# CHAPTER 4 DERIVATIVES BY THE CHAIN RULE

## 4.1 The Chain Rule (page 158)

$z = f(g(x))$ comes from $z = f(y)$ and $y = \mathbf{g(x)}$. At $x = 2$ the chain $(x^2-1)^3$ equals $\mathbf{3^3 = 27}$. Its inside function is $y = \mathbf{x^2 - 1}$, its outside function is $z = \mathbf{y^3}$. Then $dz/dx$ equals $\mathbf{3y^2 dy/dx}$. The first factor is evaluated at $y = \mathbf{x^2 - 1}$ (not at $y = x$). For $z = \sin(x^4 - 1)$ the derivative is $\mathbf{4x^3 \cos(x^4 - 1)}$. The triple chain $z = \mathbf{\cos(x+1)^2}$ has a shift and a **square** and a cosine. Then $dz/dx = \mathbf{2\cos(x+1)(-\sin(x+1))}$.

The proof of the chain rule begins with $\Delta z/\Delta x = \mathbf{(\Delta z/\Delta y)(\Delta y/\Delta x)}$ and ends with $\mathbf{dz/dx = (dz/dy)(dy/dx)}$. Changing letters, $y = \cos u(x)$ has $dy/dx = \mathbf{-\sin u(x)\frac{du}{dx}}$. The power rule for $y = [u(x)]^n$ is the chain rule $dy/dx = \mathbf{nu^{n-1}\frac{du}{dx}}$. The slope of $5g(x)$ is $\mathbf{5g'(x)}$ and the slope of $g(5x)$ is $\mathbf{5g'(5x)}$. When $f$ = cosine and $g$ = sine and $x = 0$, the numbers $f(g(x))$ and $g(f(x))$ and $f(x)g(x)$ are **1 and sin 1 and 0.**

**1** $z = y^3, y = x^2 - 3, z' = 6x(x^2-3)^2$ **3** $z = \cos y, y = x^3, z' = -3x^2 \sin x^3$
**5** $z = \sqrt{y}, y = \sin x, z' = \cos x / 2\sqrt{\sin x}$ **7** $z = \tan y + (1/\tan x), y = 1/x, z' = (\frac{-1}{x^2})\sec^2(\frac{1}{x}) - (\tan x)^{-2}\sec^2 x$
**9** $z = \cos y, y = x^2 + x + 1, z' = -(2x+1)\sin(x^2+x+1)$ **11** $17\cos 17x$ **13** $\sin(\cos x)\sin x$
**15** $x^2\cos x + 2x\sin x$ **17** $(\cos\sqrt{x+1})\frac{1}{2}(x+1)^{-1/2}$ **19** $\frac{1}{2}(1+\sin x)^{-1/2}(\cos x)$ **21** $\cos(\frac{1}{\sin x})(\frac{-\cos x}{\sin^2 x})$
**23** $8x^7 = 2(x^2)^2(2x^2)(2x)$ **25** $2(x+1) + \cos(x+\pi) = 2x + 2 - \cos x$
**27** $(x^2+1)^2 + 1; \sin U$ from 0 to sin 1; $U(\sin x)$ is 1 and 0 with period $2\pi$; $R$ from 0 to $x$; $R(\sin x)$ is half-waves.
**29** $g(x) = x + 2, h(x) = x^2 + 2; k(x) = 3$ **31** $f'(f(x))f'(x)$; no; $(-1/(1/x)^2)(-1/x^2) = 1$ and $f(f(x)) = x$
**33** $\frac{1}{2}(\frac{1}{2}x + 8) + 8; \frac{1}{8}x + 14; \frac{1}{16}$ **35** $f(g(x)) = x, g(f(y)) = y$
**37** $f(g(x)) = \frac{1}{1-x}, g(f(x)) = 1 - \frac{1}{x}, f(f(x)) = x = g(g(x)), g(f(g(x))) = \frac{x}{x-1} = f(g(f(x)))$
**39** $f(y) = y - 1, g(x) = 1$ **43** $2\cos(x^2+1) - 4x^2\sin(x^2+1); -(x^2-1)^{-3/2}; -(\cos\sqrt{x})/4x + (\sin\sqrt{x})/4x^{3/2}$
**45** $f'(u(t))u'(t)$ **47** $(\cos^2 u(x) - \sin^2 u(x))\frac{du}{dx}$ **49** $2xu(x) + x^2\frac{du}{dx}$ **51** $1/4\sqrt{1-\sqrt{1-x}}\sqrt{1-x}$
**53** $df/dt$ **55** $f'(g(x))g'(x) = 4(x^3)^3 3x^2 = 12x^{11}$ **57** $3600; \frac{1}{2}; 18$ **59** $3; \frac{1}{3}$

