CHAPTER 15 VECTOR CALCULUS

15.1 Vector Fields (page 554)

A vector field assigns a **vector** to each point (x, y) or (x, y, z). In two dimensions $\mathbf{F}(x, y) = \mathbf{M(x,y)i+N(x,y)j}$. An example is the position field $\mathbf{R} = \mathbf{x\,i} + \mathbf{y\,j}(+\mathbf{z\,k})$. Its magnitude is $|\mathbf{R}| = \mathbf{r}$ and its direction is **out from the origin**. It is the gradient field for $f = \frac{1}{2}(\mathbf{x^2} + \mathbf{y^2})$. The level curves are **circles**, and they are **perpendicular** to the vectors **R**.

Reversing this picture, the spin field is $\mathbf{S} = -\mathbf{y\,i} + \mathbf{x\,j}$. Its magnitude is $|\mathbf{S}| = \mathbf{r}$ and its direction is **around the origin**. It is not a gradient field, because no function has $\partial f/\partial x = -\mathbf{y}$ and $\partial f/\partial y = \mathbf{x}$. **S** is the velocity field for flow going **around the origin**. The streamlines or field lines or integral **curves** are **circles**. The flow field $\rho\mathbf{V}$ gives the rate at which **mass** is moved by the flow.

A gravity field from the origin is proportional to $\mathbf{F} = \mathbf{R}/\mathbf{r^3}$ which has $|\mathbf{F}| = \mathbf{1/r^2}$. This is Newton's **inverse** square law. It is a gradient field, with potential $f = \mathbf{1/r}$. The equipotential curves $f(x, y) = c$ are **circles**. They are **perpendicular** to the field lines which are **rays**. This illustrates that the **gradient** of a function $f(x, y)$ is **perpendicular** to its level curves.

The velocity field $y\,\mathbf{i} + x\,\mathbf{j}$ is the gradient of $f = \mathbf{xy}$. Its streamlines are **hyperbolas**. The slope dy/dx of a streamline equals the ratio $\mathbf{N/M}$ of velocity components. The field is **tangent** to the streamlines. Drop a leaf onto the flow, and it goes along **a streamline**.

1 $f(x, y) = x + 2y$ **3** $f(x, y) = \sin(x + y)$ **5** $f(x, y) = \ln(x^2 + y^2) = 2\ln r$
7 $\mathbf{F} = xy\mathbf{i} + \frac{x^2}{2}\mathbf{j}, f(x, y) = \frac{x^2 y}{2}$ **9** $\frac{\partial f}{\partial x} = 0$ so f cannot depend on x; streamlines are vertical (y = constant)
11 $\mathbf{F} = 3\mathbf{i} + \mathbf{j}$ **13** $\mathbf{F} = \mathbf{i} + 2y\,\mathbf{j}$ **15** $\mathbf{F} = 2x\,\mathbf{i} - 2y\,\mathbf{j}$ **17** $\mathbf{F} = e^{x-y}\,\mathbf{i} - e^{x-y}\,\mathbf{j}$
19 $\frac{dy}{dx} = -1; y = -x + C$ **21** $\frac{dy}{dx} = -\frac{x}{y}; x^2 + y^2 = C$ **23** $\frac{dy}{dx} = \frac{-x/y^2}{1/y} = \frac{-x}{y}; x^2 + y^2 = C$ **25** parallel
27 $\mathbf{F} = \frac{5x}{r}\mathbf{i} + \frac{5y}{r}\mathbf{j}$ **29** $\mathbf{F} = \frac{-mMG}{r^3}(x\mathbf{i} + y\,\mathbf{j}) - \frac{mMG}{((x-1)^2+y^2)^{3/2}}((x - 1)\,\mathbf{i} + y\,\mathbf{j})$
31 $\mathbf{F} = \frac{\sqrt{2}}{2}y\,\mathbf{i} - \frac{\sqrt{2}}{2}x\,\mathbf{j}$ **33** $\frac{dy}{dx} = \frac{-2}{x^2} = -\frac{1}{2}; \frac{dy}{dx} = \frac{x}{\sqrt{x^2-3}} = 2$
35 $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial f}{\partial r}\frac{x}{r}; \frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{y}{r}; f(r) = C$ gives circles
37 T; F (no equipotentials); T; F (not multiple of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$)
39 **F** and $\mathbf{F} + \mathbf{i}$ and $2\mathbf{F}$ have the same streamlines (different velocities) and equipotentials (different potentials). But if f is given, **F** must be grad f.

Answers 2 – 8 includes extra information about streamlines.

2 $x\mathbf{i} + \mathbf{j}$ is the gradient of $f(x, y) = \frac{1}{2}\mathbf{x^2} + \mathbf{y}$, which has **parabolas** $\frac{1}{2}x^2 + y = c$ as equipotentials (they open down). The streamlines solve $dy/dx = 1/x$ (this is N/M). So $y = \ln x + C$ gives the streamlines.

4 $\mathbf{i}/y - x\mathbf{j}/y^2$ is the gradient of $f(x, y) = \mathbf{x/y}$, which has rays $x/y = C$ as equipotentials (compare Figure 13.2; the axis $y = 0$ is omitted). The streamlines solve $dy/dx = N/M = -x/y$. So $y\,dy = -x\,dx$ and the streamlines are $\mathbf{y^2} + \mathbf{x^2} = \mathbf{constant}$ (circles).

6 $x^2\mathbf{i} + y^2\mathbf{j}$ is the gradient of $f(x, y) = \frac{1}{3}(\mathbf{x^3} + \mathbf{y^3})$, which has closed curves $x^3 + y^3$ = constant as equipotentials. The streamlines solve $dy/dx = y^2/x^2$ or $dy/y^2 = dx/x^2$ or $\mathbf{y^{-1}} = \mathbf{x^{-1}}$ + constant.

8 The potential can be $f(x, y) = \mathbf{x}\sqrt{\mathbf{y}}$. Then the field is $\nabla f = \sqrt{y}\mathbf{i} + \frac{1}{2}x\mathbf{j}/\sqrt{y}$. The equipotentials are curves

$x\sqrt{y} = C$ or $y = C^2/x^2$. The streamlines solve $dy/dx = N/M = x/2y$ so $2y\,dy = x\,dx$ or $\mathbf{y^2 - \frac{1}{2}x^2 = c}$.

10 If $\frac{\partial f}{\partial x} = -y$ then $f = \mathbf{-yx +}$ **any function** $\mathbf{C(y)}$. In this case $\frac{\partial f}{\partial y} = \mathbf{-x + \frac{dC}{dy}}$ which can't give $\frac{\partial f}{\partial y} = x$.

12 $\frac{\partial f}{\partial x} = \mathbf{1}$ and $\frac{\partial f}{\partial y} = \mathbf{-3}$; $\mathbf{F} = \mathbf{i} - 3\mathbf{j}$ has **parallel lines** $x - 3y = c$ as equipotentials.

14 $\frac{\partial f}{\partial x} = \mathbf{2x - 2}$ and $\frac{\partial f}{\partial y} = \mathbf{2y}$; $\mathbf{F} = (2x-2)\mathbf{i} + 2y\mathbf{j}$ leads to circles $(x-1)^2 + y^2 = c$ around the center (1,0).

16 $\frac{\partial f}{\partial x} = \mathbf{e^x \cos y}$ and $\frac{\partial f}{\partial y} = \mathbf{-e^x \sin y}$; $\mathbf{F} = e^x(\cos y\mathbf{i} - \sin y\mathbf{j})$ leads to curves $e^x \cos y = c$ which stay inside a strip like $|y| < \frac{\pi}{2}$. (They come in along the top, turn near the y axis, and leave along the bottom.)

18 $\frac{\partial f}{\partial x} = \mathbf{\frac{-y}{x^2}}$ and $\frac{\partial f}{\partial y} = \mathbf{\frac{1}{x}}$; $\mathbf{F} = -\frac{y}{x^2}\mathbf{i} + \frac{1}{x}\mathbf{j}$ has the **rays** $\frac{y}{x} = c$ as equipotentials (omit the axis $x = 0$).

20 $\frac{dy}{dx} = x$ gives $\mathbf{y = \frac{1}{2}x^2 + C}$ (parabolas). 22 $\frac{dy}{dx} = -\frac{x}{y}$ gives $\mathbf{y^2 + x^2 = C}$ (circles).

24 $\frac{dy}{dx} = \frac{1}{2}$ gives $\mathbf{y = \frac{1}{2}x + C}$ (parallel lines).

26 $f(x,y) = \mathbf{\frac{1}{2}\ln(x^2+y^2) = \ln\sqrt{x^2+y^2}}$. This comes from $\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2}$ or $f = \int \frac{x\,dx}{x^2+y^2}$.

