INFINITE DIMENSIONAL FLAG VARIETIES

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ABSTRACT

To every Coxeter group W, one can associate its Hecke algebra $\mathcal{H}$ over the polynomial ring $\mathbf{Z}[q]$, which can be thought of as a deformation of the group algebra over $\mathbf{Z}$ of W. In the case where W is a Weyl group, $\mathcal{H}$ can also be interpreted, using a canonical basis, as the algebra of intertwinig operators of the space of functions on the flag variety of the corresponding Chevalley group. In [KL], Kazhdan and Lusztig extend the ground ring of $\mathcal{H}$ to be $\mathbf{Z}[q^{1/2},q^{-1/2}]$, and use a different basis to study representations of $\mathcal{H}$ (and W). In case W is a Weyl group, the entries of the matrix of change of basis (the Kazhdan-Lusztig polynomials) turn out to have have far reaching interpretations in terms of the geometry and representation theory of the corresponding Lie algebra and groups.

The present work grew out of an attempt to generalize the geometric interpretations of the Kazhdan-Lusztig constructions to the case where W is a crystallographic group (i.e. the order of the product of a pair of generators is 2,3,4,6 or ∞). The link with the classical theory is the fact that crystallographic groups are precisely the 'Weyl groups' of Kac-Moody Lie algebras.

Section 1 contains a construction of Tits's $\mathbf{Z}$-form of the universal enveloping algebra of a Kac-Moody algebra $\mathfrak{g}$ (or, more appropriately, one of Tits's $\mathbf{Z}$-forms). The idea is as in [K], and it allows one to extend coefficients to an arbitrary field F (of any characteristic), and associate to $\mathfrak{g}$ a 'Kac-Moody group' G over F, which is done in section 2. In case F is finite, the adjoint group of G has already been constructed in [MT]. The construction here is basically the same as the one Peterson and Kac carry out for F of characteristic 0 ([PK]), and has the advantage over Tits's of being in the spirit of the classical theory as developped in [S3], a fact which allows a more tractable study of the structure and representations of G. In case $\mathfrak{g}$ is finite dimensional, the group G is a classical Chevalley-type finite dimensional group, while if $\mathfrak{g}$ is not, then G is infinite dimensional in the sense that it contains subgroups of arbitrarily large finite dimension. As most of the proofs of the structural facts about the group constructed in [PK] use a completion of a subalgebra of $\mathfrak{g}$ and the ability to exponentiate its elements (facts which present essential difficulties in positive characteristic), the results in §2 are weaker than those in [PK1](although the group is the same one if charF=0). However, they afford a different presentation of the flag variety, the study of which is taken up in section 3.

As in [PK], the flag variety G/B proves to be a Bruhat-type (BwB) disjoint union, indexed by W, of (finite-dimensional) affine cells C(w), each of which admits a 'Schubert variety' (still finite-dimensional) as its closure. the geometry of thesevarieties is studied by: 1) generalizing the results of Dheodar ([Dh]): after reproving the lemmas in [Dh1] in the present setting using the Birkhoff decomposition ($B_{-}wB$), the results of [Dh2-4] carry over almost verbatim; and 2) adapting a construction of Demazure's ([D]), the main tool for which is the fact that 'a

codimension-1 piece of the Borel common to two adjacent minimal parabolics acts trivially', so that one can construct a (finite-dimensional) 'resolution' Z_w as in [D1] to a closed subset of projective space which is identified, using the Borel fixed point theorem, as being $\bigcup_{y\leq w} C(y)=X_w$, for $w\in W$.

These facts are applied in section 4 to:

1) showing that X_w is always non-singular in codimension 1,

2) giving geometric interpretations to some of the Kazhdan-Lusztig constructions for crystallographic W, using the positive characteristic approach of [KL](the methods of [S] do not seem amenable to direct generalisation because the G orbits in $(G/B)^2$ are not finite dimensional),

3) a study of the case where the reduced expressions of $w\in W$ consist of distinct reflexions (e.g. X_w is then smooth),

and 4) an explanation (and proof) of a remark made in [T3], which suggest that no further generalisation (to arbitrary Coxeter groups) can be made.

Thesis Supervisor: Dr. Victor G. Kac, Professor of Mathematics.

Section 1 : Construction of the Enveloping Algebras $\mathcal{U}_{\mathbf{Z}}, \mathcal{U}_K$, and associated Modules.

Start with an NxN generalized Cartan matrix; that is, A is an NxN matrix with entries a_{ij} , $1\leq i,j\leq N$, satisfying:

$a_{ii}=2$ for all i,

$a_{ij}\in \mathbf{Z}^-$ if $i\neq j$,

with $a_{ij}=0$ if and only if $a_{ji}=0$.

We assume for simplicity that A is indecomposable, i.e. that $A\neq\begin{pmatrix} A_1 & 0 \\ 0 & A_2\end{pmatrix}$ after any reordering of the index set.

Fix a $\mathbf{Q}$-vector space h of dimension N + corank(A); it can then be shown ([K1]) that there exists subsets $\Pi\subset h^*, \Pi^\vee\subset h$, where $\Pi=\{\alpha_i, 1\leq i\leq N\}$, $\Pi^\vee=\{\alpha_i^\vee, 1\leq i\leq N\}$, such that:

Π is a linearly independent subset of h^*,

$\Pi^\vee$ is a linearly independent subset of h,

and $\langle\alpha_j,\alpha_i^\vee\rangle=a_{ij}$.

The Kač-Moody algebra $g=g_{\mathbf{Q}}(A)$ is the Lie algebra over Q generated by h and elements e_i, f_i, $1\leq i\leq N$, with "defining relations":

$[h,h]=0$,

$[e_i,f_j]=\delta_{ij}\alpha_i^\vee$,

$[h,e_i]=\langle\alpha_i,h\rangle e_i$, for all $h\in h$,

$[h,f_i]=-\langle\alpha_i,h\rangle f_i$, for all $h\in h$,

and $(\mathrm{ad}\ e_i)^{1-a_{ij}}(e_j) = (\mathrm{ad}\ f_i)^{1-a_{ij}}(f_j) = 0$ whenever $i\neq j$.

It can then be shown that, identifying h, e_i, f_i with their respective images in g, if $n_+=$ subalgebra of g generated by e_i, $1\leq i\leq N$, and $n_-=$ subalgebra of g generated by f_i, $1\leq i\leq N$, then we have ([K1]):

<u>1.0 Proposition</u> :

$g=n_-\oplus h\oplus n_+$ (the triangular decomposition),

$[g,g]=g'$ is generated by e_i, f_i , $1\leq i\leq N$,

$h\cap g'=\bigoplus_{i=1}^{N} \mathbf{Q}\alpha_i^\vee$,

h is its own centralizer in g ,

the center of g is contained in $h\cap g'$,

The abelian subalgebra h acts, by the adjoint representation, completely reducibly on $\mathfrak{g}$, so that $\mathfrak{g}=\oplus\mathfrak{g}_\alpha, \alpha\in\mathfrak{h}^*$, where $\mathfrak{g}_\alpha=\{x\in\mathfrak{g}$ such that $[h,x]=\langle\alpha,h\rangle x$ for all $h\in\mathfrak{h}\}$, and we have $\mathfrak{g}_o=\mathfrak{h}$ as noted in 1.0 .

Now, since $\mathfrak{h}$ preserves the decomposition 1.0, one sees that if $\mathfrak{g}_\alpha\neq 0$, then either $\mathfrak{g}_\alpha\subset\mathfrak{h}$, or $\mathfrak{g}_\alpha\subset\mathfrak{n}_+$, or $\mathfrak{g}_\alpha\subset\mathfrak{n}_-$. Writing $Q_+=\bigoplus_{i=1}^{N}\mathbf{N}\alpha_i\ \varepsilon\mathfrak{h}^*$, $Q_- = -Q_+$, this implies that if $\mathfrak{g}_\alpha\neq 0$, either $\alpha=0$, or $\alpha\in Q_\pm-\{0\}$, in which case $\mathfrak{g}_\alpha\varepsilon\mathfrak{n}_\pm$; so that if $\Delta=\{\alpha\varepsilon\mathfrak{h}^*|\mathfrak{g}_\alpha\neq 0\}$ is the set of roots, then $\Delta=\Delta_+\amalg\Delta_-$, where $\Delta_\pm=\Delta\cap Q_\pm$.

For each $1\le i\le N$, define $r_i\in GL(\mathfrak{h}^*)$ by $r_i(\lambda)=\lambda-\langle\lambda,\alpha_i^\vee\rangle\alpha_i$, and let $S=\{r_1, \dots, r_N\}$. The subgroup $\langle S\rangle$ of $GL(\mathfrak{h}^*)$ is called the Weyl group of $\mathfrak{g}$, and is in fact a Coxeter group; more precisely, $\langle S\rangle$ is isomorphic to the quotient W of the free group generated by $r_1, \dots, r_N$ by the subgroup generated by

$$(r_i)^2,\ 1\le i\le N,$$
$$\text{and } (r_ir_j)^{m_{ij}},\ 1\le i\neq j\le N,$$

where m_{ij} are computed by the following table ([K3]):

If	$a_{ij}a_{ji}=$	0	1	2	3	≥ 4
then	$m_{ij}=$	2	3	4	6	∞

Conversely, given a crystallographic group W (i.e. a Coxeter group as above with $m_{ij}\varepsilon\{2,3,4,6,\infty\}$), one can find a generalized Cartan matrix A such that the Weyl group of $\mathfrak{g}_{\mathbf{Q}}(A)$ is precisely W: indeed it suffices to take $a_{ij}=-4\cos^2\frac{\pi}{m_{ij}}$ if $i<j$, $a_{ij}=-1$ if $i>j$ and $a_{ji}\neq 0$;

The matrix A thus obtained will fail to be indecomposable exactly when the Coxeter graph of W is disconnected, in which case $\mathfrak{g}_{\mathbf{Q}}(A)$ is isomorphic to $\mathfrak{g}_{\mathbf{Q}}(A_1)\oplus\mathfrak{g}_{\mathbf{Q}}(A_2)$, $W\simeq W_1\times W_2$, where W_i is the Weyl group of $\mathfrak{g}_{\mathbf{Q}}(A_i)$. In any case, one gets an embedding of W in $GL(\mathfrak{h}^*)$.

Similarly, one has an action of W on $\mathfrak{h}$ given by $r_ih=h-\langle\alpha_i,h\rangle\alpha_i^\vee$; and the two representations thus obtained are contragredient (i.e. $\langle r_i\lambda,r_ih\rangle=\langle\lambda,h\rangle$ if $\lambda\varepsilon\mathfrak{h}^*$, $h\varepsilon\mathfrak{h}$).

Write $e_i^{(m)}$ for the element $\frac{e_i^m}{m!}$ of the universal enveloping algebra of $\mathfrak{g}'$, $1\le i\le N$, $m\varepsilon\mathbf{N}$, and similarly $f_i^{(m)}=\frac{1}{m!}f_i^m\ \varepsilon\ U(\mathfrak{g}')$. Define $U_{\mathbf{Z}}$ to be the Z-subalgebra of $U(\mathfrak{g}')$ generated by $\{ e_i^{(m)}, f_i^{(m)}, 1\le i\le N, m\varepsilon\mathbf{N} \}$, and set

$\mathcal{U}_+ =$ the subalgebra of $\mathcal{U}_{\mathbf{Z}}$ generated by $e_i^{(m)}$, $1 \leq i \leq N$, $m \in \mathbf{N}$,

$\mathcal{U}_- =$ the subalgebra of $\mathcal{U}_{\mathbf{Z}}$ generated by $f_i^{(m)}$, $1 \leq i \leq N$, $m \in \mathbf{N}$.

Also, if R is any $\mathbf{Q}$-algebra with 1, and $x \in R, n \in \mathbf{N}$, define $\binom{x}{n} \in R$ to be the element $\frac{x(x-1)\ldots(x-n+1)}{n!}$. Finally, set

$\mathcal{U}_o =$ the $\mathbf{Z}$-subalgebra of $\mathcal{U}(\mathfrak{g}')$ generated by $\binom{\alpha_i^\vee}{n}$, $1 \leq i \leq N$, $n \in \mathbf{N}$.

(see [T1]).

one then has

<u>1.1 Proposition</u> :

$$\mathcal{U}_o \subset \mathcal{U}_{\mathbf{Z}}, \quad \mathcal{U}_o \otimes \mathbf{Q} = \mathcal{U}(\mathfrak{h} \cap \mathfrak{g}') ,$$

$$\mathcal{U}_\pm \otimes \mathbf{Q} = \mathcal{U}(\mathfrak{n}_\pm) ,$$

$$\mathcal{U}_{\mathbf{Z}} \otimes \mathbf{Q} = \mathcal{U}(\mathfrak{g}'), \text{ and } \mathcal{U}_{\mathbf{Z}} = \mathcal{U}_- \otimes \mathcal{U}_o \otimes \mathcal{U}_+ \qquad \text{(see [T])}.$$

proof:

Given $n \in \mathbf{N}$,
$$e_i^{(n)} f_i^{(n)} = \sum_{k=0}^{n} f_i^{(k)} \binom{\alpha_i^\vee - 2k}{n-k} e_i^{(k)} \qquad \text{([S2])}$$
$$= \binom{\alpha_i^\vee}{n} + \sum_{k=1}^{n} f_i^{(k)} \binom{\alpha_i^\vee - 2k}{n-k} e_i^{(k)} .$$

Now we can find integers $a_m^{(k)}$ such that

$$\binom{\alpha_i^\vee - 2k}{n-k} = \sum_{m=0}^{n-k} a_m^{(k)} \binom{\alpha_i^\vee}{m} \qquad \text{([S2])}$$

so that we actually have

$$\underbrace{e_i^{(n)} f_i^{(n)}}_{\in\, \mathcal{U}_{\mathbf{Z}}} = \binom{\alpha_i^\vee}{n} + \sum_{k=1}^{n} \sum_{m=0}^{n-k} a_m^{(k)} \underbrace{f_i^{(k)} \binom{\alpha_i^\vee}{m} e_i^{(k)}}_{\in\, \mathcal{U}_{\mathbf{Z}}}$$

and each $\binom{\alpha_i^\vee}{n}$ is seen to lie in $\mathcal{U}_{\mathbf{Z}}$ by induction on n, hence $\mathcal{U}_o \subset \mathcal{U}_{\mathbf{Z}}$; that $\mathcal{U}_o \otimes \mathbf{Q} = \mathcal{U}(\mathfrak{h} \cap \mathfrak{g}')$ is clear as $\mathfrak{h} \cap \mathfrak{g}' = \bigoplus_{i=1}^{N} \mathbf{Q}\alpha_i^\vee$ so that $\mathcal{U}(\mathfrak{h} \cap \mathfrak{g}')$ is the symmetric algebra in $\alpha_1^\vee, \ldots, \alpha_N^\vee$ over $\mathbf{Q}$ and the first statement is proved.

The second statement is clear by the Birkhoff-Witt theorem applied to $\mathfrak{n}_\pm$, so let's verify the last statement:

we have an injection

$$I_1 : \mathcal{U}_{\mathbf{Z}} \otimes \mathbf{Q} \longrightarrow \mathcal{U}(\mathfrak{g}'), \quad I_1(u \otimes c) = c \cdot u .$$

By the Birkhoff-Witt theorem applied to 1.0, the map

$$I_2 : \mathcal{U}(n_-)\otimes\mathcal{U}(h\cap g')\otimes\mathcal{U}(n_+) \to \mathcal{U}(g')$$

given by $I_2(u_-\otimes u_o\otimes u_+) = u_-u_ou_+$ is an isomorphism, so we get an isomorphism

$$I_2^{-1} : \mathcal{U}(g') \to \mathcal{U}(n_-)\otimes\mathcal{U}(h\cap g')\otimes\mathcal{U}(n_+)$$

Using the first two statements just proved, we also obtain a map

$$I_3 : \mathcal{U}(n_-)\otimes\mathcal{U}(h\cap g')\otimes\mathcal{U}(n_+) \to \mathcal{U}_-\otimes\mathcal{U}_o\otimes\mathcal{U}_+\otimes\mathbf{Q}$$

and a map

$$I_4 : \mathcal{U}_-\otimes\mathcal{U}_o\otimes\mathcal{U}_+\otimes\mathbf{Q} \to \mathcal{U}_{\mathbf{Z}}\otimes\mathbf{Q} \quad , \quad I_4(u_-\otimes u_o\otimes u_+\otimes c) = cu_-u_ou_+ .$$

I_1 shows that $\mathcal{U}_{\mathbf{Z}}\otimes\mathbf{Q} \subset \mathcal{U}(g')$, and $I_4 \circ I_3 \circ I_2^{-1}$ proves the reverse inclusion.

Finally, it is clear that $I_2(\mathcal{U}_-\otimes\mathcal{U}_o\otimes\mathcal{U}_+) \subset \mathcal{U}_{\mathbf{Z}}$, so that $I_2 : \mathcal{U}_-\otimes\mathcal{U}_o\otimes\mathcal{U}_+ \to \mathcal{U}_{\mathbf{Z}}$ is an injective map, and we need to check that $I_2^{-1}(\mathcal{U}_{\mathbf{Z}}) \subset \mathcal{U}_-\otimes\mathcal{U}_o\otimes\mathcal{U}_+$, which follows directly from the formulas

$$e_i^{(n)} f_i^{(m)} = \sum_{j=0}^{\min(m,n)} f^{(m-j)} \underbrace{\binom{\alpha_i^\vee - m - n + 2j}{j}}_{\varepsilon\,\mathcal{U}_o \text{ as before}} e_i^{(n-j)} ,$$

$$e_j^{(m)} f_i^{(n)} = f_i^{(n)} e_j^{(m)} \quad \text{if } i \neq j ,$$

$$\binom{\alpha_j^\vee}{n} f_i^{(m)} = f_i^{(m)} \underbrace{\binom{\alpha_j^\vee - m\langle\alpha_i, \alpha_j^\vee\rangle}{n}}_{\varepsilon\,\mathcal{U}_o}$$

$$\text{and } e_i^{(m)} \binom{\alpha_j^\vee}{n} = \underbrace{\binom{\alpha_j^\vee + m\langle\alpha_i, \alpha_j^\vee\rangle}{n}}_{\varepsilon\,\mathcal{U}_o} e_i^{(m)} ,$$

all of which can be checked by induction as in [S2].

The decomposition 1.1 can be made more explicit in the case of a classical Cartan matrix (i.e. when g is a finite dimensional simple Lie algebra once one extends coefficients to $\mathbf{C}$), and one can then exhibit a $\mathbf{Z}$-basis for $\mathcal{U}_{\mathbf{Z}}$ ([K],[S1]). One problem with generalizing this result is that the root spaces do not in general have any canonical bases, indeed they are not necessarily one-dimensional. The ones that are do however play a central role in the theory, and they can be

easily obtained as follows:
the elements e_i, f_i , $1\le i\le N$ act, by the adjoint representation, locally nilpotently on $\mathcal{U}_{\mathbf{Z}}$ (indeed, on $\mathcal{U}(\mathfrak{g})$), i.e. for every $u\in\mathcal{U}$, there exists $M(u)\in\mathbf{N}$ such that $e_i^{(m)}.u = 0$ for all $m\ge M(u)$, and similarly for f_i : that is so because of the equality

$$\underline{1.2}\quad \operatorname{ad} y^m.(x_1\ldots x_n) = \sum_{j_1+\ldots j_n=m} \binom{m}{j_1,\ldots,j_n}(\operatorname{ad} y^{j_1}.x_1)\ldots(\operatorname{ad} y^{j_n}.x_n)$$

which holds in the envelope of any Lie algebra $\mathfrak{a}$, when $y,x_1,\ldots,x_n \in \mathfrak{a}$, and $\binom{m}{j_1,\ldots j_n} = \dfrac{m!}{j_1!\ldots j_n!}$, and can be easily checked by induction on m and $n\in\mathbf{N}$; using 1.2 with $y=e_i$, one gets

$$\underline{1.3}\quad \operatorname{ad} e_i^{(m)}(x_1\ldots x_n) = \sum_{j_1+\ldots+j_n=m} (\operatorname{ad} e_i^{(j_1)}.x_1)\ldots(\operatorname{ad} e_i^{(j_n)}.x_n)$$

from which one concludes that local nilpotence of e_i (or f_i) on $\mathcal{U}$ follows from local nilpotence on the generators of $\mathcal{U}$, and that is an immediate consequence of the defining relations for $\mathfrak{g}$.

So for $x \in \{e_i,f_i\ ,\ 1\le i\le N\}$, one can define elements $\exp x$, $\exp -x$ in $\operatorname{End}(\mathcal{U}_{\mathbf{Z}})$ by

$$\underline{1.4}\qquad \exp x\ .\ v=\sum_m \operatorname{ad}x^{(m)}.v \qquad ,\in \mathcal{U}_{\mathbf{Z}}\ \text{ if } v\in \mathcal{U}_{\mathbf{Z}}$$

$$\text{and } \exp{-x}\ .\ v=\sum_m (-1)^m\operatorname{ad}x^{(m)}.v\ ,\in \mathcal{U}_{\mathbf{Z}}\ \text{ if } v\in \mathcal{U}_{\mathbf{Z}}\ ,$$

each sun being of course finite as each term beyond $M(v)$ chosen as above is 0.

The decomposition 1.1 and formula 1.4 show that $\exp\pm x$ are actually endomorphisms of $\mathcal{U}(\mathfrak{g}')$ and that if $v\in \mathfrak{g}'\subset\mathcal{U}(\mathfrak{g}')$, then $\exp x.v\in \mathfrak{g}'$again. Now, using 1.3, one easily checks that $\exp{-x} = (\exp x)^{-1}$, and that if $y_1,y_2\in\mathfrak{g}'$ then $(\exp(\varepsilon x).[y_1,y_2])=[\exp(\varepsilon x).y_1,\exp(\varepsilon x).y_2]$, $\varepsilon=\pm1$, so that $\exp\pm x$ defined as above are Lie algebra automorphisms.

Let's define, for $1\le i\le N$, $\dot{r}_i\in\operatorname{Aut}(\mathcal{U}_{\mathbf{Z}})$ by $\dot{r}_i=\exp e_i\circ\exp{-f_i}\circ\exp e_i$. A classical computation shows that $\dot{r}_i|_{\mathfrak{h}}=r_i$ (see, e.g., [K2]), and $\dot{r}_i$ is a Lie algebra automorphism extending r_i. We proceed to define for every $w\in W\hookrightarrow GL(\mathfrak{h})$ such an extension $\dot{w}$ by choosing a reduced expression $w=r_{i_1}\ldots r_{i_k}$ for w, and setting $\dot{w}=\dot{r}_{i_1}\ldots\dot{r}_{i_k}$; to check that this does indeed define $\dot{w}$ uniquely, one needs to verify that

<u>1.5 Proposition</u> :

If $w=r_{i_1}\ldots r_{i_k}=r_{j_1}\ldots r_{j_k}$, and w has length k, then $\dot{r}_{i_1}\ldots\dot{r}_{i_k} = \dot{r}_{j_1}\ldots\dot{r}_{j_k}$.

proof:

Proceed by induction as follows:

If k=1, there is nothing to prove. Otherwise, the exchange condition ([K2]) implies that for some $0\leq\alpha\leq k-1$, we have $r_{j_1}\ldots r_{j_k} = r_{j_1}\ldots r_{j_\alpha} r_{j_{\alpha+2}}\ldots r_{j_k} r_{i_k}$.

If $\alpha\neq 0$, the induction hypothesis applied to

$$r_{j_{\alpha+1}}\ldots r_{j_k} = r_{j_{\alpha+2}}\ldots r_{j_k} r_{i_k}$$

gives

1.6 $\dot{r}_{j_{\alpha+1}}\ldots\dot{r}_{j_k} = \dot{r}_{j_{\alpha+2}}\ldots\dot{r}_{j_k}\dot{r}_{i_k}$,

and, applied to $r_{i_1}\ldots r_{i_{k-1}} = r_{j_1}\ldots r_{j_\alpha} r_{j_{\alpha+2}}\ldots r_{j_k}$ gives

1.6' $\dot{r}_{i_1}\ldots\dot{r}_{i_{k-1}} = \dot{r}_{j_1}\ldots\dot{r}_{j_\alpha}\dot{r}_{j_{\alpha+2}}\ldots\dot{r}_{j_k}$,

hence $\dot{r}_{j_1}\ldots\dot{r}_{j_k} = \dot{r}_{j_1}\ldots\dot{r}_{j_\alpha}\dot{r}_{j_{\alpha+2}}\ldots\dot{r}_{j_k}\dot{r}_{i_k}$ by 1.6,

$= \dot{r}_{i_1}\ldots\dot{r}_{i_{k-1}}\dot{r}_{i_k}$ by 1.6' ;

If $\alpha=0$, i.e. $w=r_{j_2}\ldots r_{j_k} r_{i_k}$, then we know by induction that

1.7 $\dot{r}_{i_1}\ldots\dot{r}_{i_k} = \dot{r}_{j_2}\ldots\dot{r}_{j_k}\dot{r}_{i_k}$.

