tag{35}$$

where

$$g_\Delta(t) = \begin{cases} t^2 & ; |t| \leq \Delta \\ 2\Delta(t-\frac{\Delta}{2}) & ; t > \Delta \\ -2\Delta(t+\frac{\Delta}{2}) & ; t < -\Delta \end{cases} \tag{36}$$

r, Δ are constants to be specified later.

To prove stability we first note that $V_K \geqslant N_K$. We then show that V_K satisfies the hypothesis of section 2.2.2 ((20), (21), (22)) with $V_K = N_K$ and appropriate choices for ε_0, α, and Z. (Note that V_K can be viewed as a Liapunov function). This will immediately imply stability and we can compute numbers $\bar{N}$, K, η satisfying (3), (4) from (23) and (24).

To begin the proof let us first make some definitions. The following parameters in (37) are arbitrary.

$$0 < \rho_0 < 1 \tag{37a}$$

$$0 < \rho_1 < 1 \tag{37b}$$

$$0 < \rho_2 < 1 \tag{37c}$$

$$\sigma = \sum_{0}^{2} \rho_i < 1 \tag{37d}$$

$$0 < s < 1 \tag{37e}$$

$$0 < s' < 1 \tag{37f}$$

$$0 < q \tag{37g}$$

Let

$$W = \bar{e}^{1} - \lambda \tag{38}$$

$$\chi = \ln\left(1 + \frac{\sqrt{W(1-\sigma)}}{1-\lambda}\right) \tag{39}$$

$$\tilde{m} = 1 - \exp\left[\frac{-(1-\lambda)\sqrt{W(1-\sigma)}}{1-\lambda+\sqrt{W(1-\sigma)}}\right] \tag{40}$$

$$K_1 = \bar{e}^{1} + \bar{e}^{2} + \frac{2p_{00} + e^{2}}{p_{00}^{2}} \tag{40.5}$$

$$\bar{\gamma} = \frac{2\chi(1-s)\tilde{m}\rho_1 W}{(\lambda+\rho_0 W)(K_1+2\chi s\tilde{m}) + K_1\rho_1 W} \tag{41}$$

$$0 < \gamma \leq \bar{\gamma} \tag{42}$$

$$r = \frac{\rho_1 W}{\gamma^2 (K_1 + 2\chi s\tilde{m})} \tag{43}$$

$$a_1 = \max\left\{\frac{11}{\gamma^2 s\tilde{m}} + 1,\ \frac{62\bar{e}^{3}}{\rho_2 W}\right\} \tag{43.3}$$

$$K_2 = 4 + 5\bar{e}^{1} + \bar{e}^{2} \tag{43.7}$$

$$\Delta = \max\left\{ \ln\left(\frac{8}{\gamma s' \tilde{m}(1-\lambda)}\right), \frac{rK_2 + \rho_0 W + \lambda}{2r(1-s')\tilde{m}\gamma} \right\} \tag{44}$$

$$a_2 = \Delta^2 r \tag{45}$$

$$\alpha = a_1 + a_2 \tag{46}$$

We will show in section 2.2.3.3 that with these definitions

$$E\left[V_{k+1} - V_k + \rho_0 W \,;\, V_k > \alpha \,\middle|\, \mathcal{F}_k \right] \leq 0 \tag{47}$$

where $\mathcal{F}_k$ is the σ-field generated by $(N_i, f_i)_{i\leq k}$.

In section 2.2.3.3 we also show

$$\left(|V_{k+1} - V_k| \,\middle|\, \mathcal{F}_k \right) < \mu_0 + \mu_1 U \tag{48}$$

where

$$U \sim \mathrm{Poi}(\lambda) \tag{49}$$

$$\mu_0 = 1 + 4r\Delta \tag{50}$$

$$\mu_1 = 1 + 2r\Delta \tag{51}$$

In section 2.2.3.4 we combine the results of sections 2.2.2, 2.2.3.3 to compute $\bar{N}$, K , and η satisfying (3) and (4). In the next section we present several lemmas which will be useful in section 2.2.3.3.

2.2.3.2 Facts to be used in section 2.2.3.3

In lemmas 2-5 we state miscellaneous relationships and properties of the quantities defined in section 2.2.3.1 which will be useful later.

Lemma 2

a) $\chi < 1$ (52)

$$\chi = \ln\left(1 + \frac{\sqrt{W(1-\sigma)}}{1-\lambda}\right) \leq \frac{\sqrt{W(1-\sigma)}}{1-\lambda} \leq \sqrt{W} < 1$$

b) $\tilde{m} < 1$ (53)

c) $\gamma < 1$ (54)

$$\gamma \leq \bar{\gamma} = \frac{2\chi(1-s)\tilde{m}}{\left(\frac{\lambda + p_0 W}{p_1 W}\right)(K_1 + 2\chi s\tilde{m}) + K_1} \leq \frac{2\chi(1-s)\tilde{m}}{K_1} < \frac{2}{K_1} < 1$$

d) $\gamma\left(\frac{e}{p_{00}} - 1\right) < 1$ (55)

$$\gamma < \frac{2}{K_1} \leq \frac{2}{\left(\frac{e}{p_{00}}\right)^2} \leq \frac{2}{\left(\frac{e}{p_{00}} - 1\right)\left(\frac{e}{p_{00}}\right)} \leq \frac{2e^{-1}}{\left(\frac{e}{p_{00}} - 1\right)} < \frac{1}{\left(\frac{e}{p_{00}} - 1\right)}$$

$$e)\ \max_i \alpha^\gamma(i) = e^{\gamma\left(\frac{e}{p_{00}}-1\right)} < e \tag{56}$$

<u>Lemma 3</u> For all φ satisfying $|\varphi - \varphi^*| < \chi$ we have

$$\lambda - (\lambda + e^\varphi)\exp(-\lambda - e^\varphi) \leq -\sigma w \tag{57}$$

<u>Proof</u>

$$\chi = \ln(1-\lambda+\sqrt{w(1-\sigma)}) - \ln(1-\lambda) \leq \ln(1-\lambda) - \ln(1-\lambda-\sqrt{w(1-\sigma)})$$

Thus if $|\varphi - \varphi^*| < \chi$ we have

$$\ln(1-\lambda-\sqrt{w(1-\sigma)}) - \ln(1-\lambda) < \varphi - \varphi^* < \ln(1-\lambda+\sqrt{w(1-\sigma)}) - \ln(1-\lambda)$$

Hence

$$1 - \sqrt{w(1-\sigma)} < \lambda + e^\varphi < 1 + \sqrt{w(1-\sigma)}$$

Equivalently

$$(1-\lambda-e^\varphi)^2 < w(1-\sigma)$$

or

$$\lambda + \sigma w < e^{-1} - (1-\lambda-e^\varphi)^2 \tag{58}$$

Taylor's theorem gives

$$h e^{-h} \geqslant e^{-1} - (h-1)^2 \qquad \forall h \in \mathbb{R} \tag{59}$$

Combining (58) and (59) gives (57).

Lemma 4 a). If $\varphi > \varphi^* + \chi$ then $m(\varphi) \leq -\tilde{m}$.

b). If $\varphi < \varphi^* - \chi$ then $m(\varphi) \geqslant \tilde{m}$.

Proof From (13), (16) we have

$$m(\varphi) = e^{1-G} - 1 \tag{60}$$

where $G = e^{\varphi} + \lambda$.

First note that $m(\varphi)$ is a decreasing function of φ. Now

$$-m(\varphi^* + \chi) = 1 - e^{-\sqrt{w(1-\sigma)}}$$

$$\geqslant 1 - \exp\left[\frac{-(1-\lambda)\sqrt{w(1-\sigma)}}{1-\lambda+\sqrt{w(1-\sigma)}}\right] = \tilde{m}$$

$$m(\varphi^* - \chi) = \exp\left[\frac{(1-\lambda)\sqrt{w(1-\sigma)}}{1-\lambda+\sqrt{w(1-\sigma)}}\right] - 1$$

$$\geqslant \tilde{m}$$

Lemma 5

$$\lambda + \rho_0 W \leq r\left[2\chi(1-s)\tilde{m}\gamma - K_1\gamma^2\right] \tag{61}$$

Proof From (41), (42) we have

$$\gamma^2[(\lambda+\rho_0 W)(K_1+2\chi s\tilde{m})+K_1\rho_1 W] \le \gamma[2\chi(1-s)\tilde{m}\rho_1 W] \tag{62}$$

or equivalently

$$\left(\frac{\lambda+\rho_0 W}{\rho_1 W}\right) \le \frac{2\chi(1-s)\tilde{m}\gamma - K_1\gamma^2}{\gamma^2(K_1+2\chi s\tilde{m})} \tag{63}$$

From (43) we have

$$\rho_1 W = r\gamma^2(K_1+2\chi s\tilde{m}) \tag{64}$$

Multiplying (63) and (64) we obtain (61).

Lemma 6 bounds the error in approximating $\mathrm{Prob}(X=0)$ and $\mathrm{Prob}(X=1)$ by $\mathrm{Prob}(\tilde{X}=0)$ and $\mathrm{Prob}(\tilde{X}=1)$ respectively, where X is a binominal random variable with parameters n and f, and $\tilde{X}$ is a Poisson random variable with mean nf.