28 The gradient $3x^2\mathbf{i} + 3y^2\mathbf{j}$ is perpendicular. For unit length take $\mathbf{F}$ (or $\mathbf{V}$) as $\mathbf{(x^2i + y^2j)/\sqrt{x^4+y^4}}$.

30 The field is a multiple of $\mathbf{i}+\mathbf{j}$. To have speed 4 take $\mathbf{F}$ (or $\mathbf{V}$) as $\mathbf{\sqrt{8}(i+j)}$.

32 From the gradient of $y - x^2$, $\mathbf{F}$ must be $\mathbf{-2xi + j}$ (or this is $-\mathbf{F}$).

34 The slope $\frac{dy}{dx}$ is $\mathbf{-f_x/f_y}$ from the first equation. The field is $f_x\mathbf{i} + f_y\mathbf{j}$ so this slope is $-M/N$. The product with the streamline slope N/M is $\mathbf{-1}$, so level curves are perpendicular to streamlines.

36 $\mathbf{F}$ is the gradient of $f = \mathbf{\frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2}$. The equipotentials are **ellipses** if $ac > b^2$ and **hyperbolas** if $ac < b^2$. (If $ac = b^2$ we get straight lines.)

40 (a) $\mathbf{R}+\mathbf{S} = (x-y)\mathbf{i} + (y+x)\mathbf{j}$ has **magnitude** $\mathbf{\sqrt{2}\,r}$. (b) The magnitude is now $\mathbf{\sqrt{2}}$ (difference of perpendicular unit vectors). (c) The direction stays parallel to $\mathbf{i}+\mathbf{j}$ (at $\mathbf{45°}$).

(d) $y\mathbf{i}$ is a **shear field**, pointing in the x direction and growing in the y direction.

15.2 Line Integrals (page 562)

Work is the **integral** of $\mathbf{F}\cdot d\mathbf{R}$. Here $\mathbf{F}$ is the **force** and $\mathbf{R}$ is the **position**. The **dot** product finds the component of $\mathbf{F}$ in the direction of movement $d\mathbf{R} = dx\,\mathbf{i} + dy\,\mathbf{j}$. The straight path $(x,y) = (t,2t)$ goes from **(0,0)** at $t=0$ to **(1,2)** at $t=1$ with $d\mathbf{R} = dt\,\mathbf{i} + \mathbf{2dt}\,\mathbf{j}$.

Another form of $d\mathbf{R}$ is $\mathbf{T}ds$, where $\mathbf{T}$ is the **unit tangent** vector to the path and the arc length has $ds = \mathbf{\sqrt{(dx/dt)^2 + (dy/dt)^2}}$. For the path $(t,2t)$, the unit vector $\mathbf{T}$ is $\mathbf{(i+2j)/\sqrt{5}}$ and $ds = \mathbf{\sqrt{5}}dt$. For $\mathbf{F} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{F}\cdot\mathbf{T}\,ds$ is still $\mathbf{5}dt$. This $\mathbf{F}$ is the gradient of $f = \mathbf{3x+y}$. The change in $f = 3x+y$ from (0,0) to (1,2) is **5**.

When $\mathbf{F} = \text{grad } f$, the dot product $\mathbf{F}\cdot d\mathbf{R}$ is $(\partial f/\partial x)dx + \mathbf{(\partial f/\partial y)dy} = df$. The work integral from P to Q is $\int df = \mathbf{f(Q) - f(P)}$. In this case the work depends on the **endpoints** but not on the **path**. Around a closed path the work is **zero**. The field is called **conservative**. $\mathbf{F} = (1+y)\,\mathbf{i} + x\,\mathbf{j}$ is the gradient of $f = \mathbf{x + xy}$. The work from (0,0) to (1,2) is **3**, the change in potential.

For the spin field $\mathbf{S} = \mathbf{-y\,i + x\,j}$, the work does depend on the path. The path $(x,y) = (3\cos t, 3\sin t)$ is a circle with $\mathbf{S}\cdot d\mathbf{R} = \mathbf{-y\,dx + x\,dy = 9\,dt}$. The work is $\mathbf{18\pi}$ around the complete circle. Formally $\int g(x,y)ds$ is the limit of the sum $\mathbf{\sum g(x_i, y_i)\Delta s_i}$.

The four equivalent properties of a conservative field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ are **A: zero work around closed paths,**

B: work depends only on endpoints, C: gradient field, D: $\partial M/\partial y = \partial N/\partial x$. Test **D** is passed by $\mathbf{F} = (y+1)\mathbf{i} + x\mathbf{j}$. The work $\int \mathbf{F} \cdot d\mathbf{R}$ around the circle $(\cos t, \sin t)$ is **zero**. The work on the upper semicircle equals the work on **the lower semicircle (clockwise)**. This field is the gradient of $f = \mathbf{x} + \mathbf{xy}$, so the work to $(-1,0)$ is $\mathbf{-1}$ **starting from (0,0)**.

1 $\int_0^1 \sqrt{1^2+2^2}dt = \sqrt{5}; \int_0^1 2\,dt = 2$ **3** $\int_0^1 t^2\sqrt{2}dt + \int_1^2 1\cdot(2-t)dt = \frac{\sqrt{2}}{3} + \frac{1}{2}$
5 $\int_0^{2\pi}(-3\sin t)dt = 0$ (gradient field); $\int_0^{2\pi} -9\sin^2 t\,dt = -9\pi = -$ area
7 No, $xy\,\mathbf{j}$ is not a gradient field; take line $x = t, y = t$ from (0,0) to (1,1) and $\int t^2 dt \neq \frac{1}{2}$
9 No, for a circle $(2\pi r)^2 \neq 0^2 + 0^2$ **11** $f = x + \frac{1}{2}y^2; f(0,1) - f(1,0) = -\frac{1}{2}$
13 $f = \frac{1}{2}x^2y^2; f(0,1) - f(1,0) = 0$ **15** $f = r = \sqrt{x^2+y^2}; f(0,1) - f(1,0) = 0$
17 Gradient for $n = 2$; after calculation $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{n-2}{r^n}$
19 $x = a\cos t, z = a\sin t, ds = a\,dt, M = \int_0^{2\pi}(a + a\sin t)a\,dt = 2\pi a^2$
21 $x = a\cos t, y = a\sin t, ds = a\,dt, M = \int_0^{2\pi} a^3\cos^2 t\,dt = \pi a^3, (\bar{x},\bar{y}) = (0,0)$ by symmetry
23 $\mathbf{T} = \frac{2\mathbf{i}+2t\mathbf{j}}{\sqrt{4+4t^2}} = \frac{\mathbf{i}+t\mathbf{j}}{\sqrt{1+t^2}}; \mathbf{F} = 3x\,\mathbf{i} + 4\,\mathbf{j} = 6t\,\mathbf{i} + 4\,\mathbf{j}, ds = 2\sqrt{1+t^2}dt, \mathbf{F}\cdot\mathbf{T}ds = (6t\,\mathbf{i}+4\,\mathbf{j})\cdot(\frac{\mathbf{i}+t\mathbf{j}}{\sqrt{1+t^2}})2\sqrt{1+t^2}dt =$
$20t\,dt; \mathbf{F}\cdot d\mathbf{R} = (6t\mathbf{i}+4\mathbf{j})\cdot(2\,dt\mathbf{i} + 2t\,dt\,\mathbf{j}) = 20t\,dt$; work $= \int_1^2 20t\,dt = 30$
25 If $\frac{\partial M(y)}{\partial y} = \frac{\partial N(x)}{\partial x}$ then $M = ay + b, N = ax + c$, constants a, b, c
27 $\mathbf{F} = 4x\mathbf{j}$ (work $= 4$ from (1,0) up to (1,1)) **29** $f = [x - 2y]_{(0,0)}^{(1,1)} = -1$ **31** $f = [xy^2]_{(0,0)}^{(1,1)} = 1$
33 Not conservative; $\int_0^1(t\mathbf{i} - t\mathbf{j})\cdot(dt\,\mathbf{i} + dt\,\mathbf{j}) = \int 0\,dt = 0; \int_0^1(t^2\mathbf{i} - t\mathbf{j})\cdot(dt\,\mathbf{i} + 2t\,dt\,\mathbf{j}) = \int_0^1 -t^2dt = -\frac{1}{3}$
35 $\frac{\partial M}{\partial y} = ax, \frac{\partial N}{\partial x} = 2x + b$, so $a = 2, b$ is arbitrary **37** $\frac{\partial M}{\partial y} = 2ye^{-x} = \frac{\partial N}{\partial x}; f = -y^2e^{-x}$
39 $\frac{\partial M}{\partial y} = \frac{-xy}{r^3} = \frac{\partial N}{\partial x}; f = r = \sqrt{x^2+y^2} = |x\mathbf{i} + y\mathbf{j}|$
41 $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j}$ has $\frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = 1$, no f **43** $2\pi; 0; 0$