**2** $f(y) = y^2; g(x) = x^3 - 3; \frac{dz}{dx} = 6x^2(x^3-3)$ **4** $f(y) = \tan y; g(x) = 2x; \frac{dz}{dx} = 2\sec^2 2x$
**6** $f(y) = \sin y; g(x) = \sqrt{x}; \frac{dz}{dx} = \frac{\cos x}{2\sqrt{x}}$ **8** $f(y) = \sin y; g(x) = \cos x; \frac{dz}{dx} = -\sin x \cos(\cos x)$
**10** $f(y) = \sqrt{y}; g(x) = x^2; \frac{dz}{dx} = (\frac{1}{2\sqrt{y}})(2x) = 1$ **12** $\frac{dz}{dx} = \sec^2(x+1)$ **14** $\frac{dz}{dx} = 3x^2$ **16** $\frac{dz}{dx} = \frac{27}{2}\sqrt{9x+4}$
**18** $\frac{dz}{dx} = \frac{\cos(x+1)}{2\sqrt{\sin(x+1)}}$ **20** $\frac{dz}{dx} = \frac{\cos(\sqrt{x}+1)}{2\sqrt{x}}$ **22** $\frac{dz}{dx} = 4x(\sin x^2)(\cos x^2)$
**24** $\frac{dz}{dx} = \mathbf{3(3x)^2(3)}$ or $z = 27x^3$ and $\frac{dz}{dx} = \mathbf{81x^2}$ **26** $\frac{dz}{dx} = \frac{2\cos x \sin x}{2\sqrt{1-\cos^2 x}} = \mathbf{\cos x}$ or $z = \sin x$ and $\frac{dz}{dx} = \mathbf{\cos x}$
**28** $f(y) = y + 1; h(y) = \sqrt[3]{y}; k(y) \equiv 1$ **30** $f(y) = \sqrt{y}, g(x) = 1 - x^2; f(y) = \sqrt{1-y}, g(x) = x^2$
**32** (a) 22 (b) $4f'(5)$ (c) 8 (d) 4 **34** $C = \mathbf{16}$ because this solves $C = \frac{1}{2}C + 8$ (fixed point)
**36** $f(y), g(x), |f(g(x)) - 9| < \epsilon$
**38** For $g(g(x)) = x$ the graph of $g$ should be **symmetric across the 45° line:** If the point $(x, y)$ is on the graph so is $(y, x)$. Examples: $g(x) = -\frac{1}{x}$ or $-x$ or $\sqrt[3]{1-x^3}$.

**40 False** (The chain rule produces $-1$ : so derivatives of even functions are odd functions)
**False** (The derivative of $f(x) = x$ is $f'(x) = 1$) **False** (The derivative of $f(1/x)$ is $f'(1/x)$ times $-1/x^2$)
**True** (The factor from the chain rule is 1) **False** (see equation (8)).

**42** From $x = \frac{\pi}{4}$ go up to $y = \sin\frac{\pi}{4}$. Then go **across** to the parabola $z = y^2$. Read off $z = \sqrt{\sin\frac{\pi}{4}}$ on the horizontal $z$ axis.

**44** This is the chain rule applied to $\frac{dz}{dy}$ (a function of $y$). Its $x$ derivative is its $y$ derivative $(\frac{d^2z}{dy^2})$ times $\frac{dy}{dx}$.
If $z = y^2$ and $y = x^3$ then $\frac{dz}{dy} = 2y$ and $\frac{d^2z}{dy^2}\frac{dy}{dx} = \mathbf{2(3x^2)}$. Check another way: $\frac{dz}{dy} = 2x^3$ and $\frac{d}{dx}(\frac{dz}{dy}) = \mathbf{6x^2}$.

**46** $\frac{dz}{dx} = (3u^2)(3x^2) = 9x^8$ **48** $\frac{dy}{dt} = \frac{1}{2\sqrt{u(t)}}\frac{du}{dt}$ **50** $\frac{dy}{dx} = 2xf'(x^2) + 2f(x)\frac{df}{dx}$

**52** $\frac{dz}{dt} = -nu(t)^{-n-1}\frac{du}{dt}$ **54** $\frac{dy}{dt} = -\frac{1}{t^2}$ **56** $\cos(\sin x)\cos x$

**58** (a) 53 (sum rule for derivatives) (b) 60 (chain rule)

**60** Note that $G' = \cos(\sin x)\cos x$ and $G'' = -\cos(\sin x)\sin x - \sin(\sin x)\cos^2 x$. We were told that $H(x) = \cos(\cos x)$ should be included too.