Lemma 6

For $0 \le f < 1/2$, $n > 0$ we have

a)
$$\left| e^{-nf} - (1-f)^n \right| \le \frac{8e^{-2}}{n}$$

b)

$$\left| nf e^{-nf} - nf(1-f)^{n-1} \right| \le \frac{54 e^{-3}}{n}$$

Proof

a)

$$0 \le e^{-nf} - (1-f)^n = e^{-nf} - e^{n\ln(1-f)}$$

$$\le e^{-nf}(-nf - n\ln(1-f))$$

$$\le 2nf^2 e^{-nf}$$

$$\le \frac{8e^{-2}}{n}$$

The first inequality follows by noting $e^a - e^b \le e^a(a-b)$ if $a \geq b$. The second inequality follows from Taylor's theorem and using $f \le 1/2$. Finally, the third inequality follows from maximizing $2nf^2 e^{-nf}$ over f.

b)

$$nf e^{-nf} - nf(1-f)^{n-1} \le nf(e^{-nf} - (1-f)^n)$$

$$\le 2n^2 f^3 e^{-nf}$$

$$\le \frac{54 e^{-3}}{n}$$

$$nf(1-f)^{n-1} - nfe^{-nf} = \frac{nf}{1-f}(1-f)^n - nfe^{-nf}$$

$$\leq nfe^{-nf}\left(\frac{1}{1-f} - 1\right)$$

$$= \frac{nf^2}{1-f}e^{-nf}$$

$$\leq 2nf^2e^{-nf} \leq \frac{8e^{-2}}{n} < \frac{54e^{-3}}{n}$$

The next lemma bounds the same quantities as in the previous lemma, but the bounds are in terms of f.

<u>Lemma 7</u>

Let f be given. Suppose

$$0 \leq g_i \leq f \leq 1/2 \qquad i = 1, 2, \ldots, n$$

Then a)

$$\left| e^{-\sum_1^n g_i} - \prod_1^n (1-g_i) \right| \leq \frac{2f}{e}$$

and

b)

$$\left| \left(\sum_1^n g_i\right) e^{-\sum_1^n g_i} - \sum_{i=1}^n g_i \prod_{j \neq i} (1-g_j) \right| \leq 2f$$

Proof

a) From Taylor's theorem,

$$-g_i - \frac{g_i^2}{2(1-f)^2} \leq \ln(1-g_i) \leq -g_i$$

$$e^{-\sum_1^n g_i - \frac{g_i^2}{2(1-f)^2}} \leq \prod_1^n (1-g_i) \leq e^{-\sum_1^n g_i}$$

$$0 \geq \prod_1^n (1-g_i) - e^{-\sum_1^n g_i} \geq e^{-\sum_1^n g_i}\left[e^{-\sum_1^n \frac{g_i^2}{2(1-f)^2}} - 1\right]$$

$$\geq e^{-\sum_1^n g_i}\left[-\sum_1^n \frac{g_i^2}{2(1-f)^2}\right]$$

$$\geq \left(\sum_1^n g_i\right)e^{-\sum_1^n g_i}\left[\frac{-f}{2(1-f)^2}\right] \geq \frac{-f}{2e(1-f)^2} \geq \frac{-2f}{e}$$

b) Now

$$\frac{1}{1-f}\sum_{i=1}^n g_i \prod_{j=1}^n (1-g_j) \geq \sum_{i=1}^n g_i \prod_{j\neq i} (1-g_j) \geq \sum_{i=1}^n g_i \prod_{j=1}^n (1-g_j) \qquad (65)$$

Hence

$$\sum_{i=1}^n g_i \prod_{j\neq i} (1-g_j) - \left(\sum_{i=1}^n g_i\right)e^{-\sum_1^n g_i} \geq \sum_{i=1}^n g_i\left[\prod_{i=1}^n (1-g_i) - e^{-\sum_1^n g_i}\right] \qquad (66)$$

$$\geq \sum_{i=1}^n g_i\left[e^{-\sum_1^n g_i}\sum_{i=1}^n g_i\left(\frac{-f}{2(1-f^2)}\right)\right] \qquad (67)$$

$$\geq \frac{-2}{e^2} \frac{f}{(1-f)^2} \tag{68}$$

$$\geq \frac{-8}{e^2} f > -2f$$

(67) follows from part a). (68) follows by noting that $t^2 e^{-t} \leq 4e^{-2}$ for all t.

Also from (65) we have

$$\left(\sum_1^n g_i\right) e^{-\sum_1^n g_i} - \sum_{i=1}^n g_i \prod_{j\neq i} (1-g_j) \geq \left(\sum_1^n g_i\right) e^{-\sum_1^n g_i} - \frac{1}{1-f} \sum_1^n g_i \left[\prod_1^n (1-g_i)\right]$$

$$= \sum_1^n g_i \left[e^{-\sum_1^n g_i} - \left(1 + \frac{f}{1-f}\right) \prod_1^n (1-g_i) \right]$$

$$= \sum_1^n g_i \left[e^{-\sum_1^n g_i} - \prod_1^n (1-g_i) \right] - \sum_1^n g_i \prod_1^n (1-g_i) \frac{f}{1-f}$$

$$\geq -\sum_1^n g_i \prod_1^n (1-g_i) \frac{f}{1-f}$$

$$\geq \frac{-f}{1-f} \geq -2f$$

This completes the proof of lemma 7.

Let

$$\pi(n,f) = (\pi_0, \pi_1, \pi_2)$$

$$= \left(e^{-nf-\lambda}, (\lambda+nf)e^{-\lambda-nf}, 1-(1+nf+\lambda)e^{-\lambda-nf}\right)$$

$$\tilde{\pi}(n,f) = (\tilde{\pi}_0, \tilde{\pi}_1, \tilde{\pi}_2)$$

$$= \left(e^{-\lambda}(1-f)^n,\ \lambda e^{-\lambda}(1-f)^n + e^{-\lambda} nf(1-f)^{n-1},\ 1 - e^{-\lambda}\left((1-f)^n + \lambda(1-f)^n + nf(1-f)^{n-1}\right)\right)$$

Note that $\tilde{\pi}_0$, $\tilde{\pi}_1$, $\tilde{\pi}_2$ are the probabilities that a Poisson random variable with mean $\lambda + nf$ is zero, one, or greater than one respectively. Also note that

$$\tilde{\pi}_i = \mathrm{Prob}(\tilde{Z}_k = i \mid N_k = n, f_k = f) \qquad i = 0,1,2$$

The next lemma bounds the error in approximating $\tilde{\pi}(n,f)$ by $\pi(n,f)$. The inequalties in the lemma are meant in a component wise sense, and the proof follows easily from lemmas 6, 7 and the facts $e^{-\lambda} \leq 1$, $\lambda e^{-\lambda} \leq e^{-1}$.

Lemma 8

For $n > 0$, $0 \leq f \leq 1/2$ we have

a)

$$\begin{bmatrix} |\pi_0 - \tilde{\pi}_0| \\ |\pi_1 - \tilde{\pi}_1| \\ |\pi_2 - \tilde{\pi}_2| \end{bmatrix} \leq \frac{1}{n} \begin{bmatrix} 8e^{-2} \\ 62e^{-3} \\ 8e^{-2} + 62e^{-3} \end{bmatrix} \qquad (69)$$

b)

$$\begin{bmatrix} |\pi_0 - \tilde{\pi}_0| \\ \\ |\pi_1 - \tilde{\pi}_1| \\ \\ |\pi_2 - \tilde{\pi}_2| \end{bmatrix} \leq f \begin{bmatrix} 2e^{-1} \\ \\ 2e^{-2}+2 \\ \\ 2(e^{-1}+e^{-2}+1) \end{bmatrix} \tag{70}$$

We now use lemmas 6 and 7 to show

<u>Lemma 9</u>

If $N_K > a_1$, then

$$\left| E[N_{K+1} - N_K | \mathcal{F}_K] - \lambda + G e^{-G} \right| \leq \rho_2 W \tag{71}$$

where $G = N_K f_K + \lambda$.

<u>Proof</u>

$$\left| E[N_{K+1} - N_K | \mathcal{F}_K] - \lambda + G e^{-G} \right|$$

$$= \left| -e^{-\lambda} N_K f_K (1-f_K)^{N_K - 1} + \lambda e^{-\lambda}(1-(1-f_K)^{N_K}) + \sum_{K=2}^{\infty} K \frac{\lambda^K}{K!} e^{-\lambda} - \lambda + G e^{-G} \right|$$

$$= \left| -e^{-\lambda} N_K f_K (1-f_K)^{N_K - 1} - \lambda e^{-\lambda}(1-f_K)^{N_K} + G e^{-G} \right|$$

$$\leq \bar{e}^{\lambda}\left|N_k f_k \bar{e}^{N_k f_k} - N_k f_k (1-f_k)^{N_k - 1}\right| + \lambda \bar{e}^{\lambda}\left|e^{-N_k f_k} - (1-f_k)^{N_k}\right|$$

$$\leq \frac{54\bar{e}^3}{N_k} + \frac{1}{e}\cdot\frac{8\bar{e}^2}{N_k}$$

$$= \frac{62\bar{e}^3}{N_k} \tag{72}$$

(71) follows immediately from (72) since $N_k > a_1 > \frac{62\bar{e}^3}{\rho_2 w}$.

The next two lemmas bound the error of the approximation in (12).

Lemma 10 If $N_k > a_1$, and $\varphi_k > \varphi^*$ then

$$E[\varphi_{k+1} - \varphi_k \mid \mathcal{F}_k] - \gamma m(\varphi_k) \leq \gamma^2 s \tilde{m} \tag{73}$$

Proof

$$E[\varphi_{k+1} - \varphi_k \mid \mathcal{F}_k] \leq E\left[\ln \frac{N_{k+1} \vee 1}{N_k \vee 1} \,\middle|\, \mathcal{F}_k\right] + \gamma \tilde{\pi}(N_k, f_k) P \underline{c} \tag{74}$$

$$E[\varphi_{k+1} - \varphi_k \mid \mathcal{F}_k] - \gamma m(\varphi_k) \leq E\left[\ln \frac{N_{k+1} \vee 1}{N_k \vee 1} \,\middle|\, \mathcal{F}_k\right] + \gamma(\tilde{\pi} - \pi)(N_k, f_k) P \underline{c} \tag{75}$$

$$\leq E\left[\frac{N_{k+1} - N_k}{N_k} \,\middle|\, \mathcal{F}_k\right] + \gamma(\tilde{\pi} - \pi)(N_k, f_k)\begin{bmatrix} e-1 \\ -1 \\ -1 \end{bmatrix} \tag{76}$$

$$\leq \frac{1}{N_k}\left[1 + (e-1)8\bar{e}^2 + 62\bar{e}^3 + 8\bar{e}^2 + 62\bar{e}^3\right] \tag{77}$$

The last inequaltiy follows from $E[N_{k+1}-N_k|\mathcal{F}_k] \le \lambda < 1$, $\gamma < 1$, and from lemma 8 a). Since $N_K > a_1 > \frac{11}{\gamma^2 s \tilde{m}}$ and $(1+8e^{-1}+124\,e^{-3}) < 11$ we conclude (73) holds.

