CHAPTER 5

Integrals

5.1 The Idea of the Integral

This chapter is about the ***idea*** of integration, and also about the ***technique*** of integration. We explain how it is done *in principle*, and then how it is done *in practice*. Integration is a problem of adding up infinitely many things, each of which is infinitesimally small. Doing the addition is not recommended. The whole point of calculus is to offer a better way.

The problem of integration is to find a limit of sums. The key is to work backward from a limit of differences (which is the derivative). ***We can integrate*** $v(x)$ ***if it turns up as the derivative of another function*** $f(x)$. The integral of $v = \cos x$ is $f = \sin x$. The integral of $v = x$ is $f = \frac{1}{2}x^2$. Basically, $f(x)$ is an "***antiderivative***". The list of f's will grow much longer (Section 5.4 is crucial). A selection is inside the cover of this book. If we don't find a suitable $f(x)$, numerical integration can still give an excellent answer.

I could go directly to the formulas for integrals, which allow you to compute areas under the most amazing curves. (Area is the clearest example of adding up infinitely many infinitely thin rectangles, so it always comes first. It is certainly not the only problem that integral calculus can solve.) But I am really unwilling just to write down formulas, and skip over all the ideas. Newton and Leibniz had an absolutely brilliant intuition, and there is no reason why we can't share it.

They started with something simple. We will do the same.

SUMS AND DIFFERENCES

Integrals and derivatives can be mostly explained by working (very briefly) with sums and differences. Instead of functions, we have n ordinary numbers. The key idea is nothing more than a basic fact of algebra. In the limit as $n \to \infty$, it becomes the basic fact of calculus. The step of "going to the limit" is the essential difference between algebra and calculus! It has to be taken, in order to add up infinitely many infinitesimals—but we start out this side of it.

To see what happens before the limiting step, we need ***two sets of n numbers***. The first set will be $v_1, v_2, \ldots, v_n$, where v suggests velocity. The second set of numbers will be $f_1, f_2, \ldots, f_n$, where f recalls the idea of distance. You might think d would be a better symbol for distance, but that is needed for the dx and dy of calculus.

A first example has $n = 4$:

$$v_1, v_2, v_3, v_4 = 1, 2, 3, 4 \qquad f_1, f_2, f_3, f_4 = 1, 3, 6, 10.$$

The relation between the v's and f's is seen in that example. When you are given 1, 3, 6, 10, how do you produce 1, 2, 3, 4? ***By taking differences***. The difference between 10 and 6 is 4. Subtracting $6 - 3$ is 3. The difference $f_2 - f_1 = 3 - 1$ is $v_2 = 2$. Each v is the difference between two f's:

$$v_j \textit{ is the difference } f_j - f_{j-1}.$$

This is the discrete form of the derivative. I admit to a small difficulty at $j = 1$, from the fact that there is no f_0. The first v should be $f_1 - f_0$, and the natural idea is to agree that f_0 is zero. This need for a starting point will come back to haunt us (or help us) in calculus.

Now look again at those same numbers—but start with v. From $v = 1, 2, 3, 4$ how do you produce $f = 1, 3, 6, 10$? ***By taking sums***. The first two v's add to 3, which is f_2. The first three v's add to $f_3 = 6$. The sum of all four v's is $1 + 2 + 3 + 4 = 10$. ***Taking sums is the opposite of taking differences.***

That idea from algebra is the key to calculus. The sum f_j involves all the numbers $v_1 + v_2 + \cdots + v_j$. The difference v_j involves only the *two* numbers $f_j - f_{j-1}$. The fact that one reverses the other is the "Fundamental Theorem." Calculus will change sums to integrals and differences to derivatives—but why not let the key idea come through now?

> **5A** ***Fundamental Theorem of Calculus*** (before limits):
>
> $$\textit{If each } v_j = f_j - f_{j-1}, \textit{ then } v_1 + v_2 + \cdots + v_n = f_n - f_0.$$

The differences of the f's add up to $f_n - f_0$. All f's in between are canceled, leaving only the last f_n and the starting f_0. *The sum "telescopes"*:

$$v_1 + v_2 + v_3 + \cdots + v_n = (f_1 - f_0) + (f_2 - f_1) + (f_3 - f_2) + \cdots + (f_n - f_{n-1}).$$

The number f_1 is canceled by $-f_1$. Similarly $-f_2$ cancels f_2 and $-f_3$ cancels f_3. Eventually f_n and $-f_0$ are left. When f_0 is zero, the sum is the final f_n.

That completes the algebra. ***We add the*** v's ***by finding the*** f's.

Question How do you add the odd numbers $1 + 3 + 5 + \cdots + 99$ (the v's)?
Answer They are the differences between 0, 1, 4, 9, ... These f's are squares. By the Fundamental Theorem, the sum of 50 odd numbers is $(50)^2$.

The tricky part is to discover the right f's! Their differences must produce the v's. In calculus, the tricky part is to find the right $f(x)$. Its derivative must produce $v(x)$. It is remarkable how often f can be found—more often for integrals than for sums. Our next step is to understand how ***the integral is a limit of sums***.

SUMS APPROACH INTEGRALS

Suppose you start a successful company. The rate of income is increasing. After x years, the income per year is $\sqrt{x}$ million dollars. In the first four years you reach $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{4}$ million dollars. Those numbers are displayed in a bar graph (Figure 5.1a, for investors). I realize that most start-up companies make losses, but your company is an exception. If the example is too good to be true, please keep reading.

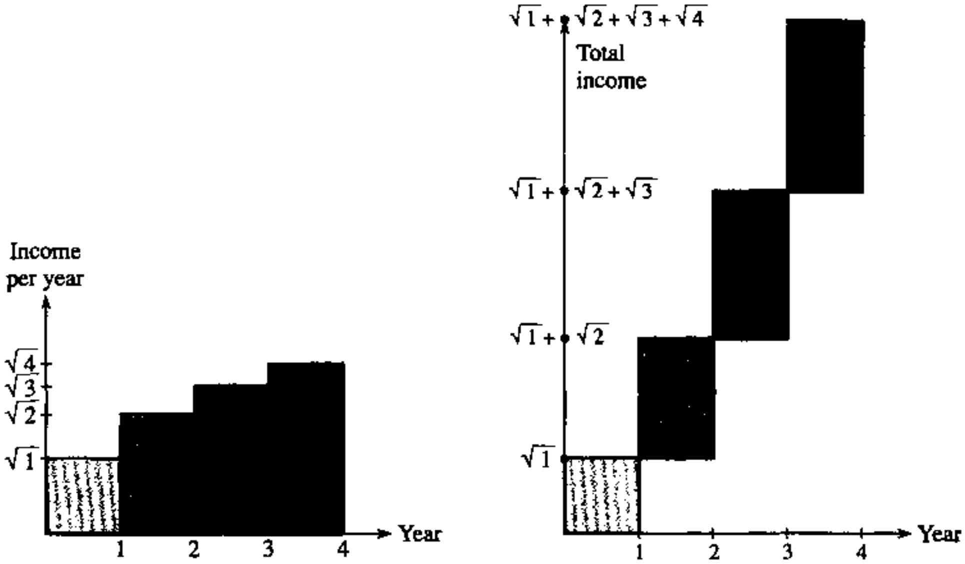


Fig. 5.1 Total income = total area of rectangles = 6.15.

The graph shows four rectangles, of heights $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$. Since the base of each rectangle is one year, those numbers are also the ***areas*** of the rectangles. One investor, possibly weak in arithmetic, asks a simple question: ***What is the total income for all four years?*** There are two ways to answer, and I will give both.

The first answer is $\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}$. Addition gives 6.15 million dollars. Figure 5.1b shows this total—which is reached at year 4. This is exactly like velocities and distances, but now v is the ***income per year*** and f is the ***total income***. Algebraically, f_j is still $v_1 + \cdots + v_j$.

The second answer comes from geometry. ***The total income is the total area of the rectangles.*** We are emphasizing the correspondence between ***addition*** and ***area***. That point may seem obvious, but it becomes important when a second investor (smarter than the first) asks a harder question.

Here is the problem. ***The incomes as stated are false.*** The company did not make a million dollars the first year. After three months, when x was 1/4, the rate of income was only $\sqrt{x} = 1/2$. The bar graph showed $\sqrt{1} = 1$ for the whole year, but that was an overstatement. The income in three months was not more than 1/2 times 1/4, the rate multiplied by the time.

All other quarters and years were also overstated. Figure 5.2a is closer to reality, with 4 years divided into 16 quarters. It gives a new estimate for total income.

Again there are two ways to find the total. We add $\sqrt{1/4}+\sqrt{2/4}+\cdots+\sqrt{16/4}$, remembering to multiply them all by 1/4 (because each rate applies to 1/4 year). This is also the area of the 16 rectangles. The area approach is better because the 1/4 is automatic. Each rectangle has base 1/4, so that factor enters each area. The total area is now 5.56 million dollars, closer to the truth.

You see what is coming. The next step divides time into weeks. After one week the rate $\sqrt{x}$ is only $\sqrt{1/52}$. That is the height of the first rectangle—its base is $\Delta x = 1/52$. There is a rectangle for every week. Then a hard-working investor divides time into days, and the base of each rectangle is $\Delta x = 1/365$. At that point there are $4 \times 365 = 1460$ rectangles, or 1461 because of leap year, with a total area below $5\frac{1}{2}$

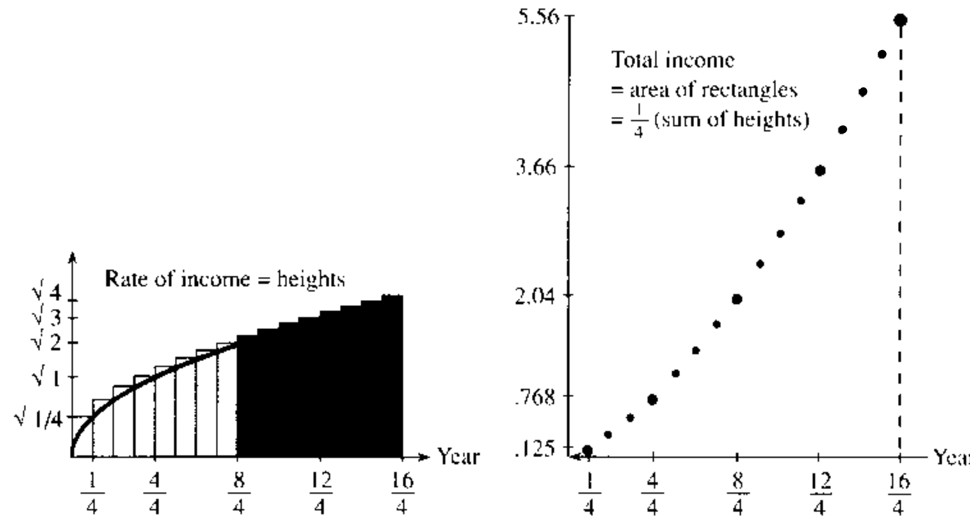


Fig. 5.2 Income = sum of areas (not heights)

$$=\frac{1}{4}\left(\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{4}}+\cdots+\sqrt{\frac{16}{4}}\right)$$

million dollars. The calculation is elementary but depressing—adding up thousands of square roots, each multiplied by Δx from the base. There has to be a better way.

The better way, in fact the best way, is calculus. The whole idea is to allow for ***continuous change. The geometry problem is to find the area under the square root curve.*** That question cannot be answered by arithmetic, because it involves a *limit*. The rectangles have base Δx and heights $\sqrt{\Delta x}$, $\sqrt{2\Delta x}$, ..., $\sqrt{4}$. There are $4/\Delta x$ rectangles—more and more terms from thinner and thinner rectangles. ***The area is the limit of the sum as*** $\Delta x \to 0$.

This limiting area is the "integral." We are looking for a number below $5\frac{1}{2}$.