We shall also assume that $w=r_{i_2}\ldots r_{i_k} r_{j_k}$ (otherwise, by symmetry, we are back in the case $\alpha\neq 0$, with i and j exchanged). We then have similarly

1.7' $\dot{r}_{j_1}\ldots\dot{r}_{j_k} = \dot{r}_{i_2}\ldots\dot{r}_{i_k}\dot{r}_{j_k}$,

and it suffices to show that the right hand sides of 1.7, 1.7' are equal, i.e. that

$$(\dot{r}_{i_2}\ldots\dot{r}_{i_{k-1}})\dot{r}_{i_k}\dot{r}_{j_k} = (\dot{r}_{j_2}\ldots\dot{r}_{j_{k-1}})\dot{r}_{j_k}\dot{r}_{i_k}.$$

Proceeding by induction, this reduces to proving that

$$\dot{r}_{i_k}\dot{r}_{j_k}\dot{r}_{i_k}\ldots = \dot{r}_{j_k}\dot{r}_{i_k}\dot{r}_{j_k}\ldots$$

whenever $r_{i_k} r_{j_k} r_{i_k}\ldots = r_{j_k} r_{i_k} r_{j_k}\ldots$.

Now one can compute directly that

if $m_{ij}=2$, then $\dot{r}_i.x_j=x_j$,

if $m_{ij}=3$, then $\dot{r}_i\dot{r}_j.x_i=x_j$,

if $m_{ij}=4$, then $\dot{r}_i\dot{r}_j\dot{r}_i.x_j=x_j$,

if $m_{ij}=6$, then $\dot{r}_i\dot{r}_j\dot{r}_i\dot{r}_j\dot{r}_i x_j=x_j$

with $x\,\varepsilon\,\{e,f\}$ (using classical formulas, as in the proof of 1.1), from which it follows easily that

if $m_{ij}=2$, then $\dot{r}_i\mathrm{o}\exp(\varepsilon x_j)\mathrm{o}\dot{r}_i^{-1}=\exp(\varepsilon x_j)$,

if $m_{ij}=3$, then $\dot{r}_i\dot{r}_j\mathrm{o}\exp(\varepsilon x_i)\mathrm{o}\dot{r}_j^{-1}\dot{r}_i^{-1}=\exp(\varepsilon x_j)$, $(\varepsilon=\pm 1)$

and similarly for $m_{ij}=4,6$, and one concludes that

if $m_{ij}=2$, then $\dot{r}_i\dot{r}_j\dot{r}_i^{-1}=\dot{r}_j$,

if $m_{ij}=3$, then $\dot{r}_i\dot{r}_j\dot{r}_i\dot{r}_j^{-1}\dot{r}_i^{-1}=\dot{r}_j$,

etc... as needed.

The proof of 1.5 is now complete,

so that $\dot{w}$ is indeed well-defined.

We thus get a map $W \to \mathrm{End}(\mathcal{U}(g'))$, $w \mapsto \dot{w}$, satisfying

1) $\dot{w}$ is an automorphism of $\mathcal{U}(g')$, leaving $\mathcal{U}_{\mathbf{Z}}$ and g' stable,

2) $\dot{w}|_h = w\,\varepsilon\,W \hookrightarrow GL(h)$,

3) If $x,y\,\varepsilon\,g'$, then $\dot{w}.[x,y]=[\dot{w}x,\dot{w}y]$.

Suppose then that $\lambda\,\varepsilon\,\Delta$, $x\,\varepsilon\,g_\lambda \subset g'$, $h\,\varepsilon\,h$:

we have
$$\begin{aligned}[h,\dot{w}.x] &= \dot{w}.[\dot{w}^{-1}.h,x]\\ &= \dot{w}.[w^{-1}.h,x]\\ &= \dot{w}.(\langle\lambda,w^{-1}.h\rangle x)\\ &= \dot{w}.(\langle w.\lambda,h\rangle x)\\ &= \langle w.\lambda,h\rangle\dot{w}.x\ ,\end{aligned}$$

i.e. $\dot{w}$ restricts to a bijection $g_\lambda \to g_{w.\lambda}$: in particular, Δ is W-invariant (W acts on h^* as above), and if $\lambda\,\varepsilon\,\Delta$, $\dim g_\lambda=\dim g_{w.\lambda}$. We know from 1.0 that if $\lambda\,\varepsilon\,\Pi$ (i,e. $\lambda=\alpha_i$ for some i), then $\dim g_\lambda=1$ (in fact, $g_\lambda=\mathbb{Q}e_i$), and similarly for $\lambda\,\varepsilon\,-\Pi$. So one is naturally led to define subsets Δ^{re}, Δ^{re}_+, Δ^{re}_- of Δ as follows (see [K3]) :

$$\Delta^{re}= (W.\Pi)\ ,\ \Delta^{re}_+= W.\Pi\cap\Delta_+\ ,\ \Delta^{re}_-= \Delta^{re}\cap\Delta_-\ .$$

If $\lambda\,\varepsilon\,\Delta^{re}$, then, as in the finite dimensional case,

$\Delta\cap\mathbf{Z}\lambda=\{\pm\lambda\}$, and $\dim\mathfrak{g}_\lambda=1$.

If moreover $\lambda=\alpha_i$, we know that $\mathfrak{g}_\lambda=\mathbf{Q}e_i$, and that $\mathfrak{g}_\lambda\cap\mathcal{U}_\mathbf{Z}=\mathbf{Z}e_i$ (the intersection taken in $\mathcal{U}(\mathfrak{g}')$). We construct such a basis for $\mathfrak{g}_\lambda$, any $\lambda\in\Delta^{re}$, as follows:

Let $X(\lambda) = \{ \pm\dot{w}.e_i \;,\; w\in W$ and $1\le i\le N$ such that $w.\alpha_i=\lambda \}$.

1.8 Proposition :

$X(\lambda)$ has exactly two elements, for any λ.

proof:

If $w\in W$, $1\le i\le N$, then $\dot{w}.e_i\in\mathcal{U}_\mathbf{Z}\cap\mathfrak{g}_\mathbf{Q}$, in fact $\dot{w}.e_i\in\mathfrak{g}_{w\alpha_i}=\mathfrak{g}_\lambda$ which is one dimensional. So if $w_1.\alpha_j=w_2.\alpha_k=\lambda$, then $\dot{w}_1.e_j=c\,\dot{w}_2.e_k$, for some $c\in\mathbf{Q}$; hence

$$\dot{w}_1^{-1}.\dot{w}_2.e_k=\frac{1}{c}e_j.$$

But $\dot{w}_2.e_k\in\mathcal{U}_\mathbf{Z}\cap\mathfrak{g}_{w_2.\alpha_k}$, so $\dot{w}_1^{-1}\dot{w}_2.e_k\in\mathcal{U}_\mathbf{Z}\cap\mathfrak{g}_{w_1^{-1}w_2\alpha_k}=\mathcal{U}_\mathbf{Z}\cap\mathfrak{g}_{\alpha_j}=\mathbf{Z}e_j$, hence $c=\pm1$.

In other words, if $w_1\alpha_j=w_2\alpha_k=\lambda$, then $\dot{w}_1e_j=\pm\dot{w}_2e_k\in\mathfrak{g}_\lambda\cap\mathcal{U}_\mathbf{Z}$, and $X(\lambda)$ does indeed consist of two elements, each the negative of the other.

So we fix an arbitrary $1\le i\le N$ and $w_\lambda\in W$ satisfying $w_\lambda.\alpha_i=\lambda$ (we take $w_\lambda=1$ and $i=k$ if $\lambda=\alpha_k$), and define $e_\lambda=\dot{w}_\lambda.e_i$, $f_\lambda=\dot{w}_\lambda.f_i$.

The elements of Δ^{re} are called real roots ([K3]); so for every real root λ, say $\lambda\in\Delta_+^{re}$, we now have root vectors e_λ, f_λ such that $\mathfrak{g}_\lambda=\mathbf{Q}e_\lambda$, $\mathfrak{g}_{-\lambda}=\mathbf{Q}f_\lambda$. Also, if $x\in\mathfrak{g}_\lambda$, say $x=c.e_\lambda$ with $c\in\mathbf{Q}$, and if $x\in\mathcal{U}_\mathbf{Z}$, then

$$\dot{w}_\lambda^{-1}.x = c\dot{w}_\lambda^{-1}\dot{w}_\lambda e_i = ce_i \;;$$

As $\dot{w}_\lambda^{-1}.x\in\mathfrak{g}_{\alpha_i}\cap\mathcal{U}_\mathbf{Z}=\mathbf{Z}e_i$, one concludes that $c\in\mathbf{Z}$. Therefore, we also have:

1.9 $\qquad \mathfrak{g}_\lambda\cap\mathcal{U}_\mathbf{Z}=\mathbf{Z}e_\lambda$, $\mathfrak{g}_\lambda\cap\mathcal{U}_\mathbf{Z}=\mathbf{Z}f_\lambda$, for any $\lambda\in\Delta_+^{re}$

Finally, with λ as above, define $\lambda^\vee=w_\lambda.\alpha_i^\vee\in\mathfrak{h}$. Then $\mathbf{Q}f_\lambda+\mathbf{Q}\lambda^\vee+\mathbf{Q}e_\lambda$ is isomorphic to $\mathfrak{sl}_2(\mathbf{Q})$ and one easily proves as in 1.1 that $\binom{\lambda^\vee}{n}\in\mathcal{U}_o$ for all $n\in\mathbf{N}$, $\lambda\in\Delta_+^{re}$, and that $e_\lambda^{(m)}$, $f_\lambda^{(m)}\in\mathcal{U}_\mathbf{Z}$ (where $x_\lambda^{(m)}=\dot{w}_\lambda.x^{(m)}$ for $x\in\{e,f\}$).

Suppose now that K is a field, so that we may form $\mathcal{U}_K=\mathcal{U}_\mathbf{Z}\otimes_\mathbf{Z}K$. It follows from 1.1 that if $\mathcal{U}_\sigma(K)=\mathcal{U}_\sigma\otimes_\mathbf{Z}K$ with $\sigma\in\{+,-,0\}$, then

$$\mathcal{U}_K=\mathcal{U}_-(K)\otimes_\mathbf{Z}\mathcal{U}_o(K)\otimes_\mathbf{Z}\mathcal{U}_+(K) \;,$$

and 1.9 shows that for all $\lambda\in\Delta_+^{re}$, $e_\lambda\otimes1\neq0$, $f_\lambda\otimes1\neq0$ in $\mathcal{U}_K$.

Being the tensor product of two $\mathbf{Z}$-algebras, $\mathcal{U}_K$ carries a structure of associative algebra, with $(x\otimes t).(y\otimes s)= xy\otimes ts$. The product is K-linear so that $\mathcal{U}_K$ is also a K-Lie algebra. The same holds for $\mathcal{U}_\sigma(K)$ as above, so that we have the Birkhoff-Witt decomposition $\mathcal{U}_K=\mathcal{U}_-(K)\otimes\mathcal{U}_o(K)\otimes\mathcal{U}_+(K)$.

Finally, one has a map $W \to \mathrm{Aut}(\mathcal{U}_K)$, satisfying $w.(u\otimes t)=(\dot{w}.u)\otimes t$; this map is again injective, for of $w\neq w' \in W$, one can find $\alpha_i \in \Pi$ such that $w\alpha_i\neq w'\alpha_i$, in which case $\dot{w}e_i=\varepsilon e_{w\alpha_i}$ while $\dot{w}'e_i=\varepsilon' e_{w'\alpha_i}$ (with $\varepsilon,\varepsilon'=\pm1$) so that $w.(e_i\otimes 1)\neq w'.(e_i\otimes 1)$, as needed.

A few basic facts about the representation theory of $\mathcal{U}$ will be needed. To distinguish the "good" characters of $\mathcal{U}_o(K)$, let's make the following definition: for $a \in \mathbf{Z}$, $n \in \mathbf{N}$, write

$$\binom{a}{n} = \begin{cases} \dfrac{a!}{n!(a-n)!} & \text{if } a\geq n, \text{ as usual }, \\ 0 & \text{if } 0\leq a<n \\ (-1)^n \dfrac{(n-a-1)!}{n!(-a-1)!} & \text{if } a<0 \text{ ;} \end{cases}$$

so that $\binom{a}{n}$ is a well-defined integer.

It is now easy to check that one obtains a homomorphism

$$(Q^\vee)^* \longrightarrow \mathrm{Hom}(\mathcal{U}_o(K),K),$$

where $(Q^\vee)^*$ denotes the $\mathbf{Z}$-dual, in h^*, of $Q^\vee=\bigoplus_{i=1}^{N}\mathbf{Z}\alpha_i^\vee$, satisfying

$$\Lambda(\binom{\lambda^\vee}{n}\otimes t) = t.\binom{\langle\Lambda,\lambda^\vee\rangle}{n} \quad ,$$

for all $\Lambda\in (Q^\vee)^*$, $\lambda^\vee\in Q^\vee$, $n\in\mathbf{N}$, $t\in K$. Call the image of this map the characters of $\mathcal{U}_o(K)$ (see [T1]).

A $\mathcal{U}_{\mathbf{Z}}$-module M is called integrable if all e_i,f_i act locally nilpotently on M, i.e. for each $v\in M$, there exists a positive integer $m(v)$ such that if $m(v)\leq m \in \mathbf{N}$, then $e_i^{(m)}.v=f_i^{(m)}.v=0$. A $\mathcal{U}_K$-module is integrable if all $e_i\otimes 1,f_i\otimes 1$ act locally nilpotently and $\mathcal{U}_o(K)$ acts by characters as above. Finally, a $g=g_Q(A)$-module M is integrable if all e_i,f_i act locally nilpotently, and M is completely reducible as an h-module, all weight spaces being finite dimensional.

A large class of integrable g-modules has been constructed by V. Kač ([K4]): it consists of certain highest- and lowest-weight modules. To construct these modules, start with a 'weight' $\Lambda\in h^*$ such that $\langle\Lambda,\alpha_i^\vee\rangle\in\mathbf{N}$ for all i. Let $J(\Lambda)$ be the ideal of $\mathcal{U}(g)$ generated by n_+

together with all $(h-\Lambda(h))$, $h\varepsilon h$. The g-module $U(g)/J(\Lambda)$, on which g acts by left-multiplication, can be shown to have a unique proper maximal submodule, such that the quotient $L(\Lambda)$ is an integrable g-module in the above sense (see [K5]; $L(\Lambda)$ is irreducible by construction).

$L(\Lambda)$ has highest weight Λ, i.e. if $v^+\varepsilon L(\Lambda)$ is the image under the quotient maps of $1\varepsilon U(g)$, then

1.10
$$U(n_+).v^+=0$$
$$h.v^+-\langle\Lambda,h\rangle v^+=0,\text{ for all } h\varepsilon h,$$
$$\text{and } L(\Lambda)=U(n_-).v^+.$$

One then knows ([K5]) that each $\dot{r}_i$ acts on $L(\Lambda)$ as before, that $\dim_Q L(\Lambda)_{r_i\Lambda}=\dim_Q L(\Lambda)_\Lambda=1$,(where if M is a g-module and $\lambda\varepsilon h^*$, then $M_\lambda=\{m\varepsilon M$ such that $h.m=\langle\lambda,h\rangle m$, for all $h\varepsilon h\}$) and that, in fact, $\dot{r}_i.v^+\,\varepsilon\,(L(\Lambda)_{r_i\Lambda}\cap U_Z.v^+)-\{0\}$.

Since we also know, by construction, that $(U_Z.v^+\cap L(\Lambda)_\Lambda)=Zv^+$, one can show as in 1.5-1.7 that if $r_{i_1}\ldots r_{i_k}=r_{j_1}\ldots r_{j_k}=w\,\varepsilon\,W$ with length(w)=k, $\dot{r}_{i_1}.(\ldots(\dot{r}_{i_k}.v^+)..)=\dot{r}_{j_1}.(\ldots(\dot{r}_{j_k}.v^+)..)$, so that for each $w\,\varepsilon\,W$ we have a well defined element $\dot{w}.v^+\varepsilon\, L_{w\Lambda}\cap U_Z.v^+$. Finally, one knows that if $L(\Lambda)_\lambda\neq 0$, then $\dim_Q L(\Lambda)_\lambda<\infty$,
$$\Lambda-\lambda\,\varepsilon\,\mathbf{N}.\Pi,$$
$$\text{and } L(\Lambda)=\sum_\lambda L(\Lambda)_\lambda.$$

Given a highest weight g-module M as above, one obtains canonically a lowest weight module M^* as follows:

If ω is the involution of g given by $\omega e_i=f_i$, $\omega|_h=-\mathrm{Id}_h$,define a new g-module structure on M by requiring
$$\underset{(\text{old})}{x.m}=\underset{(\text{new})}{\omega(x).m}\quad,\text{ for all } x\,\varepsilon\, g,\ m\,\varepsilon\, M.$$

Writing M^* for M with this new g-module structure, and v^- for v^+, one obtains a g-module satisfying 1.10 with all ± signs reversed.

Let M be an integrable g-module with highest weight Λ, and let's prove:

1.11 Proposition :

a) $(U_Z.v^+\cap M_\lambda)\neq 0$ if $M_\lambda\neq 0$ (in fact, $(U_Z.v^+\cap M_\lambda)\otimes Q=M_\lambda$);

b) If $v=\sum_\lambda v_\lambda\,\varepsilon U_Z.v^+$, with $v_\lambda\varepsilon M_\lambda$, then $v_\lambda\varepsilon U_Z.v^+$ for every λ.

c) $M_{w\Lambda} \cap \mathcal{U}_{\mathbf{Z}}.v^+ = \mathbf{Z}\dot{w}.v^+$.

d) $(\mathcal{U}_{\mathbf{Z}}.v^+) \otimes_{\mathbf{Z}} K$ is an integrable $\mathcal{U}_K$-module.

the same statements being true of lowest weight modules.

proof:

To prove a), one needs to observe that $\mathcal{U}(\mathfrak{g}).v^+ = \mathcal{U}(\mathfrak{g}').v^+$ as both are equal to $\mathcal{U}(n_-).v^+$. Now $\mathcal{U}(n_-) = \mathcal{U}_- \otimes \mathbf{Q}$ by 1.1, so if $v \in M_\lambda$, then $v = cu.v^+$ with $c \in \mathbf{Q}$, $u \in \mathcal{U}_-$ hence $\frac{v}{c} \in \mathcal{U}_{\mathbf{Z}}.v^+$, and the reverse inclusion is clear.

Choose a lattice in h^* of rank $d = M + \operatorname{corank} A = \dim h$ containing Λ and Π, and let S_Λ be its $\mathbf{Z}$-dual in h, say $S_\Lambda = \oplus_i \mathbf{Z}h_i$. If $h \in S_\Lambda$ and $n \in \mathbf{N}$, the element $\operatorname{ad}\binom{h}{n}$ of $\operatorname{End}_{\mathbf{Q}}(\mathcal{U}(\mathfrak{g}))$ stabilizes $\mathcal{U}_{\mathbf{Z}}$: in fact,

$$\left.\begin{aligned} \operatorname{ad}\binom{h}{n}.e_i^{(m)} &= m\binom{\langle\alpha_i,h\rangle}{n}e_i^{(m)} \\ \text{while } \operatorname{ad}\binom{h}{n}.f_i^{(m)} &= m\binom{-\langle\alpha_i,h\rangle}{n} f_i^{(m)} \end{aligned}\right\} \in \mathcal{U}_{\mathbf{Z}}$$

Define a map height: $Q_+ \to \mathbf{N}$, by $\operatorname{height}(\Sigma n_i\alpha_i) = \Sigma n_i$. For every $\lambda, \mu \in Q_+$, we then have $\operatorname{height}(\lambda+\mu) = \operatorname{height}(\lambda) + \operatorname{height}(\mu)$. and one defines a gradation on the algebra $\mathcal{U}(n_+)$ so that if $u \in \mathcal{U}(n_+)$ is in the ν-weight space of the adjoint action of h on $\mathcal{U}(n_+)$ then $\operatorname{degree}(u) = \operatorname{height}(\nu)$. (this is the principal gradation as in [K1]). Let $\mathcal{U}_j = \{u \in \mathcal{U}(n_+) \mid \deg(u) = j\}$, and let's first show that $\mathcal{U}_+$ is homogeneous with respect to this gradation:

Choose $h_o \in S_\Lambda$ such that $\langle\alpha_i, h_o\rangle = 1$ for all i. If $u \in \mathcal{U}_+$, say $u = u_{j_1} + \ldots + u_{j_k}$ with $u_{j_i} \in \mathcal{U}_{j_i}$ and $j_1 < \ldots < j_k$, then

$$\operatorname{ad}\binom{h_o}{j_k}.u = \sum_{i=1}^{k-1} \binom{j_i}{j_k} u_{j_i} + u_{j_k}$$

$$= u_{j_k}.$$

Proceeding by induction, we find that $u_{j_i} \in \mathcal{U}_{\mathbf{Z}}$ for all i; hence $\mathcal{U}_+ = \sum_{j \in \mathbf{N}} (\mathcal{U}_+ \cap \mathcal{U}_j)$.

One defines a gradation $\sum_{j \in \mathbf{N}} \mathcal{U}_{-j}$ of $\mathcal{U}(n_-)$ similarly, and we have $\mathcal{U}_- = \sum_{j \in \mathbf{N}} (\mathcal{U}_- \cap \mathcal{U}_{-j})$.

Again, if $h \in S_\Lambda$, and $n \in \mathbf{N}$, then $\binom{h}{n} \in \mathcal{U}(\mathfrak{g})$, so $\binom{h}{n}$ acts on M. We can now show that $\binom{h}{n}$ stabilizes $\mathcal{U}_{\mathbf{Z}}.v^+$:

Indeed, suppose $v=u_-.v^+$, and assume $u_-=\Sigma u_j$ with $u_j \in \mathcal{U}_{-j}$.

An easy calculation shows that $\binom{h}{n}u_j.v^+$ is of the form $\sum_{m_i} \binom{<\Lambda-m_1\alpha_1-\dots-m_N\alpha_N,h>}{n} u_j v^+$, the sum being over some N-tuples $(m_1,\dots,m_N)$ satisfying $m_1+\dots+m_N=j$. In any case, this shows that $\binom{h}{n}.v \in \mathcal{U}_{\mathbf{Z}}.v^+$.

Assume now that $v=\sum_{\lambda\in A} v_\lambda$, where A is a (finite) subset of weights, and let $\mu \in A$. One knows that the subring of $Q[X_1,\dots,X_d]$ (polynomials in d variables with coefficients in Q) generated by all monomials of the form $\binom{X_i}{n_i}$, $1\leq i\leq N$, $n_i \in \mathbf{N}$, separates points in $\mathbf{Z}^d$ ([S1]). So for each $\lambda\in A-\{\mu\}$, find such a polynomial $P_\lambda(X_1,\dots,X_d)$ satisfying

$$P_\lambda(<\lambda,h_1>,\dots,<\lambda,h_d>)=0,$$
$$P_\lambda(<\mu,h_1>,\dots,<\mu,h_d>)=1,$$

and let $F(X_1,\dots,X_d) = \prod_{\lambda\in A-\{\mu\}} P_\lambda$.

If $u = F(h_1,\dots,h_d)$, then $u\in\mathcal{U}(\mathfrak{g})$, u stabilizes $\mathcal{U}_{\mathbf{Z}}.v^+$ by our choice of polynomials P_λ, and

$$\begin{aligned} u.v = \sum_{\lambda\in A} u.v_\lambda &= \sum_{\lambda\in A-\{\mu\}} F(<\lambda,h_1>,\dots,<\lambda,h_d>)v_\lambda \\ &\quad +F(<\mu,h_1>,\dots,<\mu,h_d>)v_\mu \\ &= 0 + v_\mu \\ &= v_\mu. \end{aligned}$$

As $v\in\mathcal{U}_{\mathbf{Z}}.v^+$, we must have $v \in \mathcal{U}_{\mathbf{Z}}$, which proves b).
The proof of c) is identical to that of 1.9 (note that $\dim M_{w\Lambda}=1$).
proof of d): If $v = \sum_i (u_i.v^+)\otimes t_i$, then for every i, there exists a positive integer m_i such that $f_1^{(m)}.(u_i.v^+)=0$ and $e_1^{(m)}.(u_i.v^+)=0$ if $m\geq m_i$. So if $m\geq m(v)=\max_i(m_i)$, then $e_1^{(m)}.v = \Sigma e_1^{(m)}.(u_i.v^+)\otimes t=0$, and similarly for $e_2,\dots$.
So we need only check the action of $\mathcal{U}_o(K)$ on M :

By b), it is sufficient to check that if m is of the form $m=(u.v^+)\otimes s$ with $u.v^+ \varepsilon M_\mu$ and $u \varepsilon \mathcal{U}_{\mathbf{Z}}$, then $u_o.m=\chi_m(u_o)m$ for all $u_o \,\varepsilon\, \mathcal{U}_o(K)$, where χ_m is some character of $\mathcal{U}_o(K)$ as above. Without loss of generality, we may take $u_o=\binom{\lambda^v}{n}$, $\lambda^v \,\varepsilon\, Q^v$, and compute:

$$\begin{aligned}(\binom{\lambda^v}{n}\otimes t).m &= \binom{\lambda^v}{n}.u.v^+\otimes st\\ &= t\binom{<\mu,\lambda^v>}{n}u.v^+\otimes s\\ &= t\binom{<\mu,\lambda^v>}{n}m\end{aligned}$$

so that χ_m is the image character of $\mu \,\varepsilon\, (Q^v)^*$.