## 4.2 Implicit Differentiation and Related Rates (page 163)

For $x^3 + y^3 = 2$ the derivative $dy/dx$ comes from **implicit** differentiation. We don't have to solve for **y**. Term by term the derivative is $3x^2 + \mathbf{3y^2\frac{dy}{dx}} = 0$. Solving for $dy/dx$ gives $\mathbf{-x^2/y^2}$. At $x = y = 1$ this slope is $\mathbf{-1}$. The equation of the tangent line is $y - 1 = \mathbf{-1(x-1)}$.

A second example is $y^2 = x$. The $x$ derivative of this equation is $\mathbf{2y\frac{dy}{dx}} = 1$. Therefore $dy/dx = \mathbf{1/2y}$. Replacing $y$ by $\sqrt{x}$ this is $dy/dx = \mathbf{1/2\sqrt{x}}$.

In related rates, we are given $dg/dt$ and we want $df/dt$. We need a relation between $f$ and **g**. If $f = g^2$, then $(df/dt) = \mathbf{2g}(dg/dt)$. If $f^2 + g^2 = 1$, then $df/dt = \mathbf{-\frac{g}{f}\frac{dg}{dt}}$. If the sides of a cube grow by $ds/dt = 2$, then its volume grows by $dV/dt = \mathbf{3s^2(2)} = \mathbf{6s^2}$. To find a number (8 is wrong), you also need to know **s**.

**1** $-x^{n-1}/y^{n-1}$ **3** $\frac{dy}{dx} = 1$ **5** $\frac{dy}{dx} = \frac{1}{F'(y)}$ **7** $(y^2 - 2xy)/(x^2 - 2xy)$ or 1 **9** $\frac{1}{\sec^2 y}$ or $\frac{1}{1+x^2}$

**11** First $\frac{dy}{dx} = -\frac{y}{x}$, second $\frac{dy}{dx} = \frac{x}{y}$ **13** Faster, faster **15** $2zz' = 2yy' \rightarrow z' = \frac{y}{x}y' = y'\sin\theta$

**17** $\sec^2\theta = \frac{c}{200\pi}$ **19** $500\frac{df}{dx}; 500\sqrt{1 + (\frac{df}{dx})^2}$ **21** $\frac{dy}{dt} = -\frac{8}{3}; \frac{dy}{dt} = -2\sqrt{3}; \infty$ then 0

**23** $V = \pi r^2 h; \frac{dh}{dt} = \frac{1}{4\pi}\frac{dV}{dt} = -\frac{1}{4\pi}$ in/sec **25** $A = \frac{1}{2}ab\sin\theta, \frac{dA}{dt} = 7$ **27** 1.6 m/sec; 9 m/sec; 12.8 m/sec

**29** $-\frac{7}{5}$ **31** $\frac{dz}{dt} = \frac{\sqrt{2}}{2}\frac{dy}{dt}; \frac{d\theta}{dt} = \frac{1}{10}\cos^2\theta\frac{d\theta}{dt}; \theta'' = \frac{\cos\theta}{10}y'' - \frac{1}{50}\cos^3\theta\sin\theta(y')^2$

**2** $\frac{dy}{dx} = -\mathbf{\frac{y^2+2xy}{x^2+2xy}}$ **4** $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} = -\mathbf{\frac{1}{2}}$ **6** $f'(x) + F'(y)\frac{dy}{dx} = y + x\frac{dy}{dx}$ so $\frac{dy}{dx} = \mathbf{\frac{y-f'(x)}{F'(y)-x}}$

**8** $1 = \cos y\frac{dy}{dx}$ so $\frac{dy}{dx} = \mathbf{\frac{1}{\cos y}} = \mathbf{\frac{1}{\sqrt{1-x^2}}}$. **10** $ny^{n-1}\frac{dy}{dx} = 1$ so $\frac{dy}{dx} = \mathbf{\frac{1}{n}}$

**12** $2(x-2) + 2y\frac{dy}{dx} = 0$ gives $\frac{dy}{dx} = \mathbf{1}$ at (1,1); $2x + 2(y-2)\frac{dy}{dx} = 0$ also gives $\frac{dy}{dx} = \mathbf{1}$.