2 Note $ds = \sqrt{\sin^2 t + \cos^2 t}\,dt = dt$. Then $\int x\,ds = \int_0^{\pi/2}\cos t\,dt = \mathbf{1}$ and $\int xy\,ds = \int_0^{\pi/2}\sin t\cos t\,dt = \mathbf{\frac{1}{2}}$.
4 Around the square $0 \le x, y \le 3, \int_3^0 y\,dx = -9$ along the top (backwards) and $\int_0^3 -x\,dy = -9$ up the right side. All other integrals are zero: answer $\mathbf{-18}$. By Section 15.3 this integral is always $-2\times$ area.
6 $\int \frac{ds}{dt}dt = \int ds =$ arc length $= \mathbf{5}$.
8 **Yes** The field $x\mathbf{i}$ is the gradient of $f = \frac{1}{2}x^2$. Here $M = x$ and $N = 0$ so we have $\int_P^Q M dx + N dy = f(Q) - f(P)$. More directly: up and down movement has no effect on $\int x\,dx$.
10 **Not much.** Certainly the limit of $\Sigma(\Delta s)^2$ is zero.
12 $\frac{\partial N}{\partial x} = 0$ and $\frac{\partial M}{\partial y} = 1$; **not conservative**, take straight path $x = 1 - t, y = t : \int \mathbf{F}\cdot d\mathbf{R} = \int y\,dx + dy =$ $\int_0^1 t(-dt) + dt = \mathbf{\frac{1}{2}}$.
14 $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ and $\mathbf{F}$ is the gradient of $f = xe^y$. Then $\int \mathbf{F}\cdot d\mathbf{R} = f(Q) - f(P) = \mathbf{-1}$.
16 $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$; **not conservative**, choose straight path $x = 1-t, y = t : \int -y^2dx + x^2dy = \int t^2dt + (1-t)^2dt = \mathbf{\frac{2}{3}}$.
18 $\frac{\mathbf{R}}{r^n}$ has $M = \frac{x}{(x^2+y^2)^{n/2}}$ and $\frac{\partial M}{\partial y} = -xny(x^2+y^2)^{-(n/2)-1}$. This agrees with $\frac{\partial N}{\partial x}$ so $\frac{\mathbf{R}}{r^n}$ is a **gradient field for all n. The potential is** $\mathbf{f} = \frac{\mathbf{r^{2-n}}}{\mathbf{2-n}}$ or $\mathbf{f} = \ln \mathbf{r}$ when $n = 2$.
20 The semicircle has $x = a\cos t, y = a\sin t, ds = adt, 0 \le t \le \pi$. The mass is $M = \int \rho ds = \int \rho a dt = \rho a\pi$. The moment is $M_x = \int \rho y\,ds = \int \rho a^2 \sin t\,dt = 2\rho a^2$. Then $\bar{x} = 0$ (by symmetry) and $\bar{y} = \frac{2\rho a^2}{\rho a\pi} = \frac{2a}{\pi}$.
22 (a) For a gradient field $\int \mathbf{F}\cdot d\mathbf{R} = f(Q) - f(P)$. Here $Q = (1,1,1)$ and $P = (0,0,0)$ so $f(Q) - f(P) = \mathbf{2}$.
(b) $\int M dx + N dy + P dz = \int t^2dt - t(2t\,dt) + t^3(3t^2dt) = \frac{1}{6}$.
24 $P = 0$ means $\frac{\partial f}{\partial z} = 0$. So f is $f(x,y)$. So $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$ cannot depend on z.
26 (a) $\int y^3dx + 3xy^2dy = \int_0^1(yt)^3(x\,dt) + 3xt(yt)^2(ydt) = \mathbf{xy^3}$. Then $\frac{\partial W}{\partial x} = y^3$ and $\frac{\partial W}{\partial y} = 3xy^2$ (conservative).
(b) $W = \int_0^1(xt)^3(x\,dt) + 3(yt)(xt)^2(y\,dt) = \mathbf{\frac{1}{4}(x^4 + 3y^2x^2)}$. But $\frac{\partial W}{\partial x} \neq M$ (not conservative).

(c) $W = \int_0^1 \frac{xt}{yt}(x\,dt) + \frac{yt}{xt}(y\,dt) = \frac{\mathbf{x^2}}{\mathbf{y}} + \frac{\mathbf{y^2}}{\mathbf{x}}$. But $\frac{\partial W}{\partial x} \neq M$ (not conservative).

(d) $W = \int_0^1 e^{xt+yt}(x\,dt + y\,dt) = e^{x+y} - 1$. Then $\frac{\partial W}{\partial x} = e^{x+y}$ and $\frac{\partial W}{\partial y} = e^{x+y}$ **(conservative)**.

28 $\mathbf{F} = x^2\mathbf{j}$ on the circle $x = \cos t, y = \sin t$ has $\int \mathbf{F} \cdot d\mathbf{R} = \int_0^{2\pi} \cos^2 t(\cos t\,dt) = \mathbf{0}$.

30 $\int x^2 dy = \int_0^1 t^2 dt = \frac{1}{3}$ but $\int_0^1 t^2(2t\,dt) = \frac{1}{2}$.

32 $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$ (not conservative): $\int x^2 y\,dx + xy^2 dy = \int_0^1 2t^3 dt = \frac{1}{2}$ but $\int_0^1 t^2(t^2)dt + t(t^2)^2(2t\,dt) = \frac{17}{35}$.

34 The potential is $f = \frac{1}{2}\ln(\mathbf{x^2} + \mathbf{y^2} + 1)$. Then $f(1,1) - f(0,0) = \frac{1}{2}\ln 3$.

36 $\int_0^1 -t^2(-2t\,dt) + (1 - t^2)(2t\,dt) = \mathbf{1}$ (as before). On the quarter-circle ending at $t = \frac{\pi}{4}$:
$\int_0^{\pi/4}(-\sin 2t)\,(-2\sin 2t\,dt) + (\cos 2t)(2\cos 2t\,dt) = 2\frac{\pi}{4} = \frac{\pi}{2}$ as before.

38 $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -2ye^x - 2ye^x \neq 0$. **No potential** $f(x,y)$.

40 $\mathbf{F} = \frac{y\mathbf{i}+x\mathbf{j}}{\sqrt{y^2+x^2}}$ has $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$.

42 $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ if and only if $b = c$. Then $f(x,y) = \frac{1}{2}ax^2 + bxy + \frac{1}{2}dy^2$.

44 **True** because $\int \mathbf{F} \cdot d\mathbf{R} = \int y\,dx$. **False** because $\mathbf{F} = y\mathbf{i}$ is not conservative. (The area underneath depends on the curve.) **True** because the area is π (and $\int y\,dx = \int_0^{2\pi} \sin t(\sin t\,dt) = \pi$.)

15.3 Green's Theorem (page 571)

The work integral $\oint M\,dx + N\,dy$ equals the double integral $\iint(\mathbf{N_x} - \mathbf{M_y})\mathbf{dx\,dy}$ by **Green's Theorem**. For $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j}$ the work is **zero**. For $\mathbf{F} = \mathbf{x}\mathbf{j}$ and $-\mathbf{y}\mathbf{i}$ the work equals the area of R. When $M = \partial f/\partial x$ and $N = \partial f/\partial y$, the double integral is zero because $\mathbf{f_{xy}} = \mathbf{f_{yx}}$. The line integral is zero because $\mathbf{f(Q)} = \mathbf{f(P)}$ **when** $\mathbf{Q} = \mathbf{P}$ **(closed curve)**. An example is $\mathbf{F} = \mathbf{y}\,\mathbf{i} + \mathbf{x}\,\mathbf{j}$. The direction on C is **counterclockwise** around the outside and **clockwise** around the boundary of a hole. If R is broken into very simple pieces with crosscuts between them, the integrals of $\mathbf{M\,dx} + \mathbf{N\,dy}$ cancel along the crosscuts.

Test D for gradient fields is $\partial\mathbf{M}/\partial\mathbf{y} = \partial\mathbf{N}/\partial\mathbf{x}$. A field that passes this test has $\oint \mathbf{F} \cdot d\mathbf{R} = \mathbf{0}$. There is a solution to $f_x = \mathbf{M}$ and $f_y = \mathbf{N}$. Then $df = M\,dx + N\,dy$ is an **exact** differential. The spin field $\mathbf{S}/r^2$ passes test **D** except at $\mathbf{r} = \mathbf{0}$. Its potential $f = \theta$ increases by $\mathbf{2\pi}$ going around the origin. The integral $\iint(N_x - M_y)dx\,dy$ is not zero but $\mathbf{2\pi}$.