> ***Algebra*** (*area of n rectangles*): Compute $v_1 + \cdots + v_n$ by finding f's.
> *Key idea*: If $v_j = f_j - f_{j-1}$ then the sum is $f_n - f_0$.
>
> ***Calculus*** (*area under curve*): *Compute the limit of* $\Delta x[v(\Delta x) + v(2\Delta x) + \cdots]$.
> *Key idea*: ***If*** $v(x) = df/dx$ ***then area = integral to be explained next.***

5.1 EXERCISES

Read-through questions

The problem of summation is to add $v_1 + \cdots + v_n$. It is solved if we find f's such that $v_j =$ __a__. Then $v_1 + \cdots + v_n$ equals __b__. The cancellation in $(f_1 - f_0) + (f_2 - f_1) + \cdots + (f_n - f_{n-1})$ leaves only __c__. Taking sums is the __d__ of taking differences.

The differences between 0, 1, 4, 9 are $v_1, v_2, v_3 =$ __e__. For $f_j = j^2$ the difference between f_{10} and f_9 is $v_{10} =$ __f__. From this pattern $1 + 3 + 5 + \cdots + 19$ equals __g__.

For functions, finding the integral is the reverse of __h__. If the derivative of $f(x)$ is $v(x)$, then the __i__ of $v(x)$ is $f(x)$. If $v(x) = 10x$ then $f(x) =$ __j__. This is the __k__ of a triangle with base x and height $10x$.

Integrals begin with sums. The triangle under $v = 10x$ out to $x = 4$ has area __l__. It is approximated by four rectangles of heights 10, 20, 30, 40 and area __m__. It is better approximated by eight rectangles of heights __n__ and area __o__. For n rectangles covering the triangle the area is the sum of __p__. As $n \to \infty$ this sum should approach the number __q__. *That is the integral of* $v = 10x$ *from* 0 *to* 4.

Problems 1–6 are about sums f_j and differences v_j.

1 With $v = 1, 2, 4, 8$, the formula for v_j is ________ (not 2^j). Find f_1, f_2, f_3, f_4 starting from $f_0 = 0$. What is f_7?

2 The same $v = 1, 2, 4, 8, \ldots$ are the differences between $f = 1, 2, 4, 8, 16, \ldots$. Now $f_0 = 1$ and $f_j = 2^j$. (a) Check that $2^5 - 2^4$ equals v_5. (b) What is $1 + 2 + 4 + 8 + 16$?

3 The differences between $f = 1, 1/2, 1/4, 1/8$ are $v = -1/2, -1/4, -1/8$. These negative v's do not add up to these positive f's. Verify that $v_1 + v_2 + v_3 = f_4 - f_0$ is still true.

4 Any constant C can be added to the antiderivative $f(x)$ because the ________ of a constant is zero. Any C can be added to $f_0, f_1, \ldots$ because the ________ between the f's is not changed.

5 Show that $f_j = r^j/(r-1)$ has $f_j - f_{j-1} = r^{j-1}$. Therefore the geometric series $1 + r + \cdots + r^{j-1}$ adds up to ________ (remember to subtract f_0).

6 The sums $f_j = (r^j - 1)/(r - 1)$ also have $f_j - f_{j-1} = r^{j-1}$. Now $f_0 =$ ________. Therefore $1 + r + \cdots + r^{j-1}$ adds up to f_j. The sum $1 + r + \cdots + r^n$ equals ________.

7 Suppose $v(x) = 3$ for $x < 1$ and $v(x) = 7$ for $x > 1$. Find the area $f(x)$ from 0 to x, under the graph of $v(x)$. (Two pieces.)

8 If $v = 1, -2, 3, -4, \ldots$, write down the f's starting from $f_0 = 0$. Find formulas for v_j and f_j when j is odd and j is even.

Problems 9–16 are about the company earning $\sqrt{x}$ per year.

9 When time is divided into weeks there are $4 \times 52 = 208$ rectangles. Write down the first area, the 208th area, and the jth area.

10 How do you know that the sum over 208 weeks is smaller than the sum over 16 quarters?

11 A pessimist would use $\sqrt{x}$ at the *beginning* of each time period as the income rate for that period. Redraw Figure 5.1 (both parts) using heights $\sqrt{0}, \sqrt{1}, \sqrt{2}, \sqrt{3}$. How much lower is the estimate of total income?

12 The same pessimist would redraw Figure 5.2 with heights $0, \sqrt{1/4}, \ldots$. What is the height of the last rectangle? How much does this change reduce the total rectangular area 5.56?

13 At every step from years to weeks to days to hours, the pessimist's area goes ________ and the optimist's area goes ________. The difference between them is the area of the last ________.

14 The optimist and pessimist arrive at the same limit as years are divided into weeks, days, hours, seconds. Draw the $\sqrt{x}$ curve between the rectangles to show why the pessimist is always too low and the optimist is too high.

15 (Important) Let $f(x)$ be the area under the $\sqrt{x}$ curve, above the interval from 0 to x. The area to $x + \Delta x$ is $f(x + \Delta x)$. The extra area is $\Delta f =$ ________. This is almost a rectangle with base ________ and height $\sqrt{x}$. So $\Delta f/\Delta x$ is close to ________. As $\Delta x \to 0$ we suspect that $df/dx =$ ________.

16 Draw the $\sqrt{x}$ curve from $x = 0$ to 4 and put triangles below to prove that the area under it is more than 5. Look left and right from the point where $\sqrt{1} = 1$.

Problems 17–22 are about a company whose expense rate $v(x) = 6 - x$ is decreasing.

17 The expenses drop to zero at $x =$ ________. The total expense during those years equals ________. This is the area of ________.

18 The rectangles of heights 6, 5, 4, 3, 2, 1 give a total estimated expense of ________. Draw them enclosing the triangle to show why this total is too high.

19 How many rectangles (enclosing the triangle) would you need before their areas are within 1 of the correct triangular area?

20 The accountant uses 2-year intervals and computes $v = 5, 3, 1$ at the midpoints (the odd-numbered years). What is her estimate, how accurate is it, and why?

21 What is the area $f(x)$ under the line $v(x) = 6 - x$ above the interval from 2 to x? What is the derivative of this $f(x)$?

22 What is the area $f(x)$ under the line $v(x) = 6 - x$ above the interval from x to 6? What is the derivative of this $f(x)$?

23 With $\Delta x = 1/3$, find the area of the three rectangles that enclose the graph of $v(x) = x^2$.

24 Draw graphs of $v = \sqrt{x}$ and $v = x^2$ from 0 to 1. Which areas add to 1? The same is true for $v = x^3$ and $v =$ ________.

25 From x to $x + \Delta x$, the area under $v = x^2$ is Δf. This is almost a rectangle with base Δx and height ________. So $\Delta f/\Delta x$ is close to ________. In the limit we find $df/dx = x^2$ and $f(x) =$ ________.

26 Compute the area of 208 rectangles under $v(x) = \sqrt{x}$ from $x = 0$ to $x = 4$.

5.2 Antiderivatives

The symbol $\int$ was invented by Leibniz to represent the integral. It is a stretched-out **S**, from the Latin word for sum. This symbol is a powerful reminder of the whole construction: ***Sum approaches integral*, S *approaches* $\int$, *and rectangular area approaches curved area***:

$$\textit{curved area} = \int v(x)\,dx = \int \sqrt{x}\,dx. \tag{1}$$

The rectangles of base Δx lead to this limit—the integral of $\sqrt{x}$. The "dx" indicates that Δx approaches zero. The heights v_j of the rectangles are the heights $v(x)$ of the curve. The sum of v_j times Δx approaches "*the integral of v of x dx*." You can imagine an infinitely thin rectangle above every point, instead of ordinary rectangles above special points.

We now find the area under the square root curve. The "***limits of integration***" are 0 and 4. The lower limit is $x = 0$, where the area begins. (*The start could be any point* $x = a$.) The upper limit is $x = 4$, since we stop after four years. (*The finish could be any point* $x = b$.) The area of the rectangles is a sum of base Δx times heights $\sqrt{x}$. The curved area is the limit of this sum. ***That limit is the integral of $\sqrt{x}$ from* 0 *to* 4**:

$$\lim_{\Delta x \to 0}\left[(\sqrt{\Delta x})(\Delta x) + (\sqrt{2\Delta x})(\Delta x) + \cdots + (\sqrt{4})(\Delta x)\right] = \int_{x=0}^{x=4} \sqrt{x}\,dx. \tag{2}$$

The outstanding problem of integral calculus is still to be solved. *What is this limiting area?* We have a symbol for the answer, involving $\int$ and $\sqrt{x}$ and dx—but we don't have a number.

THE ANTIDERIVATIVE

I wish I knew who discovered the area under the graph of $\sqrt{x}$. It may have been Newton. The answer was available earlier, but the key idea was shared by Newton and Leibniz. They understood the parallels between sums and integrals, and between differences and derivatives. I can give the answer, by following that analogy. I can't give the proof (yet)—it is the Fundamental Theorem of Calculus.

In algebra the difference $f_j - f_{j-1}$ is v_j. When we add, the sum of the v's is $f_n - f_0$. ***In calculus the derivative of $f(x)$ is $v(x)$.*** When we integrate, ***the area under the $v(x)$ curve is $f(x)$ minus $f(0)$.*** Our problem asks for the area out to $x = 4$:

> **5B** (Discrete vs. continuous, rectangles vs. curved areas, addition vs. integration) ***The integral of $v(x)$ is the difference in $f(x)$***:
>
> $$\textit{If } df/dx = \sqrt{x} \textit{ then area} = \int_{x=0}^{x=4} \sqrt{x}\,dx = f(4) - f(0). \tag{3}$$

What is $f(x)$? Instead of the derivative of $\sqrt{x}$, we need its "***antiderivative***." We have to find a function $f(x)$ whose derivative is $\sqrt{x}$. It is the opposite of Chapters 2–4, and requires us to ***work backwards***. The derivative of x^n is nx^{n-1}—now we need the antiderivative. The quick formula is $f(x) = x^{n+1}/(n+1)$—we aim to understand it.

Solution Since the derivative lowers the exponent, the antiderivative *raises* it. We go from $x^{1/2}$ to $x^{3/2}$. But then the derivative is $(3/2)x^{1/2}$. It contains an unwanted factor $3/2$. *To cancel that factor, put $2/3$ into the antiderivative*:

$$f(x) = \tfrac{2}{3}x^{3/2} \textit{ has the required derivative } v(x) = x^{1/2} = \sqrt{x}.$$

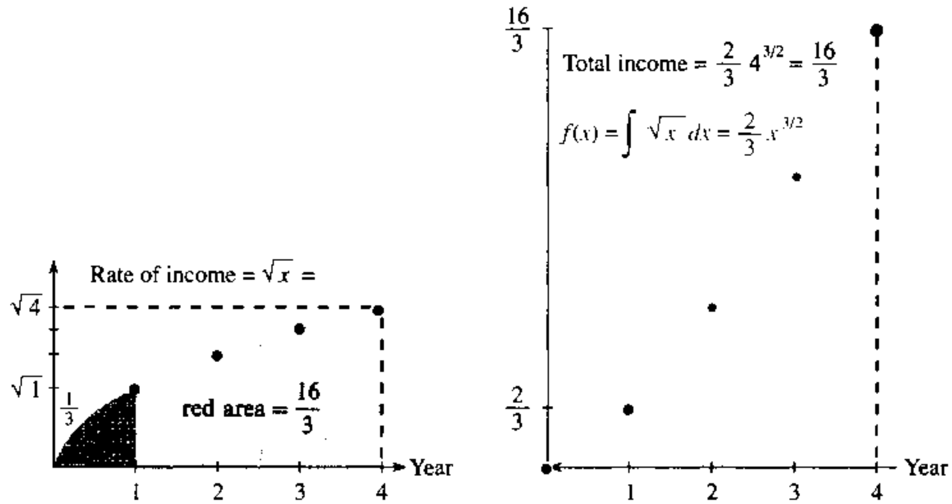


Fig. 5.3 The integral of $v(x) = \sqrt{x}$ is the exact area 16/3 under the curve.