Finally, we show that

1.12 Proposition :

> If K has characteristic p>0, then $\mathcal{U}_K$ carries a Frobenius map $F:\mathcal{U}_K \to \mathcal{U}_K$ satisfying $F(x\otimes t)=x\otimes t^p$ whenever $x \varepsilon \{\alpha_i^v, e_i, f_i,\ 1\le i\le N\}$, and every $\mathcal{U}_K$-module $\mathcal{U}_{\mathbf{Z}}.v^{\pm}\otimes K$ carries a Frobenius such that $F(u.v)=F(u).F(v)$ whenever $v \varepsilon \mathcal{U}_{\mathbf{Z}}.v^{\pm}\otimes K$, $u \varepsilon \mathcal{U}_K$.

proof:

If $\mathcal{U}_j(K) = \{\ \sum_i x_i\otimes t_i,\ x_i \,\varepsilon\, \mathcal{U}_+$ with degree$(x_i)=j\ \}$, then $\mathcal{U}_j(K)$ is a $Q^v\otimes K$-stable finite dimensional subspace of $\mathcal{U}_+(K)$. Using the $\mathbf{Z}$-basis Π^v for Q^v, we obtain a Frobenius F on $Q^v\otimes K$, and F extends uniquely to each $\mathcal{U}_j(K)$ in such a way that $F(\Sigma x_i\otimes t_i)=\Sigma x_i\otimes t_i^p$ ([J]), hence to all of $\mathcal{U}_+(K)$. Define F on $\mathcal{U}_-$ similarly, and extend the Frobenius on $Q^v\otimes K$ to $\mathcal{U}_o(K)$ by identifying the latter with the symmetric algebra of $Q^v\otimes K$. We finally obtain $F:\mathcal{U}_K \to \mathcal{U}_K$ by using 1.1 so that $F(u_-\otimes u_o\otimes u_+)=F(u_-)\otimes F(u_o)\otimes F(u_+)$, with $u_\sigma \,\varepsilon\, \mathcal{U}_\sigma(K)$.

Now, if $v \,\varepsilon\, M$, say $v=\sum_i v_i\otimes t_i$ with $v_i \varepsilon\ \mathcal{U}_{\mathbf{Z}}.v^+$, $t_i \varepsilon K$, then each v_i is of the form $\sum_\lambda v_i^\lambda$, where $v_i \varepsilon M_\lambda \cap \mathcal{U}_{\mathbf{Z}}.v^+$ by 1.11, so that, defining F on $(M_\lambda \cap \mathcal{U}_{\mathbf{Z}}.v^+)\otimes K$ by $F(\sum_i v_i^\lambda\otimes t_i)=\sum_i v_i^\lambda\otimes t_i^p$ (recall that $\dim_K(M_\lambda \cap \mathcal{U}_{\mathbf{Z}}.v^+)\otimes K \le \dim_Q(M_\lambda) < \infty$), one sees that $F(v)=\sum_i v_i\otimes t_i^p$ does satisfy $F(u.v)=F(u).F(v)$.

Example 1: One always has $\dot{r}_i e_i = -f_i$; assume that $N \geq 2$, and $a_{12} = -2$. Then $\dot{r}_1 e_2 = e_1^{(2)} \cdot e_2 + e_2 e_1^{(2)} - e_1 e_2 e_1$. In particular, if $N=2$, then $\dot{r}_1^{\ 2}$ is the identity on $U_{\mathbf{Z}}$.

More generally, one has $\dot{r}_i e_j = \mathrm{ad}\, e_i^{(-a_{ij})} . e_j$,

and $\dot{r}_i f_j = (-1)^{a_{ij}} \mathrm{ad}\, f_i^{(-a_{ij})} . f_j$

if $i \neq j$.

Section 2 : Construction of the group.

Fix a field K. For every $\alpha \in \Delta_+^{re}$, $t \in K$, let $X_\alpha(t)$ be the formal sum $\sum_{n=0}^{\infty} e_\alpha^{(n)} \otimes t^n$, and $X_\alpha(K)$ the set $\{X_\alpha(t),\ t \in K\}$.

Defining a product in X_α by $X_\alpha(t).X_\alpha(s)=X_\alpha(t+s)$, one can easily see that X_α becomes a group, isomorphic to $\mathbf{G}_a$, and such that if M is any integrable $\mathcal{U}_K$-module, there exists a homomorphism $X_\alpha \to \mathrm{Aut}_K(M)$, with $X_\alpha(t).v=\sum_n t^n e_\alpha^{(n)} \otimes 1.v$, the sum being finite by integrability.

Similarly, let $Y_\alpha(K)=\{Y_\alpha(t)=\sum_{n=0} f_\alpha^{(n)} \otimes t^n,\ t \in K\}$, defined for any $\alpha \in \Delta_+^{re}$, and consider the group G_K^*= free product of all X_α, Y_β, over all $\alpha, \beta \in \Delta_+^{re}$, so that G_K^* acts on all integrable $\mathcal{U}_K$-modules as above.

If I_o^*= intersection, taken over all $\mathcal{U}_Z.v^{\pm} \otimes K$ as in 1.11, of the kernels of the representation of G_K^* on $\mathcal{U}_Z.v^{\pm} \otimes K$, and I_1^*= kernel of the action of G_K^* on $\mathcal{U}_K$, set $I^* = I_o^* \cap I_1^*$, and let $G(K)=G_K^*/I^*$: G(K) is the Kač-Moody group associated to the generalized Cartan matrix A (or the crystallographic group W). If A were a classical Cartan matrix, G(K) would be the universal (simply-connected) group associated to A. Using I_1^* instead of I^* would yield the adjoint group .

In order to study the structure of G, let $x_\alpha(t)$, $y_\alpha(t)$, x_α, y_α be the images in G(K) of $X_\alpha(t), Y_\alpha(t), X_\alpha, Y_\alpha$ in G_K^*, let A^i be the subgroup of G generated by $X_{\alpha_i}, Y_{\alpha_i}$, $1 \leq i \leq N$, and call $\mathrm{Ad}: G(K) \to \mathrm{Aut}(\mathcal{U}_K)$ the representation of G(K) on $\mathcal{U}_K$.

It is easy to see that $\mathrm{Ad}(A^i)$ stabilizes the Lie subalgebra of $\mathcal{U}_K$ spanned by the ordered basis ($e_i \otimes 1$, $\alpha_i^\vee \otimes 1$, $f_i \otimes 1$) and that with respect to this basis,

$$\mathrm{Ad}x_\alpha(t) \text{ has matrix } \begin{pmatrix} 1 & -2t & -t^2 \\ 0 & 0 & t \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{and } \mathrm{Ad}y_\alpha(t) \text{ has matrix } \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ -t^2 & 2t & 1 \end{pmatrix},$$

so that: a) the maps $X_{\alpha_i} \to G(K)$, $Y_{\alpha_i} \to G(K)$

$$X_{\alpha_i}(t) \mapsto x_{\alpha_i}(t), \quad Y_{\alpha_i}(t) \mapsto y_{\alpha_i}(t)$$

are injective ,

and b) $\mathrm{Ad}\, A^i|_{Ke_i\otimes 1+K\alpha_i^\vee\otimes 1+Kf_i\otimes 1} \simeq PSL_2(K)$

On the other hand, one knows that given any integrable module M for the algebra $\mathcal{U}_K(s\ell_2)$ defined as above, there exists a representation $SL_2(K) \longrightarrow \mathrm{Aut}(M)$ satisfying $\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}.v = X(t).v$, $\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}.v = Y(t).v$, and $\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}.v = t^{\langle\nu,\alpha^\vee\rangle}v$ if $\mathcal{U}_o(s\ell_2)$ acts on v by the character ν ([T2]).

Fixing i and applying this theorem to the $s\ell_2$ sub-Lie algebra of $\mathcal{U}_K$ spanned by $e_i\otimes 1$, $\alpha_i^\vee\otimes 1$, $f_i\otimes 1$, one gets a homomorphism $\Phi_i : SL_2(K)\longrightarrow G(K)$ such that $\Phi_i(SL_2(K))=A^i$; as we also have $\mathrm{Ad}\, A^i \simeq PSL_2(K)$, we can assert that A^i is a group of type A_1.

Before obtaining commutation relations and structural facts about G, let's observe that if char(K)=p>0, and if the map $K\longrightarrow K$, $t\mapsto t^p$ is invertible, then G(K) carries a Frobenius map such that $Fx_\alpha(t)=x_\alpha(t^p)$: indeed, the map $\tilde{F}:G_K^*\longrightarrow G_K^*$ given by $\tilde{F}(z_\alpha(t))=z_\alpha(t^p)$ for $z\varepsilon\{x,y\}$ satisfies $\tilde{F}(I^*)=I^*$ by 1.12, and hence factors to the desired map on G itself.

Let's denote by $H_i=\{H_i(t)=\Phi_i\begin{pmatrix}t & 0\\ 0 & 1/t\end{pmatrix}, t\in K^\times\}\subset A^i$, M_i= normalizer of H_i in A^i, H= subgroup of G generated by $H_1,\dots,H_N$, M= subgroup of G generated by M_i, $1\le i\le N$, U_+= subgroup of G generated by all $x_\alpha, \alpha\in\Delta_+^{re}$, U_-= subgroup of G generated by all $y_\alpha, \alpha\in\Delta_+^{re}$, $\dot{r}_i(t)=x_{\alpha_i}(t)y_{\alpha_i}(-1/t)x_{\alpha_i}(t)$.

<u>2.1 Theorem :</u> (see [PK2])

a) If M is an integrable $\mathcal{U}_K$-module, and if $u\in\mathcal{U}_K$, then $(\mathrm{Ad}x_\alpha(t)(u)) = x_\alpha(t)\circ u\circ x_{-\alpha}(t)$, in End(M)

b) If $\alpha,\beta\in\Delta_+^{re}$, & $\mathbb{N}\alpha+\mathbb{N}\beta\cap\Delta_+=\mathbb{N}\alpha+\mathbb{N}\beta\cap\Delta_+^{re}\subset\{\alpha,\beta,\alpha+\beta\}$ then for some $n_{\alpha,\beta}\in\mathbb{Z}$, and all $t,s\in K$, $(x_\alpha(t),x_\beta(s))=x_{\alpha+\beta}(-n_{\alpha,\beta}ts)$.

c) $\dot{r}_i(1)x_\alpha\dot{r}_i(-1)=x_{r_i\alpha}$ if $\alpha\in\Delta_+^{re}-\{\alpha_i\}$.

d) $(H_i,H_j)=1$ for all i,j.

e) $hU_+h^{-1}=U_+$ if $h\in H$.

f) $W\simeq M/H$

proof:

To prove a) we may assume that $u=u_o\otimes s$, $u_o\in\mathcal{U}_Z$, $s\in K$, $v\in M$, so that $(\mathrm{Ad}\, x_\alpha(t).(u)).v = \sum_m t^m(\mathrm{ad}e_\alpha^{(m)}.u_o\otimes s).v$, while

$$x_\alpha(t)\circ u\circ x_\alpha(-t).v = \sum_{i,j} (-t)^i t^j (e_\alpha^{(j)}\otimes 1.(u.((e_\alpha^{(i)}\otimes 1).v))$$
$$= \sum_{i,j} (-1)^i t^{i+j}((e_\alpha^{(j)} u_o e_\alpha^{(i)})\otimes s).v$$
$$= \sum_{i,j} t^{2i+j}(e_\alpha^{(j)} u_o e_\alpha^{(2i)})\otimes s.v - t^{2i+j+1}(e_\alpha^{(j)} u_o e_\alpha^{(2i+1)})\otimes s.v$$
$$= \sum_m t^m \Big(\sum_i e_\alpha^{(m-2i)} u_o e_\alpha^{(2i)}\otimes s.v - e_\alpha^{(m-2i-1)} u_o e_\alpha^{(2i+1)}\otimes s.v \Big)$$

so that a) is proved if we know that

2.2 $$\mathrm{ad}\, e_\alpha^{(m)}.u_o = \sum_i e_\alpha^{(m-2i)} u_o e_\alpha^{(2i)} - e_\alpha^{(m-2i-1)} u_o e_\alpha^{(2i+1)}$$

But one can easily prove by induction, starting with the formula $\mathrm{ad} e_i.u = e_i u - u e_i$ that

$\mathrm{ad}\, e_i^m.u = \sum_{j=0} (-1)^j \binom{m}{j} e_i^{m-j} u e_i^j$, which implies 2.2 if we let $u=(\dot{w}^\alpha)^{-1}u_o$, and apply $\dot{w}^\alpha$ to the above equality, with i chosen so that $\dot{w}^\alpha.\alpha_i=\alpha$.

Now to check all equalities A=B with $A,B \in G$, we use a) to verify that $AB^{-1} \in I^*$ where A and B^{-1} are interpreted as elements of G^*, i.e.

To prove b): let's first compute $n_{\alpha,\beta}$: if $\alpha+\beta \notin \Delta_+^{re}$; set $n_{\alpha,\beta}=0$ for we know that in this case e_α and e_β commute. Otherwise, as in 1.9, we know that $(\dot{w}^{\alpha+\beta})^{-1}.[e_\beta,e_\alpha]$ is an element of $\mathcal{U}_{\mathbf{Z}}\cap \mathfrak{g}_{\alpha_i} = \mathbf{Z}e_i$, where i is such that $\dot{w}^{\alpha+\beta}\alpha_i=\alpha+\beta$, hence $(\dot{w}^{\alpha+\beta})^{-1}.[e_\beta,e_\alpha] = ne_i$ for some $n \in \mathbf{Z}$, and we let $n_{\alpha,\beta}=n$.

We now check that $(X_\alpha(t),X_\beta(s)).X_{\alpha+\beta}(n_{\alpha,\beta}ts) \in I_o^*$:

Fix $t,s \in K$, and set $R=\mathbf{Z}[T,S]$, the ring of polynomials in two variables with integer coefficients, and let $\mathcal{U}_R$ denote $\mathcal{U}_{\mathbf{Z}}\otimes R$. let $v_o \in \mathcal{U}_{\mathbf{Z}}$, and let $D_T \in \mathrm{End}\,\mathcal{U}_R$ be the operator $T\frac{d}{dT}$.

As e_λ acts locally nilpotently on $\mathcal{U}_{\mathbf{Z}}$ in the adjoint representation, for any $\lambda \in \Delta_+^{re}$, we can define elements $E(ce_\lambda) = \sum_k \mathrm{ad}\, e_\lambda^{(k)}\otimes c \in \mathrm{End}\,\mathcal{U}_R$, for any $c \in \mathbf{Z}[T,S]$. Now write

2.3 $$f(T,S)=E(Te_\alpha)E(Se_\beta)E(-Te_\alpha)E(-Se_\beta)E(n_{\alpha,\beta}TSe_{\alpha+\beta}).v_o\otimes 1\ ,$$

and let's compute $D_T f = T(\ (\mathrm{ad}(e\otimes 1))f\ )$

$$+T.E(Te_\alpha)\{E(Se_\beta)(-\mathrm{ad}(e_\alpha\otimes 1))E(-Te_\alpha)E(-Se_\beta)E(n_{\alpha,\beta}TSe_{\alpha+\beta}).v_o\otimes 1$$
$$+E(Se_\beta)E(-Te_\alpha)E(-Se_\beta)(\mathrm{ad}e_{\alpha+\beta}\otimes n_{\alpha,\beta}S)E(n_{\alpha,\beta}TSe_{\alpha+\beta}).v_o\otimes 1\}$$

(using the "product rule"),

$$= T(\mathrm{ad}e_\alpha\otimes 1)f - T\mathrm{ad}(E(Te_\alpha)E(Se_\beta).e_\alpha\otimes 1)f$$
$$+ n_{\alpha,\beta}ST\mathrm{ad}(E(Te_\alpha)E(Se_\beta)E(-Te_\alpha)E(-Se_\beta).e_{\alpha+\beta}\otimes 1)f$$

(using a)),

$$= T.\mathrm{ad}(e_\alpha\otimes 1-e_\alpha\otimes 1-S[e_\beta,e_\alpha]\otimes 1+n_{\alpha,\beta}Se_{\alpha+\beta}\otimes 1)f$$

(as with our assumptions on α,β, e_α and e_β commute with $e_{\alpha+\beta}$)

$= 0$, so that

2.4 $$f(T,S)=f(0,S)=v_o\otimes 1.$$

Now we form the tensor product $\mathcal{U}_R\otimes_R K\simeq\mathcal{U}_K$, where K has the $\mathbf{Z}[T,S]$-module structure obtained by mapping $T\mapsto t$, $S\mapsto s$: Then equations 2.3 and 2.4 give

$$(X_\alpha(t),X_\beta(s))X_{\alpha+\beta}(n_{\alpha,\beta}st).v_o\otimes 1 = v_o\otimes 1,$$

which is verified for all $v_o\varepsilon\mathcal{U}_{\mathbf{Z}}$, from which one concludes that

$$(X_\alpha(t),X_\beta(s))X_{\alpha+\beta}(n_{\alpha,\beta}st)\ \varepsilon\ I_1^*\ .$$

To check that $(X_\alpha(t),X_\beta(s))X_{\alpha+\beta}(n_{\alpha,\beta}st)\ \varepsilon\ I_o^*$ one proceeds exactly as above, and b) is proved.

proof of c): given $\alpha\,\varepsilon\,\Delta_+^{re}-\{\alpha_i\}$, $1\le i\le N$, we already know that $\dot{r}_i.e_\alpha\,\varepsilon\,\mathcal{U}_{\mathbf{Z}}\cap g_{r_i\alpha}=\mathbf{Z}e_{r_i\alpha}$, so that there exists an integer $k_{i,\alpha}\varepsilon\,\mathbf{Z}$ with

$$\dot{r}_i e_\alpha=k_{i,\alpha}e_{r_i\alpha}.$$

In fact, $\dot{r}_i^{-1}e_{r_i\alpha}\varepsilon\mathcal{U}_{\mathbf{Z}}\cap g_\alpha$, hence $k_{i,\alpha}=\pm 1$.

Let's show that $\dot{r}_i(1)X_\alpha(t)\dot{r}_i(-1)X_{r_i\alpha}(-k_{i,\alpha}t)\ \varepsilon\ I^*$:

If M is an integrable $\mathcal{U}_K$-module, and $v\,\varepsilon\,M$, then

$$\dot{r}_i(1)X_\alpha(t)\dot{r}_i(-1).v = \sum_m t^m\dot{r}_i(1).(e_\alpha^{(m)}\otimes 1).\dot{r}_i(-1).v$$

$= \sum_m t^m (\mathrm{Ad}\dot{r}_i(1).(e_\alpha^{(m)} \otimes 1)).v$, by a),

$= \sum_m t^m (\dot{r}_i . e_\alpha^{(m)} \otimes 1).v$, by the definition of $\dot{r}_i$,

$= \sum_m t^m k_{i,\alpha}^m e_{r_i\alpha}^{(m)} \otimes 1.v$

$= X_{r_i\alpha}(k_{i,\alpha} t).v$.

As $X_{r_i\alpha}(k_{i,\alpha}t) = X_{r_i\alpha}(-k_{i,\alpha}t)^{-1}$, we obtain

$$\dot{r}_i(1) x_\alpha(t) \dot{r}_i(-1) = x_{r_i\alpha}(\pm t) \text{ , as needed.}$$

proof of d): it is a priori clear that $(H_i, H_i) = 1$. Now assume that M is an integrable module as in 1.11, so that for all $v \in M$, $v = \sum_\lambda v_\lambda \otimes t_\lambda$, with $v_\lambda \in \mathcal{U}_{\mathbf{Z}}.v^+ \cap M_\lambda$, $t_\lambda \in K$. We know that $H_i(s).v = \sum_\lambda s^{\langle\lambda,\alpha_i^\vee\rangle} v_\lambda \otimes t_\lambda$, so that if $1 \le i \le N$, and s_i, s_j are elements of K, then

$$H_i(s_i)H_j(s_j)H_i(s_i^{-1})H_j(s_j^{-1}).v = \sum_\lambda s_i^{\langle\lambda,\alpha_i^\vee\rangle} s_j^{\langle\lambda,\alpha_j^\vee\rangle} s_i^{-\langle\lambda,\alpha_i^\vee\rangle} . s_j^{-\langle\lambda,\alpha_j^\vee\rangle} v_\lambda \otimes t_\lambda$$

$$= \sum v_\lambda \otimes t_\lambda = v$$

and hence $H_i(s_i)H_j(s_j)H_i(s^{-1})H_j(s_j^{-1}) \in I_o^*$.
The proof that $H_i(s_i)H_j(s_j)H_i(s_i^{-1})H_j(s_j^{-1}) \in I_1^*$ is similar, and as we already know that $H_k(s^{-1}) = H_k(s)^{-1}$, we obtain $(H_i, H_j) = 1$ in G, for all i,j.

To prove e), we keep the notation of d), and find that

$$H_i(t)X_\alpha(s)H_i(t^{-1}).v = \sum_m H_i(t).e_\alpha^{(m)} \otimes s^m . H_i(t^{-1}).v$$

$$= \sum_m (\mathrm{Ad}H_i(t).(e_\alpha^{(m)} \otimes s^m)).v$$

Now $\mathcal{U}_o(K)$ acts by the image character of m on $e_\alpha^{(m)} \otimes s^m$ so that, using 1.11, $\mathrm{Ad}H_i(t).(e^{(m)} \otimes s^m) = t^{\langle m\alpha,\alpha_i^\vee\rangle} e^{(m)} \otimes s^m$

$$= e_\alpha^{(m)} \otimes (t^{\langle\alpha,\alpha_i^\vee\rangle} s)^m.$$

Plugging this expression back into the equation above, one finds that

$$H_i(t)X\ (s)H_i(t^{-1}).v = \sum_m e_\alpha^{(m)} \otimes (t^{\langle\alpha,\alpha_i^\vee\rangle} s)^m.v$$

$$= X_\alpha(t^{\langle\alpha,\alpha_i^\vee\rangle} s).v \text{ , as needed.}$$

Proof of f): As each M_i is generated by $\dot{r}_i(1)$ and H_i, M/H is generated by the images under the quotient maps of $\{\dot{r}_i(1),\ 1\leq i\leq N\}$, and a classical computation shows that $\dot{r}_i(1)^2 \varepsilon H_i$, so that each generator of M/H has order 2. Now fix $i\neq j$, and assume that m_{ij} is finite, so that the matrix $A_{ij} = \begin{pmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{pmatrix}$ is a Cartan matrix in the usual sense. We wish to check that

2.5 $$(\underbrace{\dot{r}_i(1)\dot{r}_j(1)\ldots}_{m_{ij}\text{ terms}})(\underbrace{\dot{r}_j(1)\dot{r}_i(1)\ldots}_{m_{ij}\text{ terms}})^{-1} = 1.$$

To this end, let $\mathfrak{a}_{ij}$ be the subalgebra of $\mathfrak{g}_Q(A)$ generated by $\{e_i,e_j,f_i,f_j\}$; letting Σ_{ij} be the positive root system $\Delta^+(\mathfrak{a}_{ij}, Q\alpha_i^\vee+Q\alpha_j^\vee)$, $\mathfrak{a}_{ij}\otimes\mathbf{C}$ is the Lie algebra associated to the matrix A_{ij} and has $\{e_\lambda, f_\lambda, \lambda\,\varepsilon\,\Sigma_{ij}\}\cup\{\alpha_i^\vee,\alpha_j^\vee\}$ for Chevalley basis. Order the set Σ_{ij}, say $\Sigma_{ij}=\{\lambda_1,\ldots,\lambda_n\}$. Then the Kostant Z-form $\mathcal{U}_{ij}$ of the universal envelope of $\mathfrak{a}_{ij}\otimes\mathbf{C}$ has Z-basis $\{f_{\lambda_1}^{(p_1)}\ldots f_{\lambda_n}^{(p_n)}\binom{\lambda_1^\vee}{q_1}\binom{\lambda_2^\vee}{q_2}e_{\lambda_1}^{(r_1)}\ldots e_{\lambda_n}^{(r_n)}\}$, and $\mathcal{U}_{ij}\subset\mathcal{U}_{\mathbf{Z}}$ by 1.9.