The flow form of Green's Theorem is $\oint_{\mathbf{C}} \mathbf{M\,dy} - \mathbf{N\,dx} = \iint_{\mathbf{R}}(\mathbf{M_x} + \mathbf{N_y})\mathbf{dx\,dy}$. The normal vector in $\mathbf{F}\cdot\mathbf{n}ds$ points **out across C** and $|\mathbf{n}| = \mathbf{1}$ and $\mathbf{n}\,ds$ equals $dy\,\mathbf{i} - dx\,\mathbf{j}$. The divergence of $M\mathbf{i} + N\mathbf{j}$ is $\mathbf{M_x} + \mathbf{N_y}$. For $\mathbf{F} = x\mathbf{i}$ the double integral is $\iint \mathbf{1\,dt} =$ **area**. There is a source. For $\mathbf{F} = y\,\mathbf{i}$ the divergence is **zero**. The divergence of $\mathbf{R}/r^2$ is **zero** except at $\mathbf{r} = \mathbf{0}$. This field has a **point** source.

A field with no source has properties $\mathbf{E} =$ **zero flux through C**, $\mathbf{F} =$ **equal flux across all paths from P to Q**, $\mathbf{G} =$ **existence of stream function**, **H**= zero divergence. The stream function g satisfies the equations $\partial\mathbf{g}/\partial\mathbf{y} = \mathbf{M}$ **and** $\partial\mathbf{g}/\partial\mathbf{x} = -\mathbf{N}$. Then $\partial M/\partial x + \partial N/\partial y = 0$ because $\partial^2 g/\partial x\partial y = \partial^2\mathbf{g}/\partial\mathbf{y}\,\partial\mathbf{x}$. The example $\mathbf{F} = y\mathbf{i}$ has $g = \frac{1}{2}\mathbf{y^2}$. There is not a potential function. The example $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$ has $g = \mathbf{xy}$ and also $f = \frac{1}{2}\mathbf{x^2} - \frac{1}{2}\mathbf{y^2}$. This f satisfies Laplace's equation $\mathbf{f_{xx}} + \mathbf{f_{yy}} = \mathbf{0}$, because the field $\mathbf{F}$ is both **conservative and source-free**. The functions f and g are connected by the **Cauchy-Riemann** equations

$\partial f/\partial x = \partial g/\partial y$ and $\partial \mathbf{f}/\partial \mathbf{y} = -\partial \mathbf{g}/\partial \mathbf{x}$.

1 $\int_0^{2\pi}(a\cos t)a\cos t\,dt = \pi a^2; N_x - M_y = 1, \iint dx\,dy = \text{area } \pi a^2$

3 $\int_0^1 x\,dx + \int_1^0 x\,dx = 0, N_x - M_y = 0, \iint 0\,dx\,dy = 0$

5 $\int x^2 y\,dx = \int_0^{2\pi}(a\cos t)^2(a\sin t)(-a\sin t\,dt) = -\frac{a^4}{4}\int_0^{2\pi}(\sin 2t)^2 dt = -\frac{\pi a^4}{4}$;
$N_x - M_y = -x^2, \iint(-x^2)dx\,dy = \int_0^{2\pi}\int_0^a -r^2\cos^2\theta\, r\,dr\,d\theta = -\frac{\pi a^4}{4}$

7 $\int x\,dy - y\,dx = \int_0^{\pi}(\cos^2 t + \sin^2 t)\,dt = \pi; \iint(1+1)dx\,dy = 2\text{ (area)} = \pi; \int x^2 dy - xy\,dx = \frac{1}{2} + 1$;
$\int_0^1\int_0^1(2x+x)dx\,dy = \frac{3}{2}$

9 $\frac{1}{2}\int_0^{2\pi}(3\cos^4 t\sin^2 t + 3\sin^4 t\cos^2 t)dt = \frac{1}{2}\int_0^{2\pi}3\cos^2 t\sin^2 t\,dt = \frac{3}{2}\frac{\pi}{4}$ (see Answer 5)

11 $\int \mathbf{F}\cdot d\mathbf{R} = 0$ around any loop; $\mathbf{F} = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j}$ and $\int \mathbf{F}\cdot d\mathbf{R} = \int_0^{2\pi}[-\sin t\cos t + \sin t\cos t]dt = 0$;
$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ gives $\iint 0\,dx\,dy$

13 $x = \cos 2t, y = \sin 2t, t$ from 0 to $2\pi; \int_0^{2\pi} -2\sin^2 2t\,dt = -2\pi = -2$ (area);
$\int_0^{2\pi} -2dt = -4\pi = -2$ times Example 7

15 $\int M dy - N dx = \int_0^{2\pi} 2\sin t\cos t\,dt = 0; \iint(M_x + N_y)dx\,dy = \iint 0\,dx\,dy = 0$

17 $M = \frac{x}{r}, N = \frac{y}{r}, \int M dy - N dx = \int_0^{2\pi}(\cos^2 t + \sin^2 t)dt = 2\pi; \iint(M_x + N_y)dx\,dy = \iint(\frac{1}{r} - \frac{x^2}{r^3} + \frac{1}{r} - \frac{y^2}{r^3})dx\,dy =$
$\iint \frac{1}{r}dx\,dy = \iint dr\,d\theta = 2\pi$

19 $\int M dy - N dx = \int -x^2 y\,dx = \int_1^0 -x^2(1-x)dx = \frac{1}{12}; \int_0^1\int_0^{1-y} x^2 dx\,dy = \frac{1}{12}$

21 $\iint(M_x + N_y)dx\,dy = \iint \text{div}\,\mathbf{F}\,dx\,dy = 0$ between the circles

23 Work: $\int a\,dx + b\,dy = \iint(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})dx\,dy$; Flux: *same integral*

25 $g = \tan^{-1}(\frac{y}{x}) = \theta$ is undefined at (0,0) **27** Test $M_y = N_x : x^2 dx + y^2 dy$ is exact $= d(\frac{1}{3}x^3 + \frac{1}{3}y^3)$

29 $\text{div}\,\mathbf{F} = 2y - 2y = 0; g = xy^2$ **31** $\text{div}\,\mathbf{F} = 2x + 2y$; no g **33** $\text{div}\,\mathbf{F} = 0; g = e^x\sin y$

35 $\text{div}\,\mathbf{F} = 0; g = \frac{y^2}{x}$

37 $N_x - M_y = -2x, -6xy, 0, 2x - 2y, 0, -2e^{x+y}$; in **31** and **33** $f = \frac{1}{3}(x^3 + y^3)$ and $f = e^x\cos y$

39 $\mathbf{F} = (3x^2 - 3y^2)\mathbf{i} - 6xy\,\mathbf{j}; \text{div}\,\mathbf{F} = 0$ **41** $f = x^4 - 6x^2y^2 + y^4; g = 4x^3y - 4xy^3$

43 $\mathbf{F} = e^x\cos y\,\mathbf{i} - e^x\sin y\,\mathbf{j}; g = e^x\sin y$

45 $N = f(x), \int M dx + N dy = \int_0^1 f(1)dy + \int_1^0 f(0)dy = f(1) - f(0); \iint(N_x - M_y)dx\,dy =$
$\iint \frac{\partial f}{\partial x}dx\,dy = \int_0^1 \frac{\partial f}{\partial x}dx$ (Fundamental Theorem of Calculus)

2 $\oint x^2 y\,dy = \int_0^{2\pi} a^2\cos^2 t(a\sin t)(a\cos t\,dt) = \mathbf{0}; M = 0, N = x^2 y, \iint 2xy\,dx\,dy =$
$\int_0^{2\pi}\int_0^a 2r\cos\theta(r\sin\theta)r\,dr\,d\theta = \mathbf{0}$

4 $\oint y\,dx = \int_0^1 t(-dt) = -\mathbf{\frac{1}{2}}; M = y, N = 0, \iint(-1)dx\,dy = -\text{ area } = -\mathbf{\frac{1}{2}}$.

6 $\oint x^2 y\,dx = \int_0^1(1-t)^2 t(-dt) = -\mathbf{\frac{1}{12}}; M = x^2 y, N = 0, \int_0^1\int_0^{1-y} -x^2 dx\,dy = -\int_0^1 \frac{(1-y)^3}{3}dy = -\mathbf{\frac{1}{12}}$.

8 $M = xy^2$ and $N = x^2 y + 2x$ so $\oint M dx + N dy = \iint[(2xy + 2) - 2xy]dx\,dy = \mathbf{2}$ **times area.**

10 $M = by$ and $N = cx : \oint M dx + N dy = \iint(c - b)dx\,dy = (\mathbf{c} - \mathbf{b})$ **times area**;
$b = 7$ and $\mathbf{c} = \mathbf{7}$ make the integral zero.