There you see the key to integrals: Work backward from derivatives (and adjust).

Now comes a number—the exact area. At $x = 4$ we find $x^{3/2} = 8$. Multiply by 2/3 to get 16/3. Then subtract $f(0) = 0$:

$$\int_{x=0}^{x=4} \sqrt{x}\,dx = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{2}{3}(8) = \frac{16}{3} \tag{4}$$

The total income over four years is $16/3 = 5\frac{1}{3}$ ***million dollars***. This is $f(4) - f(0)$. The sum from thousands of rectangles was slowly approaching this exact area $5\frac{1}{3}$.

Other areas The income in the first year, at $x = 1$, is $\frac{2}{3}(1)^{3/2} = \frac{2}{3}$ million dollars. (The false income was 1 million dollars.) The total income after x years is $\frac{2}{3}x^{3/2}$, which is the antiderivative $f(x)$. *The square root curve covers 2/3 of the overall rectangle it sits in.* The rectangle goes out to x and up to $\sqrt{x}$, with area $x^{3/2}$, and 2/3 of that rectangle is below the curve. (1/3 is above.)

Other antiderivatives The derivative of x^5 is $5x^4$. Therefore the antiderivative of x^4 is $x^5/5$. Divide by 5 (or $n + 1$) to cancel the 5 (or $n + 1$) from the derivative. And don't allow $n + 1 = 0$:

The derivative $v(x) = x^n$ ***has the antiderivative*** $f(x) = x^{n+1}/(n + 1)$.

EXAMPLE 1 The antiderivative of x^2 is $\frac{1}{3}x^3$. This is the area under the parabola $v(x) = x^2$. The area out to $x = 1$ is $\frac{1}{3}(1)^3 - \frac{1}{3}(0)^3$, or 1/3.

Remark on $\sqrt{x}$ *and* x^2 The 2/3 from $\sqrt{x}$ and the 1/3 from x^2 add to 1. Those are the areas below and above the $\sqrt{x}$ curve, in the corner of Figure 5.3. If you turn the curve by 90°, it becomes the parabola. The functions $y = \sqrt{x}$ and $x = y^2$ are inverses! The areas for these inverse functions add to a square of area 1.

AREA UNDER A STRAIGHT LINE

You already know the area of a triangle. The region is below the diagonal line $v = x$ in Figure 5.4. The base is 4, the height is 4, and the area is $\frac{1}{2}(4)(4) = 8$. Integration is

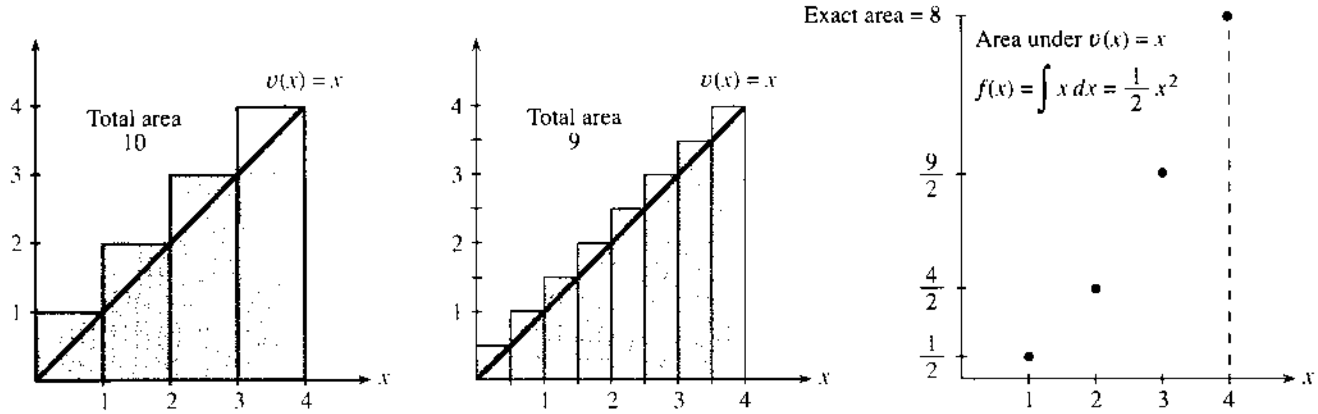


Fig. 5.4 Triangular area 8 as the limit of rectangular areas 10, 9, $8\frac{1}{2}$, ...

not required! But if you allow calculus to repeat that answer, and build up the integral $f(x) = \frac{1}{2}x^2$ as the limiting area of many rectangles, you will have the beginning of something important.

The four rectangles have area $1 + 2 + 3 + 4 = 10$. That is greater than 8, because the triangle is inside. 10 is a first approximation to the triangular area 8, and to improve it we need more rectangles.

The next rectangles will be thinner, of width $\Delta x = 1/2$ instead of the original $\Delta x = 1$. There will be eight rectangles instead of four. They extend above the line, so the answer is still too high. The new heights are 1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4. The total area in Figure 5.4b is the sum of the base $\Delta x = 1/2$ times those heights:

$$\text{area} = \tfrac{1}{2}(\tfrac{1}{2} + 1 + \tfrac{3}{2} + 2 + \cdots + 4) = 9 \text{ (which is closer to 8).}$$

Question What is the area of 16 rectangles? Their heights are $\frac{1}{4}, \frac{1}{2}, \ldots, 4$.
Answer With base $\Delta x = \frac{1}{4}$ the area is $\frac{1}{4}(\frac{1}{4} + \frac{1}{2} + \cdots + 4) = 8\frac{1}{2}$.

The effort of doing the addition is increasing. A formula for the sums is needed, and will be established soon. (The next answer would be $8\frac{1}{4}$.) But more important than the formula is the idea. ***We are carrying out a limiting process, one step at a time.*** The area of the rectangles is approaching the area of the triangle, as Δx decreases. The same limiting process will apply to other areas, in which the region is much more complicated. Therefore we pause to comment on what is important.

Area Under a Curve

What requirements are imposed on those thinner and thinner rectangles? It is not essential that they all have the same width. And it is not required that they cover the triangle completely. The rectangles could lie *below* the curve. The limiting answer will still be 8, even if the widths Δx are unequal and the rectangles fit inside the triangle or across it. We only impose two rules:

1. The largest width Δx_{max} must approach zero.
2. The top of each rectangle must touch or cross the curve.

The area under the graph is defined to be the limit of these rectangular areas, if that limit exists. For the straight line, the limit does exist and equals 8. That limit is independent of the particular widths and heights—as we absolutely insist it should be.

Section 5.5 allows any continuous $v(x)$. The question will be the same—***Does the limit exist***? The answer will be the same—***Yes***. That limit will be the *integral of* $v(x)$, and it will be the area under the curve. It will be $f(x)$.

EXAMPLE 2 The triangular area from 0 to x is $\frac{1}{2}$(base)(height) $= \frac{1}{2}(x)(x)$. That is $f(x) = \frac{1}{2}x^2$. Its derivative is $v(x) = x$. But notice that $\frac{1}{2}x^2 + 1$ has the *same derivative*. So does $f = \frac{1}{2}x^2 + C$, for any constant C. There is a "***constant of integration***" in $f(x)$, which is wiped out in its derivative $v(x)$.

EXAMPLE 3 Suppose the velocity is decreasing: $v(x) = 4 - x$. If we sample v at $x = 1, 2, 3, 4$, the rectangles lie *under* the graph. Because v is decreasing, the right end of each interval gives $v_{\min}$. Then the rectangular area $3 + 2 + 1 + 0 = 6$ is less than the exact area 8. The rectangles are *inside* the triangle, and eight rectangles with base $\frac{1}{2}$ come closer:

$$\text{rectangular area} = \tfrac{1}{2}(3\tfrac{1}{2} + 3 + \cdots + \tfrac{1}{2} + 0) = 7.$$

Sixteen rectangles would have area $7\frac{1}{2}$. We repeat that the rectangles need not have the same widths Δx, but it makes these calculations easier.

What is the area out to an arbitrary point (like $x = 3$ or $x = 1$)? We could insert rectangles, but the Fundamental Theorem offers a faster way. Any antiderivative of $4 - x$ will give the area. ***We look for a function whose derivative is*** $4 - x$. The derivative of $4x$ is 4, the derivative of $\frac{1}{2}x^2$ is x, so work backward:

$$\text{to achieve } df/dx = 4 - x \text{ choose } f(x) = 4x - \tfrac{1}{2}x^2.$$

Calculus skips past the rectangles and computes $f(3) = 7\frac{1}{2}$. ***The area between*** $x = 1$ ***and*** $x = 3$ ***is the difference*** $7\frac{1}{2} - 3\frac{1}{2} = 4$. In Figure 5.5, this is the area of the trapezoid.

The f-curve flattens out when the v-curve touches zero. No new area is being added.

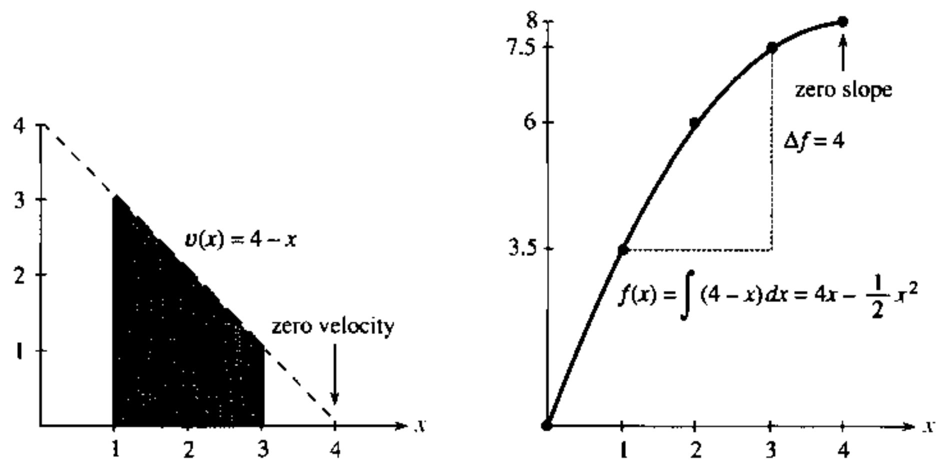


Fig. 5.5 The area is $\Delta f = 7\frac{1}{2} - 3\frac{1}{2} = 4$. Since $v(x)$ decreases, $f(x)$ bends down.

INDEFINITE INTEGRALS AND DEFINITE INTEGRALS

We have to distinguish two different kinds of integrals. They both use the antiderivative $f(x)$. The definite one involves the limits 0 and 4, the indefinite one doesn't:

The ***indefinite integral*** is a ***function*** $f(x) = 4x - \frac{1}{2}x^2$.

The ***definite integral*** from $x = 0$ to $x = 4$ is the ***number*** $f(4) - f(0)$.

The definite integral is definitely 8. But the indefinite integral is not necessarily $4x - \frac{1}{2}x^2$. ***We can change*** $f(x)$ ***by a constant without changing its derivative*** (since the derivative of a constant is zero). The following functions are also antiderivatives:

$$f(x)=4x-\tfrac{1}{2}x^2+1,\qquad f(x)=4x-\tfrac{1}{2}x^2-9,\qquad f(x)=4x-\tfrac{1}{2}x^2+C.$$

The first two are particular examples. The last is the general case. The constant C can be anything (including zero), to give all functions with the required derivative. The theory of calculus will show that there are no others. The indefinite integral is the most general antiderivative (with no limits):

$$\textbf{\textit{indefinite integral}}\ f(x)=\int v(x)\,dx=4x-\tfrac{1}{2}x^2+C. \tag{5}$$

By contrast, the definite integral is a number. It contains no arbitrary constant C. More that that, it contains no variable x. The definite integral is determined by the function $v(x)$ and the limits of integration (also known as the ***endpoints***). It is the area under the graph between those endpoints.