Now let R_{ij} be the left hand side of 2.5, and fix $1\leq k\leq N$, $s\varepsilon\mathbf{N}$. As each e_λ and f_λ act locally nilpotently on $\mathcal{U}_{\mathbf{Z}}$, the $\mathfrak{a}_{ij}\otimes\mathbf{C}$- module $\mathrm{ad}\,\mathcal{U}_{ij}.e_k^{(s)}\otimes\mathbf{C}$ is finite dimensional. Let G_{ij} be the associated Chevalley group (with respect to the lattice $\mathrm{ad}\,\mathcal{U}_{ij}.e_k^{(s)}$) over K, and let $\tilde{G}_{ij}$ be the group generated by $\mathrm{Ad}A^i, \mathrm{Ad}A^j$ in $\mathrm{Aut}(\mathcal{U}_{ij}.e_k^{(s)}\otimes K)$. By construction, $\tilde{G}_{ij}\subset G_{ij}$. If $\tilde{R}_{ij}$ is the image of R_{ij} in $\tilde{G}_{ij}$, we then know ([S3]) that $\tilde{R}_{ij}=1$, hence $\mathrm{Ad}R_{ij}.e_k^{(s)}\otimes 1=e_k^{(s)}\otimes 1$. One proves similarly that $\mathrm{Ad}R_{ij}$ fixes the remaining generators $f_k^{(s)}$, and we obtain $R_{ij}\varepsilon I_1^*$. On the other hand if $v^+\varepsilon M$ as in 1.11 and $u\varepsilon\mathcal{U}_{\mathbf{Z}}$, $t\varepsilon K$, we have

$$\begin{aligned} R_{ij}.u\otimes t.v^+\otimes 1 &= (\mathrm{Ad}R_{ij}(u\otimes t)).R_{ij}^{-1}.(v^+\otimes 1) \\ &= u\otimes t.R_{ij}^{-1}(v^+\otimes 1). \end{aligned}$$

One shows that $R_{ij}v^+\otimes 1=v^+\otimes 1$ as above (using the Chevalley group with respect to the finite dimensional module $\mathcal{U}_{ij}.v^+$) hence $R_{ij}.v=v$ for all $v\varepsilon\mathcal{U}_{\mathbf{Z}}.v^+\otimes K$, so that $R_{ij}\varepsilon I_o^*$, and the proof of 2.5 is now complete.

If m_{ij} is odd, 2.5 implies that

$$\underbrace{\dot{r}_i(1)\dot{r}_j(1)\ldots\dot{r}_i(1)\dot{r}_j(1)\dot{r}_i(1)}_{m_{ij}\text{ terms}} \cdot \underbrace{\dot{r}_j(1)\dot{r}_i(1)\ldots\dot{r}_j(1)\dot{r}_i(1)\dot{r}_j(1)}_{m_{ij}\text{ terms}}$$

$$= \underbrace{\dot{r}_j(1)\dot{r}_i(1)\ldots\dot{r}_j(1)\dot{r}_i(1)\dot{r}_j(1)}_{m_{ij}\text{ terms}} \cdot \underbrace{\dot{r}_j(1)\dot{r}_i(1)\ldots\dot{r}_j(1)\dot{r}_i(1)\dot{r}_j(1)}_{m_{ij}\text{ terms}}$$

and the right hand side is in H because $\dot{r}_i(1)^2 \varepsilon H$ and each $\dot{r}_i(1)$ normalizes H, so the equation gives

$$(\dot{r}_i(1)\dot{r}_j(1))^{m_{ij}} \varepsilon H,$$

and similarly if m_{ij} is even then $(\dot{r}_i(1)\dot{r}_j(1))^{m_{ij}} \varepsilon H$.

Writing $R_i = \dot{r}_i(1)H \,\varepsilon\, M/H$, we have proved that M/H is generated by $\{R_1,\ldots R_N\}$, and that these satisfy

$$\begin{cases} R_i^2 = 1, \\ (R_iR_j)^{m_{ij}} = 1 \text{ if } i \neq j,\ m_{ij} < \infty. \end{cases}$$

By the definition of W, one obtains a surjective homomorphism $\Psi : W \longrightarrow M/H$, with $\Psi(r_i) = R_i$.

We check that Ψ is 1-1: let $w \,\varepsilon\, W-\{1\}$ have reduced expression $w = r_{i_1}\ldots r_{i_k}$, and choose $i_o \varepsilon [1,N]$ such that $w\alpha_{i_o} \varepsilon \Delta_-^{re}$: then $Ad(\dot{r}_{i_1}(1)\ldots\dot{r}_{i_k}(1)).(e_{i_o} \otimes 1) \notin Ke_{i_o}$, hence there exists no h in H such that $\dot{r}_{i_1}(1)\ldots\dot{r}_{i_k}(1)h \,\varepsilon\, I_1^*$, so $\dot{r}_{i_1}(1)\ldots\dot{r}_{i_k}(1) \notin H$.

As $\dot{r}_{i_1}(1)\ldots\dot{r}_{i_k}(1)H = \Psi(w)$, this shows that $\Psi(w) \neq 1$.

And this completes the proof of f) and of 2.1 .

In the proof above, we used the fact that each $\dot{r}_j(1)$ normalizes H. In fact, if $s \,\varepsilon\, K^\times$, one has

$$\dot{r}_j(1)h_i(s)\dot{r}_j(-1) = h_i(s)h_j(s^{-a_{ij}}) ,$$

a formula which can be easily checked using 2.1a) as before.

<u>Remark</u>: Let's write z_λ for x_λ if $\lambda \varepsilon \Delta_+^{re}$,

y_λ if $\lambda \varepsilon \Delta_-^{re}$.

Assume that $\alpha\neq\beta\in\Delta^{re}$, $\alpha+\beta\neq 0$, let $S=((\mathbf{N}\alpha+\mathbf{N}\beta)\cap\Delta)-\{\alpha,\beta\}$, and assume that $S\subset\Delta^{re}$ and that S is finite.

Then one can show as in 2.1b) and [S2] that there exists integers c_i such that for all $s,t\in K$,

$$(z_\alpha(s),z_\beta(t))=\prod_{n_i\alpha+m_i\beta\in S} z_{n_i\alpha+m_i\beta}(c_i s^{n_i}t^{m_i})$$

the product taken in the ordering $n_i\alpha+m_i\beta<n_j\alpha+m_j\beta$ iff $n_i+m_i<n_j+m_j$ or $n_i+m_i=n_j+m_j$ and $n_i<n_j$.

Let us now fix i and write U^{α_i} for the subgroup of U_+ generated by all elements of the form xyx^{-1} with $x\in x_{\alpha_i}$, $y\in y_\beta$, $\beta\in\Delta_+^{re}-\{\alpha_i\}$.

2.6 Proposition : ([PK1])

$$U_+ = x_{\alpha_i} \ltimes U^{\alpha_i}.$$

proof:

Suppose $\beta\in\Delta_+^{re}-\{\alpha_i\}$, and let $\varepsilon=\begin{cases}1 & \text{if } <\alpha_i,\beta^\vee>\ \geq 0,\\ -1 & \text{if } <\alpha_i,\beta^\vee>\ < 0\end{cases}$

so that $\varepsilon<\alpha_i,\beta^\vee>\ \geq 0$.

We now show that there exists a subset S of $\Delta_+^{re}-\{\alpha_i\}$ such that

if $\varepsilon=1$, $(x_{\alpha_i},x_\beta)\ \prod_{\lambda\in S} x_\lambda$,

and if $\varepsilon=-1$, $(y_{\alpha_i},x_\beta)\ \prod_{\lambda\in S} x_\lambda$.

Indeed let $S=(\mathbf{N}\varepsilon\alpha_i+\mathbf{N}\beta)\ \Delta-\{\varepsilon\alpha_i,\beta\}$. Then S clearly does not contain α_i; now let $\lambda_{n,m}=(n\varepsilon\alpha_i+m\beta)$, $n,m\in\mathbf{N}$, and assume that $\lambda_{n,m}\in\Delta$: then either $\varepsilon=-1$, $n=1$, and $m=0$, in which case $\lambda_{n,m}=-\alpha_i$, or $\lambda_{n,m}\in\Delta_+$; assume that the latter is true, and let r_β be the reflexion in the real root β: we have

$$r_\beta\lambda_{n,m}=n\varepsilon\alpha_i-\underbrace{(n\varepsilon<\alpha_i,\beta^\vee>+m)}_{>0}\beta, \text{ so } r_\beta\lambda_{n,m}>0.$$

Therefore $\lambda_{n,m}\in\{\lambda\in\Delta_+ \mid r_\beta\lambda<0\}$;

Hence $S\subset\{\lambda\in\Delta_+ \mid r_\beta\lambda<0\}$, a set which is finite and which consists entirely of real roots ([K3]). By the remark made above, there exist integers $c_{n,m}$ such that if $\varepsilon=1$, then for all $s,t\in K$, $(x_{\alpha_i}(s),x_\beta(t))=\prod_{\lambda_{n,m}\in S} x_{\lambda_{n,m}}(c_{n,m}s^n t^m)$, and if $\varepsilon=-1$, then for all $s,t\in K$, $(y_{\alpha_i}(s),x_\beta(t))=\prod_{\lambda_{n,m}\in S} x_{\lambda_{n,m}}(c_{n,m}s^n t^m)$.

(Note that in case A is symmetrizable, i.e. for some diagonal matrix D, D.A is symmetric, Kač and Peterson show that $c_{n,m}=0$ if $nm\neq 1$ ([PK2]), and one computes $c_{1,1}$ as in 2.1b)).

Suppose now that $<\alpha_i,\beta^\vee>\geq 0$: we then have

$\dot{r}_i(1)x_{\alpha_i}(t)x_\beta(s)x_{\alpha_i}(-t)\dot{r}_i(-1)= \dot{r}_i(1)(x_{\alpha_i}(t),x_\beta(s))\dot{r}_i(-1).$
$\dot{r}_i(1)x_\beta(s)\dot{r}_i(-1)$

which, by the result above lies in U^{α_i},
while if $<\alpha_i,\beta^\vee><0$, and $t\in K^\times$, we have

$\dot{r}_i(1)x_{\alpha_i}(t)x_\beta(s)x_{\alpha_i}(-t)\dot{r}_i(-1)= H_i(u)\dot{r}_i(-t)x_{\alpha_i}(t)x_\beta(s)x_{\alpha_i}(-t).$
$\dot{r}_i(t)H_i(u)^{-1}$

where u is chosen so that $\dot{r}_i(-t)=\dot{r}_i(1)H_i(u)^{-1}$,
$=H_i(u)x_{\alpha_i}(-t)(y_{\alpha_i}(t^{-1}),x_\beta(s))x_\beta(s)x_{\alpha_i}(t)H_i(u)^{-1}$, which is again in U^{α_i} (using 2.1e)). This shows that $\dot{r}_i(1)$ normalizes U^{α_i}.
It is clear from the definition of U^{α_i} that x_{α_i} normalizes it, and if $g\in x_{\alpha_i}\cap U^{\alpha_i}$, then $\dot{r}_i(1)g\dot{r}_i(-1).v^+\otimes 1=v^+\otimes 1$ for all $v^+\in M$ as in 1.11 because $\dot{r}_i(1)g\dot{r}_i(-1)$ lies in $y_{\alpha_i}\cap U^{\alpha_i}$ which is contained in U_+; but the only element of y_{α_i} that fixes all such $v^+\otimes 1$ is $y_{\alpha_i}(0) = 1$, so $\dot{r}_i(1)g\dot{r}_i(-1)=1$, i.e. $g=1$ and we must have

$$x_{\alpha_i}\cap U^{\alpha_i} = 1.$$

Let's note that, in fact, $U^{\alpha_i} = U_+\cap \dot{r}_i(1)U_+\dot{r}_i(-1)$: indeed, $\dot{r}_i(1)U_+\dot{r}_i(1)=(y_{\alpha_i}\ltimes U^{\alpha_i})$by 2.6, hence

$$U^{\alpha_i}\subset U_+\cap\dot{r}_i(1)U_+\dot{r}_i(-1),$$

while if $g=x_{\alpha_i}(t)g'\in U_+$ (with $g'\in U^{\alpha_i}$, $t\in K$) is in $U_+\cap\dot{r}_i(1)U_+\dot{r}_i(-1)$, then $\dot{r}_i(1)x_{\alpha_i}(t)g'\dot{r}_i(-1) \in U_+$, i.e. $y_{\alpha_i}(-t)\dot{r}_i(1)g'\dot{r}_i(-1) \in U_+$, so $t=0$, hence $g=g'$, so $U_+ \cap \dot{r}_i(1)U_+\dot{r}_i(-1) \subset U^{\alpha_i}$, as needed.

We conclude this section by defining subgroups B_ε by $B_\varepsilon = HU_\varepsilon$, for $\varepsilon = \pm 1$, and observing that

2.7 Proposition :

a) $U_- \cap B_+ = 1$

b) If $G(K)$ carries a Frobenius map F as above, then $FB_\varepsilon \subset B_\varepsilon$, $FH \subset H$, and $\dot{r}_i(1) \in G(K)^F$ for all i.

proof:

indeed, if $g \in U_-$, and $\{g.(v^+ \otimes 1)\ ,\ v^+ \otimes 1\}$ is linearly dependent in $\mathcal{U}_{\mathbb{Z}}.v^+ \otimes K$ for all $v^+ \in M$ as in 1.11, then $g=1$, whereas if g is in B_+, then $g.v^+ \otimes 1 \in Kv^+ \otimes 1$, and a) is proved.

b) is clear.

Example 2: Suppose $\mu \in \Delta_+^{re}$, say $\mu = w\alpha_k$, and let $\mu_m = wr_k\alpha_m \in \Delta_+^{re}$; μ and μ_m are 'adjacent' roots. Let's show that x_μ and x_{μ_m} must then commute:

If $i,j \in \mathbb{N}$, then $r_k w^{-1}(i\mu_m + j\mu) = i\alpha_m - j\alpha_k$, and $i\mu_m + j\mu$ is a root if and only if $r_k w^{-1}(i\mu_m + j\mu)$ is one; however the right hand side is neither in Δ_+ nor in Δ_- unless i or j is zero. Hence $\mathbb{N}\mu_m + \mathbb{N}\mu \cap \Delta_+ = \{\mu_m, \mu\}$, so x_μ, x_{μ_m} commute by 2.1b).

Section 3 : The Bruhat decomposition and Schubert cells

We have seen that if $w \in W$ has reduced expression $w = r_{i_1} \dots r_{i_k}$, and X,Y are subgroups of G such that $H \subset X$ and H normalizes Y, then one may define subsets wX, wYw^{-1} of G by requiring that

$$wX = \{\dot{r}_{i_1}(1) \dots \dot{r}_{i_k}(1)x,\ x \in X\},$$

$$wYw^{-1} = \{\dot{r}_{i_1}(1) \dots \dot{r}_{i_k}(1) y \dot{r}_{i_k}(-1) \dots \dot{r}_{i_1}(-1),\ y \in Y\}$$

so that if $w = w_1 w_2$ with $w_i \in W$, then $wX = w_1(w_2 X)$ and $wYw^{-1} = w_1(w_2 Y w_2^{-1}) w_1^{-1}$ whenever these expressions make sense : indeed, if w_1 has reduced expression $r_{m_1} \dots r_{m_j}$ and $w_2 = r_{n_{j+1}} \dots r_{n_k}$ then as in 2.1f) one has $\Psi(w) = \Psi(w_1)\Psi(w_2)$, hence for some $h \in H$, $\dot{r}_{i_1} \dots \dot{r}_{i_k} = \dot{r}_{m_1} \dots \dot{r}_{n_k} h$.

Using this notation, one can define for each $w \in W$ a subset $B_+ w B_+$ of G, and we shall now show that

3.1 Proposition : (see [PK2])

$$G = \bigcup_{w \in W} B_+ w B_+ = \bigcup_{w \in W} B_- w B_+$$

The proof of these decompositions will be carried out in stages: the decomposition itself is proved in 3.1-3.7 and uniqueness is established by 3.11.

In what follows, we fix $w \in W$, $w = r_{j_1} \dots r_{j_k}$ a reduced expression, and write $\underline{w} = \dot{r}_{j_1}(1) \dots \dot{r}_{j_k}(1)$.

Assume now that $g \in wB_+ r_i$, for some i, so that $g = \underline{w} h x_{\alpha_i}(t) g_o \dot{r}_i(1)$, with $h \in H$, $t \in K$, $g_o \in U^{\alpha_i}$;

3.2 If $w\alpha_i \in \Delta_+$, then $\underline{w} x_{\alpha_i}(t) \dot{r}_i(1) = x_{w\alpha_i}(s) \underline{w} \dot{r}_i(1)$ for some $s \in K$,

hence $g = (\underline{w} h \underline{w}^{-1}) \underline{w} x_{\alpha_i}(t) \dot{r}_i(1) (\dot{r}_i(-1) g_o \dot{r}_i(1))$

$= h' x_{w\alpha_i}(s) \underline{w} \dot{r}_i(1) g_o'$ with $h' \in H$, $g_o' \in U_+$,

i.e. $g \in B_+ w r_i B_+$,

whereas

3.3 If $w\alpha_i \in \Delta_-$, and $t \neq 0$, then $\underline{w} x_{\alpha_i}(t) \dot{r}_i(1) = \underline{w} y_{\alpha_i}(t^{-1}) x_{\alpha_i}(t) h_o$, with $h_o \in H$,

$$=x_{-w\alpha_i}(s)x_{\alpha_i}(t)h_o \quad \text{with } s\varepsilon K,$$

hence $g=(\underline{w}h\underline{w}^{-1})\underline{w}x_{\alpha_i}(t)\dot{r}_i(1)(\dot{r}_i(-1)g_o\dot{r}_i(1))$

$=h'x_{-w\alpha_i}(s)x_{\alpha_i}(t)h_og_o'$, with $h'\varepsilon H$, $g_o'\varepsilon U_+$,

i.e. $g\,\varepsilon\, B_+wB_+$.

One concludes that $B_+wB_+r_iB_+\subset B_+wB_+\cup B_+r_iB_+$, so that, by induction on n, if $w'=r_{i_1}\ldots r_{i_n}$ with length(w')=n, then

<u>3.4</u> $\quad (B_+wB_+).(B_+w'B_+)\subset B_+wB_+\cup\bigcup_s B_+wr_{s_1}\ldots r_{s_m}B_+$

where the second union is taken over all m-tuples $(r_{s_1}\ldots r_{s_m})$ with $s_1<\ldots<s_n$, $m\leq n$.

Hence $\bigcup_{w\varepsilon W} B_+wB_+$ is a subgroup of G. Moreover, $x_\alpha\subset B_+\subset\bigcup_{w\varepsilon W} B_+wB_+$, for all $\alpha\,\varepsilon\,\Delta_+^{re}$, while if $w_\beta\varepsilon W$ is the reflexion in the positive real root β, then $y_\beta=w_\beta x_\beta w_\beta^{-1}\subset(B_+w_\beta B_+).(B_+w_\beta B_+)$, so that $\bigcup_{w\varepsilon W} B_+wB_+$ contains all y_β as well. Since the family $\{x_\alpha,\ \alpha\,\varepsilon\,\Delta_+^{re}\}\cup\{y_\alpha,\ \alpha\,\varepsilon\,\Delta_+^{re}\}$ generates G, this establishes the Bruhat decomposition $G=\bigcup_{w\varepsilon W} B_+wB_+$.

We now show that $G=\bigcup_{w\varepsilon W} B_-wB_+$ (the Birkhoff decomposition as in [KP2]):

With w as before, assume $g=\underline{w}hx_{\alpha_i}(t)g_o\dot{r}_i(1)\,\varepsilon\, wB_+r_i$.

<u>3.5</u> If $w\alpha_i\varepsilon\Delta_-$, then $\underline{w}x_{\alpha_i}(t)\dot{r}_i(1)=h_oy_{-w\alpha_i}(s)\underline{w}\dot{r}_i(1)$, with $h_o\varepsilon H$, $s\varepsilon K$,

hence $g=(\underline{w}h\underline{w}^{-1})h_oy_{-w\alpha_i}(s)\underline{w}\dot{r}_i(1)(\dot{r}_i(-1)g_o\dot{r}_i(1))$,

i.e. $g\,\varepsilon\, B_-wr_iB_+$,

whereas

<u>3.6</u> If $w\alpha_i\varepsilon\Delta_+$, $t\neq0$, then $\underline{w}x_{\alpha_i}(t)\dot{r}_i(1)=\underline{w}y_{\alpha_i}(t^{-1})x_{\alpha_i}(t)h_o$, $h_o\varepsilon H$

$$=y_{w\alpha_i}(s)\underline{w}x_{\alpha_i}(t)h_o,\ s\,\varepsilon\, K,$$

hence $g=(\underline{w}h\underline{w}^{-1})y_{w\alpha_i}(s)\underline{w}x_{\alpha_i}(t)h_o(\dot{r}_i(-1)g_o\dot{r}_i(1))$,

i.e. $g\,\varepsilon\, B_-wB_+$;

One concludes that $B_-wB_+r_iB_+\subset B_-wB_+\cup B_-wr_iB_+$, and, by induction on n=length(w'), if $A=\bigcup_{w\varepsilon W} B_-wB_+$, and $g\varepsilon B_+w'B_+$, then $Ag\subset A$. As $G=\bigcup_{w\varepsilon W} B_+wB_+$ we see that the subset A of G is invariant under right-translation by

G, hence A=G as stated above.

Let $\Phi(w)=\{\alpha\in\Delta_+^{re} \mid w^{-1}\alpha<0\}$. One can then prove by induction on k ([K3]) that $\Phi(w)=\{\alpha_{j_1}, r_{j_1}\alpha_{j_2}, \ldots, r_{j_1}\ldots r_{j_{k-1}}\alpha_{j_k}\}$. Let $U_w=U_+\cap wU_-w^{-1}$.

3.7 Proposition :

> Given $g\in U_w$, then for every $\beta\in\Phi(w)$, there is a unique $t_\beta\in K$ such that
>
> $$g=\prod_{\beta\in\Phi(w)} x_\beta(t_\beta) \qquad ,$$
>
> the product being taken in the ordering of $\Phi(w)$ as above.

proof:

This is clear if $w=r_i$, $1\leq i\leq N$, by 2.1c). Proceeding by induction on length(w), we assume the result known for $w'=r_{j_2}\ldots r_{j_k}$ and compute, for $w=r_{j_1}w'$,

$$U_w=U_+\cap r_{j_1}w'U_-w'^{-1}r_{j_1}=r_{j_1}(r_{j_1}^{-1}U_+r_{j_1}\cap w'U_-w'^{-1})r_{j_1}^{-1}$$

$$=r_{j_1}(y_{\alpha_{j_1}}\ltimes U^{\alpha_{j_1}}\cap w'U_-w'^{-1})r_{j_1}^{-1}$$

Now $w'^{-1}\alpha_{j_1}=-w'^{-1}r_{j_1}^{-1}\alpha_{j_1}=-w^{-1}\alpha_{j_1}>0$ because $\alpha_{j_1}\in\Phi(w)$,

hence 1) $U^{\alpha_{j_1}}\cap w'U_-w'^{-1}=U_+\cap w'U_-w'^{-1}=U_{w'}$,

and 2) $y_{\alpha_{j_1}}\subset w'U_-w'^{-1}$.

So for all $g\in U_w$, $g=\dot{r}_{j_1}(1)(y_{\alpha_{j_1}}(v).\prod_{\lambda\in\Phi(w')} x_\lambda(v_\lambda))\dot{r}_{j_1}(-1)$

for some uniquely determined v's in K,

$$=x_{\alpha_{j_1}}(s).\prod_{\lambda\in\Phi(w')} x_{r_{j_1}\lambda}(s_\lambda) \text{ , with } s_*\in K,$$

$$=\prod_{\beta\in\Phi(w)} x_\beta(t_\beta).$$

In particular, U_w is 'isomorphic' to $K^{\text{length}(w)}$, the k-fold product of K. In fact,

3.8 Proposition :

> U_w is, in a natural way, a unipotent algebraic group

proof:

Indeed, let n_w be the subspace of n_+ spanned by $\{e_\alpha, \alpha\varepsilon\Phi(w)\}$. If $\alpha,\beta\,\varepsilon\,\Phi(w)$, and $i,j\,\varepsilon\,\mathbf{N}$, then $w^{-1}(i\alpha+j\beta)=iw^{-1}(\alpha)+jw^{-1}(\beta)$ is negative, hence if $i\alpha+j\beta$ is a root, it must lie in $\Phi(w)$ (a fortiori, it must then be real). This implies that n_w is a Lie algebra, and moreover that if $n_w^{(m)}$ is the m-th component of the central series of n_w, then (using the principal gradation as in 1.11)

$$n_w^{(m)} \subset \sum_{j\geq m} u_j \quad .$$

So if $M = 1 + \max\{\text{length}(\lambda), \lambda\varepsilon\Phi(w)\}$, then on one hand $n_w^{(M)}$ is contained in n_w and $n_w \cap \sum_{j\geq M} u_j = 0$, while in fact $n_w^{(M)}$ is contained in $\sum_{j\geq M} u_j$. Hence $n_w^{(M)}=0$, and n_w is a nilpotent Lie algebra.