12 Let R be the square with base from a to b on the x axis. Set $\mathbf{F} = f(x)\mathbf{j}$ so $M = 0$ and $N = f(x)$. The line integral $\oint M dx + N dy$ is $(\mathbf{b} - \mathbf{a})\mathbf{f}(\mathbf{b})$ up the right side **minus** $(\mathbf{b} - \mathbf{a})\mathbf{f}(\mathbf{a})$ down the left side. The double integral is $\iint \frac{df}{dx}dx\,dy = (\mathbf{b} - \mathbf{a})\int_{\mathbf{a}}^{\mathbf{b}} \frac{\mathbf{df}}{\mathbf{dx}}\mathbf{dx}$. Green's Theorem gives equality; cancel $b - a$.

14 $\int_P^Q \mathbf{S}\cdot d\mathbf{R} = \oint -\mathbf{y}\,\mathbf{dx} + \mathbf{x}\,\mathbf{dy}$ since the integrals along the axes are zero. By Green's Theorem this is $\iint \mathbf{2dx}\,\mathbf{dy} = 2$ times area between path and axes.

16 $\oint \mathbf{F}\cdot\mathbf{n}ds = \int xy\,dy = \mathbf{\frac{1}{2}}$ up the right side of the square where $\mathbf{n} = \mathbf{i}$ (other sides give zero).
Also $\int_0^1\int_0^1(y + 0)dx\,dy = \mathbf{\frac{1}{2}}$.

18 In the double integral $M_x = \frac{\partial}{\partial x}(\frac{-y}{\sqrt{x^2+y^2}}) = \frac{xy}{(x^2+y^2)^{3/2}}$ and $N_y = \frac{\partial}{\partial y}(\frac{x}{\sqrt{x^2+y^2}}) = \frac{-xy}{(x^2+y^2)^{3/2}}$

so $M_x + N_y = 0$: Double integral = **0**. Along the bottom edge (where $y = 0$ and $\mathbf{n} = -\mathbf{j}$) the line integral is $\int \frac{\mathbf{S}}{r}\cdot\mathbf{n}\,ds = \int_0^1 \frac{-x\,dx}{\sqrt{x^2+0^2}} = -\mathbf{1}$. The right side ($x = 1$ and $\mathbf{n} = \mathbf{i}$) yields $\int_0^1 \frac{-y\,dy}{\sqrt{1^2+y^2}} = -\sqrt{1+y^2}]_0^1 = \mathbf{1} - \sqrt{\mathbf{2}}$. Back across the top ($y = 1, \mathbf{n} = \mathbf{j}$, **notice** $ds = -dx$!) $\int_1^0 \frac{-x\,dx}{\sqrt{x^2+1^2}} = \sqrt{2} - 1$. Down the left side (notice $ds = -dy$!) gives **+1**. Adding the four sides $\oint \frac{\mathbf{S}}{r}\cdot\mathbf{n}\,ds = \mathbf{0}$.

20 $\mathbf{F} = \text{grad } r = (\frac{x}{r}, \frac{y}{r})$ has $\mathbf{F}\cdot\mathbf{n} = 0$ along the x axis where $\mathbf{n} = -\mathbf{j}$ and $y = 0$. On the unit circle $\mathbf{n}$ is equal to $\mathbf{F}$ (unit vector pointing outward) so $\mathbf{F}\cdot\mathbf{n} = 1$. Around the semicircle $\oint \mathbf{F}\cdot\mathbf{n}ds = \int_0^\pi 1\,d\theta = \pi$. The double integral has $M_x = \frac{\partial}{\partial x}(\frac{x}{r}) = \frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x} = \frac{r^2}{r^3} - \frac{x^2}{r^3} = \frac{y^2}{r^3}$. Similarly $N_y = \frac{\partial}{\partial y}(\frac{y}{r}) = \frac{x^2}{r^3}$ and $M_x + N_y = \frac{r^2}{r^3} = \frac{1}{r}$. The double integral is $\int_0^\pi\int_0^1 \frac{1}{r}(r\,dr\,d\theta) = \pi$.

22 $\oint \mathbf{F}\cdot\mathbf{n}\,ds$ is the same through a square and a circle because the difference is $\iint(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y})dx\,dy = \iint \text{div }\mathbf{F}dx\,dy = 0$ over the region in between.

24 $\oint(\cos^3 y\,dy - \sin^3 x\,dx) = \iint(0-0)dx\,dy = 0$. A different example would be more revealing.

26 $\text{div }\frac{\mathbf{S}}{r^2} = \frac{\partial}{\partial x}(\frac{-y}{x^2+y^2}) + \frac{\partial}{\partial y}(\frac{x}{x^2+y^2}) = \frac{-2xy+2yx}{(x^2+y^2)^2} = 0$. Integrating $\frac{y}{x^2+y^2}$ gives $g = \frac{\mathbf{1}}{\mathbf{2}}\ln(\mathbf{x^2} + \mathbf{y^2}) = \ln\mathbf{r}$. This is infinite at $x = y = 0$.

28 $\frac{\partial g}{\partial y} = M$ and $\frac{\partial g}{\partial x} = -N$ are compatible when $M_x + N_y = \mathbf{g_{yx}} - \mathbf{g_{xy}} = \mathbf{0}$. If also $N_x = M_y$ then $\mathbf{g_{xx}} + \mathbf{g_{yy}} = -N_x + M_y = \mathbf{0}$ and g solves Laplace's equation.

30 $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 3y^2 - 3y^2 = 0$. Solve $\frac{\partial g}{\partial y} = 3xy^2$ for $g = \mathbf{xy^3}$ and check $\frac{\partial g}{\partial x} = y^3$.

32 $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0 + 0$. Solve $\frac{\partial g}{\partial y} = y^2$ for $g = \frac{1}{3}y^3 + C(x)$ and add $C(x) = \frac{1}{3}x^3$ to give $\frac{\partial g}{\partial x} = x^2$. Then $g = \frac{\mathbf{1}}{\mathbf{3}}(\mathbf{y^3} + \mathbf{x^3})$.

34 $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = e^{x+y} - e^{x+y} = 0$. Solve $\frac{\partial g}{\partial y} = e^{x+y}$ for $g = \mathbf{e^{x+y}}$ and check $\frac{\partial g}{\partial x} = e^{x+y}$.

36 $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = y + x \neq 0$ **(no stream function)**.

38 $g(Q) = \int_P^Q \mathbf{F}\cdot\mathbf{n}ds$ starting from $g(P) = 0$. Any two paths give the same integral because forward on one and back on the other gives $\oint \mathbf{F}\cdot\mathbf{n}\,ds = 0$, provided the tests $E - H$ for a stream function are passed.

40 With $M_x + N_y = 0$ we can solve $\partial g/\partial y = M = 3x^2 - 3y^2$ and $\partial g/\partial x = -M = 6xy$ to find $g = \mathbf{3x^2y} - \mathbf{y^3}$. Then $f_x = g_y = M$ and $f_y = -g_x = N$.

42 $Mdy - Ndx$ is an exact differential if $\frac{\partial M}{\partial x} = -\frac{\partial N}{\partial y}$. (Then there is a stream function g.)

44 $\oint\int \mathbf{S}\cdot d\mathbf{R} = \oint -y\,dx + x\,dy = \mathbf{2} \times \textbf{area} \neq 0$.

46 Simply connected: **2, 3, 6(?), 7.** The other regions contain circles that can't shrink to points.

15.4 Surface Integrals (page 581)

A small piece of the surface $z = f(x,y)$ is nearly **flat**. When we go across by dx, we go up by $(\partial\mathbf{z}/\partial\mathbf{x})\mathbf{dx}$. That movement is $\mathbf{A}dx$, where the vector $\mathbf{A}$ is $\mathbf{i} + \mathbf{dz/dx\,k}$. The other side of the piece is $\mathbf{B}dy$, where $\mathbf{B} = \mathbf{j} + (\partial\mathbf{z}/\partial\mathbf{y})\mathbf{k}$. The cross product $\mathbf{A}\times\mathbf{B}$ is $\mathbf{N} = -\partial\mathbf{z}/\partial\mathbf{x}\,\mathbf{i} - \partial\mathbf{z}/\partial\mathbf{y}\,\mathbf{j} + \mathbf{k}$. The area of the piece is $dS = |\mathbf{N}|dx\,dy$. For the surface $x = xy$, the vectors are $A = \iint\sqrt{\mathbf{1} + \mathbf{x^2} + \mathbf{y^2}}\mathbf{dx\,dy}$ and $N = -\mathbf{y}\,\mathbf{i} - \mathbf{x} + \mathbf{k}$. The area integral is $\iint dS = \mathbf{i} + \mathbf{y}\,\mathbf{k}$.