**14** $2 + 2y\frac{d^2y}{dx^2} + 2(\frac{dy}{dx})^2 = 0$ yields $\frac{d^2y}{dx^2} = -\frac{1}{y} - \frac{x^2}{y^3} = -\mathbf{\frac{y^2+x^2}{y^3}}$.

**16** $y$ catches up to $z$ as $\theta$ increases to $\frac{\pi}{2}$. So $y'$ should be larger than $z'$. **18** $y'$ approaches $200\pi c/200\pi = \mathbf{c}$

**20** $x$ is a constant (fixed at 7) and therefore **a change $\Delta$ x is not allowed**

**22** $x^2+y^2 = 10^2$ so $2x\frac{dx}{dt}+2y\frac{dy}{dt} = 0$ and $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -2\,\frac{x}{y} = -c$ when $x = \frac{1}{2}cy$. This means $(\frac{1}{2}cy)^2+y^2 = 10^2$ or $y = \frac{\mathbf{10}}{\sqrt{\mathbf{1+(\frac{1}{2}c)^2}}}$

**24** Distance to you is $\sqrt{x^2+8^2}$, rate of change is $\frac{x}{\sqrt{x^2+8^2}}\frac{dx}{dt}$ with $\frac{dx}{dt} = 560$. (a) Distance = 16 and $x = 8\sqrt{3}$ and rate is $\frac{8\sqrt{3}}{16}(560) = \mathbf{280\sqrt{3}}$; (b) $x = 8$ and rate is $\frac{8}{\sqrt{8^2+8^2}}(560) = \mathbf{280\sqrt{2}}$; (c) $x = 0$ and rate is **zero.**

**26** $10c(t-3) = 8t$ divided by $c(t-3) = 4$ gives $10 = 2t$. So $\mathbf{t = 5}$ and $\mathbf{c = 2}$. The $x$ and $y$ distances between ball and receiver are $2t-10$ and $12t-60$. The derivative of $\sqrt{(2t-10)^2+(12t-60)^2}$ $= \sqrt{148}|t-5|$ is $-\sqrt{\mathbf{148}}$.

**28** Volume = $\frac{4}{3}\pi r^3$ has $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. If this equals twice the surface area $4\pi r^2$ (with minus for evaporation) than $\frac{dr}{dt} = \mathbf{-2}$.

**30** $\frac{d\theta}{dt} = 4\pi$ radians/second; $0 = 2x\frac{dx}{dt} - 6\cos\theta\frac{dx}{dt} + 6x\sin\theta\frac{d\theta}{dt}$; at $\theta = \frac{\pi}{2}, x = 3\sqrt{3}$ and $6\sqrt{3}\frac{dx}{dt} + 18\sqrt{3}\frac{d\theta}{dt}$ gives $\frac{dx}{dt} = \mathbf{-12\pi}$; at $\theta = \pi, x = 0$ and $\frac{dx}{dt} = \mathbf{0}$.

## 4.3 Inverse Functions and Their Derivatives (page 170)

The functions $g(x) = x-4$ and $f(y) = y+4$ are **inverse** functions, because $f(g(x)) = \mathbf{x}$. Also $g(f(y)) = \mathbf{y}$. The notation is $f = g^{-1}$ and $g = \mathbf{f^{-1}}$. The composition **of f and $\mathbf{f^{-1}}$** is the identity function. By definition $x = g^{-1}(y)$ if and only if $y = \mathbf{g(x)}$. When $y$ is in the range of $g$, it is in the **domain** of $g^{-1}$. Similarly $x$ is in the **domain** of $g$ when it is in the **range** of $g^{-1}$. If $g$ has an inverse then $g(x_1)\neq g(x_2)$ at any two points. The function $g$ must be steadily **increasing** or steadily **decreasing.**

The chain rule applied to $f(g(x)) = x$ gives $(df/dy)(\mathbf{dg/dx}) = \mathbf{1}$. The slope of $g^{-1}$ times the slope of $g$ equals **1**. More directly $dx/dy = 1/(\mathbf{dy/dx})$. For $y = 2x+1$ and $x = \frac{1}{2}(y-1)$, the slopes are $dy/dx = \mathbf{2}$ and $dx/dy = \mathbf{\frac{1}{2}}$. For $y = x^2$ and $x = \sqrt{\mathbf{y}}$, the slopes are $dy/dx = \mathbf{2x}$ and $dx/dy = \mathbf{1/2\sqrt{y}}$. Substituting $x^2$ for $y$ gives $dx/dy = \mathbf{1/2x}$. Then $(dx/dy)(dy/dx) = \mathbf{1}$.