<u>Lemma 11</u> If $N_K > a_1$ and $\varphi_K \le \varphi^*$ then

$$\gamma m(\varphi_k) - E[\varphi_{k+1}-\varphi_k|\mathcal{F}_k] \le \gamma^2 s \tilde{m} \tag{78}$$

<u>Proof</u> If $\varphi_K \le \varphi^*$ we have $(N_k \vee 1) f_K \le 1-\lambda \le 1$. Noting that $a_1 > 2e$ we must have $f_k < \frac{1}{2e}$ in this case. Since $\beta = 1/2$, $\beta > f_k \cdot a^\gamma(2)$ so that

$$E[\varphi_{k+1}-\varphi_k|\mathcal{F}_k] = E[\ln \frac{N_{k+1}\vee 1}{N_k \vee 1}|\mathcal{F}_k] + \gamma\tilde{\pi}(N_k, f_k)P_{\subseteq} \tag{79}$$

$$\gamma m(\varphi_k) - E[\varphi_{k+1}-\varphi_k|\mathcal{F}_k] = -E[\ln \frac{N_{k+1}}{N_k}|\mathcal{F}_k] + \gamma(\pi-\tilde{\pi})(N_k, f_k)\begin{pmatrix} e-1 \\ -1 \\ -1 \end{pmatrix} \tag{80}$$

$$\le E[\ln \frac{N_k}{N_k-1}] + \gamma(\pi-\tilde{\pi})(N_k, f_k)\begin{pmatrix} e-1 \\ -1 \\ -1 \end{pmatrix} \tag{81}$$

$$\le \frac{1}{N_k-1} + \frac{1}{N_k}[(e-1)8e^{-2}+62e^{-3}+8e^{-2}+62e^{-3}] \tag{82}$$

$$\le \frac{1}{N_k-1}[1+8e^{-1}+124e^{-3}] \tag{82.5}$$

$$\le \frac{11}{N_k-1} \tag{83}$$

Since $N_K > a_1 > \frac{11}{\gamma^2 s \tilde{m}} + 1$, (78) follows from (83).

Lemmas 12 and 13 show that if φ_K is "far away" from φ^* then φ_K will drift toward φ^*.

Lemma 12

If $m_K > \Delta$ then

$$E[\varphi_{K+1} - \varphi_K \mid \mathcal{F}_K] \leq \gamma(s'-1)\tilde{m} \tag{84}$$

Proof

$$\Delta = \max\left\{ \ln \frac{8}{\gamma s' \tilde{m}(1-\lambda)}, \frac{rK_2 + \rho_0 W + \lambda}{2r(1-s')\tilde{m}\gamma} \right\}$$

$$\geq \ln \frac{8}{\gamma s' \tilde{m}(1-\lambda)}$$

Thus if $m_K = \varphi_K - \varphi^* > \Delta$ we have

$$(N_K \vee 1) f_K \geq \frac{8}{\gamma s' \tilde{m}}$$

Noting that $f_K \leq 1/2$ we obtain

$$N_K \geq \frac{16}{\gamma s' \tilde{m}} \tag{85}$$

By following a line of argument identical to (74) - (77) we obtain

$$E[\varphi_{K+1} - \varphi_K \mid \mathcal{F}_K] - \gamma m(\varphi_K) \leq \frac{11}{N_K}$$

$$\leq \frac{11}{16}\gamma s'\tilde{m}$$

$$< \gamma s'\tilde{m} \tag{86}$$

Note that

$$\Delta \geq \ln \frac{8}{\gamma s'\tilde{m}(1-\lambda)}$$

$$\geq \ln 8$$

$$> 1$$

$$> \chi \tag{87}$$

Thus $\Delta > \chi$ and

$$m(\varphi_k) \leq m(\varphi^* + \Delta)$$

$$\leq m(\varphi^* + \chi)$$

$$\leq -\tilde{m} \tag{88}$$

Combining (88) and (86) we obtain (84).

<u>Lemma 13</u> If $m_k < -\Delta$ then

$$E[\varphi_{k+1} - \varphi_k \mid \mathcal{F}_k] \geq \gamma(1-s')\tilde{m} \tag{89}$$

Proof: If $m_K < -\Delta$ then

$$m_K = \varphi_K - \varphi^* < \ln\left(\frac{\gamma s' \tilde{m}(1-\lambda)}{8}\right)$$

$$\varphi_K < \ln\left(\frac{\gamma s' \tilde{m}(1-\lambda)^2}{8}\right) < \ln\left(\frac{\gamma s' \tilde{m}}{8}\right)$$

$$(N_K \vee 1) f_K < \frac{\gamma s' \tilde{m}}{8} \tag{90}$$

$$< \frac{1}{2e} \tag{91}$$

From (90), (91) we have

$$f_K < \frac{\gamma s' \tilde{m}}{8} \tag{92}$$

and

$$f_K < \frac{1}{2e} \tag{93}$$

Since $\beta = 1/2$ and $\max_i a^\gamma(i) < e$ we observe from (93) that the min in (6) will not be taken by β. Thus for $m_K < -\Delta$

$$E[\varphi_{K+1} - \varphi_K \mid \mathcal{F}_K] = E\left[\ln \frac{N_{K+1} \vee 1}{N_K \vee 1} \mid \mathcal{F}_K\right] + \tilde{\pi}(N_K, f_K) P \leq \gamma$$

$$\geq e^{-\lambda} N_k f_k (1-f_k)^{N_k-1} \cdot \ln \frac{(N_k-1)V1}{N_k V1} + \tilde{\pi}(N_k,f_k) P \leq \gamma$$

$$\geq e^{-\lambda} N_k f_k (1-f_k)^{N_k-1} \cdot \ln(\tfrac{1}{2}) + \tilde{\pi}(N_k,f_k) P \leq \gamma$$

$$\geq N_k f_k \ln(1/2) + \tilde{\pi}(N_k,f_k) P \leq \gamma$$

$$\gamma\, m(\varphi_k) - E[\varphi_{k+1} - \varphi_k | \mathcal{F}_k] \leq N_k f_k \ln(2) + (\pi - \tilde{\pi})(N_k,f_k) P \leq \gamma \tag{94}$$

$$\leq N_k f_k \ln(2) + f_k \left[\tfrac{2}{e}(e-1) + 2e^{-2} + 2 + 2(e^{-1} + e^{-2} + 1)\right] \tag{95}$$

$$\leq \frac{\gamma s' \hat{m}}{8} [\ln 2 + 6 + 4e^{-2}] \tag{96}$$

(95) follows from (94) by lemma 7. (96) follows from (95) by (90) and (92).
Since $m_k = \varphi_k - \varphi^* < -\Delta$

$$m(\varphi_k) \geq m(\varphi^* - \Delta)$$

$$\geq m(\varphi^* - x)$$

$$\geqslant \tilde{m} \tag{97}$$

Combining (97) and (96) we obtain (89).

Lemmas 14 and 15 bound the tail of the distribution of $|\varphi_{k+1}-\varphi_k|$ given $\mathcal{F}_k$.

Lemma 14 : $(|\varphi_{k+1}-\varphi_k| \,|\, \mathcal{F}_k) \prec U+2$ (98)

where $U \sim \mathrm{Poi}(\lambda)$.

Proof

$$|\varphi_{k+1}-\varphi_k| \leq \left|\ln \frac{N_{k+1}V1}{N_kV1}\right| + \left|\ln \frac{(f_k \cdot a^\gamma(z_k)) \wedge \beta}{f_k}\right|$$

$$\leq \frac{1}{(N_k-1)V1}\left|N_{k+1}-N_k\right| + \max_i \left|\ln a^\gamma(i)\right|$$

$$= \frac{1}{(N_k-1)V1}\left|N_{k+1}-N_k\right| + \gamma\left(\frac{e}{p_{00}}-1\right) \tag{99}$$

$$\leq \left|N_{k+1}-N_k\right| + 1$$

The last inequality follows from Lemma 2d). Since

$$\left|N_{k+1}-N_k\right| \prec U+1 \tag{100}$$

the lemma is proved.

If N_k is large the previous lemma can be strengthened.

Lemma 15 If $N_K > a_1$, then

$$(|\varphi_{K+1} - \varphi_K| \,|\, \mathcal{F}_K) \prec \gamma\left(U + \frac{e}{p_{00}}\right) \tag{101}$$

Proof

$$N_K > a_1 = \max\left\{\frac{11}{\gamma^2 s \tilde{m}} + 1,\ \frac{62\bar{e}^3}{p_2 w}\right\}$$

$$\geq \frac{11}{\gamma^2 s \tilde{m}} + 1$$

$$\geq \frac{11}{\gamma^2} + 1 \geq \frac{11}{\gamma} + 1$$

$$\geq \gamma^{-1} + 1 \tag{102}$$

(101) follows from (99), (102), and (100).

2.2.3.3 Derivation of (47) and (48)

(47) is repeated here for convenience.

$$E[V_{k+1} - V_k + p_0 w\ ;\ V_k > \alpha \,|\, \mathcal{F}_k] \leq 0 \qquad (47), (103)$$

First note that

$$\{V_k > \alpha\} \subset \{N_k > a_1\} \cup \{r g_\Delta(m_k) > a_2\} \tag{104}$$

(104) is clear once we note that $\alpha = a_1 + a_2$.