With parameters u and v, a typical point on a 45° cone is $x = u\cos v, y = \mathbf{u\sin v}, z = \mathbf{u}$. A change in u moves that point by $\mathbf{A\,du} = (\mathbf{cos\,v\,i} + \mathbf{sin\,v\,j} + \mathbf{k})du$. The change in v moves the point by $\mathbf{B}dv = (-\mathbf{u\,sin v\,i} + \mathbf{u\,cos\,v\,j})\mathbf{dv}$. The normal vector is $\mathbf{N} = \mathbf{A}\times\mathbf{B} = -\mathbf{u\,cos\,v\,i} - \mathbf{u\,sin\,v\,j} + \mathbf{u\,k}$. The area is $dS = \sqrt{\mathbf{2}}\,\mathbf{u}du\,dv$. In this example $\mathbf{A}\cdot\mathbf{B} - \mathbf{0}$ so the small piece is a a **rectangle** and $dS = |\mathbf{A}||\mathbf{B}|du\,dv$.

For flux we need $\mathbf{n}dS$. The **unit normal** vector $\mathbf{n}$ is $\mathbf{N} = \mathbf{A}\times\mathbf{B}$ divided by $|\mathbf{N}|$. For a surface $z = f(x,y)$, the product $\mathbf{n}dS$ is the vector $\mathbf{N}\ dx\ dy$ (to memorize from table). The particular surface $z = xy$ has $\mathbf{n}dS = (-\mathbf{yi} - \mathbf{xj} + \mathbf{k})dx\ dy$. For $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ the flux through $z = xy$ is $\mathbf{F} \cdot \mathbf{n}dS = -\mathbf{xy}\ dx\ dy$.

On a $30°$ cone the points are $x = 2u\cos v, y = 2u\sin v, z = u$. The tangent vectors are $\mathbf{A} = \mathbf{2\cos v\, i} + \mathbf{2\sin v\, j} + \mathbf{k}$ and $\mathbf{B} = -\mathbf{2\,u\sin v\, i} + \mathbf{2\,u\cos v\, j}$. This cone has $\mathbf{n}dS = \mathbf{A} \times \mathbf{B}\ du\ dv = (-\mathbf{2u\cos v\, i} - \mathbf{2\, u\sin v\, j} + \mathbf{4\, u\, k})\mathbf{du\ dv}$. For $\mathbf{F} = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$, the flux element through the cone is $\mathbf{F} \cdot \mathbf{n}dS =$ **zero.** The reason for this answer is **that F is along the cone.** The reason we don't compute flux through a Möbius strip **is that N cannot be defined (the strip is not orientable).**

1 $\mathbf{N} = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}; dS = \sqrt{1+4x^2+4y^2}\ dx\ dy; \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\ r\ dr\ d\theta = \frac{\pi}{6}(17^{3/2} - 1)$
3 $\mathbf{N} = -\mathbf{i} + \mathbf{j} + \mathbf{k}; dS = \sqrt{3}\ dx\ dy$; area $\sqrt{3}\pi$
5 $\mathbf{N} = \frac{-x\mathbf{i} - y\mathbf{j}}{\sqrt{1-x^2-y^2}} + \mathbf{k}; dS = \frac{dx\ dy}{\sqrt{1-x^2-y^2}}; \int_0^{2\pi}\int_0^{1/\sqrt{2}} \frac{r\ dr d\theta}{\sqrt{1-r^2}} = \pi(2-\sqrt{2})$
7 $\mathbf{N} = -7\mathbf{j} + \mathbf{k}; dS = 5\sqrt{2}\ dx\ dy$; area $5\sqrt{2}A$
9 $\mathbf{N} = (y^2 - x^2)\mathbf{i} - 2xy\mathbf{j} + \mathbf{k}; dS = \sqrt{1 + (y^2-x^2)^2 + 4x^2y^2}dx\ dy = \sqrt{1+(y^2+x^2)^2}dx\ dy;$
$\int_0^{2\pi}\int_0^1 \sqrt{1+r^4}\ r\ dr\ d\theta = \frac{\pi}{\sqrt{2}} + \frac{\pi\ln(1+\sqrt{2})}{2}$
11 $\mathbf{N} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}; dS = 3dx\ dy$; 3(area of triangle with $2x + 2y \le 1$) $= \frac{3}{8}$
13 $\pi a\sqrt{a^2+h^2}$ **15** $\int_0^1\int_0^{1-y} xy(\sqrt{3}\ dx\ dy) = \frac{\sqrt{3}}{24}$
17 $\int_0^{2\pi}\int_0^{\pi/4} \sin^2\phi\cos\phi\sin\theta\cos\theta(\sin\phi\ d\phi\ d\theta) = 0$ **19** $\mathbf{A} = \mathbf{i}+\mathbf{j}+2\mathbf{k}; \mathbf{B} = \mathbf{j}+\mathbf{k}; \mathbf{N} = -\mathbf{i}-\mathbf{j}+\mathbf{k}; dS = \sqrt{3}\ du\ dv$
21 $\mathbf{A} = -\sin u(\cos v\ \mathbf{i} + \sin v\ \mathbf{j}) + \cos u\ \mathbf{k}; \mathbf{B} = -(3+\cos u)\sin v\ \mathbf{i} + (3+\cos u)\cos v\ \mathbf{j};$
$\mathbf{N} = -(3+\cos u)(\cos u\cos v\ \mathbf{i} + \cos u\sin v\ \mathbf{j} + \sin u\ \mathbf{k}); dS = (3+\cos u)du\ dv$
23 $\iint(-M\frac{\partial f}{\partial x} - N\frac{\partial f}{\partial y} + P)dx\ dy = \iint(-2x^2 - 2y^2 + z)dx\ dy = \iint -r^2(r\ dr\ d\theta) = -8\pi$
25 $\mathbf{F}\cdot\mathbf{N} = -x + y + z = 0$ on plane
27 $\mathbf{N} = -\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{F} = (v+u)\ \mathbf{i} - u\ \mathbf{j}, \iint \mathbf{F}\cdot\mathbf{N}dS = \iint -v\ du\ dv = 0$
29 $\iint dS = \int_0^{2\pi}\int_0^{2\pi}(3+\cos u)du\ dv = 12\pi^2$ **31** Yes **33** No
35 $\mathbf{A} = \mathbf{i} + f'\cos\theta\ \mathbf{j} + f'\sin\theta\ \mathbf{k}; \mathbf{B} = -f\sin\theta\ \mathbf{j} + f\cos\theta\ \mathbf{k}; \mathbf{N} = ff'\mathbf{i} - f\cos\theta\ \mathbf{j} - f\sin\theta\ \mathbf{k}; dS = |\mathbf{N}|dx\ d\theta =$
$f(x)\sqrt{1+f'^2}\ dx\ d\theta$