To see the relation of indefinite to definite, answer this question: *What is the definite integral between* $x=1$ *and* $x=3$? The indefinite integral gives $f(3)=7\frac{1}{2}+C$ and $f(1)=3\frac{1}{2}+C$. To find the area between the limits, ***subtract f at one limit from f at the other limit***:

$$\int_{x=1}^{3} v(x)\,dx=f(3)-f(1)=(7\tfrac{1}{2}+C)-(3\tfrac{1}{2}+C)=4. \tag{6}$$

The constant cancels itself! The definite integral is the *difference* between the values of the indefinite integral. C disappears in the subtraction.

The difference $f(3)-f(1)$ is like f_n-f_0. The sum of v_j from 1 to n has become "***the integral of*** $v(x)$ ***from*** 1 ***to*** 3." Section 5.3 computes other areas from sums, and 5.4 computes many more from antiderivatives. Then we come back to the definite integral and the Fundamental Theorem:

$$\int_a^b v(x)\,dx=\int_a^b \frac{df}{dx}\,dx=f(b)-f(a). \tag{7}$$

5.2 EXERCISES

Read-through questions

Integration yields the ___a___ under a curve $y=v(x)$. It starts from rectangles with base ___b___ and heights $v(x)$ and areas ___c___. As $\Delta x\to 0$ the area $v_1\Delta x+\cdots+v_n\Delta x$ becomes the ___d___ of $v(x)$. The symbol for the indefinite integral of $v(x)$ is ___e___.

The problem of integration is solved if we find $f(x)$ such that ___f___. Then f is the ___g___ of v, and $\int_2^6 v(x)\,dx$ equals ___h___ minus ___i___. The limits of integration are ___j___. This is a ___k___ integral, which is a ___l___ and not a function $f(x)$.

The example $v(x)=x$ has $f(x)=$ ___m___. It also has $f(x)=$ ___n___. The area under $v(x)$ from 2 to 6 is ___o___. The constant is canceled in computing the difference ___p___ minus ___q___. If $v(x)=x^8$ then $f(x)=$ ___r___.

The sum $v_1+\cdots+v_n=f_n-f_0$ leads to the Fundamental Theorem $\int_a^b v(x)\,dx=$ ___s___. The ___t___ integral is $f(x)$ and the ___u___ integral is $f(b)-f(a)$. Finding the ___v___ under the v-graph is the opposite of finding the ___w___ of the f-graph.

Find an antiderivative $f(x)$ for $v(x)$ in 1–14. Then compute the definite integral $\int_0^1 v(x)\,dx=f(1)-f(0)$.

1 $5x^4+4x^5$

2 $x+12x^2$

3 $1/\sqrt{x}$ (or $x^{-1/2}$)

4 $(\sqrt{x})^3$ (or $x^{3/2}$)

5 $x^{1/3}+(2x)^{1/3}$

6 $x^{1/3}/x^{2/3}$

7 $2\sin x+\sin 2x$

8 $\sec^2 x+1$

9 $x\cos x$ (by experiment)

10 $x\sin x$ (by experiment)

11 $\sin x\cos x$

12 $\sin^2 x\cos x$

13 0 (find all f)

14 -1 (find all f)

15 If $df/dx=v(x)$ then the definite integral of $v(x)$ from a to b is ________. If $f_j-f_{j-1}=v_j$ then the definite sum of $v_3+\cdots+v_7$ is ________.

16 The areas include a factor Δx, the base of each rectangle. So the sum of v's is multiplied by ________ to approach the integral. The difference of f's is divided by ________ to approach the derivative.

17 The areas of 4, 8, and 16 rectangles were 10, 9, and $8\frac{1}{2}$, containing the triangle out to $x = 4$. Find a formula for the area A_N of N rectangles and test it for $N = 3$ and $N = 6$.

18 Draw four rectangles with base 1 *below* the $y = x$ line, and find the total area. What is the area witb N rectangles?

19 Draw $y = \sin x$ from 0 to π. Three rectangles (base $\pi/3$) and six rectangles (base $\pi/6$) contain an arch of the sine function. Find the areas and guess the limit.

20 Draw an example where three lower rectangles under a curve (heights m_1, m_2, m_3) have less area than two rectangles.

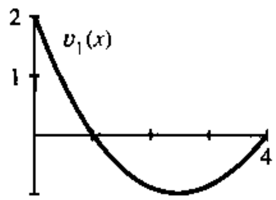
21 Draw $y = 1/x^2$ for $0 < x < 1$ with two rectangles under it (base 1/2). What is their area, and what is the area for four rectangles? Guess the limit.

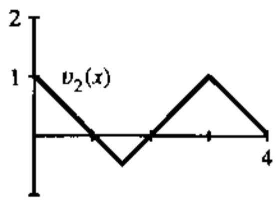
22 Repeat Problem 21 for $y = 1/x$.

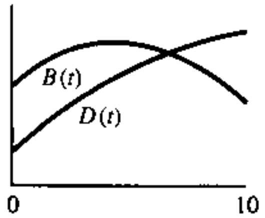
23 (with calculator) For $v(x) = 1/\sqrt{x}$ take enough rectangles over $0 \leqslant x \leqslant 1$ to convince any reasonable professor that the area is 2. Find $f(x)$ and verify that $f(1) - f(0) = 2$.

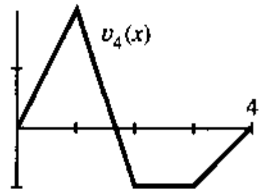
24 Find the area under the parabola $v = x^2$ from $x = 0$ to $x = 4$. Relate it to the area 16/3 below $\sqrt{x}$.

25 For v_1 and v_2 in the figure estimate the areas $f(2)$ and $f(4)$. Start witb $f(0) = 0$.









26 Draw $y = v(x)$ so that the area $f(x)$ increases until $x = 1$, stays constant to $x = 2$, and decreases to $f(3) = 1$.

27 Describe the indefinite integrals of v_1 and v_2. Do the areas increase? Increase then decrease? ...

28 For $v_4(x)$ find the area $f(4) - f(1)$. Draw $f_4(x)$.

29 The graph of $B(t)$ shows the birth rate: births per unit time at time t. $D(t)$ is the death rate. In what way do these numbers appear on the graph?

1. The change in population from $t = 0$ to $t = 10$.
2. The time T when the population was largest.
3. The time t^* when the population increased fastest.

30 Draw the graph of a function $y_4(x)$ whose area function is $v_4(x)$.

31 If $v_2(x)$ is an antiderivative of $y_2(x)$, draw $y_2(x)$.

32 Suppose $v(x)$ increases from $v(0) = 0$ to $v(3) = 4$. The area under $y = v(x)$ plus the area on the left side of $x = v^{-1}(y)$ equals ________.

33 ***True or false***, when $f(x)$ is an antiderivative of $v(x)$.

(a) $2f(x)$ is an antiderivative of $2v(x)$ (*try examples*)
(b) $f(2x)$ is an antiderivative of $v(2x)$
(c) $f(x) + 1$ is an antiderivative of $v(x) + 1$
(d) $f(x + 1)$ is an antiderivative of $v(x + 1)$.
(e) $(f(x))^2$ is an antiderivative of $(v(x))^2$.

5.3 Summation versus Integration

This section does integration the hard way. We find explicit formulas for $f_n = v_1 + \cdots + v_n$. From areas of rectangles, the limits produce the area $f(x)$ under a curve. According to the Fundamental Theorem, df/dx should return us to $v(x)$—and we verify in each case that it does.

May I recall that there is sometimes an easier way? If we can find an $f(x)$ whose derivative is $v(x)$, then the integral of v is f. Sums and limits are not required, when f is spotted directly. The next section, which explains how to look for $f(x)$, will displace this one. (If we can't find an antiderivative we fall back on summation.) Given a successful f, adding any constant produces another f—since tbe derivative of tbe constant is zero. The right constant achieves $f(0) = 0$, with no extra effort.

This section constructs $f(x)$ from sums. The next section searches for antiderivatives.

THE SIGMA NOTATION

In a section about sums, there has to be a decent way to express them. Consider $1^2 + 2^2 + 3^2 + 4^2$. The individual terms are $v_j = j^2$. Their sum can be written in ***summation notation***, using the capital Greek letter Σ (pronounced sigma):

$$1^2 + 2^2 + 3^2 + 4^2 \text{ is written } \sum_{j=1}^{4} j^2.$$

Spoken aloud, that becomes "***the sum of*** j^2 ***from*** $j = 1$ ***to*** 4." It equals 30. The limits on j (written below and above Σ) indicate where to start and stop:

$$v_1 + \cdots + v_n = \sum_{j=1}^{n} v_j \quad \text{and} \quad v_3 + \cdots + v_9 = \sum_{k=3}^{9} v_k. \tag{1}$$

The k at the end of (1) makes an additional point. There is nothing special about the letter j. That is a "*dummy variable*," no better and no worse than k (or i). Dummy variables are only on one side (the side with Σ), and they have no effect on the sum. *The upper limit n is on both sides.* Here are six sums:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n \qquad \sum_{j=1}^{4} (-1)^j = -1 + 1 - 1 + 1 = 0$$

$$\sum_{j=1}^{5} (2j-1) = 1 + 3 + 5 + 7 + 9 = 5^2 \qquad \sum_{i=0}^{0} v_i = v_0 \left[\text{only one term}\right]$$

$$\sum_{i=1}^{4} j^2 = \left[\text{meaningless?}\right] \qquad \sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2 \left[\text{infinite series}\right]$$

The numbers 1 and n or 1 and 4 (or 0 and ∞) are the ***lower limit*** and ***upper limit***. The dummy variable i or j or k is the *index* of summation. I hope it seems reasonable that the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ adds to 2. We will come back to it in Chapter 10.†

A sum like $\Sigma_{j=1}^{n} 6$ looks meaningless, but it is actually $6 + 6 + \cdots + 6 = 6n$. It follows the rules. In fact $\Sigma_{i=1}^{4} j^2$ is not meaningless either. Every term is j^2 and by the same rules, that sum is $4j^2$. However the i was probably intended to be j. Then the sum is $1 + 4 + 9 + 16 = 30$.

Question What happens to these sums when the upper limits are changed to n?
Answer *The sum depends on the stopping point n.* A formula is required (when possible). Integrals stop at x, sums stop at n, and we now look for special cases when $f(x)$ or f_n can be found.

A SPECIAL SUMMATION FORMULA

How do you add the first 100 whole numbers? The problem is to compute

$$\sum_{j=1}^{100} j = 1 + 2 + 3 + \cdots + 98 + 99 + 100 = ?$$

---

†Zeno the Greek believed it was impossible to get anywhere, since he would only go halfway and then half again and half again. Infinite series would have changed his whole life.

If you were Gauss, you would see the answer at once. (He solved this problem at a ridiculous age, which gave his friends the idea of getting him into another class.) His solution was to combine $1 + 100$, and $2 + 99$, and $3 + 98$, *always adding to* 101. There are fifty of those combinations. Thus the sum is $(50)(101) = 5050$.

The sum from 1 to n uses the same idea. The first and last terms add to $n + 1$. The next terms $n - 1$ and 2 also add to $n + 1$. If n is even (as 100 was) then there are $\frac{1}{2}n$ parts. Therefore the sum is $\frac{1}{2}n$ times $n + 1$:

$$\sum_{j=1}^{n} j = 1 + 2 + \cdots + (n - 1) + n = \frac{1}{2}n(n + 1). \tag{2}$$

The important term is $\frac{1}{2}n^2$, but the exact sum is $\frac{1}{2}n^2 + \frac{1}{2}n$.

What happens if n is an odd number (like $n = 99$)? Formula (2) remains true. The combinations $1 + 99$ and $2 + 98$ still add to $n + 1 = 100$. There are $\frac{1}{2}(99) = 49\frac{1}{2}$ such pairs, because the middle term (which is 50) has nothing to combine with. Thus $1 + 2 + \cdots + 99$ equals $49\frac{1}{2}$ times 100, or 4950.