Hence to show (103) it suffices to show the following two equations (105) and (106) hold:

$$E[V_{k+1} - V_k + \rho_0 W \,;\, N_k > a_1 \,|\, \mathcal{F}_k] \leq 0 \tag{105}$$

$$E[V_{k+1} - V_k + \rho_0 W \,;\, r g_\Delta(m_k) > a_2 \,|\, \mathcal{F}_k] \leq 0 \tag{106}$$

Note the inequality

$$g_\Delta(m_{k+1}) - g_\Delta(m_k) \leq (\varphi_{k+1} - \varphi_k)\, g'_\Delta(m_k) + (\varphi_{k+1} - \varphi_k)^2 \tag{107}$$

where

$$g'_\Delta(t) = \begin{cases} 2t & ; |t| < \Delta \\ 2\Delta & ; t \geq \Delta \\ -2\Delta & ; t \leq -\Delta \end{cases} \tag{108}$$

We now prove (105). For $N_k > a_1$, $m_k > x$ we have

$$r E[g_\Delta(m_{k+1}) - g_\Delta(m_k) | \mathcal{F}_k] \leq r E[\varphi_{k+1} - \varphi_k | \mathcal{F}_k]\, g'_\Delta(m_k) + r E[(\varphi_{k+1} - \varphi_k)^2 | \mathcal{F}_k] \tag{109}$$

$$\leq r g'_\Delta(m_k)(\gamma m(\varphi_k) + \gamma^2 s \tilde{m}) + r E[(\varphi_{k+1} - \varphi_k)^2 | \mathcal{F}_k] \tag{110}$$

$$\leq r g'_\Delta(m_k)(-\gamma \tilde{m} + \gamma s \tilde{m}) + r E[(\varphi_{k+1} - \varphi_k)^2 | \mathcal{F}_k] \tag{111}$$

$$\leq 2r\chi(-\gamma\tilde{m}+\gamma s\tilde{m}) + r\gamma^2 E[(U+\frac{e}{p_{oo}})^2] \qquad (112)$$

$$\leq 2r\chi(s-1)\gamma\tilde{m} + r\gamma^2 K_1 \qquad (113)$$

$$\leq -\lambda - \rho_o W \qquad (114)$$

(109) follows from (107). (110) follows from (109) by lemma 10. (111) follows from lemma 4 a) and $\gamma < 1$. (112) is obtained from $g'_\Delta(m_k) \geq g'_\Delta(\chi) = 2\chi$ and from lemma 15. (113) follows by noting $E[(U+\frac{e}{p_{oo}})^2] \leq K_1$. (114) follows from lemma 5.

A similar argument shows that for $N_k > a_1$, $m_k < -\chi$ we have

$$rE[g_\Delta(m_{k+1}) - g_\Delta(m_k) \mid \mathcal{F}_k] \leq -\lambda - \rho_o W \qquad (115)$$

Since we always have

$$E[N_{k+1} - N_k \mid \mathcal{F}_k] \leq \lambda \qquad (116)$$

then (116), (115), and (114) yields for $N_k > a_1$, $|m_k| > \chi$,

$$E[V_{k+1} - V_k \mid \mathcal{F}_k] \leq -\rho_o W \qquad (117)$$

Now consider $N_k > a_1 \;, 0 \le m_k \le \chi$:

$$r E[g_\Delta(m_{k+1}) - g_\Delta(m_k) | \mathcal{F}_k] \le r g_\Delta'(m_k)[\gamma m(\varphi_k) + \gamma^2 s \tilde{m}] + r\gamma^2 K_1 \tag{118}$$

$$\le 2 r \chi \gamma^2 s \tilde{m} + r \gamma^2 K_1 \tag{119}$$

$$= \rho_1 W \tag{120}$$

(118) follows from (107), lemma 10, and $E[(\varphi_{k+1} - \varphi_k)^2 | \mathcal{F}_k] \le E[(\upsilon + \frac{e}{p_{\infty}})^2] \le K_1$.
(119) follows since $m_k \le \chi$. (120) follows from (64).

(120) can be shown similarly to hold when $N_k > a_1 \;, -\chi \le m_k \le 0$.
By lemma 9 we obtain for $N_k > a_1 \,, |m_k| \le \chi$

$$E[N_{k+1} - N_k | \mathcal{F}_k] \le \lambda - (e^{\varphi_k} + \lambda)\exp(-\lambda - e^{\varphi_k}) + \rho_2 W \tag{121}$$

$$\le (\rho_2 - \sigma) W \tag{122}$$

$$\le (-\rho_0 - \rho_1) W \tag{123}$$

(122) follows from lemma 3. Combining (123), (120), (117) we obtain (105).

We now show (106). Note that since $a_2 = r\Delta^2$ then $r g_\Delta(m_k) > a_2$ implies either $m_k > \Delta$ or $m_k < -\Delta$. Thus by (107), lemma 12, lemma 13, and lemma 14 we have

$$r\,E[g_\Delta(m_{k+1})-g_\Delta(m_k)\,|\,\mathcal{F}_k] \le -2r\gamma(1-s')\tilde{m}\Delta + r\,E[(U+2)^2] \tag{124}$$

$$\le -2r\gamma(1-s')\tilde{m}\Delta + rK_2 \tag{125}$$

$$\le -\rho_0 W - \lambda \tag{126}$$

(125) follows from $E[(U+2)^2] \le K_2$. (126) follows by noting that $\Delta \ge \frac{rK_2+\rho_0 W+\lambda}{2r(1-s')\tilde{m}\gamma}$ from the definition of Δ. Combining (116) and (126) we obtain (106). This completes the proof of (47).

We now show (48). Note that

$$|g'_\Delta(t)| \le 2\Delta \quad , \quad -\infty < t < \infty \tag{127}$$

Now

$$|V_{k+1}-V_k| = |N_{k+1}-N_k + rg_\Delta(m_{k+1}) - rg_\Delta(m_k)|$$

$$\le |N_{k+1}-N_k| + r\,|g_\Delta(m_{k+1})-g_\Delta(m_k)|$$

$$\le |N_{k+1}-N_k| + 2\Delta r\,|\varphi_{k+1}-\varphi_k| \tag{128}$$

Thus by (128), (100), and Lemma 14,

$$(\,|V_{k+1}-V_k|\,|\,\mathcal{F}_k) \prec (U+1) + 2\Delta r\,(U+2)$$

which proves (48).

2.2.3.4 Computation of $\bar{N}, K, \eta$ satisfying (3) and (4)

Note that for any $q > 0$ we have

$$E[e^{q(M_0+M_1 U)}] = e^{qM_0} \cdot \sum_{K=0}^{\infty} \frac{\lambda^K}{K!} e^{-\lambda} e^{qM_1 K}$$

$$= e^{qM_0} e^{\lambda e^{qM_1}} e^{-\lambda} = e^{qM_0 + \lambda[e^{qM_1} - 1]} \tag{129}$$

Thus the correspondence between the hypothesis of section 2.2.2 and (47), (48) is

$$\varepsilon_0 = \rho_0 W \tag{130}$$

$$Z = M_0 + M_1 U \tag{131}$$

$$D = e^{qM_0 + \lambda[e^{qM_1} - 1]} \tag{132}$$

All other parameters in section 2.2.2 have the same meaning as in section 2.2.3.

From (23) we have (assuming $N_0 = 0$ and $f_0 = \beta$)

$$E[e^{\eta V_K}] \leq \rho^K e^{\eta V_0} + \frac{1-\rho^K}{1-\rho} D e^{\eta \alpha} \tag{133}$$

Using the concavity of the $\ln(\cdot)$ function we have from Jensens inequality

$$E[V_k] \leq \frac{1}{\eta}\ln\left[\rho^k e^{\eta V_0} + \frac{1-\rho^k}{1-\rho} D e^{\eta\alpha}\right] \tag{134}$$

From (134) we obtain

$$\limsup_k E[V_k] \leq \frac{1}{\eta}\ln\left(\frac{D}{1-\rho} e^{\eta\alpha}\right)$$

$$= \frac{1}{\eta}\ln\left(\frac{D}{1-\rho}\right) + \alpha \tag{135}$$

Since $V_k \geqslant N_k \quad \forall k$ we have

$$\limsup_k E[N_k] \leq \frac{1}{\eta}\ln\left(\frac{D}{1-\rho}\right) + \alpha \tag{136}$$

From (24) we have

$$\mathrm{Prob}(v_k \geqslant b) \leq \rho^k e^{\eta(V_0-b)} + \frac{1-\rho^k}{1-\rho} D e^{\eta(\alpha-b)} \tag{137}$$

From which we obtain

$$\limsup_k \mathrm{Prob}(N_k \geqslant b) \leq \frac{D e^{\eta\alpha}}{1-\rho} e^{-\eta b} \tag{138}$$

Thus K, $\bar{N}$, η in (3), (4) can be obtained from (136) and (138).

CHAPTER THREE

Stable Algorithms when P is not of the form in (5)

3.1 Introduction

In this chapter we describe algorithms that maintain stability when P is not necessarily of the form in (5) but P corresponds to a DMC with non-zero capacity. Recall that

$$m(\varphi) = (e^{-G}, Ge^{-G}, 1-(1+G)e^{-G}) \cdot (P\underline{c})^T$$

where $G = \lambda + e^{\varphi}$

$$\underline{c} = \begin{pmatrix} \ln a(0) \\ \ln a(1) \\ \ln a(2) \end{pmatrix}$$

Suppose P is such that there exists a choice of the function $a(\cdot)$ that yields the following properties for $m(\cdot)$:

$$m(\varphi^*) = 0 \tag{139}$$

$$m(\varphi) > 0 \quad \text{for} \quad \varphi < \varphi^* \tag{140}$$

$$m(\varphi) < 0 \quad \text{for} \quad \varphi > \varphi^* \tag{141}$$

Consider the algorithm identical to the one analyzed in Chapter 2 with $a(\cdot)$ chosen to satisfy (139) - (141). Examination of the proof of stability in Chapter 2 reveals that the same proof can be used to prove stability for this algorithm except for different choices for the parameters in

(40) - (45).