One also concludes from the above that $\mathbf{N}\alpha+\mathbf{N}\beta\cap\Delta_+ = \mathbf{N}\alpha+\mathbf{N}\beta\cap\Delta_+^{re}$ is finite and contained in $\Phi(w)$ for any $\alpha,\beta\,\varepsilon\,\Phi(w)$, say $\mathbf{N}\alpha+\mathbf{N}\beta\cap\Delta_+=\{i_1\alpha+j_1\beta,\ldots,i_n\alpha+j_n\beta\}$. Since we also know that each e_α, $\alpha\varepsilon\Phi(w)$, spans over $\mathbf{Z}$ the root space $g_\alpha \cap u_{\mathbf{Z}}$, one can show as in 2.1b) that there exist integers a_{i_m,j_m} such that

$$(x_\alpha(r),x_\beta(s)) = \prod_{m=1}^{n} x_{i_m\alpha+j_m\beta}(a_{i_m,j_m} r^{i_m}s^{j_m})$$

for all $r,s\,\varepsilon\,K$.

Let $\pi:K^k \longrightarrow U_w$ be the bijection

$$\pi(t_1,\ldots t_k)=x_{\alpha_{j_1}}(t_1) \;\ldots\; x_{r_{j_1}\ldots r_{j_{k-1}}\alpha_{j_k}}(t_k)$$

as in 3.7; the above then shows that the map
$((t_1,\ldots,t_k),(s_1,\ldots,s_k)) \mapsto \pi^{-1}(\pi(t_1,\ldots,t_k)^{-1}\cdot\pi(s_1,\ldots,s_k))$
is a morphism of affine k-space, hence π induces a structure of algebraic group on U_w. The fact that U_w is unipotent follows from the nilpotence of n_w as above.

Letting U^w be the subgroup of U_+ generated by $\{ax_\beta a^{-1}$, $a\varepsilon U_w$, and $\beta\,\varepsilon\,\Delta_+^{re}-\Phi(w)\}$, we see that U_w normalizes U^w, and, by 3.7, that $U_w \cup U^w$

contains all the generators of U_+, so $U_+=U_w.U^w$. One can then show by induction on length(w) as in 2.6 and 3.7 that $w^{-1}U^w w\subset U_+$, from which one deduces that in fact $U^w=U_+\cap wU_+w^{-1}$.

Before going on to more geometric considerations, let's now complete the proof of 3.1 by establishing the disjointness of the various cosets on question:

3.9 Fix Λ, $L(\Lambda)=L$, $v^+\in L_\Lambda$ as in 1.10, with Λ chosen such that if $\sigma\in W$ satisfies $\sigma\Lambda=\Lambda$, then $\sigma=1$. If $q\in\mathbf{N}$, then 1.11 implies that

$$\sum_{\substack{\lambda \\ \text{height}(\Lambda-\lambda)\leq q}} L_\lambda = (\mathcal{a}.v^+)\otimes\mathbf{Q} \quad ,$$

where $\mathcal{a}$ is the $\mathbf{Z}$-submodule of $\mathcal{U}_-$ spanned by all x with degree(x)$\leq$q (the gradation being again the principal gradation). Given a field K, if $v_w = (\dot{w}v^+)\otimes 1$, then $v_w = \dot{r}_{j_1}(1)\ldots\dot{r}_{j_k}(1).(v^+\otimes 1)$ and is a non-zero element of $(L_{w\lambda}\cap\mathcal{U}_{\mathbf{Z}}.v^+)\otimes K$. We can therefore assert that for all $g\in B_+wB_+$, $g.v^+=tv_w+\Sigma v_\lambda\otimes t_\lambda$, for some $t\in K^\times$, $t_\lambda\in K$ depending on g, the sum being over all λ with height$(\Lambda-\lambda)<$height$(w\Lambda)$, with $v_\lambda\in\mathcal{U}_{\mathbf{Z}}.v^+\cap L_\lambda$.

If $w'\in W$ has length less than w, say $w'=r_{i_1}\ldots r_{i_n}$ with $n\leq k$, then $g'=\dot{r}_{i_1}(1)\ldots\dot{r}_{i_n}(1)$ is in $B_+w'B_+$, and $g'.v^+\otimes 1=v_{w'}$, which cannot be put in the form $tv_w+\Sigma v_\lambda\otimes t_\lambda$ with $t\neq 0$. This shows that $B_+w'B_+\not\subset B_+wB_+$. As both sets are B_+-double cosets, they are either equal or disjoint; hence the latter is true.

One proves similarly that $B_-wB_+=B_-w'B_+$ if and only if $w=w'$, and the decompositions 3.1 are now completely proved. In particular,

3.10 $$G/B_+ = \bigcup_{w\in W} C(w) = \bigcup_{w\in W} C'(w)$$

where $C(w)$, $C'(w)$ are the images in G/B_+ of B_+wB_+, B_-wB_+ respectively.

We also know that

3.11 Proposition : ([PK2])

the group U_w acts simply transitively (by left translation) on $C(w)$

proof:

Indeed, if $bwB_+\in C(w)$, then $b=u_1u_2h$, with $u_1\in U_w$, $u_2\in U^w$, so that

$$bwB_+=u_1w\underbrace{(w^{-1}u_2w)(w^{-1}hw)}_{\in\, U_+}B_+$$

$$=u_1wB_+ ,$$

while if $u_1 wB_+ = wB_+$, then $\underline{w}^{-1} u_1 \underline{w} \in B_+ \cap U_- = 1$, so $u_1 = 1$.

In particular, each $C(w)$ is 'isomorphic' to U_w.

We now proceed to refine the decompositions 3.1 by decomposing $C(w)$ itself as follows ([Dh]):

Given $u \in U_w$, and $0 \le i \le k = \text{length}(w)$, let σ_i be the unique element of W such that $u\dot{r}_{j_1}(1) \ldots \dot{r}_{j_i}(1) \in B_- \sigma_i B_+$, and write $\eta(u) = (\sigma_o, \sigma_1, \ldots, \sigma_k) \in W^{k+1}$.

Note that σ_o is always 1, since $u = 1.1.u \in B_- B_+$. Let us now prove

<u>3.12 Proposition :</u>

> the sequence $\eta(u)$ satisfies $\sigma_i \in \{\sigma_{i-1}, \sigma_{i-1} r_{j_i}\}$, for all i.

proof:

Fix $i \geq 1$, and write $w_i = r_{j_i} \ldots r_{j_k}$, $\underline{w}_i = \dot{r}_{j_i}(1) \ldots \dot{r}_{j_k}(1)$. We then know that there are elements $b_i \in B_-$, $u_{i+1} \in U_{w_{i+1}}$, $v_{i+1} \in U^{w_{i+1}^{-1}}$

such that

<u>3.13</u> $$u\underline{w} = b_i \sigma_i u_{i+1} \underline{w}_{i+1} v_{i+1}$$

This is so because $u\underline{w}\underline{w}_i^{-1} = b_i \sigma_i b_+$ where $b_+ = u_{i+1} u'_{i+1}$ for some $u_{i+1} \in U_{w_{i+1}}$, $u'_{i+1} \in U^{w_{i+1}}$, and we obtain 3.11 by writing

$$v_{i+1} = \underline{w}_{i+1}^{-1} u'_{i+1} \underline{w}_{i+1} \in w_{i+1}^{-1} U^{w_{i+1}} w_{i+1} = U^{w_{i+1}^{-1}} .$$

Applying 3.11 to the index i-1, we see that

$$\begin{aligned} b_i \sigma_i u_{i+1} \underline{w}_{i+1} v_{i+1} &= b_{i-1} \sigma_{i-1} u_i \underline{w}_i v_i \\ &= b_{i-1} \sigma_{i-1} u_i \dot{r}_{j_i}(1) \underline{w}_{i+1} v_i \end{aligned}$$

hence $\quad b_i \sigma_i u_{i+1} \underline{w}_{i+1} (v_{i+1} v_i^{-1}) \underline{w}_{i+1}^{-1} = b_{i-1} \sigma_{i-1} u_i \dot{r}_{j_i}(1)$;

The left-hand-side is in $B_- \sigma_i B_+$, while the right-hand-side is in $B_- \sigma_{i-1} B_+ r_{j_i}$. By 3.5 & 3.6, if $\sigma_{i-1} \alpha_{j_i} < 0$, then

<u>3.14</u> $\quad B_- \sigma_{i-1} B_+ r_{j_i} \subset B_- \sigma_{i-1} r_{j_i} B_+$, while if $\sigma_{i-1} \alpha_{j_i} > 0$, then

$$B_- \sigma_{i-1} B_+ r_{j_i} \subset B_- \sigma_{i-1} B_+ \cup B_- \sigma_{i-1} r_{j_i} B_+,$$

so by disjointness in 3.1, $\sigma_i \in \{\sigma_{i-1}, \sigma_{i-1} r_{j_i}\}$.

Let us now define (as in [Dh2]) the subset $D \subset W^{k+1}$ by requiring

$D = \{\ \sigma=(\sigma_o,\ldots,\sigma_k)\ |\ \sigma_o=1,\ \sigma_i\varepsilon\{\sigma_{i-1},\ \sigma_{i-1}r_{j_i}\}$ and $\sigma_i\leq\sigma_{i-1}r_{j_i}$ for all $i\}$.

where $\leq$ is the Bruhat order in the Coxeter group W. We will need the following facts about $\leq$ (see e.g. [D]):

1) If $y\leq w$, then $y=r_{i_1}\ldots r_{i_n}$, for some $i_1<\ldots<i_n$, $i_m\varepsilon\{j_1,\ldots,j_k\}$

2) If $w\alpha_i>0$, then $w\leq wr_i$.

To see that for all $u\varepsilon U_w$, $\eta(u)\varepsilon D$, we still need to check that $\sigma_j\leq\sigma_{j-1}r_{j_i}$. If $\sigma_j=\sigma_{j-1}r_{j_i}$, this is obvious, while if $\sigma_j=\sigma_{j-1}$, 3.14 implies that $\sigma_{j-1}\alpha_{j_i}>0$, hence $\sigma_{j-1}\leq\sigma_{j-1}r_{j_i}$ by 2) above. We thus obtain a map $\eta:U_w\longrightarrow D$.

Now define, for each $\sigma\varepsilon D$, a subset D_σ of C(w) by setting

$$D_\sigma = \{uwB_+\ \varepsilon\ C(w),\ \text{where } u\varepsilon U_w\ |\ \eta(u)=\sigma\}:$$

By 3.11, D_σ is well-defined, and we have $C(w)=\bigcup_{\sigma\varepsilon D} D_\sigma$. Also, if $p\varepsilon C(w)$, say $p=uwB_+$, then $p=b_-\sigma_k u_{k+1}w_{k+1}v_{k+1}B_+=b_-\sigma_k u_{k+1}v_{k+1}B_+$ as $w_{k+1}=1$, so $p\varepsilon C'(\sigma_k)$, where $(\sigma_o,\ldots,\sigma_k)=\eta(u)$, hence $D_\sigma\subseteq C'(\sigma_k)$.

One knows that $\sigma_k\leq w$, for all $(\sigma_o,\ldots,\sigma_k)\varepsilon D$ ([Dh1]), so that we also have

<u>3.15</u> $$C(w)\cap C'(y) = \bigcup_{\substack{\sigma\varepsilon D\\ \sigma_k=y}} D_\sigma,$$ for all $y\varepsilon W$ (in particular, $C(w)\cap C'(y)$ is empty if $y\not\leq w$).

To study the subsets D_σ in more detail, we put an algebraic structure on C(w) by requiring the 'isomorphism' 3.11 to be an isomorphism of varieties. Note that if char$K=p\neq0$, the Frobenius $F:G(K)\longrightarrow G(K)$ factors to G/B_+, and that the isomorphism just described 'commutes' with F.

Fix $\sigma\varepsilon D$. Then for each i, one knows that either $\sigma_{i-1}>\sigma_i$, or $\sigma_{i-1}<\sigma_i$, or $\sigma_{i-1}=\sigma_i$. Set

$$K_{\sigma,i} = \begin{cases} K & \text{if } \sigma_{i-1}>\sigma_i \\ \{0\} & \text{if } \sigma_{i-1}<\sigma_i \\ K^\times & \text{if } \sigma_{i-1}=\sigma_i \end{cases}$$

and define $f_i:K_{\sigma,i}\times U_{w_{i+1}}\longrightarrow U_{w_i}$ by $f_i(t,u_{i+1})=x_{\alpha_{j_i}}(t)\dot{r}_{j_i}(1)\tilde{u}_{i+1}\dot{r}_{j_i}(1)$, where $\tilde{u}_{i+1}=u_{i+1}$ if $\sigma_{i-1}\neq\sigma_i$, while if $\sigma_{i-1}=\sigma_i$, $\tilde{u}_{i+1}$ is the element of

$U_{w_{i+1}}$ satisfying $x_{\alpha_{j_i}}(t^{-1})u_{i+1}=\tilde{u}_{i+1}v_{i+1}$ with $v_{i+1}\varepsilon U^{w_{i+1}}$ (see[Dh2]).

One can show, as in [Dh2], that f_i is injective, $\mathrm{Im}f_i$ is locally closed in U_{w_i}, and f_i is an isomorphism onto its image. In fact,

$$\mathrm{Im}f_i = \begin{cases} U_{w_i} & \text{if } \sigma_{i-1}>\sigma_i \\ \dot{r}_{j_i}U_{w_{i+1}}\dot{r}_{j_i}^{-1} & \text{if } \sigma_{i-1}<\sigma_i \\ U_{w_i}-\dot{r}_{j_i}U_{w_i}\dot{r}_{j_i}^{-1} & \text{if } \sigma_{i-1}=\sigma_i \end{cases}$$

Defining the set $V_1(\sigma)$ inductively by letting $V_{k+1}(\sigma)=1\subset U_{w_k}$, and writing $V_i(\sigma)=f_i(K_{\sigma,i}\times V_{i+1}(\sigma))\subset U_{w_i}$, we see that $V_1(\sigma)\subset U_{w_1}=U_w$, and one can show as in [Dh2] that

$$V_1(\sigma).wB_+ = D_\sigma$$

Letting $m(\sigma)=\#\{i \mid \sigma_{i-1}>\sigma_i\}$, and $n(\sigma)=\#\{i \mid \sigma_{i-1}=\sigma_i\}$, we can identify the Dheodar components D_σ explicitely, namely

$$\underline{3.16} \qquad D_\sigma \simeq K^{m(\sigma)}\times(K^\times)^{n(\sigma)}$$

For $1\leq i\leq k$, define subgroups B_i, P_i of G by letting $B_o=B_+$,

$$B_i = w_{k-i+1}^{-1}B_o w_{k-i+1}$$

and P_i = subgroup of G generated by B_i and B_{i-1}.

As $B_o=B_+$, one can easily check, using 3.2 & 3.3, that $P_1=B_o\cup B_o r_{j_k}B_o$ so, in particular, $\dot{r}_{j_k}(1)\,\varepsilon\, P_1$, so we can also write

$$\underline{3.17} \qquad P_1=r_{j_k}B_o\cup B_1B_o=r_{j_k}B_o\cup y_{\alpha_{j_k}}B_o=HA^{j_k}\ltimes U^{\alpha_{j_k}},$$

i.e. P_1 is a standard parabolic subgroup of semisimple rank 1 in case G is finite dimensional.

To obtain similar decompositions for P_i, $i\geq 2$, let $\beta_i=w_{k-i+2}^{-1}\alpha_{j_{k-i+1}}$ so that if s_i denotes reflexion in the positive real root β_i, then $s_i=w_{k-i+2}^{-1}r_{j_{k-i+1}}w_{k-i+2}\,\varepsilon\, W$. Write $\underline{s}_i=\underline{w}_{k-i+2}^{-1}\dot{r}_{j_{k-i+1}}(1)\underline{w}_{k-i+2}\,\varepsilon\, G$. By induction on i, we have $B_i=\underline{s}_iB_{i-1}\underline{s}_i^{-1}$.

We now prove, again by induction on i, that y_{β_i}, $x_{\beta_{i+1}}\subset B_i$:

This is true for i=1 by 3.17 . Assuming $x_{\beta_i}\varepsilon B_{i-1}$, we get $s_i x_{\beta_i} s_i^{-1} \subset B_i$, and $s_i x_{\beta_i} s_i^{-1} = y_{\beta_i}$, while the definition of B_i implies that $w_{k-i+1}^{-1} x_{\alpha_{j_{k-i}}} w_{k-i+1}$ is contained in B_i, and $w_{k-i+1}^{-1} x_{\alpha_{j_{k-i}}} w_{k-i+1} = x_{w_{k-i+1}^{-1}\alpha_{j_{k-i}}} = x_{\beta_{i+1}}$.

Since P_i is generated by B_{i-1}, B_i, we obtain in particular that both x_{β_i} and y_{β_i} are contained in P_i; hence $\underline{s_i} = \underline{w}_{k-i+2}^{-1} \dot{r}_{j_{k-i+1}}(1) \underline{w}_{k-i+2}$, which is equal to $x_{\beta_i}(a) y_{\beta_i}(b) x_{\beta_i}(a)$ for some $a,b \varepsilon K$, is an element of P_i. We can now prove that

3.18 Proposition :

$$B_{i+1} \subset B_i \cup B_i s_{i+1} B_i$$

proof:

Indeed, 3.18 can be rewritten

$$s_{i+1} w_{k-i+1}^{-1} B_o w_{k-i+1} s_{i+1}^{-1} \subset w_{k-i+1}^{-1} (B_o \cup B_o w_{k-i+1} s_{i+1} w_{k-i+1}^{-1} B_o) w_{k-i+1}$$

which is equivalent to

$$w_{k-i+1} s_{i+1} w_{k-i+1}^{-1} B_o w_{k-i+1} s_{i+1}^{-1} w_{k-i+1}^{-1} \subset B_o \cup B_o w_{k-i+1} s_{i+1} w_{k-i+1}^{-1} B_o ,$$

and, since $w_{k-i+1} s_{i+1} w_{k-i+1}^{-1} = w_{k-i+1} s_{i+1}^{-1} w_{k-i+1}^{-1} = r_{j_{k-i+2}}$, this inclusion is true by 3.2,3.3 as above.

Note that one consequence of 3.18 is that $B_i \cup B_i s_{i+1} B_i$ is equal to $w_{k-i+1}^{-1} (B_+ \cup B_+ r_{j_{k-i+2}} B_+) w_{k-i+1}$, and is itself a group, hence

3.19 $$P_{i+1} = B_i \cup B_i s_{i+1} B_i$$

and that $B_+ \cup B_+ r_{j_{k-i+2}} B_+ = r_{j_{k-i+2}} (r_{j_{k-i+2}} B_+ \cup y_{\alpha_{j_{k-i+2}}} B_+)$, so that, conjugating each term by w_{k-i+1}^{-1}, we also have

$$P_{i+1} = s_{i+1} B_i \cup y_{\beta_{i+1}} B_i .$$

We will use the following notation : if A is a group acting on the set B on the right and on the set C on the left, let $B \times^A C = B \times C/\sim$, where $\sim$ is the equivalence $(b,c) \sim (ba, a^{-1}c)$. If B,C are themselves groups, and A is a subgroup of both with the action being that induced by multiplication, then B (respectively C) acts on $B \times^A C$ on the left (resp. right) by multiplication. Finally, if B and C are in addition themselves subgroups of D, we get a map $B \times^A C \to D$ satisfying $(b,c) \mapsto b.c$.

Using this construction repeatedly, we let Z_w be the set (see [D1])

$$Z_w = P_k \times^{B_{k-1}} P_{k-1} \times^{B_{k-2}} P_{k-2} \cdots \times^{B_1} P_1/B_o$$

and $f_w: Z_w \longrightarrow G/B_+$ the map $f_w(p_k,\ldots,p_1) = \underline{w}\cdot p_k \cdots p_1 B_+$.

The remainder of this section will be devoted to showing that Z_w admits a natural geometric struture in which it is a complete variety of pure dimension k=length(w), that $f_w(Z_w) = \bigcup_{y \leq w} C(y)$, and using this, to put a geometric structure on $\bigcup_{y \leq w} C(y)$ compatible with the one we alredy have on C(y).

So we assume K is algebraically closed :

If we fix i and let P be the 'parabolic' subgroup of G, $P = B_+ \bigcup B_+ r_{j_{k-i+2}} B_+$,

we know, as in 3.17, that $P = r_{j_{k-i+2}} B_+ \bigcup (r_{j_{k-i+2}} B_+ r_{j_{k-i+2}}) B_+$

$$= r_{j_{k-i+2}} B_+ \bigcup y_{\alpha_{j_{k-i+2}}} B_+ = A^{j_{k-i+2}} \ltimes U^{j_{k-i+2}}$$

and therefore that $P/B_+ = A^{j_{k-i+2}}/H_{j_{k-i+2}} . x_{\alpha_{j_{k-i+2}}} \simeq \mathbb{P}^1$, in such a way that

if $\pi_1 : K \longrightarrow P/B_+$, $\pi_1(s) = y_{\alpha_{j_{k-i+2}}}(s) B_+$, and $\pi_2 : K \longrightarrow P/B_+$, $\pi_2(t) = x_{\alpha_{j_{k-i+2}}}(t) r_{j_{k-i+2}} B_+$

then each π_i is an isomorphism onto its image, and $\{\pi_1(K), \pi_2(K)\}$ forms a cover by open affine lines of the projective line P/B_+ (note that

<u>3.20</u> $\pi_1(s) = \pi_2(t)$ if and only if st=1).

Conjugating by $\underline{w}^{-1}_{k-i+2}$, we identify P_{i+1}/B_i with $\mathbb{P}^1$ using 3.19, so that the maps

$$\pi_1^i, \pi_2^i : K \longrightarrow P_{i+1}/B_i,$$

defined by $\quad \pi_1^i(t) = y_{\beta_{i+1}}(t) B_i,\ \pi_2^i(t) = x_{\beta_{i+1}}(t) s_{i+1} B_i$

give the identification. We can now describe the geometric structure on Z_w by exhibiting it as a successive fibration by projective lines:

One starts with $Z_w^1 = P_k/B_{k-1}$, covered by the open affines $\mathrm{Im}\pi_1^{k-1}$, $\mathrm{Im}\pi_2^{k-1}$. Define $g_1 : Z_w^2 = P_k \times^{B_{k-1}} P_{k-1}/B_{k-2} \rightarrow Z_w^1$ by $g_1(\overline{p_k, p_{k-1}B_{k-2}}) = p_k B_{k-1}$, where $(\overline{p_k, p_{k-1}B_{k-2}})$ denotes the image in $P_k \times^{B_{k-1}} P_{k-1}/B_{k-2}$ of the element $(p_k, p_{k-1}B_{k-2})$ of $P_k \times P_{k-1}/B_{k-2}$. Fixing some $z \in Z_w^1$, we see that if

$z \in \mathrm{Im}\pi_1^{k-1}$, then there exists a unique $t \in K$ such that $z = y_{\beta_k}(t)B_{k-1}$. Define $\nu_1 : g_1^{-1}(z) \to P_{k-1}/B_{k-2} \simeq \mathbb{P}^1$ by $\nu_1(\overline{p_k, p_{k-1}B_{k-2}}) = y_{\beta_k}(-t)p_k p_{k-1}B_{k-2}$, which is in P_{k-1}/B_{k-2} because $y_{\beta_k}(-t)p_k \in B_{k-1}$; similarly, if $z \in \mathrm{Im}\pi_2^{k-1}$, then there exists a unique $t \in K$ such that $z = x_{\beta_k}(t)\underline{s}_k B_{k-1}$, and we define $\nu_2 : g_1^{-1}(z) \to P_{k-1}/B_{k-2}$ by $\nu_2(\overline{p_k, p_{k-1}B_{k-2}}) = \underline{s}_k^{-1}x_{\beta_k}(-t)p_k p_{k-1}B_{k-2}$. It is easy to see that both ν_1 and ν_2 are well-defined and surjective. We check that ν_1 is 1-1: If $\nu_1(\overline{p_k, p_{k-1}B_{k-2}}) = \nu_1(\overline{p'_k, p'_{k-1}B_{k-2}})$, then $p_{k-1}B_{k-2} = (p_k^{-1}p'_k)p'_{k-1}B_{k-2}$, $p_k = p'_k(p_k^{-1}p'_k)^{-1}$, and $p_k^{-1}p'_k) \in B_{k-1}$, so $\overline{(p_k, p_{k-1}B_{k-2})} = \overline{(p'_k, p'_{k-1}B_{k-2})}$. And one shows similarly that ν_2 is 1-1. Hence the maps

$$(g_1 \times \nu_i) : g_1^{-1}(\mathrm{Im}\ _i^{k-1}) \to \mathrm{Im}\pi_i^{k-1} \times \mathbb{P}^1 \ , \text{ for } i \in \{1,2\},$$

are bijective, and we put a geometric structure on $g_1^{-1}(\mathrm{Im}\pi_i^{k-1})$ by requiring these maps to be isomorphisms, so that

$$g_1^{-1}(\mathrm{Im}\pi_i^{k-1}) = (\mathrm{Im}\pi_i^{k-1} \times \mathrm{Im}\pi_1^{k-2}) \cup (\mathrm{Im}\pi_i^{k-1} \times \mathrm{Im}\pi_2^{k-2})$$

is a covering by open affine sets.