Remark That sum had to be 4950, because it is 5050 minus 100. The sum up to 99 equals the sum up to 100 with the last term removed. Our key formula $f_n - f_{n-1} = v_n$ has turned up again!

EXAMPLE Find the sum $101 + 102 + \cdots + 200$ of the *second* hundred numbers.

First solution This is the sum from 1 to 200 minus the sum from 1 to 100:

$$\sum_{101}^{200} j = \sum_{1}^{200} j - \sum_{1}^{100} j. \tag{3}$$

The middle sum is $\frac{1}{2}(200)(201)$ and the last is $\frac{1}{2}(100)(101)$. Their difference is 15050. *Note*! I left out "$j =$" in the limits. It is there, but not written.

Second solution The answer 15050 is exactly the sum of the first hundred numbers (which was 5050) plus an additional 10000. Believing that a number like 10000 can never turn up by accident, we look for a reason. It is found through ***changing the limits of summation***:

$$\sum_{j=101}^{200} j \text{ is the same sum as } \sum_{k=1}^{100} (k + 100). \tag{4}$$

This is important, to be able to shift limits around. Often the lower limit is moved to zero or one, for convenience. Both sums have 100 terms (that doesn't change). The dummy variable j is replaced by another dummy variable k. They are related by $j = k + 100$ or equivalently by $k = j - 100$.

The variable must change everywhere—in the lower limit and the upper limit as well as inside the sum. If j starts at 101, then $k = j - 100$ starts at 1. If j ends at 200, k ends at 100. If j appears in the sum, it is replaced by $k + 100$ (and if j^2 appeared it would become $(k + 100)^2$).

From equation (4) you see why the answer is 15050. The sum $1 + 2 + \cdots + 100$ is 5050 as before. 100 ***is added to each of those*** 100 ***terms***. That gives 10000.

EXAMPLES OF CHANGING THE VARIABLE (and the limits)

$\sum_{i=0}^{3} 2^i$ equals $\sum_{j=1}^{4} 2^{j-1}$ (here $i = j - 1$). Both sums are $1 + 2 + 4 + 8$

$\sum_{i=3}^{n} v_i$ equals $\sum_{j=0}^{n-3} v_{j+3}$ (here $i = j + 3$ and $j = i - 3$). Both sums are $v_3 + \cdots + v_n$.

Why change n to $n-3$? Because the upper limit is $i=n$. So $j+3=n$ and $j=n-3$.

A final step is possible, and you will often see it. ***The new variable j can be changed back to i.*** Dummy variables have no meaning of their own, but at first the result looks surprising:

$$\sum_{i=0}^{5} 2^i \text{ equals } \sum_{j=1}^{6} 2^{j-1} \text{ equals } \sum_{i=1}^{6} 2^{i-1}.$$

With practice you might do that in one step, skipping the temporary letter j. Every i on the left becomes $i-1$ on the right. Then $i=0, \ldots, 5$ changes to $i=1, \ldots, 6$. (At first two steps are safer.) This may seem a minor point, but soon we will be changing the limits on *integrals* instead of sums. Integration is parallel to summation, and it is better to see a "change of variable" here first.

Note about $1+2+\cdots+n$. The good thing is that Gauss found the sum $\frac{1}{2}n(n+1)$. The bad thing is that his method looked too much like a trick. I would like to show how this fits the fundamental rule connecting sums and differences:

$$\textbf{\textit{if}}\ v_1+v_2+\cdots+v_n=f_n\ \textbf{\textit{then}}\ v_n=f_n-f_{n-1}. \tag{5}$$

Gauss says that f_n is $\frac{1}{2}n(n+1)$. Reducing n by 1, his formula for f_{n-1} is $\frac{1}{2}(n-1)n$. ***The difference*** f_n-f_{n-1} ***should be the last term n in the sum***:

$$f_n-f_{n-1}=\tfrac{1}{2}n(n+1)-\tfrac{1}{2}(n-1)n=\tfrac{1}{2}(n^2+n-n^2+n)=n. \tag{6}$$

This is the one term $v_n=n$ that is included in f_n but not in f_{n-1}.

There is a deeper point here. For any sum f_n, there are two things to check. The f's must *begin* correctly and they must *change* correctly. The underlying idea is ***mathematical induction***: ***Assume the statement is true below n. Prove it for n.***

Goal: ***To prove that*** $1+2+\cdots+n=\frac{1}{2}n(n+1)$. ***This is the guess*** f_n.

Proof by induction: ***Check*** f_1 (*it equals* 1). ***Check*** f_n-f_{n-1} (*it equals n*).

For $n=1$ the answer $\frac{1}{2}n(n+1)=\frac{1}{2}\cdot 1\cdot 2$ is correct. For $n=2$ this formula $\frac{1}{2}\cdot 2\cdot 3$ agrees with $1+2$. But that separate test is not necessary! ***If*** f_1 ***is right, and if the change*** f_n-f_{n-1} ***is right for every n, then*** f_n ***must be right.*** Equation (6) was the key test, to show that the change in f's agrees with v.

That is the logic behind mathematical induction, but I am not happy with most of the exercises that use it. There is absolutely no excitement. The answer is given by some higher power (like Gauss), and it is proved correct by some lower power (like us). It is much better when we lower powers find the answer for ourselves.† Therefore I will try to do that for the second problem, which is the ***sum of squares***.

THE SUM OF j^2 AND THE INTEGRAL OF x^2

An important calculation comes next. It is the area in Figure 5.6. One region is made up of rectangles, so its area is a sum of n pieces. The other region lies under the parabola $v=x^2$. It cannot be divided into rectangles, and calculus is needed.

The first problem is to find $f_n=1^2+2^2+3^2+\cdots+n^2$. This is a sum of squares, with $f_1=1$ and $f_2=5$ and $f_3=14$. The goal is to find the pattern in that sequence. By trying to guess f_n we are copying what will soon be done for integrals.

Calculus looks for an $f(x)$ whose derivative is $v(x)$. There f is an *antiderivative* (or

†The goal of real teaching is for the *student* to find the answer. And also the problem.

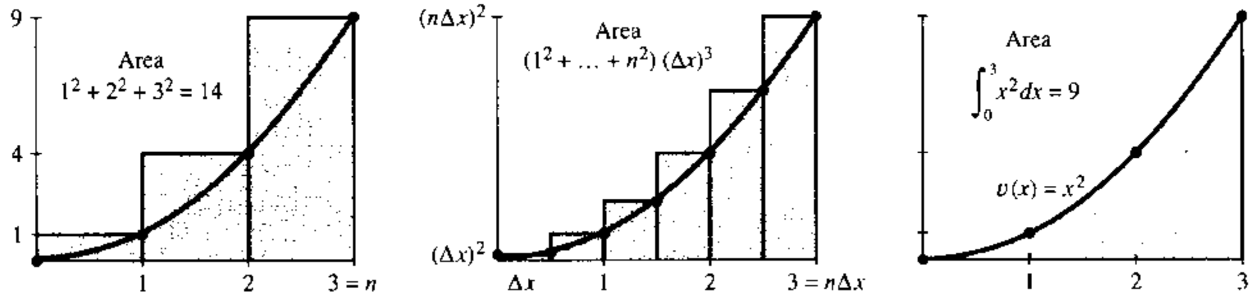


Fig. 5.6 Rectangles enclosing $v=x^2$ have area $(\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n)(\Delta x)^3 \approx \frac{1}{3}(n\Delta x)^3 = \frac{1}{3}x^3$.

an integral). Algebra looks for f_n's whose differences produce v_n. Here f_n could be called an *antidifference* (better to call it a sum).

The best start is a good guess. Copying directly from integrals, we might try $f_n = \frac{1}{3}n^3$. To test if it is right, check whether $f_n - f_{n-1}$ produces $v_n = n^2$:

$$\tfrac{1}{3}n^3 - \tfrac{1}{3}(n-1)^3 = \tfrac{1}{3}n^3 - \tfrac{1}{3}(n^3 - 3n^2 + 3n - 1) = n^2 - n + \tfrac{1}{3}.$$

We see n^2, but also $-n + \frac{1}{3}$. The guess $\frac{1}{3}n^3$ needs ***correction terms.*** To cancel $\frac{1}{3}$ in the difference, I subtract $\frac{1}{3}n$ from the sum. To put back n in the difference, I add $1+2+\cdots+n = \frac{1}{2}n(n+1)$ to the sum. The new guess (which should be right) is

$$f_n = \tfrac{1}{3}n^3 + \tfrac{1}{2}n(n+1) - \tfrac{1}{3}n = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n. \tag{7}$$

To check this answer, verify first that $f_1 = 1$. Also $f_2 = 5$ and $f_3 = 14$. To be certain, verify that $f_n - f_{n-1} = n^2$. For calculus the important term is $\frac{1}{3}n^3$:

$$\textbf{\textit{The sum}} \sum_{j=1}^{n} j^2 \textbf{\textit{ of the first n squares is }} \frac{1}{3}n^3 \textbf{\textit{ plus corrections }} \frac{1}{2}n^2 \textbf{\textit{ and }} \frac{1}{6}n.$$

In practice $\frac{1}{3}n^3$ is an excellent estimate. The sum of the first 100 squares is approximately $\frac{1}{3}(100)^3$, or a third of a million. If we need the exact answer, equation (7) is available: the sum is 338,350. Many applications (example: the number of steps to solve 100 linear equations) can settle for $\frac{1}{3}n^3$.

What is fascinating is the contrast with calculus. ***Calculus has no correction terms!*** They get washed away in the limit of thin rectangles. When the sum is replaced by the integral (the area), we get an absolutely clean answer:

$$\textbf{\textit{The integral of }} v = x^2 \textbf{\textit{ from }} x=0 \textbf{\textit{ to }} x=n \textbf{\textit{ is exactly }} \tfrac{1}{3}n^3.$$

The area under the parabola, out to the point $x = 100$, is precisely a third of a million. We have to explain why, with many rectangles.

The idea is to approach an infinite number of infinitely thin rectangles. A hundred rectangles gave an area of 338,350. Now take a thousand rectangles. Their heights are $(\frac{1}{10})^2, (\frac{2}{10})^2, \ldots$ because the curve is $v = x^2$. The base of every rectangle is $\Delta x = \frac{1}{10}$, and we add heights times base:

$$\text{area of rectangles} = \left(\frac{1}{10}\right)^2\left(\frac{1}{10}\right) + \left(\frac{2}{10}\right)^2\left(\frac{1}{10}\right) + \cdots + \left(\frac{1000}{10}\right)^2\left(\frac{1}{10}\right).$$

Factor out $(\frac{1}{10})^3$. What you have left is $1^2 + 2^2 + \cdots + 1000^2$, which fits the sum of squares formula. The exact area of the thousand rectangles is 333,833.5. I could try to guess ten thousand rectangles but I won't.

Main point: The area is approaching 333,333.333.... But the calculations are getting worse. It is time for algebra—which means that we keep "Δx" and avoid numbers.

The interval of length 100 is divided into n pieces of length Δx. (Thus $n = 100/\Delta x$.) The jth rectangle meets the curve $v = x^2$, so its height is $(j\Delta x)^2$. Its base is Δx, and we add areas:

$$\textbf{area} = (\Delta x)^2(\Delta x) + (2\Delta x)^2(\Delta x) + \cdots + (n\Delta x)^2(\Delta x) = \sum_{j=1}^{n} (j\Delta x)^2(\Delta x). \tag{8}$$

Factor out $(\Delta x)^3$, leaving a sum of n squares. The area is $(\Delta x)^3$ times f_n, and $n = \dfrac{100}{\Delta x}$:

$$(\Delta x)^3\left[\frac{1}{3}\left(\frac{100}{\Delta x}\right)^3 + \frac{1}{2}\left(\frac{100}{\Delta x}\right)^2 + \frac{1}{6}\left(\frac{100}{\Delta x}\right)\right] = \frac{1}{3}100^3 + \frac{1}{2}100^2(\Delta x) + \frac{1}{6}100(\Delta x)^2. \tag{9}$$

This equation shows what is happening. The leading term is a third of a million, as predicted. The other terms are approaching zero! They contain Δx, and as the rectangles get thinner they disappear. They only account for the small corners of rectangles that lie above the curve. The vanishing of those corners will eventually be proved for any continuous functions—*the area from the correction terms goes to zero*—but here in equation (9) you see it explicitly.