The graph of $y = g(x)$ is also the graph of $x = \mathbf{g^{-1}(y)}$, but with $x$ across and $y$ up. For an ordinary graph of $g^{-1}$, take the reflection in the line $y = x$. If (3,8) is on the graph of $g$, then its mirror image **(8,3)** is on the graph of $g^{-1}$. Those particular points satisfy $8 = 2^3$ and $3 = \log_{\mathbf{2}} \mathbf{8}$.

The inverse of the chain $z = h(g(x))$ is the chain $x = \mathbf{g^{-1}(h^{-1}(z))}$. If $g(x) = 3x$ and $h(y) = y^3$ then $z = \mathbf{(3x)^3} = \mathbf{27x^3}$. Its inverse is $x = \mathbf{\frac{1}{3}z^{1/3}}$, which is the composition of $\mathbf{g^{-1}(y)} = \mathbf{\frac{1}{3}y}$ and $\mathbf{h^{-1}(z)} = \mathbf{z^{1/3}}$.

**1** $x = \frac{y+6}{3}$ **3** $x = \sqrt{y+1}$ ($x$ unrestricted $\to$ no inverse) **5** $x = \frac{y}{y+1}$ **7** $x = (1+y)^{1/3}$

**9** ($x$ unrestricted $\to$ no inverse) **11** $y = \frac{1}{x-a}$ **13** $2 < f^{-1}(x) < 3$ **15** $f$ goes up and down

**17** $f(x)g(x)$ and $\frac{1}{f(x)}$ **19** $m \neq 0; m \geq 0; |m| \geq 1$ **21** $\frac{dy}{dx} = 5x^4, \frac{dx}{dy} = \frac{1}{5}y^{-4/5}$

**23** $\frac{dy}{dx} = 3x^2; \frac{dx}{dy} = \frac{1}{3}(1+y)^{-2/3}$ **25** $\frac{dy}{dx} = \frac{-1}{(x-1)^2}, \frac{dx}{dy} = \frac{-1}{(y-1)^2}$ **27** $y; \frac{1}{2}y^2 + C$

**29** $f(g(x)) = -1/3x^3; g^{-1}(y) = \frac{-1}{y}; g(g^{-1}(x)) = x$ **39** $2/\sqrt{3}$ **41** $1/6\cos 9$

**43** Decreasing; $\frac{dx}{dy} = \frac{1}{dy/dx} < 0$ **45** F; T; F **47** $g(x) = x^m, f(y) = y^n, x = (z^{1/n})^{1/m}$
**49** $g(x) = x^3, f(y) = y + 6, x = (z-6)^{1/3}$ **51** $g(x) = 10^x, f(y) = \log y, x = \log(10^y) = y$
**53** $y = x^3, y'' = 6x, d^2x/dy^2 = -\frac{2}{9}y^{-5/3}$; m/sec$^2$, sec/m$^2$ **55** $p = \frac{1}{\sqrt{y}} - 1; 0 < y \le 1$
**57** $\max = G = \frac{3}{8}y^{4/3}, G' = \frac{1}{2}y^{1/3}$ **59** $y^2/100$