As $Z_w^2 = \bigcup_{i,j \in \{1,2\}} (g_1 \times \nu_i)^{-1}(\mathrm{Im}\pi_i^{k-1} \times \mathrm{Im}\pi_j^{k-2})$, we will obtain a structure of prevariety on Z_w^2 provided we check that the structures on the sets $V_{j,j'} = (g_1 \times \nu_1)^{-1}(\mathrm{Im}\pi_1^{k-1} \times \mathrm{Im}\pi_j^{k-2}) \cap (g_1 \times \nu_2)^{-1}(\mathrm{Im}\pi_2^{k-1} \times \mathrm{Im}\pi_{j'}^{k-2})$, for $j,j' \in \{1,2\}$, inherited respectively from those on $(g_1 \times \nu_1)^{-1}(\mathrm{Im}\pi_1^{k-1} \times \mathbb{P}^1)$ and on $(g_1 \times \nu_2)^{-1}(\mathrm{Im}\pi_2^{k-1} \times \mathbb{P}^1)$ are compatible, which we do by showing that the map $\tau_{j,j'}$ making the diagram

$$\begin{array}{ccc}
V_{j,j'} & \xrightarrow{(g_1 \times \nu_1)} & (\mathrm{Im}\pi_1^{k-1} \cap \mathrm{Im}\pi_2^{k-1}) \times (\mathrm{Im}\pi_j^{k-2} \cap \mathrm{Im}\pi_{j'}^{k-2}) \\
\downarrow \mathrm{Id} & & \tau_{j,j'} \downarrow \\
V_{j,j'} & \xrightarrow[(g_1 \times \nu_2).]{} & (\mathrm{Im}\pi_1^{k-1} \cap \mathrm{Im}\pi_2^{k-1}) \times (\mathrm{Im}\pi_j^{k-2} \cap \mathrm{Im}\pi_{j'}^{k-2})
\end{array}$$

commute is a morphism, and this follows from the fact that $\underline{s}_k^{-1}x_{\beta_k}(-1/t)y_{\beta_k}(t) = \underline{w}_2^{-1}h_{j_1}(-\varepsilon t)x_{\beta_k}(1/t)$, that left multiplication by x_{β_k} fixes P_{k-1}/B_{k-2} pointwise, and that $w_2^{-1}H_{j_1}w_2$ acts by morphism on the projective line P_{k-1}/B_{k-2} (see 2.1, 2.6).

(Intuitively, this says that the 'fibration' $Z_w^2 \to Z_w^1$ is locally trivial over the sets $\mathrm{Im}\,\pi_i^{k-1}$).

We finally note that, as $s_{k-1}^{-1}B_{k-1}s_{k-1}\in B_{k-2}$, we also have a well-defined map $h_1\colon Z_w^1\to Z_w^2$, given by $h_1(p_kB_{k-1})=\overline{(p_k,s_{k-1}B_{k-1})}$, and satisfying $g_1\circ h_1=\mathrm{Id}_{Z_w^1}$.

Repeating the construction above inductively, we obtain fibrations $g_{i-1}\colon Z_w^i = P_k\times^{B_{k-1}}P_{k-1}\times\ldots\times^{B_{k-i+1}}P_{k-i+1}/B_{k-i}\longrightarrow Z_w^{i-1}$, with fibres equal to $P_{k-i+1}/B_{k-i}\simeq\mathbb{P}^1$, sections $h_{i-1}\colon Z_w^{i-1}\hookrightarrow Z_w^i$ given by $h_{i-1}(\overline{z})=(z,s_{k-i+1}B_{k-i})$ where $\overline{z}$ denotes the image in Z_w^i of an element z of $P_k\times^{B_{k-1}}P_{k-1}\times\ldots\times^{B_{k-i-1}}P_{k-i-1}$, and such that

3.21 $$Z_w^i = \bigcup_{n_j\in\{1,2\}} U^i_{(n_k,\ldots,n_{k-i+1})}$$

is a covering by open affines, with $U^i_{(n_k,\ldots,n_{k-i+1})}=\{\,z\in Z_w^i$ such that for some $t_k,\ldots,t_{k-i+1}\in K$, $z=\overline{(p_k(t_k),\ldots,p_{k-i+1}(t_{k-i+1})B_{k-i})}$ where $p_j(t_j)=y_{\beta_j}(t_j)$ if $n_j=1$, and $p_j(t_j)=x_{\beta_j}(t_j)s_j$ if $n_j=2$ $\}$ which is isomorphic to $\mathrm{Im}\pi^{k-1}_{n_k}\times\mathrm{Im}\pi^{k-2}_{n_{k-1}}\times\ldots\times\mathrm{Im}\pi^{k-i}_{n_{k-i+1}}$, for $1\le i\le k$.

One can now check by induction on i that the images under the restriction maps of $\mathcal{O}(U^i_{(n_k,\ldots,n_{k-i+1})})$ and $\mathcal{O}(U^i_{(n'_k,\ldots,n'_{k-i+1})})$ generate $\mathcal{O}(U^i_{(n_k,\ldots,n_{k-i+1})}\cap U^i_{(n'_k,\ldots,n'_{k-i+1})})$, so that Z_w^i is a variety, that each h_i is a morphism, and hence that Z_w^i is indeed a fibration by $\mathbb{P}^1$ over Z_w^{i-1}; in particular, each Z_w^i is projective.

If $z\in U^k_{(1,\ldots,1)}\subset Z_w^k = Z_w$, then $f_w(z)=\underline{w}y_{\beta_k}(t_k)y_{\beta_{k-1}}(t_{k-1})\ldots y_{\beta_1}(t_1)B_+$, for some $t_j\in K$,

$$=(\underline{w}y_{\beta_k}(t_k)\underline{w}^{-1})(\underline{w}y_{\beta_{k-1}}(t_{k-1})\underline{w}^{-1})\ldots(\underline{w}y_{\beta_1}(t_1)\underline{w}^{-1})wB_+.$$

Now $w\beta_i=-r_{j_1}\ldots r_{j_{k-i}}\alpha_{j_{k-i+1}}$, so $\underline{w}y_{\beta_i}\underline{w}^{-1}=x_{r_{j_1}\ldots r_{j_{k-i}}\alpha_{j_{k-i+1}}}$, hence by 3.7 we have $f_w(z)\in U_wwB_+ = C(w)$. Also, using 2.1c), 3.7, and 3.21, we see that $f_w|_{U^k_{(1,\ldots,1)}}$ gives an isomorphism of varieties

3.22 $$U^k_{(1,\ldots,1)}\simeq C(w).$$

To determine the set $X_w = \operatorname{Im} f_w \subset G/B_+$, we first show that

3.23 Proposition :

There exists a subset E_w of W such that

1) If $y \leq w$, then $y \in E_w$.

2) $X_w = \bigcup_{y \in E_w} C(y)$.

proof:

Suppose $gB_+ = f_w(\overline{p_k, \dots, p_1 B_o})$ for some $g \in G$, $p_i \in P_i$, and let $b \in B_+$; we then have

$$f_w(\overline{\underline{w}^{-1} b \underline{w} p_k, p_{k-1}, \dots, p_1 B_o}) = \underline{w}\underline{w}^{-1} b \underline{w} p_k \cdots p_1 B_+$$
$$= b . f_w(\overline{p_k, \dots, p_1 B_o})$$
$$= bgB_+ \;;$$

This implies that X_w is a union of B_+ orbits in G/B_+, i.e. $X_w = \bigcup_{y \in E_w} C(y)$, for some subset E_w of W.

Given a sequence $k \geq i_1 > i_2 > \dots > i_m \geq 1$, an easy computation shows that $ws_{i_1} s_{i_2} \dots s_{i_m} = r_{j_1} \dots \cancel{r_{j_{k-i_1+1}}} \dots \cancel{r_{j_{k-i_2+1}}} \cdots \cancel{r_{j_{k-i_m+1}}} \dots r_{j_k}$,

i.e. if $y \in W$, then $y \leq w$ if and only if $y = ws_{i_1} \dots s_{i_m}$ as above.

Fix $y = ws_{i_1} \dots s_{i_m}$, and define the point $\overline{y}$ in Z_w by $\overline{y} = (\overline{p_k, \dots, p_1})$

where $p_j = s_{i_n}$ if $j = i_n$, and $p_j = 1$ otherwise. We then have

$f_w(\overline{y}) = \underline{w} s_{i_1} \dots s_{i_m} B_+ = yB_+$, hence $C(y) \subset X_w$.

Now fix $\Lambda, L(\Lambda)$ as in 3.9, write $L = \mathcal{U}_{\mathbf{Z}} . v^+ \otimes K$, and for $q \in \mathbb{N}$

$$L_q = \sum_{\substack{\lambda \\ \text{height}(\Lambda - \lambda) \leq q}} (\mathcal{U}_{\mathbf{Z}} . v^+ \cap L(\Lambda)_\lambda) \otimes K$$

so that if $d_q = \dim_K L_q$, then $d_q < \infty$. Let $\mathbb{P}(L) = L - \{0\}/K^\times$, where $K^\times$ acts on L by scalar multiplication, let $v \longrightarrow [v]$ be the quotient map, and let $\Psi : G/B_+ \dashrightarrow \mathbb{P}(L)$ be the map $gB_+ \longrightarrow [g.(v^+ \otimes 1)]$. We then see that given $y \in W$, $\Psi(C(y)) \subset \mathbb{P}(L_q) \simeq \mathbb{P}^{d_q}$ for $q = \text{height}(\Lambda - y\Lambda)$,

$\Psi(C(y)) \not\subset \mathbb{P}(L_q)$ if $q < \text{height}(\Lambda - y\Lambda)$,

and, similarly, for q large, $\Psi(X_w) \subset \mathbb{P}(L_q)$.

3.24 Proposition :

Ψ is injective.

proof:

Given $y \in W$, then by 3.4, there exists a finite set $J_y \subset W$ such that $y^{-1}B_+y \subset \bigcup_{y' \in J_y} B_+y'B_+$. Then, as in 3.9,

$$g.(v^+ \otimes 1) \notin Kv^+ \otimes 1 \text{ if } g \in \bigcup_{y' \in J_y} B_+y'B_+ - B_+.$$

So if $\Psi(b_1yB_+)=\Psi(b_2yB_+)$, with $b_i \in B_+$, then $y^{-1}b_2^{-1}b_1y$ fixes $[v^+ \otimes 1]$, which by the above implies that $y^{-1}b_2^{-1}b_1y \in B_+$, i.e. $b_1yB_+=b_2yB_+$. Hence $\Psi|_{C(y)}$ is injective. This, combined with 3.9, shows that Ψ is itself injective.

It is clear from the definition of G that $\Psi of_w o \pi^{k-1}_{n_k} \times \ldots \times \pi^o_{n_1}$ is a morphism from $(\pi^{k-1}_{n_k} \times \ldots \times \pi^o_{n_1})^{-1}(U_{(n_k,\ldots,n_1)}) \simeq \mathbb{A}^k$ into $\mathbb{P}(L_q)$, hence $\Psi of_w\colon Z_w \longrightarrow \mathbb{P}(L_q)$ is one (for q large enough). Z_w being complete, its image must be a closed subvariety of $\mathbb{P}(L_q)$. Since $\Psi|_{X_w}$ is injective, we can thus put a geometric structure on X_w by requiring $\Psi|_{X_w}=\Psi|_{\mathrm{Im} f_w}$ to be an isomorphism onto its image, so that X_w is in fact a projective variety, $f_w\colon Z_w \longrightarrow X_w$ a morphism, and, by 3.22, $C(w)=f_w(U_{(1,\ldots,1)})$ an open subvariety. We can now identify the set E_w as being $\{y \in W \mid y \leq w\}$ as follows:

<u>3.25 Theorem</u> :

$$X_w = \bigcup_{y \leq w} C(y)$$

Proof:

The subgroup $H \simeq \mathbb{G}_m^k$ of G acts by morphisms (by 2.1e)) on Z_w and X_w, the actions being given by

$$h.\overline{(p_k,\ldots,p_1B_+)} = \overline{(hp_k,\ldots,p_1B_+)}$$
$$h.byB_+ = \underline{w}h\underline{w}^{-1}byB_+,$$

so that we have $h.f_w(z)=f_w(h.z)$ for all $z \in Z_w$, $h \in H$.

With the above action, H fixes all points $yB_+ \in X_w$, i.e. $y \in E_w$ because $\underline{w}h\underline{w}^{-1}yB_+=yy^{-1}\underline{w}h\underline{w}^{-1}yB_+$, and $y^{-1}wH(y^{-1}w)^{-1} \subset B_+$. To compute the fixed points of H on Z_w (see [D1]), assume that for some $p_i \in P_i$, $\overline{(hp_k,\ldots,p_1B_+)}=\overline{(p_k,\ldots,p_1B_+)}$ for all $h \in H$. Then for each $h \in H$, there exists elements $b_i(h) \in B_i$, $0 \leq i \leq k-1$, such that

$$1)\ hp_k=p_kb_{k-1}(h),$$

$$2)\ p_{k-1}=b_{k-1}(h)^{-1}p_{k-1}b_{k-2}(h),$$
$$\vdots$$
$$k)\ p_1=b_1(h)^{-1}p_1b_o(h).$$

The first equality implies that $p_k^{-1}Hp_k\subset B_{k-1}$, so that, using 3.19, $p_k=b_{k-1}$ or $\underline{s}_{k-1}b_{k-1}$ for some $b_{k-1}\varepsilon B_{k-1}$. Let $p_k'=p_kb_{k-1}^{-1}$, $p_{k-1}'=b_{k-1}p_{k-1}$. Then

$$\overline{(p_k',p_{k-1}',p_{k-2},\ldots,p_1B_+)}=\overline{(p_k,p_{k-1},\ldots,p_1B_+)},$$

and equations 1) and 2) become

$$1')\ hp_k'=p_k'\cdot(b_{k-1}b_{k-1}(h)b_{k-1}^{-1}),$$
$$2')\ p_{k-1}'=(b_{k-1}b_{k-1}(h)b_{k-1}^{-1})p_{k-1}'b_{k-2}(h).$$

The former implies that $b_{k-1}b_{k-1}(h)b_{k-1}^{-1}\varepsilon H$ (because $p_k'\varepsilon\{1,\underline{s}_k\}$) and in fact takes on all possible values in H as h varies, so that 2') now implies that $p_{k-1}^{-1}Hp_{k-1}\subset B_{k-2}$.

Repeating this process, we see that one actually has

$$\overline{(p_k,\ldots,p_1B_+)}=\overline{(p_k',\ldots,p_1'B_+)}$$

with each $p_i'\varepsilon\{1,\underline{s}_i\}$, so that the fixed points of H on Z_w are exactly the $\overline{(p_k',\ldots,p_1'B_+)}$ with $p_i'\varepsilon\{1,\underline{s}_i\}$.

Suppose now that $yB_+\varepsilon X_w$. Then $f_w^{-1}(yB_+)$ is a closed, non-empty, H-stable subset of Z_w. By the Borel fixed point theorem, $f_w^{-1}(yB_+)$ must contain a fixed point of Z_w under H, i.e. there exists a $\overline{(p_k',\ldots,p_1'B_+)}$ as above with $\underline{w}p_k'\cdots p_1'=y$, hence $y\leq w$.

Example 3: Suppose $w=r_{j_1}\cdots r_{j_k}$ with $j_m\neq j_n$ if $m\neq n$. Then U_w is abelian:

That is so because if $\alpha=r_{j_1}\cdots r_{j_m}\alpha_{j_{m+1}}$, and $\beta=r_{j_1}\cdots r_{j_{m+n}}\alpha_{j_{m+n+1}}$, then for any $a,b\in\mathbf{N}$, $a\alpha+b\beta=r_{j_1}\cdots r_{j_m}(a\alpha_{j_{m+1}}+br_{j_{m+1}}\cdots r_{j_{m+n}}\alpha_{j_{m+n+1}})$

$$=r_{j_1}\cdots r_{j_{m+1}}(br_{j_{m+2}}\cdots r_{j_{m+n}}\alpha_{j_{m+n+1}}-a\alpha_{j_{m+1}}).$$

Now $w'=r_{j_{m+2}}\cdots r_{j_{m+n+1}}$ is in reduced form, so $r_{j_{m+2}}\cdots r_{j_{m+n}}\alpha_{j_{m+n+1}}$ is a positive real root, and is in the $\mathbf{Z}$-span of $\{\alpha_{j_{m+2}},\ldots,\alpha_{j_{m+n+1}}\}$, hence $r_{j_{m+1}}\cdots r_{j_1}(a\alpha+b\beta)=b\sum_{i=m+2}^{m+n+1}c_i\alpha_{j_i}-a\alpha_{j_{m+1}}$, with $c_i\in\mathbf{N}$. If $a\alpha+b\beta$ is a root, then so is $r_{j_{m+1}}\cdots r_{j_1}(a\alpha+b\beta)$, but the right hand side is neither in Δ_+ or in Δ_- if $ab\neq 0$; hence for all $\alpha,\beta\in\Phi(w)$, $\mathbf{N}\alpha+\mathbf{N}\beta\cap\Delta_+$ reduces to $\{\alpha,\beta\}$.

Example 4: let $w=(r_1r_2)^3r_1$ (reduced iff $m_{12}=\infty$ which we assume). Let $\lambda_1=\alpha_1$, $\lambda_2=r_1\alpha_2$, $\ldots$,$\lambda_7=(r_1r_2)^3\alpha_1$, and let $x_i=x_{\lambda_i}$; then U_w is the group $\{x_1(t_1)\ldots x_7(t_7),\ t_i\in K\}$ with the following commutation relations:

$(x_1(t_1),x_3(t_3))=x_2(-a\delta_{b,-1}t_1t_3)$
$(x_1(t_1),x_7(t_7))=x_4(a\delta_{b,-1}\delta_{a,-4}t_1t_7)$
$(x_2(t_2),x_4(t_4))=x_3(-b\delta_{a,-1}t_2t_4)$
$(x_3(t_3),x_5(t_5))=x_4(-a\delta_{b,-1}t_3t_5)$
$(x_4(t_4),x_6(t_6))=x_5(-b\delta_{a,-1}t_4t_6)$
$(x_5(t_5),x_7(t_7))=x_6(-a\delta_{b,-1}t_5t_7)$

where $a=a_{12}$, $b=a_{21}$, δ=Kronecker delta .
(in paritcular, the central series of U_w has length≤ 2, and if $a_{12}\neq -1$ & $a_{21}\neq -1$, then U_w is abelian).

Example 5: $w=(r_1r_2)^3$, which is in reduced form iff $m_{12}\geq 6$.
If $m_{12}=\infty$, U_w is the obvious subgroup of the unipotent group in example 4.
If $m_{12}=6$, say $a_{12}=-1$, $a_{21}=-3$, then U_w has central series of length 4 ($\dim U_w=6$, $\dim U_w'=4$, $\dim U_w^{(2)}=3$, $\dim U_w^{(3)}=2$, & $U_w^{(4)}=1$)

Example 6: $w=r_1r_2r_3r_1$ (note that X_w is singular for Sl_n); then U_w is abelian unless

a) $a_{12}=a_{21}=0$, $a_{31}=-1$, and $\mathrm{char}K \nmid a_{13}$ in which case $(U_w,U_w)=x_{r_1r_2\alpha_3}\subset \mathrm{center}(U_w)$

or

b) $a_{13}=a_{31}=0$, $a_{21}=-1$, and $\mathrm{char}K \nmid a_{12}$ in which case $(U_w,U_w)=x_{r_1\alpha_2}\subset \mathrm{center}(U_w)$.

Example 7: $w=r_1r_2r_3r_2r_1$

Again X_w is singular for Sl_n, and the commutation relations in U_w reduce to

$$(x_{r_1\alpha_2}(s),x_{r_1r_2r_3\alpha_2}(t))=x_{r_1r_2\alpha_3}(-a_{23}\delta_{a_{32},-1}st)$$

and

$$(x_{\alpha_1}(s),x_{r_1r_2r_3r_2\alpha_1}(t))=\begin{cases} x_{r_1r_2\alpha_3}(-a_{13}\delta_{a_{21},0}\delta_{a_{31},-1}st) \\ \text{or} \\ x_{r_1r_2\alpha_3}(-a_{12}a_{23}\delta_{a_{31},0}\delta_{a_{32},-1}\delta_{a_{21},-1}st) \end{cases}$$

Example 8: Assume $m_{12}\neq 2$, and let $w=r_1r_2r_1$. The set E_w of 3.23 consists of $\tau_1=1$, $\tau_2=r_1$, $\tau_3=r_2$, $\tau_4=r_2r_1$, $\tau_5=r_1r_2$, $\tau_6=w$, while D has elements

	$\sigma^0=(1,1,1,1)$	$\sigma^1=(1,r_1,r_1,1)$	$\sigma^2=(1,1,1,r_1)$
$m(\sigma)=$	0	1	0
$n(\sigma)=$	3	1	2
$\sigma_3=$	τ_1	τ_1	τ_2

	$\sigma^3=(1,1,r_2,r_2)$	$\sigma^4=(1,1,r_2,r_2r_1)$	$\sigma^5=(1,r_1,r_1r_2,r_1r_2)$
$m(\sigma)=$	0	0	0
$n(\sigma)=$	2	1	1
$\sigma_3=$	τ_3	τ_4	τ_5

and $\sigma^6=(1,r_1,r_1r_2,w)$

We have $\beta_1=\alpha_1$, $\beta_2=r_1\alpha_2$, $\beta_3=r_1r_2\alpha_1>0$ as $m_{12}\neq 2$,

$ws_3s_2s_1=\tau_1$,

$ws_2s_1=\tau_2$, $ws_3s_1=\tau_3$ $(,ws_3s_2=\tau_2)$,

$ws_3=\tau_4$, $ws_1=\tau_5$ $(,ws_2=\tau_1)$,

$w=\tau_6$,

and $U_w=\{u=x_{\beta_1}(a)x_{\beta_2}(b)x_{\beta_3}(c),\ a,b,c\in K\}$ with the single nontrivial commutation relation

$$(x_{\beta_1}(a), x_{\beta_3}(c)) = x_{\beta_2}(-\delta_{a_{21},-1}{}^{a_{12}} \cdot ac).$$

We have $\eta(u)=\sigma^0$ if $a\neq 0, b\neq 0, ac\neq -1$
$=\sigma^1$ if $a=0, b\neq 0$
$=\sigma^2$ if $a\neq 0, b\neq 0, ac=-1$
$=\sigma^3$ if $a\neq 0, b=0, c\neq 0$
$=\sigma^4$ if $a\neq 0, b=0, c=0$
$=\sigma^5$ if $a=0, b=0, c\neq 0$
$=\sigma^6$ if $a=b=c=0$,

which verifies 3.16 ;

Assume now that W is the symmetric group on three letters, and A is chosen as in §1, so that $G=\mathrm{SL}_3$. One then knows that G/B_+ can be identified with the variety of flags in 3-space (in fact the map Ψ in §3 gives the isomorphism for appropriately chosen Λ). Explicitely,

fix a vector space V over K, $\dim_K V=3$, choose planes $P_+\neq P_-$, and lines $L_\pm \subset P_\pm-(P_+\cap P_-)$, so that we have

$$V=\tilde{P}\oplus L_o, \text{ where } \tilde{P}=L_+\oplus L_- \text{ and } L_o=P_+\cap P_-;$$

Labelling the generators r_1, r_2 of W in such a way that $C(r_1)=\{(V\supset P_+\supset L\supset 0),\ L\neq L_+\}$, and $C(r_2)=\{(V\supset P\supset L_+\supset 0),\ P\neq P_+\}$, we obtain:

$D_{\sigma^6}=\{(V\supset P_-\supset L_-\supset 0)\}$

$D_{\sigma^5}=\{(V\supset P_-\supset L\supset 0),\ L\notin\{L_-, L_o\}\}$

$D_{\sigma^4}=\{(V\supset P\supset L_-\supset 0),\ P\notin\{P, P_-\}\}$

and, with $F=(V\supset P\supset L\supset 0)$,

$D_{\sigma^3}=\{F \mid L\not\subset P_\pm,\ P=L+L_-\}$

$D_{\sigma^2}=\{F \mid L_\pm\not\subset P,\ L=P\cap P_-\}$

$D_{\sigma^1}=\{F \mid L\not\subset P_\pm,\ L\subset\tilde{P},\ P\not\supset L_\pm\}$

$D_{\sigma^0}=\{F \mid L\not\subset P_\pm,\ L\not\subset\tilde{P},\ P\not\supset L_\pm\}$.