If P is invertible then $a(\cdot)$ (equivalently $\underline{c}$) can obviously be chosen to satisfy (139) - (141), for example let

$$\underline{c} = P^{-1}\begin{pmatrix} e-1 \\ -1 \\ -1 \end{pmatrix}$$

Define d_i, i = 0, 1, 2 as

$$d_i = (P_{i0}, P_{i1}, P_{i2})$$

If P is not invertible but corresponds to a DMC with non-zero capacity then one of the following must hold (note $\alpha' + \beta' = 1$ in each case):

1) $d_0 \neq d_2$, $d_1 = \alpha' d_0 + \beta' d_2$, with $0 \leq \alpha'$, $0 \leq \beta'$

2a) $d_1 \neq d_2$, $d_0 = \alpha' d_1 + \beta' d_2$, with $\frac{1}{e-1} < \alpha' \leq 1$, $0 \leq \beta' < \frac{e-2}{e-1}$

2b) $d_1 \neq d_2$, $d_0 = \alpha' d_1 + \beta' d_2$, with $0 \leq \alpha' \leq \frac{1}{e-1}$, $\frac{e-2}{e-1} \leq \beta' \leq 1$

3a) $d_0 \neq d_1$, $d_2 = \alpha' d_0 + \beta' d_1$, with $0 \leq \alpha' < 1/2$, $\frac{1}{2} < \beta' \leq 1$

3b) $d_0 \neq d_1$, $d_2 = \alpha' d_0 + \beta' d_1$, with $\alpha' \neq \frac{1}{1+\lambda}$, $\frac{1}{2} \leq \alpha' \leq 1$, $0 \leq \beta' \leq 1/2$

3c) $d_0 \neq d_1$, $d_2 = \frac{1}{1+\lambda} d_0 + \frac{\lambda}{1+\lambda} d_1$

In cases 1), 2a), 3a) it turns out that $\underline{c}$ can be chosen so that

(139) - (141) holds. This is shown in section 3.2. In section 3.3, we describe a modified form of the algorithm in Chapter 2 which is stable in cases 2b), 3b), 3c).

3.2 Choice of $\underline{c}$ in cases 1), 2a), 3a)

Let

$$h_i = d_i \cdot \underline{c} \qquad i = 0, 1, 2$$

$$\ell(G) = h_0 e^{-G} + h_1 G e^{-G} + h_2 (1 - (1+G) e^{-G})$$

so that

$$m(\varphi) = \ell(e^{\varphi} + \lambda)$$

Note that

$$\frac{d}{dG} \ell(G) = e^{-G} (h_1 - h_0 + (h_2 - h_1) G) \tag{142}$$

From (142) we see that $\frac{d}{dG} \ell(G)$ has at most one root for G in the interval $[0, \infty)$. Thus if $\ell(0) > 0$ and $\lim_{G \to \infty} \ell(G) < 0$ then $\ell(G)$ has exactly one root for G in the interval $[0, \infty]$. Since $\lambda + e^{\varphi}$ is a monotone increasing function of φ sufficient conditions for (139) - (141) are

$$\ell(0) > 0 \tag{143}$$

$$\ell(1) = 0 \tag{144}$$

$$\lim_{G \to \infty} \ell(G) < 0 \tag{145}$$

(143) - (145) are equivalent to

$$d_0 \cdot \underline{c} > 0 \tag{146}$$

$$\frac{(d_0+d_1)\cdot \underline{c}}{d_2 \cdot \underline{c}} = 2-e \tag{147}$$

$$d_2 \cdot \underline{c} < 0 \tag{148}$$

To satisfy (148) we will arbitrarily set

$$d_2 \cdot \underline{c} = -1 \tag{149}$$

We will use the following lemma to show that $\underline{c}$ can be chosen to satisfy (146), (147), (149) and hence (139) - (141) in cases 1), 2a), 3a).

Lemma 16:

If $d_0 \neq d_2$ and u is a negative scalar there exists $\underline{c} \in \mathbb{R}^3$ such that the following holds:

$$d_0 \cdot \underline{c} > 0 \tag{150}$$

$$\frac{d_0 \cdot \underline{c}}{d_2 \cdot \underline{c}} = u \tag{151}$$

$$d_2 \cdot \underline{c} = -1 \tag{152}$$

Proof: If $d_0 \neq d_2$ it is obvious geometrically that (151) can be satisifed for $\underline{c}$ of the form

$$\underline{c} = y_1 d_0 + y_2 d_2 \tag{153}$$

where $y_1, y_2 \in \mathbb{R}$. One can substitute (153) into (151) to obtain y_1 in terms of y_2. y_2 can be determined from (152). (150) follows from (151) and (152). This proves the lemma.

In case 1) (147) becomes

$$\frac{[(1+\alpha')d_0 + \beta' d_2]\cdot \underline{c}}{d_2 \cdot \underline{c}} = 2-e$$

This can be rewritten as

$$\frac{d_0 \cdot \underline{c}}{d_2 \cdot \underline{c}} = \frac{2-\beta'-e}{1+\alpha'}$$

Since $\frac{2-\beta'-e}{1+\alpha'} < 0$ the lemma applies with $u = \frac{2-\beta'-e}{1+\alpha'}$ so that (146), (147), (149) can be satisfied, and hence (139) - (141). Similarly in case 2a) the lemma applies with $u = \frac{(2-e)\alpha' + \beta'}{1+\alpha'}$, and in case 3a) the lemma applies with $u = \frac{(2-e)\beta'-1}{\beta'-\alpha'}$.

3.3 A Stable Algorithm in Cases 2b), 3b), 3c)

We will show for cases 2b), 3b) (using the notation of section 3.2) there exists a $\underline{c}$ such that for some $G^* > \lambda$:

$$m(\varphi) > 0 \quad \text{for} \quad \varphi < \ln(G^* - \lambda) \tag{154}$$

$$m(\ln(G^* - \lambda)) = 0 \tag{155}$$

$$m(\varphi) < 0 \quad \text{for} \quad \varphi > \ln(G^* - \lambda) \tag{156}$$

Defining $\tilde{\varphi}^* = \ln(G^* - \lambda)$ we see that (154) – (156) imply that $\underline{s}$ can be chosen so that the algorithm "steers" φ_K towards $\tilde{\varphi}^*$. In cases 2b), 3b) it can be shown that (154) – (156) cannot be satisfied with $G^* = 1$, or equivalently we can not have $\tilde{\varphi}^* = \varphi^*$. We would like φ_K to be "steered" towards φ^* (see (7)). The algorithm we describe for cases 2b), 3b) achieves this in the following way. Divide the slots into frames of length M (a frame is M consecutive slots). Roughly speaking, the algorithm periodically "scales" the **probabilities** f_i so that if φ_K is near $\tilde{\varphi}^*$ in the first slot of a frame then φ_K will be near φ^* for the remaining $M-1$ slots of the frame. We now describe the algorithm more precisely.

Assume for the moment that there exists a choice of $\underline{s}$, say $\underline{s}_0$ such that (154) – (156) hold for some $G^* > \lambda$. Let $a_0(\cdot)$ be the corresponding choice for $a(\cdot)$, i.e.

$$\underline{s}_0 = \begin{pmatrix} \ln a_0(0) \\ \ln a_0(1) \\ \ln a_0(2) \end{pmatrix}$$

The algorithm we describe for cases 2b), 3b) is identical to that described in Chapter 2 except that the f_k's are not computed as in (6) but as follows:

$$f_0 = \beta$$

$$f_{kM+i} = \frac{1-\lambda}{G^*-\lambda} f_{kM} \qquad i = 1,2,\ldots,M-1 \tag{157}$$

$$f_{(k+1)M} = \min\{\beta, f_{kM} \cdot a_0^{\gamma}(Z_{kM})\} \tag{158}$$

Note that this algorithm requries frame synchronization and is non-stationary. Note also the algorithm only uses the output Z_k from the DMC only every M slots.

Define $\tilde{\varphi}_k$ as

$$\tilde{\varphi}_k = \ln((N_{kM} \vee 1) f_{kM})$$

Intuitively, as in Chapter 2, for N_{kM} large and f_{kM} small we have

$$E[\tilde{\varphi}_{k+1} - \tilde{\varphi}_k \mid \mathcal{F}_{kM}] \approx \gamma\, m(\tilde{\varphi}_k) \tag{159}$$

Noting (154) - (156) we see the algorithm attempts to steer $\tilde{\varphi}_k$ toward $\tilde{\varphi}^*$. If $\tilde{\varphi}_k$ is near $\tilde{\varphi}^*$ then $N_{kM} f_{kM}$ is near $G^* - \lambda$. For $i = 1,2,\ldots,M-1$

$$N_{kM+i} f_{kM+i} = N_{kM+i} f_{kM} \frac{1-\lambda}{G^*-\lambda}$$

$$\approx N_{KM} f_{KM} \cdot \frac{1-\lambda}{G^*-\lambda}$$

where the approximation holds since N_{KM} is large, f_{KM} is small. Since $N_{KM} f_{KM}$ is near $G^* - \lambda$ we conclude that $N_{KM+i} f_{KM+i}$ for $i = 1, 2, \ldots, M-1$ is near $1 - \lambda$.

The stability of this algorithm can be shown in a manner very similar to that of the proof of stability in Chapter 2. Specifically if we define

$$\tilde{V}_K = N_{KM} + \tilde{r} g_{\tilde{\Delta}} (\tilde{\phi}_K - \tilde{\phi}^*)$$

then for suitable choices of $\tilde{r}$ and $\tilde{\Delta}$ we can show that $\tilde{V}_K$ satisfies the hypothesis of Section 2.2.2. The proof would proceed in a manner almost identical to that in Chapter 2. It is seen that for a frame length of length M, then stability of this algorithm is guaranteed for $\lambda < (1-\frac{1}{M})\bar{e}^{1}$ and sufficiently small γ.