2 $\mathbf{N} = -2x\ \mathbf{i} - 2y\ \mathbf{j} + \mathbf{k}$ and $dS = \sqrt{\mathbf{1+4x^2+4y^2}}\mathbf{dx\ dy}$. Then $\iint dS = \int_0^{2\pi}\int_2^{\sqrt{8}}\sqrt{1+4r^2}\ r\ dr\ d\theta =$
$\frac{\pi}{6}(\mathbf{33^{3/2} - 17^{3/2}})$.
4 $\mathbf{N} = -\mathbf{3i} - \mathbf{4j} + \mathbf{k}$ and $dS = \sqrt{\mathbf{26}}\ \mathbf{dx\ dy}$. Then area $= \int_0^1\int_0^1 \sqrt{26}dx\ dy = \sqrt{\mathbf{26}}$.
6 $\mathbf{N} = -\frac{x\mathbf{i}}{\sqrt{1-x^2-y^2}} - \frac{y\mathbf{j}}{\sqrt{1-x^2-y^2}} + \mathbf{k}$ and $dS = \frac{\mathbf{dx\ dy}}{\sqrt{\mathbf{1-x^2-y^2}}}$. Then area $= \int_0^{2\pi}\int_{1/\sqrt{2}}^1 \frac{rdr\ d\theta}{\sqrt{1-r^2}} =$
$[-2\pi\sqrt{1-r^2}]_{1/\sqrt{2}}^1 = \sqrt{\mathbf{2}}\pi$.
8 $\mathbf{N} = -\frac{x\mathbf{i}}{r} - \frac{y\mathbf{j}}{r} + \mathbf{k}$ and $dS = \frac{x^2+y^2+r^2}{r^2}dx\ dy = \sqrt{\mathbf{2}}\ \mathbf{dx\ dy}$. Then area $= \int_0^{2\pi}\int_a^b \sqrt{2}rdr\ d\theta = \sqrt{\mathbf{2}}\pi(\mathbf{b^2 - a^2})$.
10 $\mathbf{N} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $dS = \sqrt{\mathbf{3}}\ \mathbf{dx\ dy}$. Then surface area $= \sqrt{3}$ times base area $= \mathbf{2\sqrt{3}}$.
12 $z = \sqrt{a^2-x^2}$ gives $\mathbf{N} = \frac{-x\mathbf{i}}{\sqrt{a^2-x^2}} + \mathbf{k}$ and $dS = \frac{a\ dx\ dy}{\sqrt{a^2-x^2}}$. Then area $= 4\int_0^a\int_0^{\sqrt{a^2-y^2}}\frac{a\ dx\ dy}{\sqrt{a^2-x^2}}$.
14 $\mathbf{N} = -2x\mathbf{i} + \mathbf{k}$ and $dS = \sqrt{\mathbf{1+4x^2}}\mathbf{dx\ dy}$. Area $= \int_{-2}^2\int_0^3\sqrt{1+4x^2}dx\ dy = 4\int_0^3\sqrt{1+4x^2}\ dx =$
$8\int_0^3\sqrt{(\frac{1}{2})^2 + x^2}\ dx = 8[\frac{x}{2}\sqrt{x^2+(\frac{1}{2})^2} + \frac{1}{8}\ln|x + \sqrt{x^2+(\frac{1}{2})^2}|]_0^3 = 12\sqrt{9.25} + \ln|3+\sqrt{9.25}| - (\ln\frac{1}{2}) = \mathbf{39}$.
16 On the sphere $dS = \sin\phi\ d\phi\ d\theta$ and $g = x^2 + y^2 = \sin^2\phi$. Then $\int_0^{2\pi}\int_0^{\pi/2}\sin^3\phi\ d\phi\ d\theta = 2\pi(\frac{2}{3}) = \frac{\mathbf{4\pi}}{\mathbf{3}}$.
18 $x = 2\cos v, y = 2\sin v$, and $dS = \mathbf{2du\ dv}$. Then $\iint g\ dS = \int_0^{2\pi}\int_0^3 2\cos v(2du\ dv) = \mathbf{0}$.
20 $\mathbf{A} = v\mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{B} = u\mathbf{i} + \mathbf{j} - \mathbf{k}, \mathbf{N} = \mathbf{A}\times\mathbf{B} = -2\mathbf{i} + (u+v)\mathbf{j} + (v-u)\mathbf{k}, dS = \sqrt{4+2u^2+2v^2}du\ dv$.
22 $\mathbf{A} = \cos v\ \mathbf{i} + \sin v\ \mathbf{j}, \mathbf{B} = -u\sin v\ \mathbf{i} + u\cos v\ \mathbf{j} + \mathbf{k}, \mathbf{N} = \sin v\ \mathbf{i} - \cos v\ \mathbf{j} + u\mathbf{k}, dS = \sqrt{\mathbf{2}}\ \mathbf{du\ dv}$.
24 $\iint \mathbf{F}\cdot\mathbf{n}dS = \int_0^{2\pi}\int_2^{\sqrt{8}} -r^3dr\ d\theta = -24\pi$. **26** $\iint \mathbf{F}\cdot\mathbf{n}dS = \iint 0\ dS = 0$.

28 $\mathbf{F}\cdot\mathbf{n}dS = ((u+v)\mathbf{i} - uv\mathbf{j})\cdot(-2\mathbf{i} + (u+v)\mathbf{j} + (v-u)\mathbf{k})du\,dv = (\mathbf{2u} + \mathbf{2v} - \mathbf{u^2v} - \mathbf{v^2u})\mathbf{du\,dv}$.
Integrate with $u = r\cos\theta, v = r\sin\theta : \int_0^{2\pi}\int_0^1(2r\cos\theta + 2r\sin\theta - r^3\cos^2\theta\sin\theta - r^3\sin^2\theta\cos\theta)rdrd\theta = 0$.

30 $\mathbf{A} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} - 2r\mathbf{k}, \mathbf{B} = -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j}, \mathbf{N} = \mathbf{A}\times\mathbf{B} = 2r^2\cos\theta\,\mathbf{i} + 2r^2\sin\theta\,\mathbf{j} + r\mathbf{k}$,
$\iint \mathbf{k}\cdot\mathbf{n}\,dS = \iint \mathbf{k}\cdot\mathbf{N}du\,dv = \int_0^{2\pi}\int_0^a r dr\,d\theta = \pi\mathbf{a^2}$ as in Example 12.

32 I think a "triple Möbius strip" is **orientable**.

34 The plane $z = ax + by$ has roof area $= \sqrt{a^2+b^2}$ times base area. So choose for example $a = 1$ and $b = \sqrt{2}$.

15.5 The Divergence Theorem (page 588)

In words, the basic balance law is **flow in = flow out**. The flux of $\mathbf{F}$ through a closed surface S is the double integral $\iint \mathbf{F}\cdot\mathbf{n}\mathbf{dS}$. The divergence of $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is $\mathbf{M_x + N_y + P_z}$, and it measures **the source at the point**. The total source is the triple integral $\iiint$ div $\mathbf{F}$ $\mathbf{dV}$. That equals the flux by the **Divergence Theorem**.

For $\mathbf{F} = 5z\mathbf{k}$ the divergence is **5**. If V is a cube of side a then the triple integral equals $\mathbf{5a^3}$. The top surface where $z = a$ has $\mathbf{n} = \mathbf{k}$ and $\mathbf{F}\cdot\mathbf{n} = 5a$. The bottom and sides have $\mathbf{F}\cdot\mathbf{n} =$ **zero**. The integral $\iint \mathbf{F}\cdot\mathbf{n}dS$ equals $\mathbf{5a^3}$.

The field $\mathbf{F} = \mathbf{R}/\rho^3$ has div $\mathbf{F} = 0$ except **at the origin**. $\iint \mathbf{F}\cdot\mathbf{n}dS$ equals $\mathbf{4\pi}$ over any surface around the origin. This illustrates Gauss's Law: **flux = 4π times source strength**. The field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$ has div $\mathbf{F} = \mathbf{0}$ and $\iint \mathbf{F}\cdot\mathbf{n}dS = \mathbf{0}$. For this $\mathbf{F}$, the flux out through a pyramid and in through its base are **equal**.

The symbol ∇ stands for $(\partial/\partial\mathbf{x})\mathbf{i} + (\partial/\partial\mathbf{y})\mathbf{j} + (\partial/\partial\mathbf{z})\mathbf{k}$. In this notation div $\mathbf{F}$ is $\nabla\cdot\mathbf{F}$. The gradient of f is $\nabla\mathbf{f}$. The divergence of grad f is $\nabla\cdot\nabla\mathbf{f}$ **or** $\nabla^2\mathbf{f}$. The equation div grad $f = 0$ is **Laplace's** equation.

The divergence of a product is $\text{div}(u\mathbf{V}) = \mathbf{u\,div\,V} + (\mathbf{grad\,u})\cdot\mathbf{V}$. Integration by parts in 3D is $\iiint u\,\text{div}\,\mathbf{V}dx\,dy\,dz = -\iiint \mathbf{V}\cdot\,\mathbf{grad\,u\,dx\,dy\,dz} + \iint \mathbf{u\,V}\cdot\mathbf{n\,dS}$. In two dimensions this becomes $\iint \mathbf{u}(\partial\mathbf{M}/\partial\mathbf{x} + \partial\mathbf{N}/\partial\mathbf{y})\mathbf{dx\,dy} = -\int(\mathbf{M}\,\partial\mathbf{u}/\partial\mathbf{x} + \mathbf{N}\,\partial\mathbf{u}/\partial\mathbf{y})\mathbf{dx\,dy} + \int \mathbf{u\,V}\cdot\mathbf{n\,ds}$. In one dimension it becomes **integration by parts**. For steady fluid flow the continuity equation is div $\rho\mathbf{V} = -\partial\rho/\partial\mathbf{t}$.