**2** $x = \frac{y-B}{A}$ **4** $x = \frac{y}{y-1}$ ($f^{-1}$ matches $f$) **6** no inverse **8** $x = \begin{cases} \frac{1}{3}y & y \ge 0 \\ y & y \le 0 \end{cases}$ **10** $x = y^5$
**12** The graph is a hyperbola, **symmetric across the 45° line;** $\frac{dy}{dx} = -\frac{2}{(x-1)^2}; \frac{dx}{dy} = -\frac{1}{2}(x-1)^2$ (or $-\frac{2}{(y-1)^2}$).
**14** $f^{-1}$ does not exist because $f(3)$ is the same as $f(5)$.
**16** No two $x$'s give the same $y$. **18** $y = \frac{x}{x-1}$ and $y = 2 - x$ (functions of $x + y$ and $xy$ lead to suitable $f$)
**20** The inverse of a piecewise linear function is piecewise **linear** (if the inverse exists).
**22** $\frac{dy}{dx} = -\frac{1}{(x-1)^2}; \frac{dx}{dy} = -\frac{\mathbf{1}}{\mathbf{y^2}} = -(x-1)^2$. **24** $\frac{dy}{dx} = -\frac{3}{x^4}; \frac{dx}{dy} = -\frac{\mathbf{1}}{\mathbf{3}}\mathbf{y^{-4/3}}$. **26** $\frac{dy}{dx} = \frac{ad-bc}{(cx+d)^2}; \frac{dx}{dy} = \frac{\mathbf{ad-bc}}{\mathbf{(cy-a)^2}}$.
**28** $\frac{dy}{dx} = y$. **30** jumps at $0, y_1, y_2$ to heights $x_1, x_2, x_3$; a piecewise constant function has no inverse.
**32** Hyperbola centered at $(-1, 0)$ : shift the standard hyperbola $xy = 1$.
**34** $y = -3x$ for $x \le 0$; $y = -x$ for $x \ge 0$. **36** The graph is the first quarter of the unit circle.
**38** The graph starts at (0,1) and increases with vertical asymptote at $x = 1$.
**40** $1 = \sec^2 x \frac{dx}{dy}$ so $\frac{dx}{dy} = \cos^2 x = \frac{\mathbf{1}}{\mathbf{2}}$ **42** $\frac{dy}{dx} = 1 - \cos x = 0$ so $\frac{dx}{dy} = \infty$. (The derivative does not exist.)
**44** **First proof** Suppose $y = f(x)$. We are given that $y > x$. This is the same as $y > f^{-1}(y)$.
**Second proof** The graph of $f(x)$ is above the 45° line, because $f(x) > x$. The mirror image is below the 45° line so $f^{-1}(y) < y$.
**46** $g(x) = x - 4, f(y) = 5y, g^{-1}(y) = y + 4, f^{-1}(z) = \frac{z}{5}, \mathbf{x} = \frac{\mathbf{1}}{\mathbf{5}}\mathbf{z} + \mathbf{4}$.
**48** $g(x) = x + 6, f(y) = y^3, g^{-1}(y) = y - 6, f^{-1}(z) = \sqrt[3]{z}; \mathbf{x} = \sqrt[3]{\mathbf{z}} - \mathbf{6}$
**50** $g(x) = \frac{1}{2}x + 4, f(y) = g(y), g^{-1}(y) = 2y - 8, f^{-1}(z) = g^{-1}(z); \mathbf{x} = \mathbf{2(2z - 8) - 8 = 4z - 24}$.
**52** $x^* = f^{-1}(0)$
**54** $f^{-1}(0) \approx f^{-1}(y) + (\frac{df^{-1}}{dy})(0 - y)$ is the same as $x^* \approx x + \frac{1}{df/dx}(0 - f(x))$, which gives Newton's method.
**56** $\frac{dG}{dy} = f^{-1}(y) + y\frac{df^{-1}}{dy} - F'(f^{-1}(y))\frac{df^{-1}}{dy}$. The second term cancels the third because $F'(f^{-1}(y))$ is equal to $f(f^{-1}(y)) = y$. This leaves the first term $\frac{dG}{dy} = f^{-1}(y)$. **G is the antiderivative of $\mathbf{f^{-1}}$ if $\mathbf{F' = f}$.**
**58** To maximize $yx - F(x)$ set the $x$ derivative to zero: $y = \frac{dF}{dx} = f(x)$ or $x = f^{-1}(y)$. Substitute this $x$ into $xy - F(x)$ : the maximum value is exactly $G(y)$ from Problem 56. Now maximize $xy - G(y)$. The $y$ derivative gives $x = \frac{dG}{dy}$ or by Problem 56 $x = f^{-1}(y)$. Substitute $y = f(x)$ into $xy - G(y)$ to find that the maximum value is $xf(x) - G(f(x)) = xf(x) - [f(x)x - F(f^{-1}(f(x)))] = F(x)$.
**Note:** This is the **Legendre transform** between $F(x)$ and $G(y)$ – important but not well known. Since $\frac{dF}{dx}$ is increasing (then $f^{-1}$ exists), the function $F(x)$ is convex (concave up). So is $G(y)$.

## 4.4 Inverses of Trigonometric Functions (page 175)

The relation $x = \sin^{-1} y$ means that **y** is the sine of **x**. Thus $x$ is the angle whose sine is **y**. The number $y$ lies between **−1** and **1**. The angle $x$ lies between $-\pi/2$ and $\pi/2$. (If we want the inverse to exist, there cannot be two angles with the same sine.) The cosine of the angle $\sin^{-1} y$ is $\sqrt{\mathbf{1 - y^2}}$. The derivative of $x = \sin^{-1} y$ is

$dx/dy = \mathbf{1/\sqrt{1-y^2}}$.