We now show that there exists a choice of $\underline{s}$ in cases 2b), 3b) such that (154) - (156) holds. By the same arguments used to derive (143) - (145), sufficient conditions for (154) - (156) to be satisfied for some G^* are:

$$\ell(\lambda) > 0 \tag{160}$$

$$\lim_{G\to\infty} \ell(G) < 0 \tag{161}$$

(160), (161) are equivalent to

$$(h_0-h_2)\bar{e}^{\lambda}+(h_1-h_2)\lambda\bar{e}^{\lambda}+h_2>0 \qquad (162)$$

$$h_2<0 \qquad (163)$$

Consider case 2b). By the same arguments used to prove lemma 16, it is seen that $\underline{c}$ can be chosen so that for any u_1,

$$\frac{h_1}{h_2}=\frac{d_1\cdot\underline{c}}{d_2\cdot\underline{c}}=-u_1 \qquad (164)$$

$$h_2=d_2\cdot\underline{c}=-1 \qquad (165)$$

(165) implies (163). By substituting (164), (165) into the left hand side of (162) we have

$$(\alpha' u_1+1-\beta')\bar{e}^{\lambda}+(u_1+1)\lambda\bar{e}^{\lambda}-1 \qquad (166)$$

For sufficiently large u_1 the expression in (166) can be made positive. Thus for case 2b) there exists a choice of $\underline{c}$ satisfying (162) and (163) and hence (154) - (156) for some $G^*>\lambda$.

Consider case 3b). For any u_2, $\underline{c}$ can be chosen so that

$$\frac{h_1}{h_0}=\frac{d_1\cdot\underline{c}}{d_0\cdot\underline{c}}=u_2 \qquad (167)$$

$$h_2 = (\alpha' + \beta' u_2) h_0 = (\alpha' + \beta' u_2) d_0 \cdot \underline{c} = -1 \tag{168}$$

(168) implies (163). By substituting (167) and (168) into the left hand side of (162) we have

$$e^{-\lambda} + \lambda e^{-\lambda} - \frac{e^{-\lambda} + \lambda e^{-\lambda} u_2}{\alpha' + \beta' u_2} - 1 \tag{169}$$

Since $\alpha' \neq \frac{1}{1+\lambda}$ in case 3b) it is seen that the expression in (169) can be made positive for some choice of u_2. Thus in case 3b) there exists a choice of $\underline{c}$ so that (162), (163) hold and hence (154) - (156) for some $G^* > \lambda$.

In case 3c) the expression in (169) is independent of u_2 and is negative. Thus no choice of $\underline{c}$ exists satisfying (162), (163). However we can modify the algorithm so that effectively $\lambda = 0$ in the expression in (169). This is done as follows. When a packet is generated at a station that packet is transmitted in the slot in which it was generated, except if it was generated in a slot numbered $kM - 1$ (k is any positive integer). If a packet is generated in a slot numbered $kM - 1$ then the packet is transmitted in slot $kM + 1$. The result of this is that no new packets are transmitted in slots kM, $k = 1, 2, 3, \ldots$. Packets that have collided retransmit in slot k with probability f_k. The f_k's are computed as

$$f_0 = \beta \tag{170}$$

$$f_{KM+1} = \frac{1-2\lambda}{\tilde{G}^*} f_{KM} \tag{171}$$

$$f_{KM+i} = \frac{1-\lambda}{\tilde{G}^*} f_{KM} \qquad i=2,3,..,M-1 \tag{172}$$

$$f_{(K+1)M} = \min\{\beta, f_{KM} \cdot a^{\gamma}(z_{KM})\} \tag{173}$$

In (171), (172) $\tilde{G}^*$ is a constant which is described below. We have for large N_{KM} small f_{KM},

$$E[\tilde{\varphi}_{K+1} - \tilde{\varphi}_K \mid \mathcal{F}_{KM}] \approx \hat{m}(\tilde{\varphi}_K) \tag{174}$$

where $\hat{m}(\varphi) = \ell(e^{\varphi})$. In case 3c) it is easily verified that $\underline{c}$ can be chosen so that

$$\ell(0) > 0 \tag{175}$$

$$\lim_{G\to\infty} \ell(G) < 0 \tag{176}$$

By the arguments preceding (146), this implies that $\ell(G)$ has exactly one root for $G \in (0,\infty)$. Let $\tilde{G}^*$ be that root. Defining $\hat{\varphi}^* = \ln \tilde{G}^*$ we have:

$$\hat{m}(\varphi) > 0 \qquad \text{for} \quad \varphi < \hat{\varphi}^*$$

$$\hat{m}(\hat{\varphi}^*)=0$$

$$\hat{m}(\varphi)<0 \quad \text{for } \varphi > \hat{\varphi}^*$$

Thus by (174) $\tilde{\varphi}_K$ is steered towards $\hat{\varphi}^*$. The "scaling" factors $\frac{1-2\lambda}{\tilde{G}^*}$, $\frac{1-\lambda}{\tilde{G}^*}$ in (171), (172) have been appropriately choosen so that if $\tilde{\varphi}_K$ is near $\hat{\varphi}^*$, $N_{KM+i}\, f_{KM+i}$ will be near their desired values for $i=1,2,\ldots,M-1$.

Thus, in conclusion, there exist stable algorithms when the DMC described by P has non-zero capacity.

CHAPTER FOUR

A More Restrictive Feedback Model

Consider the following model for feedback to the stations. At the end of slot K, $K=1,2,3,\ldots$ each station observes $\tilde{Z}_K$ through a DMC. The DMC's for any distinct stations are independent. That is, the feedback received in slot K by individual stations are independent conditioned on the value of $\tilde{Z}_K$. The channel error probability matrix for each station is identical. We make the additional restriction that stations do not listen to the channel output prior to the arrival of a message to be transmitted (we assume a station generates at most one message during its lifetime), so that stations cannot use the same retransmission probability, as assumed in Chapters 2 and 3.

We now describe an algorithm which the stations can follow for this feedback model. The algorithm is identical to the one described in Chapter 2 except that when a station transmits a packet for the first time and that message collides, the initial retransmission probability for that station is set at β. There-after the station uses (6) to compute the retransmission probabilties.

Suppose we index the stations which will have a packet to transmit by the integers. Let $f_{K,i}$ be retransmission probability for station i in slot K (define $f_{K,i} = 0$ if station i does not have a packet to transmit in slot K).

Let

$$S_K = \sum_i f_{K,i}$$

where the summation includes all i such that station i is active at time K.

Intuitively, we would like to choose the function $a(\cdot)$ in (6) to steer S_K towards $1-\lambda$. If the algorithm is to be stable then when N_K gets "large" the algorithm must be able to keep S_K "near" $1-\lambda$ for "long" amounts of time, "long" enough so that N_K will decrease to the point where it is no longer "large." Thus β must be small enough to guarantee that S_K will not change significantly when a message leaves the system, or when a message enters the system.

We had initially hoped to prove stability of this algorithm by using a Liapunov function of the form in (35) with m_K replaced by:

$$\ln(S_K \vee \underline{s}) - \ln(1-\lambda) \quad \text{for some constant } \underline{s}.$$

The reason this fails is that because of the max in (6) S_K will not increase if all the retransmission proabilities $f_{K,i}$ at time K are saturated at β. Thus we cannot guarantee that S_K increases when $S_K < 1-\lambda$. Perhaps it is possible to construct a Liapunov function to demonstrate stability, but it is evident that the Liapunov function must depend not only on S_K, but also on the individual retransmission probabilties $f_{K,i}$. It seems reasonable that the algorithm described above is stable, although we have been unsuccessful in proving so.

CHAPTER FIVE

A Bound of One-Half for the Throughput of First Come First Served Protocols

5.1 Introduction

An upper bound of one-half is established for the throughput of a time slotted multi-access broadcast channel subject to an infinite population of user stations (whose message generation times are modelled by a Poisson process) using (0), (1), (e) - feedback to denote a slot with none, one, or at least two packets, respectively. The upper bound applies to any algorithm which has the property that packets are successfully transmitted in the order that they are generated.

5.2 Model

The generation of information packets by remote stations will be modelled by a Poisson point process $\{0 \le \alpha_1 < \cdots < \alpha_W \le T\}$ on the time interval [0, T] with intensity $\lambda > 0$. Each of the packets can thus be identified with the time that it was generated. A conflict resolution algorithm (CRA) is a protocol that the remote stations follow to access a central broadcast channel. Time is divided into slots numbered $K = 1, 2, \ldots$. At the beginning of the K^{th} slot, the CRA designates a subset Θ_K of [0, T]. All packets in Θ_K that were not successfully transmitted in slots 1,2, ..., k - 1 are then transmitted during that slot. A transmitted packet is successfully transmitted only if no other packet is transmitted in the same slot. If more than one packet is transmitted on a slot, the packets "collide" and must be retransmitted.

It is assumed that, by listening to the channel output during slot K, each station learns $\hat{z}_K$ where $\hat{z}_K = 0$, $\hat{z}_K = 1$, or $\hat{z}_K = e$ depending on whether zero, one, or more than one packet was transmitted during the K^{th} slot. The set θ_K specified by the CRA is required to be a function of the past channel information $(\hat{z}_1, \ldots, \hat{z}_{K-1})$ so that the algorithm can be implemented in a distributed fashion.

A CRA is completed when it becomes known that all packets have been successfully transmitted. Let Υ be the (random) number of slots until completion. The efficiency of a CRA is defined to be $\eta = \frac{E[W]}{E[\Upsilon]}$. A CRA is called first come-first serve (FCFS) if packets are always successfully transmitted in the order that they were generated. For example, see [5] and [6] in which FCFS CRA's are described with efficiencies of about .488. We will prove the following theorem.

Theorem:

$\eta \leq 1/2$ for any FCFS CRA.