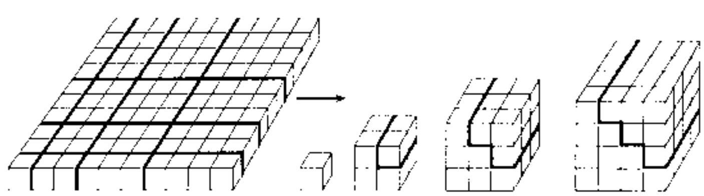
The area under the curve came from the central idea of integration: $100/\Delta x$ rectangles of width Δx approach the limiting area $= \frac{1}{3}(100)^3$. ***The rectangular area is*** $\Sigma\, v_j\, \Delta x$. ***The exact area is*** $\int v(x)\,dx$. ***In the limit*** Σ ***becomes*** $\int$ ***and*** v_j ***becomes*** $v(x)$ ***and*** Δx ***becomes*** dx.

That completes the calculation for a parabola. It used the formula for a sum of squares, which was special. But the underlying idea is much more general. The limit of the sums agrees with the antiderivative: ***The antiderivative of*** $v(x) = x^2$ ***is*** $f(x) = \frac{1}{3}x^3$. According to the Fundamental Theorem, the area under $v(x)$ is $f(x)$:

$$\int_0^{100} v(x)\,dx = f(100) - f(0) = \tfrac{1}{3}(100)^3.$$

That Fundamental Theorem is not yet proved! I mean it is not proved by us. Whether Leibniz or Newton managed to prove it, I am not quite sure. But it can be done. Starting from sums of differences, the difficulty is that we have too many limits at once. The sums of $v_j\Delta x$ are approaching the integral. The differences $\Delta f/\Delta x$ approach the derivative. A real proof has to separate those steps, and Section 5.7 will do it.

Proved or not, you are seeing the main point. What was true for the numbers f_j and v_j is true in the limit for $v(x)$ and $f(x)$. Now $v(x)$ can vary continuously, but it is still the slope of $f(x)$. ***The reverse of slope is area.***



$$\mathbf{(1+2+3+4)^2 = 1^3 + 2^3 + 3^3 + 4^3}$$

Proof without words by Roger Nelsen (*Mathematics Magazine* 1990).

Finally we review the area under $v = x$. The sum of $1 + 2 + \cdots + n$ is $\frac{1}{2}n^2 + \frac{1}{2}n$. This gives the area of $n = 4/\Delta x$ rectangles, going out to $x = 4$. The heights are $j\Delta x$, the bases are Δx, and we add areas:

$$\sum_{j=1}^{4/\Delta x} (j\Delta x)(\Delta x) = (\Delta x)^2\left[\frac{1}{2}\left(\frac{4}{\Delta x}\right)^2 + \frac{1}{2}\left(\frac{4}{\Delta x}\right)\right] = 8 + 2\Delta x. \tag{10}$$

With $\Delta x = 1$ the area is $1 + 2 + 3 + 4 = 10$. With eight rectangles and $\Delta x = \frac{1}{2}$, the area was $8 + 2\Delta x = 9$. Sixteen rectangles of width $\frac{1}{4}$ brought the correction $2\Delta x$ down to $\frac{1}{2}$. The exact area is 8. ***The error is proportional to*** Δx.

Important note There you see a question in applied mathematics. If there is an error, what size is it? How does it behave as $\Delta x \to 0$? The Δx term disappears in the limit, and $(\Delta x)^2$ disappears faster. But to get an error of 10^{-6} we need ***eight million rectangles***:

$$2\Delta x = 2 \cdot 4/8{,}000{,}000 = 10^{-6}.$$

That is horrifying! The numbers $10, 9, 8\frac{1}{2}, 8\frac{1}{4}, \ldots$ seem to approach the area 8 in a satisfactory way, but the convergence is ***much too slow***. It takes twice as much work to get one more binary digit in the answer—which is absolutely unacceptable. Somehow the Δx term must be removed. If the correction is $(\Delta x)^2$ instead of Δx, then a thousand rectangles will reach an accuracy of 10^{-6}.

The problem is that the rectangles are unbalanced. Their right sides touch the graph of v, but their left sides are much too high. The best is to cross the graph in the ***middle*** of the interval—this is the ***midpoint rule***. Then the rectangle sits halfway across the line $v = x$, and the error is zero. Section 5.8 comes back to this rule—and to Simpson's rule that fits parabolas and removes the $(\Delta x)^2$ term and is built into many calculators.

Finally we try the quick way. The area under $v = x$ is $f = \frac{1}{2}x^2$, because df/dx is v. The area out to $x = 4$ is $\frac{1}{2}(4)^2 = 8$. Done.

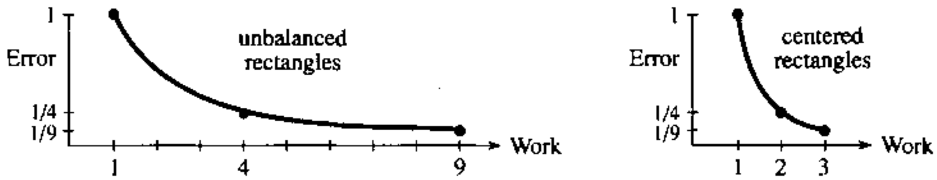


Fig. 5.7 Endpoint rules: error $\sim 1/(\text{work}) \sim 1/n$. Midpoint rule is better: error $\sim 1/(\text{work})^2$.

Optional: pth ***powers*** Our sums are following a pattern. First, $1 + \cdots + n$ is $\frac{1}{2}n^2$ plus $\frac{1}{2}n$. The sum of squares is $\frac{1}{3}n^3$ plus correction terms. *The sum of* pth *powers is*

$$1^p + 2^p + \cdots + n^p = \frac{1}{p+1} n^{p+1} \textit{ plus correction terms.} \tag{11}$$

The correction involves lower powers of n, and you know what is coming. ***Those corrections disappear in calculus***. The area under $v = x^p$ from 0 to n is

$$\int_{x=0}^{n} x^p \, dx = \lim_{\Delta x \to 0} \sum_{j=1}^{n/\Delta x} (j\Delta x)^p(\Delta x) = \frac{1}{p+1} n^{p+1}. \tag{12}$$

Calculus doesn't care if the upper limit n is an integer, and it doesn't care if the power p is an integer. We only need $p + 1 > 0$ to be sure n^{p+1} is genuinely the leading term. ***The antiderivative of*** $v = x^p$ ***is*** $f = x^{p+1}/(p+1)$.

We are close to interesting experiments. The correction terms disappear and the sum approaches the integral. Here are actual numbers for $p = 1$, when the sum and integral are easy: $S_n = 1 + \cdots + n$ and $I_n = \int x \, dx = \frac{1}{2}n^2$. The difference is $D_n = \frac{1}{2}n$. The thing to watch is the ***relative error*** $E_n = D_n/I_n$:

n	S_n	I_n	$D_n = S_n - I_n$	$E_n = D_n/I_n$
100	5050	5000	50	.010
200	20100	20000	100	.005

The number 20100 is $\frac{1}{2}(200)(201)$. Please write down the next line $n = 400$, ***and please find a formula for E_n***. You can guess E_n from the table, or you can derive it from knowing S_n and I_n. The formula should show that E_n goes to zero. More important, it should show how quick (or slow) that convergence will be.

One more number—a third of a million—was mentioned earlier. It came from integrating x^2 from 0 to 100, which compares to the sum S_{100} of 100 squares:

n	p	S_n	$I_n = \frac{1}{3}n^3$	$D = S - I$	$E = D/I$
100	2	338350	$333333\frac{1}{3}$	$5016\frac{2}{3}$	.01505
200	2	2686700	$2666666\frac{2}{3}$	$20033\frac{1}{3}$	.0075125

These numbers suggest a new idea, ***to keep n fixed and change p***. The computer can find sums without a formula! With its help we go to fourth powers and square roots:

n	p	$S = 1^p + \cdots + n^p$	$I = n^{p+1}/(p+1)$	$D = S - I$	$E_{n,p} = D/I$
100	4	2050333330	$\frac{1}{5}(100)^5$	50333330	0.0252
100	$\frac{1}{2}$	671.4629	$\frac{2}{3}(100)^{3/2}$	4.7963	0.0072

In this and future tables we don't expect exact values. The last entries are rounded off, and the goal is to see the pattern. The errors $E_{n,p}$ are sure to obey a systematic rule—they are proportional to $1/n$ and to an unknown number $C(p)$ that depends on p. I hope you can push the experiments far enough to discover $C(p)$. This is not an exercise with an answer in the back of the book—it is mathematics.

5.3 EXERCISES

Read-through questions

The Greek letter __a__ indicates summation. In $\Sigma_1^n v_j$ the dummy variable is __b__. The limits are __c__, so the first term is __d__ and the last term is __e__. When $v_j = j$ this sum equals __f__. For $n = 100$ the leading term is __g__. The correction term is __h__. The leading term equals the integral of $v = x$ from 0 to 100, which is written __i__. The sum is the total __j__ of 100 rectangles. The correction term is the area between the __k__ and the __l__.

The sum $\Sigma_{i=3}^6 i^2$ is the same as $\Sigma_{j=1}^4$ __m__ and equals __n__. The sum $\Sigma_{i=4}^5 v_i$ is the same as __o__ v_{i+4} and equals __p__. For $f_n = \Sigma_{j=1}^n v_j$ the difference $f_n - f_{n-1}$ equals __q__.

The formula for $1^2 + 2^2 + \cdots + n^2$ is $f_n =$ __r__. To prove it by mathematical induction, check $f_1 =$ __s__ and check $f_n - f_{n-1} =$ __t__. The area under the parabola $v = x^2$ from $x = 0$ to $x = 9$ is __u__. This is close to the area of __v__ rectangles of base Δx. The correction terms approach zero very __w__.

1 Compute the numbers $\sum_{n=1}^4 1/n$ and $\sum_{i=2}^5 (2i-3)$.

2 Compute $\sum_{j=0}^3 (j^2 - j)$ and $\sum_{j=1}^n 1/2^j$.

3 Evaluate the sum $\sum_{i=0}^6 2^i$ and $\sum_{i=0}^n 2^i$.

4 Evaluate $\sum_{i=1}^6 (-1)^i i$ and $\sum_{j=1}^n (-1)^j j$.

5 Write these sums in sigma notation and compute them:

$2 + 4 + 6 + \cdots + 100 \quad 1 + 3 + 5 + \cdots + 199 \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

6 Express these sums in sigma notation:

$v_1 - v_2 + v_3 - v_4 \quad v_1 w_1 + v_2 w_2 + \cdots + v_n w_n \quad v_1 + v_3 + v_5$

7 Convert these sums to sigma notation:

$$a_0 + a_1 x + \cdots + a_n x^n \qquad \sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \cdots + \sin 2\pi$$

8 The binomial formula uses coefficients $\binom{n}{j} = \dfrac{n!}{j!(n-j)!}$:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}b^n = \sum_{j=0}^n \underline{\qquad\qquad} b^j.$$

9 With electronic help compute $\sum_1^{100} 1/j$ and $\sum_1^{1000} 1/j$.

10 On a computer find $\sum_0^{10} (-1)^j/j!$ times $\sum_0^{10} 1/j!$

11 Simplify $\sum_{i=1}^{n}(a_i+b_i)^2+\sum_{i=1}^{n}(a_i-b_i)^2$ to $\sum_{i=1}^{n}$ _______.

12 Show that $\left(\sum_{i=1}^{n} a_i\right)^2 \neq \sum_{i=1}^{n} a_i^2$ and $\sum_{i=1}^{n} a_i b_i \neq \sum_{j=1}^{n} a_j \sum_{k=1}^{n} b_k$.