Section 4 : Applications

We start with the following fact:

1) $\underline{X_w \text{ is non-singular in codimension 1:}}$

Suppose that $w'<w$, and $\text{length}(w')=\text{length}(w)-1$. We then know that there exists a reduced expression $w=r_{t_1}\cdots r_{t_k}$ such that ([D1])

$$w'=r_{t_1}\cdots r_{t_{k-i_o}}\cdot r_{t_{k-i_o+2}}\cdots r_{t_k}$$

As in §3, set $w_i=r_{t_i}\cdots r_{t_k}$, $\beta_i=w^{-1}_{k-i+2}\alpha_{t_{k-i+1}}$. With our choice of reduced expressions, we then have

(*) $$r_{\beta_{i_o}}\cdot\beta_j>0 \quad \text{if } j>i_o$$

Let $V_{w'}=\underbrace{U_{(1,\dots,1)}}_{U^1}\cup \underbrace{U_{(1,\dots,1,\overset{\uparrow}{2},1,\dots,1)}}_{U^2}\subset Z_w$, where the arrow marks n_{i_o},

$$N_{w'}=\mathrm{Im}\pi_1^{k-1}\times\dots\times\mathrm{Im}\pi_1^{k-i_o}\times\pi_2^{k-i_o+1}(\{0\})\times\mathrm{Im}\pi_1^{k-i_o+2}\times\dots\times\mathrm{Im}\pi_1^{o}\subset U^2,$$

so that we have $U^2-N_{w'}\subset U^1$. We shall show that ([BGG])

a) $f_w(N_{w'})=C(w')$

b) $f_w(U^2-N_{w'})\subset C(w)$

c) $f_w|_{U^2}$ is an isomorphism onto a dense subset of X_w.

proof of a):

From the definition of f_w, we have $p\in f_w(N_{w'})$ if and only if

$$p=\underline{w}y_{\beta_k}(t_k)\cdots y_{\beta_{i_o+1}}(t_{i_o+1})r_{\beta_{i_o}}y_{\beta_{i_o-1}}(t_{i_o-1})\cdots y_{\beta_1}(t_1)B_+$$

for some $t_j\in K$.

The right hand side is equal to

$$\underline{w}r_{\beta_{i_o}}(r^{-1}_{\beta_{i_o}}y_{\beta_k}(t_k)\cdots y_{\beta_{i_o+1}}(t_{i_o+1})r_{\beta_{i_o}})y_{\beta_{i_o-1}}(t_{i_o-1})\cdots y_{\beta_1}(t_1)B_+.$$

Let $\beta'_{k-1}=r_{\beta_{i_o}}\beta_k$, $\beta'_{k-2}=r_{\beta_{i_o}}\beta_{k-1}$, $\dots$, $\beta'_{i_o}=r_{\beta_{i_o}}\beta_{i_o+1}$,

$\beta'_{i_o-1}=\beta_{i_o-1}$, $\dots$, $\beta'_1=\beta_1$,

and let us also renumber $r_{t_1}\cdots r_{t_{k-i_o}}\cdot r_{t_{k-i_o+2}}\cdots r_{t_k}$ into

$$\begin{cases} j_1=t_1, \ \dots \ , \ j_{k-i_o}=t_{k-i_o}, \\ j_{k-i_o+1}=t_{k-i_o+2}, \ \dots \ , \ j_{k-1}=t_k \end{cases}$$

If we now let $w_i'=r_{j_i}\dots r_{j_{k-1}}$, an easy computation will show that

$\beta_i'=w'^{-1}_{(k-1)-i+2}\alpha_{j_{(k-1)-i+1}}$ for all i, and (from the proof of 3.23a)),

$wr_{\beta_{i_o}} = w'$. Hence the right hand side above is $\underline{w}'y_{\beta'_{k-1}}\dots y_{\beta'_1}B_+$;

Applying 3.22 to $w'=r_{j_1}\dots r_{j_{k-1}}$, we find

$$f_w(N_{w'})=C(w').$$

b) is obvious because $U^2-N_{w'}\subset U^1$.

proof of c):

We check that $f_w|_{U^2}$ is injective: assume that

$$y_{\beta_k}(t_k)\dots y_{\beta_{i_o+1}}(t_{i_o+1})x_{\beta_{i_o}}(t_{i_o})\underline{r}_{\beta_{i_o}}y_{\beta_{i_o-1}}(t_{i_o-1})\dots y_{\beta_1}(t_1)B_+$$
$$= y_{\beta_k}(t_k')\dots y_{\beta_{i_o+1}}(t'_{i_1+1})x_{\beta_{i_o}}(t'_{i_o})\underline{r}_{\beta_{i_o}}y_{\beta_{i_o-1}}(t'_{i_o-1})\dots y_{\beta_1}(t_1')B_+;$$

write y_j for $y_{\beta_j}(t_j)$, y_j' for $y_{\beta_j}(t_j')$, if $j\neq i_o$,

y_{i_o} for $y_{\beta_{i_o}}(-t_{i_o})$, y'_{i_o} for $y_{\beta_{i_o}}(-t'_{i_o})$.

Then we must have

$y_1^{-1}\dots y_{i_o}^{-1}(\underline{r}_{\beta_{i_o}}y_{i_o+1}^{-1}\dots y_k^{-1}y_k'\dots y'_{i_o+1}r_{\beta_{i_o}})y'_{i_o}\dots y_1'B_+ = B_+$; the thing in parentheses is in U_- by (*), hence, as $U_-\cap B_+=1$, we must have

$$y_k\dots y_{i_o+1}\underline{r}_{\beta_{i_o}}y_{i_o}\dots y_1 = y_k'\dots y'_{i_o+1}\underline{r}_{\beta_{i_o}}y'_{i_o}\dots y_1' \qquad \text{(no } B_+\text{'s)}$$

This equation implies that

$$\underline{w}_jy_k\underline{w}_j^{-1}\cdot \ \dots \ \cdot \underline{w}_jy_{i_o+1}\underline{w}_j^{-1}\cdot\underline{w}_jr_{\beta_{i_o}}y_{i_o}\dots y_1$$
$$= \underline{w}_jy_k'\underline{w}_j^{-1}\cdot \ \dots \ \cdot \underline{w}_jy'_{i_o+1}\underline{w}_j^{-1}\cdot\underline{w}_jr_{\beta_{i_o}}y'_{i_o}\dots y_1'$$

Set $j=k-i_o+1$: then $w_j\beta_m > 0$ if $m\geq i_o+1$
< 0 if $m\leq i_o$,

and we have

$$\overbrace{(\underline{w}_jy_k\underline{w}_j^{-1})\dots(\underline{w}_jy_{i_o+1}\underline{w}_j^{-1})}^{(=u_-)}\underline{w}_jr_{\beta_{i_o}}\underline{w}_j^{-1}(\underline{w}_jy_{i_o}\underline{w}_j^{-1})\dots(\underline{w}_jy_1\underline{w}_j^{-1}) =$$

$$= \underbrace{(\underline{w}_j y'_k \underline{w}_j^{-1}) \ldots (\underline{w}_j y'_{i_o+1} \underline{w}_j^{-1})}_{(=u'_-)} \underbrace{\underline{w}_j r_{\beta_{i_o}} \underline{w}_j^{-1}}_{(=r_{t_{k-i_o+1}})} (\underline{w}_j y'_{i_o} \underline{w}_j^{-1}) \ldots (\underline{w}_j y'_1 \underline{w}_j^{-1})$$

which implies that

$$u_- y_{i_o}^{-1} r_{t_{k-i_o+1}} \underbrace{(\underline{w}_j y_{i_o-1} \underline{w}_j^{-1} \cdot \ldots \cdot \underline{w}_j y_1 \underline{w}_j^{-1})}_{(=u_+)} = u'_- y'^{-1}_{i_o} r_{t_{k-i_o+1}} \underbrace{(\underline{w}_j y'_{i_o-1} \underline{w}_j^{-1} \cdot \underline{w}_j y_1 \underline{w}_j^{-1})}_{(=u'_+)}$$

i.e. $\underbrace{y_{i_o} u_-^{-1} u'_- y'^{-1}_{i_o}}_{\varepsilon\, U_-} = r_{t_{k-i_o+1}} (u_+ u'^{-1}_+)\, r_{t_{k-i_o+1}}$,

and the right hand side is in U_+ (because $w_j \beta_m = -r_{t_{k-i_o+1}} \ldots r_{t_{k-m}} \alpha_{t_{k-m+1}}$ if $m<i_o$). So we must have

$$y_{\beta_k}(t_k) \cdot\cdot y_{\beta_{i_o+1}}(t_{i_o+1}) y_{\beta_{i_o}}(-t_{i_o}) = y_{\beta_k}(t'_k) \ldots y_{\beta_{i_o+1}}(t'_{i_o+1}) y_{\beta_{i_o}}(-t'_{i_o})$$

$$\text{and } y_{\beta_{i_o-1}}(t_{i_o-1}) \ldots y_{\beta_1}(t_1) = y_{\beta_{i_o-1}}(t'_{i_o-1}) \ldots y_{\beta_1}(t'_1)$$

which, by 3.7 , implies that $t_j = t'_j$ for all j.

2) Application to the Hecke algebra:

Define the Hecke algebra $\mathcal{H}$ of W to be the algebra
$$\mathcal{H} = \bigoplus_{y\in W} \mathbf{Z}[q^{1/2},q^{-1/2}].T_y$$
generated by the ring of polynomials in the indeterminates $q^{1/2},q^{-1/2}$ over $\mathbf{Z}$, and generators T_y, where the product is defined by
$$T_{r_i}.T_y=\begin{cases} T_{r_i y} & \text{if length}(r_i y)=\text{length}(y)+1 \\ q.T_{r_i y}+(q-1).T_y & \text{if length}(r_i y)=\text{length}(y)-1 \end{cases}$$
for $1\leq i\leq N$, so that with $w=r_{j_1}\ldots r_{j_k}$ and length(w)=k, we have

$T_w.T_y=T_{r_{j_1}}\ldots T_{r_{j_k}}.T_y$ for any $y\in W$.

Note that if we formally set $q^{1/2}=1$, then $\mathcal{H}$ is simply the group algebra $\mathbf{Z}[W]$.

Fix an algebraically closed field K of characteristic p>0. By 2.7, the Frobenius F on G(K) factors to a map $F:G(K)/B_+\to G(K)/B_+$ which satisfies $F(C(y))=C(y)$, $F(yB_+)=yB_+$, $F(C'(y))=C'(y)$, for all $y\in W$. If $X\subset G(K)$ or $G(K)/B_+$, and $n\in\mathbf{N}$, let's write X^{F^n} for the set $\{x\in X|F^n x=x\}$.

Given $n\in\mathbf{N}$, let L be a field of characteristic 0 containing a (2n)th root of p, and fix such a $\sqrt[2n]{p}$ L; then the map $\mathbf{Z}[q^{1/2},q^{-1/2}]\to L$, $f(q^{1/2},q^{-1/2})\to f(\sqrt[2n]{p}, (\sqrt[2n]{p})^{-1})$ allows one to define the algebra $\mathcal{H}_n=\mathcal{H}\otimes_{\mathbf{Z}[q^{1/2},q^{-1/2}]}L$. We now show that

| there is a natural embedding $\mathcal{H}_n\hookrightarrow \mathrm{End}_{G(K)^{F^n}}(\{f:G(K)^{F^n}/B_+^{F^n}\to L\})$

Call the algebra on the right hand side $\tilde{\mathcal{H}}$, and define $\{\tilde{T}_y, y\in W\}\subset\tilde{\mathcal{H}}$ by
$$(\tilde{T}_y.f)(xB_+^{F^n})=\sum_{z\in x.C(y)^{F^n}} f(z),$$
the sum being finite as $\#C(y)^{F^n}=p^{n.\text{length}(y)}$. Assume that for some $\lambda_w\in L$, $\sum_{w\in A}\lambda_w\tilde{T}_w=0$, and let $f_o\in\tilde{\mathcal{H}}$ be the map $f_o(zB_+^{F^n})=1$ if $z\in B_+^{F^n}$, 0 otherwise. We must then have $\sum_w\lambda_w\tilde{T}_w f_o(z)=0$ for all $z\in G(K)^{F^n}/B_+^{F^n}$. Plugging in $z=w^{-1}B_+^{F^n}$ for the various $w\in A$, one finds that $\lambda_w=0$ for all $w\in A$, hence the $\tilde{T}_y$'s are linearly independent. We thus obtain an L-linear isomorphism $\mathcal{H}_n\to\tilde{\mathcal{H}}$, $T_y\otimes 1\mapsto\tilde{T}_y$; to check that this is

an algebra isomorphism, we compute

$$\tilde{T}_y.\tilde{T}_{y'}.f\ (xB_+^{F^n}) = \sum_{z\varepsilon G(K)^{F^n}/B_+^{F^n}} \#\{u\varepsilon G(K)^{F^n}/B_+^{F^n} \mid \&{}^{x^{-1}u\varepsilon C(y)}_{u^{-1}z\varepsilon C(y')}\}^{F^n}.f(z)$$

$$= \sum_{z\varepsilon G(K)^{F^n}/B_+^{F^n}} \#(C(y)\cap x^{-1}zC(y'^{-1}))^{F^n}.f(z)$$

$$= \sum_{z\varepsilon G(K)^{F^n}/B_+^{F^n}} \sum_{\substack{y''\varepsilon W \\ x^{-1}z\varepsilon C(y'')}} \#(C(y)\cap y''C(y'^{-1}))^{F^n}.f(z)$$

$$= \sum_{y''\varepsilon W} \#(C(y)\cap y''.C(y'^{-1}))^{F^n}.\tilde{T}_{y''}f(z),$$

so that

4.1 $$\tilde{T}_y.\tilde{T}_{y'} = \sum_{y''\varepsilon W} \#(C(y)\cap y''.C(y'^{-1}))^{F^n}.\tilde{T}_{y''},$$

the sum being finite by 3.4 and 3.15; assume now that $y=r_i$:
By 3.2,3.3,3.15, we have $y''^{-1}.C(r_i)\cap C(y'^{-1})=\emptyset$ unless

$$y''=r_iy' \quad \text{in case length}(r_iy')=\text{length}(y')+1$$
$$\text{or } y''\varepsilon\{r_iy',y'\} \text{ in case length}(r_iy')=\text{length}(y')-1$$

In the first case, the intersection is $\{y_{y'^{-1}.\alpha_i}(t)y^{-1}B_+\}/B_+\ C(y')$, which reduces to the single point $y^{-1}B_+$ so that in this case $\#(y''^{-1}C(r_i)\cap C(y'^{-1}))^{F^n}=1\ \ (y^{-1}\varepsilon G(K)^{F^n})$,
and in the second case,

if $y''=y'$, the intersection is $\{y'^{-1}x_{\alpha_i}(t)r_iB_+\}/B_+\ C(y'^{-1})$

$$=\{y'^{-1}y_{\alpha_i}(t^{-1})B_+\}/B_+\ C(y'^{-1}), \text{ so that}$$

$\#(y''^{-1}C(r_i)\cap C(y'^{-1}))^{F^n}=(p^n-1)$,

while if $y''=r_iy'$, the intersection is $\{x_{y^{-1}.\alpha_i}(t)y^{-1}B_+\}/B_+\ C(y')$

so that $\#(y''^{-1}C(r_i)\cap C(y'^{-1}))^{F^n}=p^n$.

Now 4.1 implies that

$$\tilde{T}_{r_i}.\tilde{T}_{y'} = \sum_{y''\varepsilon W} \#(C(y)\cap y''C(y'^{-1}))^{F^n}.\tilde{T}_{y''}$$

$$= \sum_{y''\varepsilon W} \#(y''^{-1}C(y)\cap C(y'^{-1}))^{F^n}.\tilde{T}_{y''} \quad \text{(because } y''\varepsilon G(K)^{F^n})$$

$$= \begin{cases} \tilde{T}_{r_iy'} & \text{if length}(r_iy')=\text{length}(y')+1 \\ p^n.\tilde{T}_{r_iy'}+(p^n-1).\tilde{T}_{y'} & \text{if length}(r_iy')=\text{length}(y')-1 \end{cases}$$

by the computations just made. Hence the map $T_y \otimes 1 \mapsto \tilde{T}_y$ is an algebra isomorphism of $\mathcal{H}_n$ into $\tilde{\mathcal{H}}$.

One can check by induction on the length of y that all $T_y \in \mathcal{H}$ are invertible. Define polynomials $R_{z,y} \in \mathbf{Z}[q^{1/2},q^{-1/2}]$ by requiring that

$$\underline{4.2} \qquad (T_{y^{-1}})^{-1} = \sum_{z\in W} (-1)^{\text{length}(y)+\text{length}(z)} R_{z,y} \cdot q^{-\text{length}(y)} \cdot T_y \ .$$

The sum in 4.2 is finite; in fact, it only involves those elements z which satisfy $z\leq y$, as one can prove that $R_{z,y}\neq 0$ iff $z\leq y$. In [Dh4], the formula

$$R_{z,w} = \sum_{\substack{\sigma\in D\\ \sigma_k=z}} q^{m(\sigma)}(q-1)^{n(\sigma)}$$

is shown to be valid in any Coxeter group (the notation being as in 3.15) so that $R_{z,y}$ is in fact in $\mathbf{Z}[q]$. Applying 3.16 to the present setting, and using the fact that $\#(K)^{F^n}=p^n$, $\#(K^\times)^{F^n}=p^n-1$, one finds that

$$\#(C(w)\cap C'(y))^{F^n} = \sum_{\substack{\sigma\\ \sigma_k=y}} (p^n)^{m(\sigma)}(p^n-1)^{n(\sigma)}$$

$$\underline{4.3} \qquad = R_{y,w}(p^n).$$

The Kazhdan-Lusztig polynomials can now be defined as follows: let $x\mapsto\bar{x}$ be the 'anti'-automorphism of $\mathcal{H}$ extending the map $q^{1/2}\mapsto q^{-1/2}$, $T_y\mapsto T^{-1}_{y^{-1}}$. In [KL], it is shown that there exists a unique basis $\{S_y,\ y\in W\}$ of satisfying

1) $\bar{S}_y=S_y$

2) $S_y = q^{-\text{length}(y)/2} \sum_{z\leq y} P_{z,y} \cdot T_z$

with $P_{z,y}\in\mathbf{Z}[q]$, and degree$(P_{z,y})\leq\frac{(\text{length}(y)-\text{length}(z)-1)}{2}$ if $z<y$, $P_{y,y}$ being the constant 1.

Condition 1) is equivalent to the equality

$$q^{\text{length}(y)-\text{length}(z)}\bar{P}_{z,y} = \sum_{z\leq x\leq y} R_{z,x}P_{x,y}$$

and Kazhdan & Lusztig prove that if W is a (finite) Weyl group, then $P_{y,w}(q) = \sum_i \dim H^{2i}_{yB_+}(IC_w)\cdot q^i$, where H is the stalk cohomology of Deligne's middle-intersection-cohomology complex of sheaves IC_w on the variety X_w. One can generalize their proof ([KL4]) to arbitrary crystallographic groups as follows:

Assume that W is a crystallographic group, and define

$$\tilde{P}_{y,w} = \sum_j \dim H^j_{yB_+}(IC_w)q^{j/2}$$

for $y\varepsilon W$, $yB_+\varepsilon X_w$ as above. With the notation of 3.23-3.25, fix Λ, $v^+\varepsilon L(\Lambda)$, $L=\mathcal{U}_{\mathbf{Z}}.v^+\otimes K$, $L^\lambda=L(\Lambda)_\lambda\otimes K$. Let $V_y=\{v\varepsilon L \mid$ the $L^{y\Lambda}$-coordinate of v is non-zero$\}$, $V'_y=\{v\varepsilon V_y \mid$ the L^λ-coordinate of v is 0 if height$(y\Lambda-\lambda)\leq 0$ and $\lambda\neq y\Lambda\}$. Now let $V_{y,w}=V_y\cap L_{\text{height}(\Lambda-w\Lambda)}$, $V'_{y,w}=V'_y\cap L_{\text{height}(\Lambda-w\Lambda)}$ for $y\leq w\,\varepsilon\, W$. Finally, define an action of $\mathbf{G}_m$ on L by $t\longrightarrow\rho(t)$, where $\rho(t).v=t^{\text{height}(\Lambda-\lambda)}v$ whenever $v\varepsilon L^\lambda$. We can now show that

4.4 a) $\Psi(X_w\cap y.C'(1))=\Psi(X_w)\cap[V_{y,w}]$ and $\Psi(X_w\cap C'(y))=\Psi(X_w)\cap[V'_{y,w}]$, so that both $X_w\cap y.C'(1)$ and $X_w\cap C'(y)$ can be regarded as subvarieties of affine space.

b) The $\mathbf{G}_m$-action defined above induces an action of $K^\times$ on $[V_{y,w}]$ such that $\Psi(X_w\cap yC'(1))$ and $\Psi(X_w\cap C'(y))$ are $K^\times$-stable

c) In the natural identification $[V_{y,w}]\simeq K^M$, the point $[yv^+\otimes 1]$ maps to the origin, and the $K^\times$-action on $[V_{y,w}]$ decomposes into a sum of positive characters.

proof:

a) Assume $v\varepsilon V'_y$, say $v=yv^+\otimes t_{y\Lambda}+\sum_{\substack{\mu\\ \text{height}(y\Lambda-\mu)>0}} v_\mu\otimes t_\mu$ with $t_\mu\varepsilon K$, $v_\mu\varepsilon L(\Lambda)_\mu$, and fix $t\varepsilon K$, $\alpha\,\varepsilon\,\Delta^{re}_+$: then

$$y_\alpha(t).v=yv^+\otimes t_{y\Lambda}+\underbrace{\sum_m (f_\alpha^{(m)}.yv^+)\otimes t^m t_{y\Lambda}}_{\varepsilon\, L^{y\Lambda-\alpha}} + \sum_{\substack{\mu'\\ \text{height}(y\Lambda-\mu')>0}} v'_\mu\otimes t'_\mu$$

which lies again in V'_y. Hence V'_y is B_--invariant, and one concludes that for $w\varepsilon W$,

$$\Psi(C'(w))\subset[V'_y] \text{ if and only if } [wv^+\otimes 1]\,\varepsilon\,[V'_y].$$

It is clear from the definition that the latter happens only when w=y, so we have

(*) $\Psi(C'(w))\subset[V'_y]$ if and only if w=y.

Using the disjointness of the Birkhoff decomposition, and the fact that $X_w\subset\mathbf{P}(L_{\text{height}(\Lambda-w\Lambda)})$, this proves that $\Psi(X_w\cap C'(y))=\Psi(X_w)\cap[V'_{y,w}]$.

As $V_y=y.V_1$, one can also conclude from (*) that

$\Psi(y.C'(1)) \cap [V_y]$. Now assume that $p \in \Psi(X_w) \cap [V_y]$: as $p \in \Psi(X_w)$, we must have $y^{-1}.p \in \Psi(G/B_+)$, hence, using the disjointness of the Birkhoff decomposition and the injectivity of Ψ, there exists a unique $z \in W$ such that $y^{-1}p \in \Psi(C'(z))$. As the $L^{y\Lambda}$-coordinate of p is non-zero, the L^{Λ}-coordinate of $y^{-1}.p$ is non-zero, i.e., by the above, z must be 1, and $p \in \Psi(y.C'(1))$. This completes the proof of a).

proof of b): one need only observe that

$$\rho(t) o x_\alpha(s) o \rho(t)^{-1} = x_\alpha(st^{-\mathrm{height}(\alpha)}),$$
$$\rho(t) o y_\alpha(s) o \rho(t)^{-1} = y_\alpha(st^{\mathrm{height}(\alpha)}).$$

c) is clear.