The relation $x = \cos^{-1} y$ means that $y$ equals $\cos x$. Again the number $y$ lies between $-1$ and $1$. This time the angle $x$ lies between $0$ and $\pi$ (so that each $y$ comes from only one angle $x$). The sum $\sin^{-1} y + \cos^{-1} y = \pi/2$. (The angles are called **complementary**, and they add to a **right** angle.) Therefore the derivative of $x = \cos^{-1} y$ is $dx/dy = \mathbf{-1/\sqrt{1-y^2}}$, the same as for $\sin^{-1} y$ except for a **minus** sign.

The relation $x = \tan^{-1} y$ means that $y = \tan x$. The number $y$ lies between $-\infty$ and $\infty$. The angle $x$ lies between $-\pi/2$ and $\pi/2$. The derivative is $dx/dy = \mathbf{1/(1+y^2)}$. Since $\tan^{-1} y + \cot^{-1} y = \pi/2$, the derivative of $\cot^{-1} y$ is the same except for a **minus** sign.

The relation $x = \sec^{-1} y$ means that $\mathbf{y = \sec x}$. The number $y$ *never* lies between $-1$ and $1$. The angle $x$ lies between $0$ and $\pi$, but never at $x = \pi/2$. The derivative of $x = \sec^{-1} y$ is $dx/dy = \mathbf{1/\mid y \mid \sqrt{y^2-1}}$.

**1** $0, \frac{\pi}{2}, 0$ **3** $\frac{\pi}{2}, 0, \frac{\pi}{4}$ **5** $\pi$ is outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$ **7** $y = -\sqrt{3}/2$ and $\sqrt{3}/2$
**9** $\sin x = \sqrt{1-y^2}$; $\sqrt{1-y^2}$ and 1 **11** $\frac{d(\sin^{-1} y)}{dy} \cos x = 1 \rightarrow \frac{d(\sin^{-1} y)}{dy} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-y^2}}$
**13** $y = 0: 1, -1, 1; y = 1: 0, 0, \frac{1}{2}$ **15** F; F; T; T; F; F **17** $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ **19** $\frac{dz}{dx} = 3$
**21** $\frac{dz}{dx} = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}$ **23** $1 - \frac{y \sin^{-1} y}{\sqrt{1-y^2}}$ **25** $\frac{dx}{dy} = \frac{1}{|y+1|\sqrt{y^2+2y}}$ **27** $u = 1$ so $\frac{du}{dy} = 0$ **31** $\sec x = \sqrt{y^2+1}$
**33** $\frac{1}{10}, 1, \frac{1}{2}$ **35** $-y/\sqrt{1-y^2}$ **37** $\frac{1}{2}\sec\frac{x}{2}\tan\frac{x}{2}$ **39** $\frac{nx^{n-1}}{|x^n|\sqrt{x^{2n}-1}}$ **41** $\frac{dy}{dx} = \frac{1}{1+x^2}$
**43** $\frac{dy}{dx} = \frac{1}{1+x^2}$ **47** $u = 4\sin^{-1} y$ **49** $\pi$ **51** $-\pi/4$