13 "*Telescope*" *the sums* $\sum_{k=1}^{n}(2^k-2^{k-1})$ and $\sum_{j=1}^{10}\left(\frac{1}{j+1}-\frac{1}{j}\right)$. All but two terms cancel.

14 Simplify the sums $\sum_{j=1}^{n}(f_j-f_{j-1})$ and $\sum_{j=3}^{12}(f_{j+1}-f_j)$.

15 ***True or false***: (a) $\sum_{j=4}^{8} v_j=\sum_{i=2}^{6} v_{i-2}$ (b) $\sum_{i=1}^{9} v_i=\sum_{i=3}^{11} v_{i-2}$

16 $\sum_{i=1}^{n} v_i=\sum_{j=0}^{n-1}$ _______ and $\sum_{i=0}^{6} i^2=\sum_{i=2}^{8}$ _______.

17 The antiderivative of d^2f/dx^2 is df/dx. What is the sum $(f_2-2f_1+f_0)+(f_3-2f_2+f_1)+\cdots+(f_9-2f_8+f_7)$?

18 ***Induction***: Verify that $1^2+2^2+\cdots+n^2$ is $f_n=n(n+1)(2n+1)/6$ by checking that f_1 is correct and $f_n-f_{n-1}=n^2$.

19 Prove by induction: $1+3+\cdots+(2n-1)=n^2$.

20 Verify that $1^3+2^3+\cdots+n^3$ is $f_n=\frac{1}{4}n^2(n+1)^2$ by checking f_1 and f_n-f_{n-1}. The text has a *proof without words.*

21 Suppose f_n has the form $an+bn^2+cn^3$. If you know $f_1=1$, $f_2=5$, $f_3=14$, turn those into three equations for a, b, c. The solutions $a=\frac{1}{6}$, $b=\frac{1}{2}$, $c=\frac{1}{3}$ give what formula?

22 Find q in the formula $1^8+\cdots+n^8=qn^9+\text{correction}$.

23 Add $n=400$ to the table for $S_n=1+\cdots+n$ and find the relative error E_n. Guess and prove a formula for E_n.

24 Add $n=50$ to the table for $S_n=1^2+\cdots+n^2$ and compute E_{50}. Find an approximate formula for E_n.

25 Add $p=\frac{1}{2}$ and $p=3$ to the table for $S_{100,p}=1^p+\cdots+100^p$. Guess an approximate formula for $E_{100,p}$.

26 Guess $C(p)$ in the formula $E_{n,p}\approx C(p)/n$.

27 Show that $|1-5|<|1|+|-5|$. Always $|v_1+v_2|<|v_1|+|v_2|$ unless _______.

28 Let S be the sum $1+x+x^2+\cdots$ of the (infinite) geometric series. Then $xS=x+x^2+x^3+\cdots$ is the same as S minus _______. Therefore $S=$ _______. None of this makes sense if $x=2$ because _______.

29 The ***double sum*** $\sum_{i=1}^{2}\left[\sum_{j=1}^{3}(i+j)\right]$ is $v_1=\sum_{j=1}^{3}(1+j)$ plus $v_2=\sum_{j=1}^{3}(2+j)$. Compute v_1 and v_2 and the double sum.

30 The double sum $\sum_{i=1}^{2}\left(\sum_{j=1}^{3} w_{i,j}\right)$ is $(w_{1,1}+w_{1,2}+w_{1,3})+$ _______. The double sum $\sum_{j=1}^{3}\left(\sum_{i=1}^{2} w_{i,j}\right)$ is $(w_{1,1}+w_{2,1})+(w_{1,2}+w_{2,2})+$ _______. Compare.

31 Find the flaw in the proof that $2^n=1$ for every $n=0,1,2,\ldots$. For $n=0$ we have $2^0=1$. If $2^n=1$ for every $n<N$, then $2^N=2^{N-1}\cdot 2^{N-1}/2^{N-2}=1\cdot 1/1=1$.

32 Write out all terms to see why the following are true:

$$\sum_{1}^{3} 4v_j=4\sum_{1}^{3} v_j \qquad \sum_{i=1}^{2}\left(\sum_{j=1}^{3} u_i v_j\right)=\left(\sum_{1}^{2} u_i\right)\left(\sum_{1}^{3} v_j\right)$$

33 The average of 6, 11, 4 is $\bar{v}=\frac{1}{3}(6+11+4)$. Then $(6-\bar{v})+(11-\bar{v})+(4-\bar{v})=$ _______. The average of $v_1,\ldots,v_n$ is $\bar{v}=$ _______. Prove that $\Sigma(v_i-\bar{v})=0$.

34 The *Schwarz inequality* is $\left(\sum_{1}^{n} a_i b_i\right)^2 \leqslant \left(\sum_{1}^{n} a_i^2\right)\left(\sum_{1}^{n} b_i^2\right)$. Compute both sides if $a_1=2$, $a_2=3$, $b_1=1$, $b_2=4$. Then compute both sides for any a_1, a_2, b_1, b_2. The proof in Section 11.1 uses vectors.

35 Suppose n rectangles with base Δx touch the graph of $v(x)$ at the points $x=\Delta x, 2\Delta x, \ldots, n\Delta x$. Express the total rectangular area in sigma notation.

36 If $1/\Delta x$ rectangles with base Δx touch the graph of $v(x)$ at the *left* end of each interval (thus at $x=0, \Delta x, 2\Delta x, \ldots$) express the total area in sigma notation.

37 The sum $\Delta x \sum_{j=1}^{1/\Delta x}\frac{f(j\Delta x)-f((j-1)\Delta x)}{\Delta x}$ equals _______. In the limit this becomes $\int_0^1$ _______ $dx=$ _______.

5.4 Indefinite Integrals and Substitutions

This section integrates the easy way, by looking for antiderivatives. We leave aside sums of rectangular areas, and their limits as $\Delta x \to 0$. Instead we search for an $f(x)$ with the required derivative $v(x)$. In practice, this approach is more or less independent of the approach through sums—but it gives the same answer. And also, *the* *search for an antiderivative may not succeed.* We may not find f. In that case we go back to rectangles, or on to something better in Section 5.8.

A computer is ready to integrate v, but not by discovering f. It integrates between specified limits, to obtain a ***number*** (the definite integral). Here we hope to find a ***function*** (the indefinite integral). That requires a symbolic integration code like MACSYMA or *Mathematica* or MAPLE, or a reasonably nice $v(x)$, or both. An expression for $f(x)$ can have tremendous advantages over a list of numbers.

Thus our goal is to find antiderivatives and use them. The techniques will be further developed in Chapter 7—this section is short but good. First we write down what we know. *On each line, $f(x)$ is an antiderivative of $v(x)$ because $df/dx = v(x)$.*

Known pairs	***Function*** $v(x)$	***Antiderivative*** $f(x)$
Powers of x	x^n	$x^{n+1}/(n+1)+C$

$n=-1$ is not included, because $n+1$ would be zero. $v=x^{-1}$ will lead us to $f=\ln x$.

Trigonometric functions	$\cos x$	$\sin x + C$
	$\sin x$	$-\cos x + C$
	$\sec^2 x$	$\tan x + C$
	$\csc^2 x$	$-\cot x + C$
	$\sec x \tan x$	$\sec x + C$
	$\csc x \cot x$	$-\csc x + C$
Inverse functions	$1/\sqrt{1-x^2}$	$\sin^{-1} x + C$
	$1/(1+x^2)$	$\tan^{-1} x + C$
	$1/\lvert x\rvert\sqrt{x^2-1}$	$\sec^{-1} x + C$

You recognize that each integration formula came directly from a differentiation formula. The integral of the cosine is the sine, because the derivative of the sine is the cosine. For emphasis we list three derivatives above three integrals:

$$\frac{d}{dx}(\text{constant}) = 0 \qquad \frac{d}{dx}(x) = 1 \qquad \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$$

$$\int 0\,dx = C \qquad \int 1\,dx = x + C \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

There are two ways to make this list longer. One is to find the derivative of a new $f(x)$. Then f goes in one column and $v = df/dx$ goes in the other column.† The other possibility is to use rules for derivatives to find rules for integrals. That is the way to extend the list, enormously and easily.

RULES FOR INTEGRALS

Among the rules for derivatives, three were of supreme importance. They were ***linearity***, the ***product rule***, and the ***chain rule***. Everything flowed from those three. In the

†We will soon meet e^x, which goes in *both columns*. It is $f(x)$ and also $v(x)$.

reverse direction (from v to f) this is still true. The three basic methods of differential calculus also dominate integral calculus:

linearity of derivatives → linearity of integrals

product rule for derivatives → integration by parts

chain rule for derivatives → integrals by substitution

The easiest is linearity, which comes first. Integration by parts will be left for Section 7.1. This section starts on substitutions, reversing the chain rule to make an integral simpler.

LINEARITY OF INTEGRALS

What is the integral of $v(x) - w(x)$? Add the two separate integrals. The graph of $v + w$ has two regions below it, the area under v and the area from v to $v + w$. Adding areas gives the sum rule. Suppose f and g are antiderivatives of v and w:

sum rule:	$f + g$	is an antiderivative of	$v + w$
constant rule:	cf	is an antiderivative of	cv
linearity:	$af + bg$	is an antiderivative of	$av + bw$

This is a case of overkill. The first two rules are special cases of the third, so logically the last rule is enough. However it is so important to deal quickly with constants—just "*factor them out*"—that the rule $cv \leftrightarrow cf$ is stated separately. The proofs come from the linearity of derivatives: $(af + bg)'$ *equals* $af' + bg'$ *which equals* $av + bw$. The rules can be restated with integral signs:

sum rule: $$\int \left[v(x) + w(x) \right] dx = \int v(x)\, dx + \int w(x)\, dx$$

constant rule: $$\int cv(x)\, dx = c \int v(x)\, dx$$

linearity: $$\int \left[av(x) - bw(x) \right] dx = a \int v(x)\, dx + b \int w(x)\, dx$$

Note about the constant in $f(x) - C$. All antiderivatives allow the addition of a constant. For a combination like $av(x) + bw(x)$, the antiderivative is $af(x) + bg(x) + C$. *The constants for each part combine into a single constant.* To give all possible antiderivatives of a function, just remember to write "$+ C$" after one of them. The real problem is to find that one antiderivative.

EXAMPLE 1 The antiderivative of $v = x^2 + x^{-2}$ is $f = x^3/3 + (x^{-1})/(-1) + C$.

EXAMPLE 2 The antiderivative of $6 \cos t + 7 \sin t$ is $6 \sin t - 7 \cos t + C$.

EXAMPLE 3 Rewrite $\dfrac{1}{1 + \sin x}$ as $\dfrac{1 - \sin x}{1 - \sin^2 x} = \dfrac{1 - \sin x}{\cos^2 x} = \sec^2 x - \sec x \tan x.$

The antiderivative is $\tan x - \sec x + C$. That rewriting is done by a symbolic algebra code (or by you). Differentiation is often simple, so most people check that $df/dx = v(x)$.

Question How to integrate $\tan^2 x$?
Method Write it as $\sec^2 x - 1$. Answer $\tan x - x + C$.