One can now prove that $\tilde{P}_{y,w} = P_{y,w}$ exactly as in [KL4]:

Call an $\mathbb{F}_p$-variety X pure (resp. very pure) if for all $x \in X^{F^n}$ and all $i \in \mathbb{N}$, the eigenvalues of F^n on $H^i_x(IC)$ have absolute value $p^{ni/2}$ (resp. $= p^{ni/2}$). The main facts needed to complete the proof are:

4.5 a) If Y is a closed $\mathbb{F}_p$-subvariety of K^M (some M), which is stable under a diagonal action of $\mathbb{G}_m$ on K^M, the latter action being a direct sum of positive characters of $\mathbb{G}_m$, then

if Y-{origin} is very pure, then so is Y

b) With Y as above, we have $\mathbb{H}^i(Y, IC) = H^i_{\mathrm{origin}}(IC)$ (where $\mathbb{H}$ denotes hypercohomology).

Fix $w \in W$. For $z \leq w$, let Q(z) be the property: "for all $x \in C(z)^{F^n}$, $H^i_x(IC_w) = 0$ if i is odd, and the eigenvalues of F^n on $H^i_x(IC_w)$ are $p^{ni/2}$ if i is even". The main lemma (see [KL4]) is that

4.6 Q(z) holds for all $z \leq w$.

proof: Q(w) is clear. Assume that Q(z) is true for $y \leq z \leq w$, and let's prove it for y:

We know by our induction hypothesis that $(X_w \cap y.C'(1)) - C(y)$ is very pure. It is easy to see that $X_w \cap y.C'(1)$ is isomorphic to $(X_w \cap C'(y)) \times C(y)$, and that in this isomorphism $(X_w \cap yC'(1)) - C(y)$ corresponds to $((X_w \cap C'(y)) - \{yB_+\}) \times C(y)$. As $C(y)$ is smooth, our initial remark implies that $(X_w \cap C'(y)) - \{yB_+\}$ is very pure. Using 4.4 and 4.5a), we conclude that $X_w \cap C'(y)$ is itself very pure, and, going backwards, that $X_w \cap y.C'(1)$ is very pure.

We now apply the Lefschetz fixed point formula to the Frobenius map on the variety $X=X_w\cap y.C'(1)$ to obtain:

$$(*)\quad tr_{\mathbb{H}_c^*(X,IC_w)}F^n = \sum_{x\in X^{F^n}} tr_{H_x^*(IC_w)}F^n$$

where $\mathbb{H}_c$ denotes hypercohomology with compact support, and $tr_{C^*}=\sum_i (-1)^i tr_{C^i}$. The right hand side of (*) equals

$$\sum_{\substack{z\in W\\ y\le z\le w}} \sum_{x\in(C(z)\cap y.C'(1))^{F^n}} tr_{H_x^*(IC_w)}F^n \qquad (\text{as } X=X_w\cap y.C'(1) = \bigcup_{y\le z\le w} C(z)\cap yC'(1))$$

$$= \sum_{y\le z\le w} p^{n\,\mathrm{length}(y)} R_{y,z}(p^n)\, tr_{H^*_{zB_+}(IC_w)}F^n \qquad (\text{using } 4.3)$$

On the other hand, the left hand side equals

$$p^{n\,\mathrm{length}(w)}\, tr_{\mathbb{H}^*(X,IC_w)}F^{-n} \quad (\text{by Poincaré duality})$$

$$= p^{n\,\mathrm{length}(w)}\, tr_{H^i_{yB_+}(IC_w)}F^{-n} \quad \text{by 4.4 and 4.5b).}$$

Hence (*) becomes

$$(**)\quad p^{n\,\mathrm{length}(w)}\, tr_{H^*_{yB_+}(IC_w)}F^{-n} = p^{n\,\mathrm{length}(y)} \sum_{y\le z\le w} R_{y,z}(p^n)\, tr_{H^*_{zB_+}(IC_w)}F^n$$

After a careful comparison of degrees in (**) using the induction hypothesis and the fact that X is very pure, one concludes as in [KL1] and [KL4] that Q(y) is true.

To see that $\tilde{P}_{y,w}=P_{y,w}$, one rewrites (**) using 4.6 to get

$$p^{n(l(w)-l(y))}\sum_i(-1)^i p^{-ni/2}\dim H^i_{yB_+}(IC_w) = \sum_{y\le z\le w} R_{y,z}(p^n)\sum_i(-1)^i p^{ni/2}\dim H^i_{yB_+}(IC_w)$$

which is equivalent to $q^{l(w)-l(y)}\tilde{P}_{y,w} = \sum_{y\le z\le w} R_{y,z}\tilde{P}_{z,w}$ (here, $l(x)$ denotes the length of the element x of W). Using the characteristic properties of IC, it is easy to check that the $\tilde{P}_{y,w}$'s satisfy $\deg(\tilde{P}_{y,w})\le\frac{1}{2}(l(w)-l(y)-1)$, with $\tilde{P}_{w,w}=1$, hence $\tilde{P}_{y,w}=P_{y,w}$.

Added in proof: I have recently been informed that G. Lusztig has a much simpler proof, based on his paper <u>Characters of reductive groups over a finite field</u>, IHES, 1982, of this generalization of the results of [KL4].

3) <u>The case of elements of Coxeter type</u>:

Assume that $w=r_{j_1}\ldots r_{j_k}$ is such that $j_m \neq j_n$ if $m\neq n$. A direct computation shows that $P_{y,w}=1$ for all $y\leq w$, hence one can conclude that if W is a (finite) Weyl group, then the variety X_w is rationally smooth ([KL1]).

In fact, if W is any crystallographic group, and w is as above, then X_w is smooth. One can show directly that

a- if we identify $C(w)$ with K^k using 3.22, and $y=r_{j_{t_1}}\ldots r_{j_{t_m}} \leq w$, and $t_1<\ldots<t_m$, then
$C(w)\cap y.C'(1) = \{(\lambda_1,\ldots,\lambda_k) \mid \lambda_{t_i}\in K^{\times}\}$ (see [Dh3]),

and $C(w)\cap C'(y) = \{(\lambda_1,\ldots,\lambda_k) \mid \lambda_{t_i}=0\}$.

b- $X_w\cap y.C'(1) = f_w(U_{(n_k,\ldots,n_1)})$, with $n_j=\begin{cases}1 & \text{if } j\in\{t_1,\ldots t_m\}\\ 2 & \text{otherwise.}\end{cases}$
and $f_w|_{U_{(n_k,\ldots,n_1)}}$ is injective.

Let us prove instead that $f_w\colon Z_w\longrightarrow X_w$ is an isomorphism:

First note that if $z\in Z_w-U_{(1,\ldots,1)}$, then there exists $\lambda_j\in K$ such that $z=\overline{(p_k(z),\ldots,p_1(z))}$, with $p_j(z)\in\{y_{\beta_j}(\lambda_j),\underline{s}_j\}$, and for some j_o, $\lambda_{j_o}=\underline{s}_{j_o}$.

We now show that

given two sequences $s_1<\ldots<s_n$, $t_1<\ldots<t_n$, with $s_i,t_i\in\{1,\ldots,k\}$
if $r_{j_{s_1}}\ldots r_{j_{s_n}} = r_{j_{t_1}}\ldots r_{j_{t_n}} = w'$, then $s_i=t_i$ for all i,

by induction on n: If $n=1$, there is nothing to prove; otherwise, the exchange condition, applied to $w'.\alpha_{j_{s_n}}<0$ gives

(1) $w'= r_{j_{t_1}}\ldots r_{j_{t_{a-1}}}\,.\,r_{j_{t_{a+1}}}\ldots\, r_{j_{t_n}} r_{j_{s_n}}$ for some a,

and applied to $w'.\alpha_{j_{t_n}}<0$ gives

(2) $w'= r_{j_{s_1}}\ldots r_{j_{s_{b-1}}}\,.\,r_{j_{s_{b+1}}}\ldots\, r_{j_{s_n}} r_{j_{t_n}}$ for some b.

Assuming, without loss of generality, that $a\leq b$, one sees that if $b\neq n$ and $a\neq n$ then (1) and (2) imply that $t_n=s_{n-1}=t_{n-2}$, which is impossible because

$t_{n-2} < t_n$ so that $j_{t_{n-2}} \neq j_{t_n}$. Hence we must have b=a=n, in which case (2) implies that $w' = r_{j_{s_1}} \cdots r_{j_{s_{n-1}}} . r_{j_{t_n}}$. Combining this equation with $w' = r_{j_{s_1}} \cdots r_{j_{s_n}}$, we see that $r_{j_{t_n}} = r_{j_{s_n}}$, i.e. $t_n = s_n$, and the induction hypothesis gives $t_{n-1} = s_{n-1}, \ldots, t_1 = s_1$.

The notation being as above, if $z \in Z_w - U_{(1,\ldots,1)}$, then $f_w(z) = \underline{w} p_k(z) \ldots p_1(z) B_+$. As $w r_{\beta_{i_1}} \ldots r_{\beta_{i_n}} . \beta_m < 0$ if $i_1 > \ldots > i_n > m$ (that is so because $w r_{\beta_{i_1}} \ldots r_{\beta_{i_n}} . \beta_m = -(r_{j_1} \ldots r_{j_{k-i_1+1}} \ldots r_{j_{k-i_n+1}} r_{j_{k-m}} . \alpha_{j_{k-m+1}}$

$$= -\alpha_{j_{k-m+1}} + \sum_{i \in A} c_i \alpha_i \quad \text{where the subset A of } \{1,..,N\}$$

does not contain j_{k-m+1}), one sees that $f_w(z)$ is of the form $b w r_{\beta_{i_1}} \ldots r_{\beta_{i_p}} B_+$ with $b \in B_+$ and the indices $i_1, \ldots, i_p$ are such that $p_j(z) = \underline{s}_j$ if and only if j is in the set $\omega(z) = \{i_1, \ldots, i_p\}$. In particular, by the disjointness of the Bruhat decomposition and the lemma proved above, if $f_w(z_1) = f_w(z_2)$, then $\omega(z_1) = \omega(z_2)$ (with $\omega(z) = \emptyset$ if $z \in U_{(1,\ldots,1)}$). One now proves as in 4.1) (using again the idea in example 3)) that if

$$y_{\beta_k}(\lambda_k) \ldots y_{\beta_{i_1+1}}(\lambda_{i_1+1}) \underline{r}_{\beta_{i_1}} . y_{\beta_{i_1-1}}(\lambda_{i_1-1}) \ldots y_{\beta_{i_p+1}}(\lambda_{i_p+1}) \underline{r}_{\beta_{i_p}} . y_{\beta_{i_p-1}}(\lambda_{i_p-1}) \ldots y_{\beta_1}(\lambda_1) B_+$$

equals

$$y_{\beta_k}(\lambda'_k) \ldots y_{\beta_{i_1+1}}(\lambda'_{i_1+1}) \underline{r}_{\beta_{i_1}} . y_{\beta_{i_1-1}}(\lambda'_{i_1-1}) \ldots y_{\beta_{i_p+1}}(\lambda'_{i_p+1}) \underline{r}_{\beta_{i_p}} . y_{\beta_{i_p-1}}(\lambda'_{i_p-1}) \ldots y_{\beta_1}(\lambda'_1) B_+$$

then $\lambda_j = \lambda'_j$ for all j (alternately, multiplying this last equation by $\underline{w}$ gives an equality in $C(w r_{\beta_{i_1}} \ldots r_{\beta_{i_n}})$ on which one can use 3.7). Hence f_w is injective.

In particular, $X_w \cap y.C'(1)$ is open, and one can check, using appropriate coordinates, that $f_w^*|_{\mathcal{O}(X_w \cap yC'(1))}$ is an isomorphism: indeed, with Λ as

in 3.9, and for all $\alpha\varepsilon\Delta_{+}^{re}$, if r_α denotes reflexion in α, then $r_\alpha \varepsilon W$, so that $L(\Lambda)_{r_\alpha \Lambda} \neq 0$, hence $\Lambda - r_\alpha\Lambda \varepsilon Q_+$; but $\Lambda - r_\alpha \Lambda = \langle\Lambda,\alpha^\vee\rangle\alpha$, so $\langle\Lambda,\alpha^\vee\rangle > 0$; therefore, by [K5], $L(\Lambda)_{\Lambda-\alpha} \neq 0$. Now using this and the fact that all r_{j_i}'s are distinct, one can see, after a tedious but straightforward computation, that $f_w|_{U_{(n_k,\dots,n_1)}}$ always looks, in the local coordinates like a mapping $K^k \to K^M$ with $(t_1,\dots,t_k) \mapsto (t_1,\dots,t_k,p_1(t_1,\dots,t_k),\dots$
$\dots,p_{M-k}(t_1,\dots,t_k))$

$t_i \varepsilon K$, for some $M\varepsilon\mathbf{N}$, where each p_j is a homogeneous polynomial of degree at least 2. Hence f_w^* gives an isomorphism of the rings of functions, and f_w is indeed an isomorphism.

Example 9: If k=2, set $u_{m,n} = f^{(m)}_{r_{j_2}\alpha_{j_1}} . f^{(n)}_{j_2} . v^+ \otimes 1 \;\varepsilon\; \mathcal{U}_{\mathbf{Z}} . v^+ \otimes K$;

Then $f_w|_{U_{(1,1)}}$ is the map

$$\overline{(y_{r_{j_2}\alpha_{j_1}}(t_1),\ y_{j_2}(t_2)B_+)} \mapsto [wv^+\otimes 1 + t_1 . \underline{w}u_{1,0} + t_2 . \underline{w}u_{0,1} + \sum_{m,n\geq 2} t_1^m t_2^n . \underline{w}u_{m,n}]$$

Similarly, $f_w|_{U_{(2,1)}}$ is the map

$$\overline{(x_{r_{j_2}\alpha_{j_1}}(t_1) r_{j_2} r_{j_1} r_{j_2},\ y_{j_2}(t_2)B_+)} \mapsto [r_j v^+ \otimes 1 + t_1 . (-r_{j_2} u_{1,0}) + t_2 . r_{j_2} u_{0,1} + \sum_{m,n\geq 2} (-t_1)^m (t_2)^n . r_{j_2} u_{m,n}]$$

etc...

4) Geometric interpretation of the matrix A:

Suppose k=length(w)=2, so that $w=r_i r_j$ with $i\neq j$, hence $Z_w \to X_w$ is an isomorphism.

We have $Z_w^1 = P_2/B_1 \simeq \mathbb{P}^1$, and $g: Z_w \to Z_w^1$ the projection so Z_w is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. In fact, we also have a global section $h: Z_w^1 \to Z_w$, so Z_w is a ruled surface.

If $L = Z_w - \mathrm{Im}\, h$, then $g|_L : L \to Z_w^1$ is a line bundle on $\mathbb{P}^1$. We shall show that the degree of L is $-a_{ij}$, that $\mathrm{Im}\, h$ is the unique rigid section of $Z_w \to Z_w^1$, hence the self-intersection number of $(\mathrm{Im}\, h)$ is a_{ij}, i.e. the invariant of the ruled surface X_w is $-a_{ij}$ ([T3]). In particular, the product of the invariant of $X_{r_i r_j}$ by that of $X_{r_j r_i}$ is $4(\cos\frac{\pi}{m_{ij}})^2$. Since the latter is an integer only when $m_{ij} \in \{2,3,4,6,\infty\}$, one concludes that no 'nice' flag variety can be attached to a Coxeter group unless that group is crystallographic.

So we have $Z_w = Z^2_{r_i r_j} = P_2 \times^{B_1} P_1/B_o$, $Z_w^1 = P_2/B_1 \simeq \mathbb{P}^1$, and $h: Z_w^1 \to Z_w$ given by $h(p_2 B_1) = \overline{(p_2, r_j B_o)}$. Here, $B_1 = r_j^{-1} B_+ r_j$, $B_2 = w^{-1} B_+ w$, $P_1 = B_+ \cup B_+ r_j B_+$ $= r_j B_+ \cup y_{\alpha_j} B_+$, $P_2 = B_1 \cup B_1 r_j r_i r_j B_1 = r_j r_i r_j B_1 \cup y_{r_j \alpha_i} B_1$, $L = U^2_{(1,1)} \cup U^2_{(2,1)}$.

Given a map $f: P_2 \to (P_1/B_o - \{r_j B_o\}) = y_{\alpha_j} B_o$ satisfying

$$(*) \qquad f(p_2 b_1) = b_1^{-1} f(p_2)$$

we can define $\tilde{f}: Z_w^1 \to L$ by $\tilde{f}(p_2 B_1) = \overline{(p_2, f(p_2))}$, and conversely, given $\tilde{f}: Z_w^1 \to L$ satisfying $g\tilde{f}(p) = p$, we can define a (set) map $f: P_2 \to (y_{\alpha_j} B_o)/B_o$ by requiring $f(p_2)$ to be that element of P_1/B_o for which $\tilde{f}(p_2 B_1)$ can be written in the form $(p_2, f(p_2))$ (this determines f uniquely).

Now $B_1 = (H \ltimes y_{\alpha_j}) \ltimes U^{\alpha_j} = L_j \ltimes U^{\alpha_j}$, and left translation by B_1 induces the following action of L_j on the affine line $y_{\alpha_j} B_+/B_+$:

(If $b = u \in U^{\alpha_j}$, then $b^{-1} y_{\alpha_j}(t) B_+ = y_{\alpha_j}(t) B_+$)

If $b = y_{\alpha_j}(s)$, then $b^{-1} y_{\alpha_j}(t) B_+ = y_{\alpha_j}(t-s) B_+$

If $b = h_1(z_1) \dots h_N(z_N)$, then $b^{-1} y_{\alpha_j}(t) B_+ = y_{\alpha_j}(z_1^{a_{1j}} \dots z_N^{a_{Nj}} t) B_+$.

Let $\chi, \Psi: K \to P_2$ be the maps $\chi(t) = x_{r_j \alpha_i}(t) \dot{r}_j(1) \dot{r}_i(1) \dot{r}_j(1)$, $\Psi(t) = y_{r_j \alpha_i}(t)$

so that given $p_2 \varepsilon P_2$, either there exists a unique $t \varepsilon K$ and $b_1 \varepsilon B_1$ such that $p_2 = \chi(t) b_1$, or there exists a unique $t \varepsilon K$ and $b_1 \varepsilon B_1$ such that $p_2 = \Psi(t) b_1$. Given $f: P_2 \to (y_{\alpha_j} B_+)/B_+$, consider $f_1 = f \circ \chi$, $f_2 = f \circ \Psi$, so that $f_i: K \to y_{\alpha_j} B_+$. If $c \neq 0$, then $\chi(c) = \Psi(c^{-1}) h_i(-c) h_j((-c)^{-a_{ij}}) u$ with $u \varepsilon U^{\alpha_j}$, and if f satisfies (*), then

$$(**) \qquad f_1(c) = (h_i(-c) h_j((-c)^{-a_{ij}})^{-1} . f_2(c^{-1})$$

for all $c \neq 0$. Conversely, given $f_1, f_2: K \to y_{\alpha_j} B_+$ satisfying (**), one can construct $f: P_2 \to P_1/B_+ - \{r_j B_+\}$, and such a map f will automatically satisfy (*).

We thus obtain a 1-1 correspondence between maps $\tilde{f}: P_2/B_1 \to L$ satisfying $g\tilde{f}$=identity, and pairs $f_1, f_2: K \to (y_{\alpha_j} B_+)/B_+$ satisfying (**). One can check that, in this correspondence, the map $\tilde{f} : Z_w^1 \to L$ is a morphism if and only if the maps f_1, f_2 are morphisms when considered as functions on $\mathbb{A}^1$. Let's see when the latter is true:

If f_2 is a morphism, then $f_2(t) = y_{\alpha_j}(f_2^*(t)) B_+$, when f_2^* is a polynomial $\varepsilon K[t]$: (**) then becomes $f_1(c) = h_j((-c)^{a_{ij}}) h_i(-c^{-1}) y_{\alpha_j}(f_2^*(c^{-1})) B_+$, and the right hand side is equal to $y_{\alpha_j}((-c)^{a_{ij}} (-c)^{-2a_{ij}} f_2^*(c^{-1}) B_+$

$$= y_{\alpha_j}((-1)^{a_{ij}} c^{-a_{ij}} f_2^*(c^{-1})) B_+$$

so we must have $f_1(c) = y_{\alpha_j}((-1)^{a_{ij}} c^{-a_{ij}} f_2^*(c^{-1})) B_+$ for all $c \varepsilon K^\times$. If in addition f_1 is to be a morphism, then $t^{-a_{ij}} f_2^*(t^{-1})$ has to be a polynomial also, hence degree$(f_2^*) \leq -a_{ij}$. Conversely , any polynomial $f^*(t) \varepsilon K[t]$ of degree$\leq -a_{ij}$ yields a section of $L \to Z_w^1$ simply by reversing the process described above. This correspondence is clearly linear, so one concludes that $\dim H^0(Z_w^1, L) = \dim\{$polynomials in K[t] of degree$\leq -a_{ij}\} = 1 - a_{ij}$; thus, the degree of L is $-a_{ij}$.

We now examine the coordinate chart on Z_w^2 more closely:

$U_{(1,1)} = \{\ a(x_1, y_1) = (\ \dot{r}_j(1) y_{\alpha_i}(x_1) \dot{r}_j(-1)\ ,\ y_{\alpha_j}(y_1)\ ),\ x_1, y_1 \varepsilon K\ \}$

$U_{(2,1)} = \{\ b(x_2, y_2) = (\ \dot{r}_j(1) x_{\alpha_i}(x_2) \dot{r}_i(1) \dot{r}_j(-1)\ ,\ y_{\alpha_j}(y_2)\ ),\ x_2, y_2 \varepsilon K\ \}$

$U_{(1,2)} = \{\ c(x_3, y_3) = (\ \dot{r}_j(1) y_{\alpha_i}(x_3) \dot{r}_j(-1)\ ,\ x_{\alpha_j}(y_3) \dot{r}_j(1)\ ),\ x_3, y_3 \varepsilon K\ \}$

$U_{(2,2)} = \{\ d(x_4, y_4) = (\ \dot{r}_j(1) x_{\alpha_i}(x_4) \dot{r}_i(1) \dot{r}_j(-1),\ x_{\alpha_j}(y_4) \dot{r}_j(1)\ ),\ x_4, y_4 \varepsilon K\ \}$

Using the relations

$$\dot{r}_i(-t)h_i(-t)=\dot{r}_i(1)$$

$$\dot{r}_k(1)h_m(t)\dot{r}_k(-1)=h_m(t)h_k(t^{-a_{mk}})$$

$$h_k(s)x_\alpha(t)h_k(s^{-1})=x_\alpha(s^{\langle\alpha,\alpha_k^\vee\rangle}t)$$

one can determine the intersections $U_{(m,n)}\cap U_{(m',n')}$ as follows:

$$a(x_1,y_1)=b(x_2,y_2) \text{ if \& only if } x_1x_2=1 \ \& \ y_1=y_2(-x_2)^{-a_{ij}}$$

$$a(x_1,y_1)=c(x_3,y_3) \text{ if \& only if } x_1=x_3 \ \& \ y_1y_3=1$$

$$b(x_2,y_2)=d(x_4,y_4) \text{ if \& only if } x_2=x_4 \ \& \ y_2y_4=1$$

$$c(x_3,y_3)=d(x_4,y_4) \text{ if \& only if } x_3x_4=1 \ \& \ y_3=y_4(-x_4)^{-a_{ij}}$$

$$a(x_1,y_1)=d(x_4,y_4) \text{ if \& only if } x_1x_4=1 \ \& \ y_1y_4=(-x_4)^{a_{ij}}$$

$$c(x_3,y_3)=b(x_2,y_2) \text{ if \& only if } x_3x_2=1 \ \& \ y_3y_2=(-x_2)^{-a_{ij}}$$

and $\operatorname{Im} h=\{c(x_3,0)\}\cup\{d(x_4,0)\}$.

Using the coordinates $X\ (=x_1=x_3)$, $Y\ (=y_1)$, $Z\ (=y_2)$, $1/X(=x_2=x_4)$, $1/Y(=y_3)$, $1/Z(=y_4)$, the equations on the right reduce to the single equation

$$Y = Z(-X)^{-a_{ij}}$$

for $X_{r_ir_j}$, and $\operatorname{Im} h=\{Z=\infty,\ Y=\infty\}$, from which it is easy to see that $\operatorname{Im} h$ is rigid (e.g.: as $a_{ij}\leq 0$, deformations of the form $Z(-X)^{-a_{ij}}=\text{constant}$ will contain, in the limit, the fiber $X=\infty$ as well as $\operatorname{Im} h$; and deformations of the form $Y(-X)^{a_{ij}}=\text{constant}$ will tend to $\operatorname{Im} h\cup\{X=0\}$).

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