**2** $\sin^{-1}(-1) = -\frac{\pi}{2}; \cos^{-1}(-1) = \pi; \tan^{-1}(-1) = -\frac{\pi}{4}$. Note that $-\frac{\pi}{2}, \pi, -\frac{\pi}{4}$ are in the required ranges.
**4** $\sin^{-1}\sqrt{3}$ doesn't exist; $\cos^{-1}\sqrt{3}$ doesn't exist; $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.
**6** The range of $\sin^{-1}(y)$ is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Note that $\sin 2\pi = 0$ but $2\pi$ is not $\sin^{-1} 0$.
**8** $\frac{dx}{dy} = \frac{1}{2\sqrt{1-y^2/4}} = \frac{1}{\sqrt{4-y^2}}$. The graph goes from $y = -\pi$ to $y = \pi$.
**10** The sides of the triangle are $y, \sqrt{1-y^2}$, and 1. The tangent is $\mathbf{\frac{y}{\sqrt{1-y^2}}}$.
**12** $\frac{d\sin^{-1}}{dy}(\sin x)\cos x$ equals $\frac{1}{\sqrt{1-\sin^2 x}}\cos x = \mathbf{1}$ as required.
**14** $\frac{d(\sin^{-1} y)}{dy}|_{x=0} = \mathbf{1}$; $\frac{d(\cos^{-1} y)}{dy}|_{x=0} = -\infty$; $\frac{d(\tan^{-1} y)}{dy}|_{x=0} = \mathbf{1}$; $\frac{d(\sin^{-1} y)}{dy}|_{x=1} = \mathbf{\frac{1}{\cos 1}}$; $\frac{d(\cos^{-1} y)}{dy}|_{x=1} = -\mathbf{\frac{1}{\sin 1}}$; $\frac{d(\tan^{-1} y)}{dy}|_{x=1} = \mathbf{\frac{1}{\sec^2 1}}$.
**16** $\cos^{-1}(\sin x)$ is the complementary angle $\frac{\pi}{2} - x$. The tangent of that angle is $\mathbf{\frac{\cos x}{\sin x} = \cot x}$.
**18** $\frac{du}{dx} = \frac{1}{1+(2x)^2}(2) = \mathbf{\frac{2}{1+4x^2}}$. **20** $\frac{du}{dx} = \frac{1}{\sqrt{1-(\cos x)^2}}(-\sin x) = \mathbf{-1}$. Check: $z = \frac{\pi}{2} - x$ so $\frac{dz}{dx} = -1$.
**22** $\frac{dz}{dx} = -1(\sin^{-1} x)^{-2}\frac{1}{\sqrt{1-x^2}}$. **24** $\frac{dz}{dx} = 2x\tan^{-1} x + (1+x^2)\frac{1}{1+x^2} = \mathbf{2x\tan^{-1} x + 1}$.
**26** $u = x^2$ so $\frac{du}{dx} = \mathbf{2x}$. **28** $\frac{du}{dy} = \mathbf{\frac{1}{1+y^2}}$. The range of this function is $0 \le y \le \frac{\pi}{2}$.
**30** The right triangle has far side $y$ and near side 1. Then the near angle is $\tan^{-1} y$. That angle is also $\cot^{-1}(\frac{1}{y})$.
**34** The requirement is $u' = \frac{1}{1+t^2}$. To satisfy this requirement take $u = \mathbf{\tan^{-1} t}$.
**36** $u = \tan^{-1} y$ has $\frac{du}{dy} = \frac{1}{1+y^2}$ and $\frac{d^2u}{dy^2} = \mathbf{\frac{-2y}{(1+y^2)^2}}$. **38** $\frac{du}{dy} = \frac{2}{|2y|\sqrt{(2y)^2-1}} = \frac{1}{|y|\sqrt{4y^2-1}}$.

**40** By the chain rule $\frac{du}{dx} = \frac{1}{|\tan x|\sqrt{\tan^2 x - 1}}(\sec^2 x)$.

**42** By the product rule $\frac{dz}{dx} = (\cos x)(\sin^{-1} x) + (\sin x)\frac{1}{\sqrt{1-x^2}}$. Note that $z \neq x$ and $\frac{dz}{dx} \neq 1$.

**44** $\frac{dz}{dx} = \cos(\cos^{-1} x)(\frac{-1}{\sqrt{1-x^2}}) + \sin(\sin^{-1} x)(\frac{1}{\sqrt{1-x^2}}) = \frac{-x+x}{\sqrt{1-x^2}} = \mathbf{0}$.

**46** Domain $|y| \geq 1$; range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ with $x = 0$ deleted.

**48** $u(x) = \mathbf{\frac{1}{2}\tan^{-1}2x}$ (need $\frac{1}{2}$ to cancel 2 from the chain rule).

**50** $u(x) = \frac{x-1}{x+1}$ has $\frac{du}{dx} = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$. Then $\frac{d}{dx}\tan^{-1} u(x) = \frac{1}{1+u^2}\frac{du}{dx} = \frac{1}{1+(\frac{x-1}{x+1})^2}\frac{2}{(x+1)^2} = \frac{2}{(x+1)^2+(x-1)^2} = \mathbf{\frac{1}{x^2+1}}$. This is also the derivative of $\mathbf{\tan^{-1}x}$! So $\tan^{-1} u(x)$ minus $\tan^{-1} x$ is a **constant**.

**52** Problem 51 finds $u(0) = -1$ and $\tan^{-1} u(0) = -\frac{\pi}{4}$ and $\tan^{-1} 0 = 0$ and therefore $\tan^{-1} u(x) - \tan^{-1} x$ should have the constant value $-\frac{\pi}{4} - 0$. But as $x \to -\infty$ we now find $u \to 1$ and $\tan^{-1} u \to \frac{\pi}{4}$ and the difference is $\frac{\pi}{4} - (-\frac{\pi}{2}) = \frac{3\pi}{4}$. **The "constant" has changed!** It happened when $x$ passed $-1$ and $u$ became infinite and the angle $\tan^{-1} u$ jumped.

Resource: Calculus Online Textbook
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