INTEGRALS BY SUBSTITUTION

We now present the most valuable technique in this section—***substitution***. To see the idea, you have to remember the chain rule:

$$\begin{array}{rll} f(g(x)) & \text{has derivative} & f'(g(x))(dg/dx) \\ \sin x^2 & \text{has derivative} & (\cos x^2)(2x) \\ (x^3+1)^5 & \text{has derivative} & 5(x^3+1)^4(3x^2) \end{array}$$

If the function on the right is given, the function on the left is its antiderivative! There are two points to emphasize right away:

1. ***Constants are no problem—they can always be fixed.*** Divide by 2 or 15:

$$\int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + C \qquad \int x^2(x^3+1)^4\,dx = \frac{1}{15}(x^3+1)^5 + C$$

Notice the 2 from x^2, the 5 from the fifth power, and the 3 from x^3.

2. ***Choosing the inside function*** g (or u) ***commits us to its derivative***:

$$\begin{array}{ll} \text{the integral of } 2x\cos x^2 \text{ is } \sin x^2 + C & (g = x^2,\ dg/dx = 2x) \\ \text{the integral of } \cos x^2 \text{ is } (\textit{failure}) & (\text{no } dg/dx) \\ \text{the integral of } x^2\cos x^2 \text{ is } (\textit{failure}) & (\text{wrong } dg/dx) \end{array}$$

To substitute g for x^2, we need its derivative. The trick is to spot an inside function whose derivative is present. We can fix constants like 2 or 15, but otherwise dg/dx has to be there. *Very often the inside function g is written u.* We use that letter to state the ***substitution rule***, when f is the integral of v:

$$\int v(u(x))\,\frac{du}{dx}\,dx = f(u(x)) + C. \qquad (1)$$

EXAMPLE 4 $\int \sin x\cos x\,dx = \frac{1}{2}(\sin x)^2 + C \qquad u = \sin x$ (compare Example 6)

EXAMPLE 5 $\int \sin^2 x\cos x\,dx = \frac{1}{3}(\sin x)^3 + C \qquad u = \sin x$

EXAMPLE 6 $\int \cos x\sin x\,dx = -\frac{1}{2}(\cos x)^2 + C \qquad u = \cos x$ (compare Example 4)

EXAMPLE 7 $\int \tan^4 x\sec^2 x\,dx = \frac{1}{5}(\tan x)^5 + C \qquad u = \tan x$

The next example has $u = x^2 - 1$ and $du/dx = 2x$. The key step is choosing u:

EXAMPLE 8 $\int x\,dx/\sqrt{x^2-1} = \sqrt{x^2-1} + C \qquad \int x\sqrt{x^2-1}\,dx = \frac{1}{3}(x^2-1)^{3/2} + C$

A ***shift*** of x (to $x+2$) or a ***multiple*** of x (rescaling to $2x$) is particularly easy:

EXAMPLES 9–10 $\int (x+2)^3\,dx = \frac{1}{4}(x+2)^4 + C \qquad \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$

You will soon be able to do those in your sleep. Officially the derivative of $(x+2)^4$ uses the chain rule. But the inside function $u = x + 2$ has $du/dx = 1$. The "1" is there automatically, and the graph shifts over—as in Figure 5.8b.

For Example 10 the inside function is $u = 2x$. Its derivative is $du/dx = 2$. This

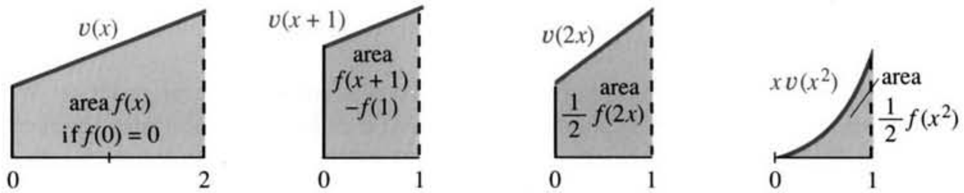


Fig. 5.8 Substituting $u = x + 1$ and $u = 2x$ and $u = x^2$. The last graph has half of $du/dx = 2x$.

required factor 2 is missing in $\int \cos 2x\, dx$, but we put it there by multiplying and dividing by 2. Check the derivative of $\frac{1}{2}\sin 2x$: the 2 from the chain rule cancels the $\frac{1}{2}$. The rule for any nonzero constant is similar:

$$\int v(x+c)\, dx = f(x+c) \qquad \text{and} \qquad \int v(cx)\, dx = \frac{1}{c} f(cx). \tag{2}$$

Squeezing the graph by c divides the area by c. Now $3x + 7$ rescales *and* shifts:

EXAMPLE 11 $\int \cos(3x+7)\, dx = \frac{1}{3}\sin(3x+7) + C \quad \int (3x+7)^2\, dx = \frac{1}{3}\cdot\frac{1}{3}(3x+7)^3 + C$

Remark on writing down the steps When the substitution is complicated, it is a good idea to get du/dx where you need it. Here $3x^2 + 1$ needs $6x$:

$$\int 7x(3x^2+1)^4\, dx = \frac{7}{6}\int (3x^2+1)^4 6x\, dx = \frac{7}{6}\int u^4 \frac{du}{dx}\, dx$$

$$\text{Now integrate:} \qquad \frac{7}{6}\frac{u^5}{5} + C = \frac{7}{6}\frac{(3x^2+1)^5}{5} + C. \tag{3}$$

Check the derivative at the end. The exponent 5 cancels 5 in the denominator, $6x$ from the chain rule cancels 6, and $7x$ is what we started with.

Remark on differentials In place of $(du/dx)\, dx$, many people just write du:

$$\int (3x^2+1)^4 6x\, dx = \int u^4\, du = \tfrac{1}{5}u^5 + C. \tag{4}$$

This really shows how substitution works. *We switch from x to u,* ***and we also switch from*** *dx* ***to*** *du.* The most common mistake is to confuse dx with du. The factor du/dx from the chain rule is absolutely needed, to reach du. The change of variables (dummy variables anyway!) leaves an easy integral, and then u turns back into $3x^2 + 1$. Here are the four steps to substitute u for x:

1. Choose $u(x)$ and compute du/dx
2. Locate $v(u)$ times du/dx times dx, or $v(u)$ times du
3. Integrate $\int v(u)\, du$ to find $f(u) + C$
4. Substitute $u(x)$ back into this antiderivative f.

EXAMPLE 12 $\int (\cos\sqrt{x})\, dx/2\sqrt{x} = \int \cos u\, du = \sin u + C = \sin\sqrt{x} + C$

(***put in*** u) (***integrate***) (***put back*** x)

The choice of u must be right, to change everything from x to u. With ingenuity, some remarkable integrals are possible. But most will remain impossible forever. The functions $\cos x^2$ and $1/\sqrt{4 - \sin^2 x}$ have no "elementary" antiderivative. Those integrals are well defined and they come up in applications—the latter gives the distance around an ellipse. That can be computed to tremendous accuracy, but not to perfect accuracy.

The exercises concentrate on substitutions, which need and deserve practice. We give a ***nonexample***—$\int (x^2+1)^2\,dx$ *does not equal* $\frac{1}{3}(x^2+1)^3$—to emphasize the need for du/dx. Since $2x$ is missing, $u = x^2+1$ does not work. But we can fix up π:

$$\int \sin \pi x\,dx = \int \sin u \frac{du}{\pi} = -\frac{1}{\pi}\cos u + C = -\frac{1}{\pi}\cos \pi x + C.$$

5.4 EXERCISES

Read-through questions

Finding integrals by substitution is the reverse of the __a__ rule. The derivative of $(\sin x)^3$ is __b__. Therefore the antiderivative of __c__ is __d__. To compute $\int (1+\sin x)^2 \cos x\,dx$, substitute $u =$ __e__. Then $du/dx =$ __f__ so substitute $du =$ __g__. In terms of u the integral is $\int$ __h__ = __i__. Returning to x gives the final answer.

The best substitutions for $\int \tan(x+3)\sec^2(x+3)dx$ and $\int (x^2+1)^{10}x\,dx$ are $u =$ __j__ and $u =$ __k__. Then $du =$ __l__ and __m__. The answers are __n__ and __o__. The antiderivative of $v\,dv/dx$ is __p__. $\int 2x\,dx/(1+x^2)$ leads to $\int$ __q__, which we don't yet know. The integral $\int dx/(1+x^2)$ is known immediately as __r__.

Find the indefinite integrals in 1–20.

1 $\int \sqrt{2+x}\,dx$ (add $+C$)
2 $\int \sqrt{3-x}\,dx$ (always $+C$)
3 $\int (x+1)^n dx$
4 $\int (x+1)^{-n} dx$
5 $\int (x^2+1)^5 x\,dx$
6 $\int \sqrt{1-3x}\,dx$
7 $\int \cos^3 x \sin x\,dx$
8 $\int \cos x\,dx/\sin^3 x$
9 $\int \cos^3 2x \sin 2x\,dx$
10 $\int \cos^3 x \sin 2x\,dx$
11 $\int dt/\sqrt{1-t^2}$
12 $\int t\sqrt{1-t^2}\,dt$
13 $\int t^3\,dt/\sqrt{1+t^2}$
14 $\int t^3\sqrt{1-t^2}\,dt$
15 $\int (1+\sqrt{x})\,dx/\sqrt{x}$
16 $\int (1+x^{3/2})\sqrt{x}\,dx$
17 $\int \sec x \tan x\,dx$
18 $\int \sec^2 x \tan^2 x\,dx$
19 $\int \cos x \tan x\,dx$
20 $\int \sin^3 x\,dx$

In 21–32 find a function $y(x)$ that solves the differential equation.

21 $dy/dx = x^2 + \sqrt{x}$
22 $dy/dx = y^2$ (try $y = cx^n$)
23 $dy/dx = \sqrt{1-2x}$
24 $dy/dx = 1/\sqrt{1-2x}$
25 $dy/dx = 1/y$
26 $dy/dx = x/y$
27 $d^2y/dx^2 = 1$
28 $d^5y/dx^5 = 1$
29 $d^2y/dx^2 = -y$
30 $dy/dx = \sqrt{xy}$
31 $d^2y/dx^2 = \sqrt{x}$
32 $(dy/dx)^2 = \sqrt{x}$

33 ***True or false***, when f is an antiderivative of v:
(a) $\int v(u(x))\,dx = f(u(x)) + C$
(b) $\int v^2(x)\,dx = \frac{1}{3}f^3(x) + C$
(c) $\int v(x)(du/dx)\,dx = f(u(x)) + C$
(d) $\int v(x)(dv/dx)\,dx = \frac{1}{2}f^2(x) + C$

34 ***True or false***, when f is an antiderivative of v:
(a) $\int f(x)(dv/dx)\,dx = \frac{1}{2}f^2(x) + C$
(b) $\int v(v(x))(dv/dx)\,dx = f(v(x)) + C$
(c) Integral is inverse to derivative so $f(v(x)) = x$
(d) Integral is inverse to derivative so $\int (df/dx)\,dx = f(x)$

35 If $df/dx = v(x)$ then $\int v(x-1)\,dx =$ ______ and $\int v(x/2)\,dx =$ ______.

36 If $df/dx = v(x)$ then $\int v(2x-1)\,dx =$ ______ and $\int v(x^2)x\,dx =$ ______.

37 $\dfrac{x^2}{1+x^2} = 1 - \dfrac{1}{1+x^2}$ so $\displaystyle\int \frac{x^2\,dx}{1+x^2} =$ ______.

38 $\int (x^2+1)^2\,dx$ is not $\frac{1}{3}(x^2+1)^3$ but ______.

39 $\int 2x\,dx/(x^2+1)$ is $\int$ ______ du which will soon be $\ln u$.

40 Show that $\int 2x^3\,dx/(1+x^2)^3 = \int (u-1)\,du/u^3 =$ ______.

41 The acceleration $d^2f/dt^2 = 9.8$ gives $f(t) =$ ______ (*two* integration constants).

42 The solution to $d^4y/dx^4 = 0$ is ______ (*four* constants).

43 If $f(t)$ is an antiderivative of $v(t)$, find antiderivatives of
(a) $v(t+3)$ (b) $v(t)+3$ (c) $3v(t)$ (d) $v(3t)$.