The tightest known upper bound for any CRA (not necessarily FCFS) is about .587 [14]. The proof of the theorem proceeds in the same manner as the proof of the theorem in [13]. One idea of the proof, suggested first by Molle [12], is to consider CRA's which use certain additional information which is not provided under the original channel model. More specifically, at the end of slot K all stations are told a subset a_K of the set $\{\alpha_1, \ldots, \alpha_W\}$ of packet locations, where a_K is determined according to the rules specified below. We will then allow the set θ_K chosen by a CRA to depend on $\hat{z}_1, \ldots, \hat{z}_{K-1}, a_1, \ldots, a_{K-1}$. Since the auxiliary information could be ignored, any upperbound on the efficiency of such possibly unrealizable algorithms is also an upper bound on the efficiency of the original class of algorithms.

5.3 Description of the Auxiliary Information

Let Ψ_K, $\Psi_K \subset \{\alpha_1, \ldots, \alpha_W\}$, be the arrival times of all packets which still remain to be transmitted at the end of K-1 slots. For example, $\Psi_1 = \{\alpha_1, \ldots, \alpha_W\}$. For any set A, $A \subset [0, T]$, let $N_K(A)$ be the cardinality of the set $A \cap \Psi_K$. We define $\mathcal{F}_K = \sigma(\hat{z}_1, \ldots, \hat{z}_K, a_1, \ldots, a_K)$ to be the σ-field generated by $\hat{z}_1, \ldots, \hat{z}_K, a_1, \ldots, a_K$.

The rules for specifiying the sets a_i have been chosen to satisfy the following important property. For any FCFS CRA and at the beginning of the K^{th} slot, the conditional distribution of Ψ_K given $\mathcal{F}_{K-1}$ is of the following form:

i). The elements of Ψ_K are known to lie in $\beta \cup \sigma_i \cup F$, where β, σ_i, and F are disjoint sets.

ii). The arrival times of m packets are known. β consists of the arrival times of these m packets.

iii). If σ_i is not empty then $\mu(\sigma_i) > 0$ and the arrival times in σ_i form a Poisson point process with intensity λ conditioned on $N_K(\sigma_i) \geq i$. The permissible values for i are i = 1 or i = 2 ($\mu(\cdot)$ denotes Lebesque measure).

iv). Each of the m known arrival times are greater than any arrival time in σ_i (See Figure 1). Any arrival time in F is greater than any arrival time in σ_i.

v). The arrival times contained in F form a Poisson point process with intensity λ.

vi). Arrival times contained in disjoint sets σ_i , F are independent.

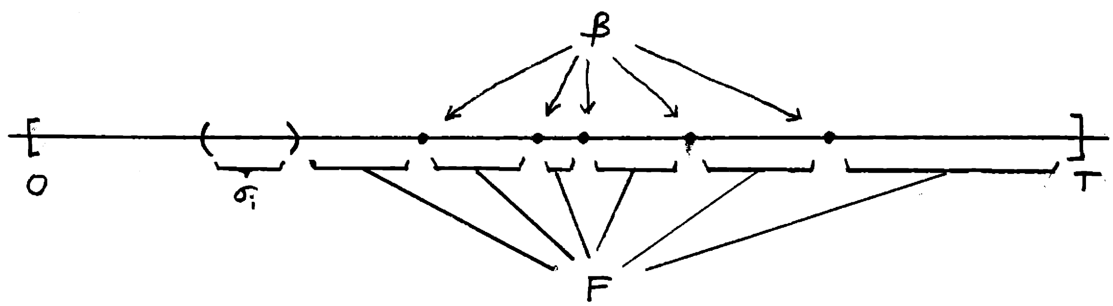

Fig. 1. Illustration of the relationship between σ_i , F , and β.

After each slot we can describe the "state" of any FCFS CRA by a variable γ_K , where γ_K is defined below. In the definition $m = |\beta|$ and β , σ_i are as above.

$$\gamma_k = \begin{cases} m & \text{— if the conditional distribution of } \Psi_k \text{ given } \mathcal{F}_{k-1} \text{ is such that the location of } m \text{ arrival times are known, and the remaining elements of } \Psi_k \text{ are distributed as a Poisson point process.} \\ ((x,i),m) & \text{— if the conditional distribution of } \Psi_k \text{ given } \mathcal{F}_{k-1} \text{ is such that the location of } m \text{ arrival times are known, the arrival times in } \sigma_i \text{ are a Poisson point process conditioned on } N_k(\sigma_i) \geq i,\ x = \lambda\cdot\mu(\sigma_i) \text{ and the remaining elements of } \Psi_k \text{ are distributed as a Poisson point process.} \end{cases}$$

It is seen that γ_k is simply a parametric representation of the conditional distribution of Ψ_k given $\mathcal{F}_{k-1}$.

Any set Θ_k which a CRA can specify for slot k can be expressed as:

$$\Theta_k = \beta' \cup \sigma_i' \cup F'$$

where

$$\beta' \subset \beta$$

$$F' \subset F$$

$$\sigma_i' \subset \sigma_i$$

For any FCFS CRA the set Θ_k must satisfy at least one of the following:

i) Θ_k is an interval with left endpoint $\inf_x (x \in \beta \cup \sigma_i \cup F)$

ii) $\sigma_i' = \sigma_i$

iii) $N_k(\beta') \geq 2$

The rules which specify the sets a_i are stated below. Note that a_k depends on γ_{k-1}, $\hat{z}_k$, and Θ_k. For convenience to the reader in verifying that the conditional distribution of ψ_k given $\mathcal{F}_{k-1}$ is of the form stated above we also state the value of γ_k for each case below. Note that $\gamma_0 = 0$. Let $t = \lambda \cdot \mu(\theta_k)$.

Case 1: $\gamma_{k-1} = m$

A) If $N_k(\beta') \geq 2$ then $a_k = \phi$. Note we must have $\hat{z}_k = e$, thus $\gamma_k = \gamma_{k-1}$.

B) If $N_k(\beta') = 1$ then

i) if $\hat{z}_k = 1$ then $a_k = \phi$ and $\gamma_k = m-1$

ii) if $\hat{z}_k = e$ then $a_k = \{\min(\Theta_k \cap (\psi_k - \beta'))\}$ and $\gamma_k = m+1$

C) If $N_k(\beta') = 0$ then $a_k = \phi$

i) $\gamma_k = \gamma_{k-1}$ if $\hat{z}_k = 0$ or $\hat{z}_k = 1$

ii) $\gamma_k = ((\lambda \cdot \mu(\theta_k), 2), m)$ if $\hat{z}_k = e$

Case 2: $\gamma_{k-1} = ((x,1), m)$

A. If $N_k(\beta') > 0$ then $a_k = \phi$. If $N_k(\beta') = 1$ we must have $\Theta_k \supset \sigma_1$ for any FCFS CRA. Thus $\hat{z}_k = e$ and $\gamma_k = \gamma_{k-1}$.

B). If $N_k(\beta') = 0$ then

i) If $t \leq x$ then $a_k = \phi$ (see Figure 2)

a) if $\hat{z}_k = 0$ then $\gamma_k = ((x-t, 1), m)$

b) if $\hat{z}_K = 1$ then $\gamma_K = m$

c) if $\hat{z}_K = e$ then $\gamma_K = ((t,2),m)$

ii) If $t > x$ then

a) if $\hat{z}_K = 1$ then $a_K = \phi$ and $\gamma_K = m$

b) if $\hat{z}_K = e$ then $a_K = \{\min((\Theta_K - \sigma_1) \cap \Psi_K)\}$

1) $\gamma_K = ((x,2),m)$ if a_K is empty

2) $\gamma_K = ((x,1),m+1)$ if a_K is not empty.

Case 3: $\gamma_{K-1} = ((x,2),m)$

A) If $t \geq x$ then $a_K = \phi$. Note that Θ_K must contain σ_2 for any FCFS CRA, thus $\hat{z}_K = e$ and $\gamma_K = \gamma_{K-1}$.

B) If $t < x$ then $a_K = \phi$

i) if $N_K(\beta') = 0$ then (see Figure 2)

a) if $\hat{z}_K = 0$ then $\gamma_K = ((x-t,2),m)$

b) if $\hat{z}_K = 1$ then $\gamma_K = ((x-t,1),m)$

c) if $\hat{z}_K = e$ then $\gamma_K = ((t,2),m)$

ii) if $N_K(\beta') > 0$ then for any FCFS CRA we must have $N_K(\beta') \geq 2$. Thus $\hat{z}_K = e$ and $\gamma_K = \gamma_{K-1}$.

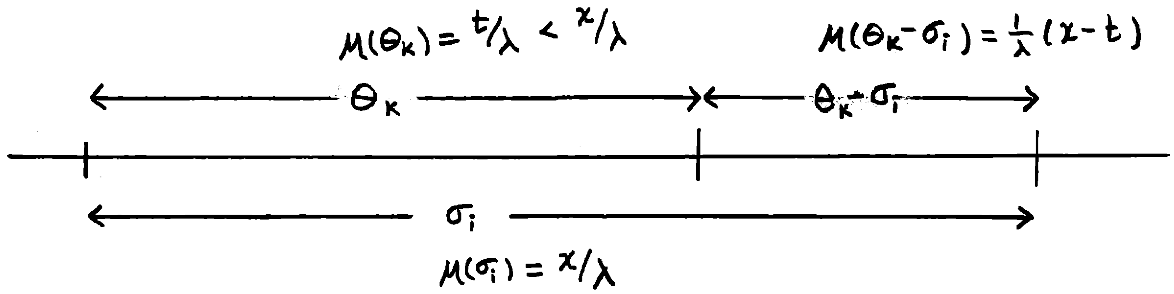

Fig. 2. Illustration of the sets σ_i, Θ_K for Cases 2 and 3 with $N_K(\beta') = 0$ and $t < x$

The bound of one-half we obtain is the best possible bound for FCFS CRA's which have the previously defined auxiliary information available, as there exists a FCFS CRA with efficiency approaching one-half as $T \to \infty$. The following idea can be used to construct a CRA with an asymptotic efficiency of one-half. Suppose the state at time K is $\gamma_K = ((x,1), m)$ and the CRA chooses $\Theta_K = \sigma_1 \cup F$, with $\frac{\mu(F)}{T} \approx 1$. If F contains any arrival times, the set a_K will contain the first arrival time in F. This process can be repeated until all arrival times after those in σ_1 are known. After the messages in σ_1 are successfully transmitted, the Θ_K's can be chosen by the CRA so that a successful transmission occurs in every slot until the algorithm terminates.

5.4 Proof of the Theorem

Define a real valued function V defined on the set of values that γ_K can take as follows:

$$V(\gamma) = \begin{cases} V_1(x) + \frac{1}{2}m & , \text{if } \gamma = ((x,1),m) \\ V_2(x) + \frac{1}{2}m & , \text{if } \gamma = ((x,2),m) \\ \frac{1}{2}m & , \text{if } \gamma = m \end{cases}$$

where

$$V_1(x) = \begin{cases} \dfrac{xe^{-x}}{2(1-e^{-x})} & , \text{if } 0 < x \le 1 \\[2ex] \dfrac{1}{2e(1-e^{-x})} & , \text{if } x > 1 \end{cases}$$

$$V_2(x) = \max\left[\sup_{0<y<x} \left(\frac{ye^{-y}(1-e^{y-x}) + y(x-y)e^{-x}}{2(1-(1+x)e^{-x})} \right), \right.$$

$$\left. \sup_{0<y<x-1} \left(\frac{ye^{-y}(1-e^{y-x}) + ye^{-y}e^{-1}}{2(1-(1+x)e^{-x})} \right) \right]$$

We define the supremum over an empty set to be zero. Now suppose a particular

FCFS CRA is chosen. The algorithm can be extended to slots beyond τ by setting $\Theta_K = \phi$ for $K > \tau$.

Let

$$Y_K = V(\gamma_K) + \sum_{j=1}^{K} I_{\{\hat{z}_j = 1\}}$$

where I_A denotes the indicator random variable of the event A.

Lemma: For all $K \geq 1$, $E[Y_K - Y_{K-1} \mid \mathcal{F}_{K-1}] \leq 1/2$.

Assuming the lemma for a moment, we first establish the theorem. Clearly $Y_\tau = W$ so that $E[Y_\tau] = E[W]$. On the other hand,

$$\begin{aligned}
E[Y_\tau] &= E[\sum_{j=1}^{\tau} Y_j - Y_{j-1}] \\
&= E[\sum_{j=1}^{\infty} (Y_j - Y_{j-1}) I_{\{j \leq \tau\}}] \\
&= E[\sum_{j=1}^{\infty} E[(Y_j - Y_{j-1}) I_{\{j \leq \tau\}} \mid \mathcal{F}_{j-1}]] \\
&= E[\sum_{j=1}^{\infty} E[(Y_j - Y_{j-1}) \mid \mathcal{F}_{j-1}]\, I_{\{j \leq \tau\}}] \\
&\leq E[\sum_{j=1}^{\infty} \tfrac{1}{2} I_{\{j \leq \tau\}}] \\
&= \tfrac{1}{2} \sum_{j=1}^{\infty} \mathrm{Prob}(j \leq \tau) \\
&= \tfrac{1}{2} E[\tau]
\end{aligned}$$

The fourth equality is justified by the fact that $\{ j \leq \tau \}$ is an $\mathcal{F}_{j-1}$ measurable event since $\{ j \leq \tau \} = \{ \tau \leq j-1 \}^c$. Hence

$$\frac{E[W]}{E[\tau]} \leq \frac{1}{2}$$

which proves the theorem.

Proof of Lemma

Let $\Delta = E[Y_k - Y_{k-1} \mid \mathcal{F}_{k-1}]$, $t = \lambda \cdot \mu(\theta_k)$

Case 1: $\gamma_{k-1} = m$

A) $N_k(\beta') \geq 2 \ : \ \Delta = 0$

B) $N_k(\beta') = 1 \ : \ \Delta = \frac{1}{2} e^{-t} + \frac{1}{2}(1 - e^{-t}) = 1/2$

C) $N_k(\beta') = 0 \ :$

$$\Delta = te^{-t} + (1 - (1+t)e^{-t}) V_2(t)$$

$$= \max \Big[\sup_{0<y<t} \left(te^{-t} + \tfrac{1}{2} y e^{-y} (1 - e^{y-t}) + \tfrac{1}{2} y (t-y) e^{-t} \right),$$

$$\sup_{0<y<t-1} \left(te^{-t} + \tfrac{1}{2} y e^{-y} (1 - e^{y-t}) + \tfrac{1}{2} y e^{-y} e^{-1} \right) \Big]$$

To evaluate $\sup_{0<y<t} \left(te^{-t} + \frac{1}{2} y e^{-y} (1 - e^{y-t}) + \frac{1}{2} y(t-y) e^{-t} \right)$ we fix y and examine the t , if any, that maximizes $f(t,y) \triangleq te^{t} + \frac{1}{2} y e^{-y} (1 - e^{y-t}) + \frac{1}{2} y (t-y) e^{-t}$.

$$\frac{\partial}{\partial t} f(t,y) = \frac{\partial}{\partial t}\left[e^{-t}\left(t(1+\tfrac{1}{2}y) - \tfrac{1}{2}(y+y^2)\right)\right]$$

$$= 0 \text{ only for } t = 1 + \frac{y+y^2}{y+2} > y$$

Hence we have

$$\sup_{\substack{y>0\\ t>y}} f(t,y) \le \max\left\{\sup_{y>0} f\left(1+\tfrac{y+y^2}{y+2}, y\right), \sup_{y>0} f(y,y), \sup_{y>0} f(\infty,y)\right\}$$

where $f(\infty,y) = \lim_{t\to\infty} f(t,y)$. Now $\sup_{y>0} f(1+\frac{y+y^2}{y+2}, y)$ can be easily evaluated numerically. This yields

$$\sup_{y>0} f\left(1+\frac{y+y^2}{y+2}, y\right) \approx .4953 \le 1/2$$

$$\arg\sup_{y>0} f\left(1+\frac{y+y^2}{y+2}, y\right) \approx .61$$

Also it is easily established that $\sup_{y>0} f(y,y) = e^{-1} \le 1/2$ whereas $\sup_{y>0} f(\infty,y) = \frac{1}{2e} \le 1/2$ so that $\sup_{0<y<t} f(t,y) \le 1/2$.

Similarly, letting $g(t,y) = te^{-t} + \frac{1}{2}ye^{-y}(1-e^{y-t}) + \frac{1}{2}ye^{-y}e^{-1}$ we find

$$\sup_{0<y<t-1} g(t,y) = \max\left[\sup_{y>0} g(y+1,y), \sup_{y>0} \lim_{t\to\infty} f(t,y)\right]$$

$$= \sup_{y>0} g(y+1,y)$$

$$= \left(\tfrac{1}{2} + e^{-1}\right) e^{-\left(\frac{1}{1+2e^{-1}}\right)} \approx .4878 \le 1/2$$

Case 2: $\gamma_{k-1} = ((x,1),m)$

Fact: 1) $V_1(x) \le 1/2 \quad \forall x > 0$

2) $V_2(x) \le 1/2 \quad \forall x > 0$

1) is easily seen. 2) Follows easily once we note that

$$V_2(x) \le \sup_{0<y<x} \left(\frac{y e^{-y}(1-e^{y-x})}{1-(1+x)e^{-x}} \right)$$

A) $N_k(\beta') > 0 \quad : \Delta = 0$

B) $N_k(\beta') = 0$

i) $t \le x$:

$$\Delta = \frac{e^{-t}(1-e^{t-x})}{1-e^{-x}} V_1(x-t) + \frac{t e^{-t}}{1-e^{-x}} + \frac{1-(1+t)e^{-t}}{1-e^{-x}} V_2(t) - V_1(x)$$

$$\le \frac{1}{2} + \frac{1}{2} \cdot \frac{t e^{-t}}{1-e^{-x}} - V_1(x)$$

$$\le 1/2$$

The first inequality follows from the facts above.

ii) $t > x$:

$$\Delta = \frac{x e^{-x} e^{x-t}}{1-e^{-x}} \left[1 - V_1(x)\right] + \frac{1}{2}\left[1 - e^{x-t}\right] + \frac{(1-(1+x)e^{-x})e^{x-t}}{1-e^{-x}} \left[V_2(x) - V_1(x)\right]$$

$$\frac{\partial}{\partial t}\Delta = e^{x-t}\left[\tfrac{1}{2} - \frac{xe^{-x}}{1-e^{-x}}(1-V_1(x)) - \left(1-\frac{xe^{-x}}{1-e^{-x}}\right)\left(V_2(x)-V_1(x)\right)\right]$$

$$\geq e^{x-t}\left[\tfrac{1}{2} - \frac{xe^{-x}}{1-e^{-x}}(1-V_1(x)) - \left(1-\frac{xe^{-x}}{1-e^{-x}}\right)\left(\tfrac{1}{2}-V_1(x)\right)\right]$$

$$= e^{x-t}\left[V_1(x) - \tfrac{1}{2}\cdot\frac{xe^{-x}}{1-e^{-x}}\right]$$

$$\geq 0$$

But $\lim_{t\to\infty}\Delta = 1/2$

Case 3:

A) $t \geq x$: $\Delta = 0$

B) $t < x$:

i) $N_K(\beta') = 0$:

$$\Delta = \frac{e^{-t}(1-(1+x-t)e^{t-x})}{1-(1+x)e^{-x}}V_2(x-t) + \frac{te^{-t}(1-e^{t-x})}{1-(1+x)e^{-x}}(1+V_1(x-t)) + \frac{1-(1+t)e^{-t}}{1-(1+x)e^{-x}}V_2(t) - V_1(x)$$

$$\leq \frac{1}{2} + \frac{1}{2}\,\frac{te^{-t}(1-e^{t-x})}{1-(1+x)e^{-x}} + \frac{te^{-t}(1-e^{t-x})}{1-(1+x)e^{-x}}V_1(x-t) - V_1(x)$$

$$\leq 1/2$$

The first inequality follows from fact 2) above.

ii) $N_K(\beta') > 0$: $\Delta = 0$

QED

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