1 div $\mathbf{F} = 1, \iiint dV = \frac{4\pi}{3}$ **3** div $\mathbf{F} = 2x + 2y + 2z, \iiint \text{div}\,\mathbf{F}\,dV = 0$ **5** div $\mathbf{F} = 3, \iint 3dV = \frac{3}{6} = \frac{1}{2}$

7 $\mathbf{F}\cdot\mathbf{N} = \rho^2, \iint_{\rho=a}\rho^2 dS = 4\pi a^4$ **9** div $\mathbf{F} = 2z, \int_0^{2\pi}\int_0^{\pi/2}\int_0^a 2\rho\cos\phi(\rho^2\sin\phi\,d\rho\,d\phi\,d\theta) = \frac{1}{2}\pi a^4$

11 $\int_0^a\int_0^a\int_0^a(2x+1)dx\,dy\,dz = a^4 + a^3; -2a^2 + 2a^2 + 0 + a^4 + 0 + a^3$

13 div $\mathbf{F} = \frac{z}{\rho}, \iiint \frac{z}{\rho}dV = 0; \mathbf{F}\cdot\mathbf{n} = x, \iint xdS = 0$ **15** div $\mathbf{F} = 1; \iiint 1dV = \frac{\pi}{3}; \iiint 1\,dV = \frac{1}{6}$

17 div $(\frac{\mathbf{R}}{\rho^7}) = \frac{\text{div}\,\mathbf{R}}{\rho^7} + \mathbf{R}\cdot\text{grad}\frac{1}{\rho^7} = \frac{3}{\rho^7} - \frac{7}{\rho^8}\mathbf{R}\cdot\text{grad}\rho$

19 Two spheres, $\mathbf{n}$ radial out, $\mathbf{n}$ radial in, $\mathbf{n} = \mathbf{k}$ on top, $\mathbf{n} = -\mathbf{k}$ on bottom, $\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ on side;
$\mathbf{n} = -\mathbf{i}, -\mathbf{j}, -\mathbf{k}, \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ on 4 faces; $\mathbf{n} = \mathbf{k}$ on top, $\mathbf{n} = \frac{1}{\sqrt{2}}(\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} - \mathbf{k})$ on cone

21 $V =$ cylinder, $\iiint \text{div}\,\mathbf{F}\,dV = \iint(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y})dx\,dy$ (z integral $= 1$); $\iint \mathbf{F}\cdot\mathbf{n}dS =$
$\int Mdy - Ndx, z$ integral $= 1$ on side, $\mathbf{F}\cdot\mathbf{n} = 0$ top and bottom; Green's flux theorem.

23 div $\mathbf{F} = \frac{-3GM}{a^3} = -4\pi G$; at the center; $\mathbf{F} = 2\mathbf{R}$ inside, $\mathbf{F} = 2(\frac{a}{\rho})^3\mathbf{R}$ outside

25 div $\mathbf{u}_r = \frac{2}{\rho}, q = \frac{2\epsilon_0}{\rho}, \iint \mathbf{E}\cdot\mathbf{n}\,dS = \iint 1dS = 4\pi$ **27** F (div $\mathbf{F} = 0$); F; T($\mathbf{F}\cdot\mathbf{n} \le 1$); F

29 Plane circle; top half of sphere; div $\mathbf{F} = 0$

2 $\iint \mathbf{F} \cdot \mathbf{n} dS = \iiint 0 \, dV = 0$
4 $\iint \mathbf{F} \cdot \mathbf{n} dS = \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z) dx \, dy \, dz = 1 + 1 + 1 = 3$.
6 $\iint \mathbf{F} \cdot \mathbf{n} dS =$ (directly) $\iint dS = 4\pi \mathbf{a}^2$. By the Divergence Theorem: $\int_0^{2\pi} \int_0^{\pi} \int_0^{a} \frac{2}{\rho} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 4\pi a^2$
8 $\iint \mathbf{F} \cdot \mathbf{n} dS = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} 3\rho^4 \sin\phi d\rho \, d\phi \, d\theta = \frac{12\pi}{5} \mathbf{a}^5$.
10 div $\mathbf{F} = 0 + xe^y \sin z - xe^y \sin z = 0$ so $\iint \mathbf{F} \cdot \mathbf{n} dS = 0$.
12 An integral over a box with small side a is near ca^3. Here div $\mathbf{F} = 2x + 1$ has integral $a^4 + a^3$, which is near a^3 because a is small. Then $\mathbf{c} = 1$, which equals div $\mathbf{F}$ on the plane $x = 0$.
14 $\mathbf{R} \cdot \mathbf{n} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{i} = x = 1$ on one face of the box. On the five other faces $\mathbf{R} \cdot \mathbf{n} = 2, 3, 0, 0, 0$. The integral is $\int_0^3 \int_0^2 1 dy \, dz + \int_0^3 \int_0^1 2 dx \, dz + \int_0^2 \int_0^1 3 dx \, dy = 18$. Also div $\mathbf{R} = 1 + 1 + 1 = 3$ and $\int_0^3 \int_0^2 \int_0^1 3 dx \, dy \, dz = 18$.
16 The normal vectors to the cube are $\mathbf{n} = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$. Then $\iint \mathbf{F} \cdot \mathbf{n} dS = \int_0^1 \int_0^1 x \, dx \, dy +$ $\int_0^1 \int_0^1 (-x) dx \, dy + \int_0^1 \int_0^1 x \, dx \, dz + \int_0^1 \int_0^1 (-x) dx \, dx + \int_0^1 \int_0^1 0 dy \, dz + \int_0^1 \int_0^1 1 dy \, dz = 1$. Also $\iiint$ div $\mathbf{F} dV = \int_0^1 \int_0^1 \int_0^1 1 dx \, dy \, dz = 1$.
18 grad $f \cdot \mathbf{n}$ is the **directional** derivative in the normal direction $\mathbf{n}$ (also written $\frac{\partial f}{\partial n}$). The Divergence Theorem gives $\iiint$ div (grad f) $dV = \iint$ grad $f \cdot \mathbf{n} dS = \iint \frac{\partial f}{\partial n} dS$. But we are given that div (grad f) $= f_{xx} + f_{yy} + f_{zz}$ is zero.
20 Suppose $\mathbf{F}$ is perpendicular to $\mathbf{n}$ on the surface; then $\iint \mathbf{F} \cdot \mathbf{n} dS = 0$. Example on the unit sphere: $\mathbf{F}$ is any $q(x, y, z)$ times the spin field $-y\mathbf{i} + x\mathbf{j}$.
22 The spin field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ has div $\mathbf{F} = 0$ and $\mathbf{F} \cdot \mathbf{n} = 0$ on the unit sphere.
24 The flux of $\mathbf{F} = \mathbf{R}/\rho^3$ through an area A on a sphere of radius ρ is A/ρ^2, because $|\mathbf{F}| = 1/\rho^2$ and $\mathbf{F}$ is outward. The spherical box has $A/\rho^2 = \sin\phi d\phi \, d\theta$ on both faces (minus sign for face pointing in). No flow through sides of box perpendicular to $\mathbf{F}$. So net flow $=$ zero.
26 When the density ρ is constant (incompressible flow), the continuity equation becomes div $\mathbf{V} = 0$. If the flow is irrotational then $\mathbf{F} =$ grad f and the continuity equation is div (ρ grad f) $= -d\rho/dt$. If also $\rho =$ constant, then **div grad f = 0**: Laplace's equation for the "potential."
28 Extend E-F-G-H in Section 15.3 to 3 dimensions: **E** The total flux $\iint \mathbf{F} \cdot \mathbf{n} dS$ through every closed surface is zero **F**. Through all surfaces with the same boundary $\iint \mathbf{F} \cdot \mathbf{n} dS$ is the same **G** There is a **stream field g** for which $\mathbf{F} =$ curl $\mathbf{g}$ **H**. The divergence of $\mathbf{F}$ is zero (this is the quick test).
30 The boundary of a solid ball is a sphere. A sphere has no boundary. Similarly for a cube or a cylinder – the boundary is a closed surface and **that surface's boundary** is empty. This is a crucial fact in topology.

15.6 Stokes' Theorem and the Curl of F (page 595)

The curl of $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the vector $(\mathbf{P_y} - \mathbf{N_z})\mathbf{i} + (\mathbf{M_z} - \mathbf{P_x})\mathbf{j} + (\mathbf{N_x} - \mathbf{M_y})\mathbf{k}$. It equals the 3 by 3 determinant $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \mathbf{M} & \mathbf{N} & \mathbf{P} \end{vmatrix}$ The curl of $x^2\mathbf{i} + z^2\mathbf{k}$ is **zero**. For $\mathbf{S} = y\mathbf{i} - (x + z)\mathbf{j} + y\mathbf{k}$ the curl is $2\mathbf{i} - 2\mathbf{k}$. This $\mathbf{S}$ is a **spin** field $\mathbf{a} \times \mathbf{R} = \frac{1}{2}($ curl $\mathbf{F}) \times \mathbf{R}$, with axis vector $\mathbf{a} = \mathbf{i} - \mathbf{k}$. For any gradient field $f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$ the curl is **zero**. That is the important identity curl grad $f =$ **zero**. It is based on $f_{xy} = f_{yx}$ and

$\mathbf{f_{xz}} = \mathbf{f_{zx}}$ and $\mathbf{f_{yz}} = \mathbf{f_{zy}}$. The twin identity is **div curl F = 0.**

The curl measures the **spin (or turning)** of a vector field. A paddlewheel in the field with its axis along **n** has turning speed $\frac{1}{2}\mathbf{n}\cdot$ **curl F**. The spin is greatest when **n** is in the direction of **curl F**. Then the angular velocity is $\frac{1}{2}|$ **curl F**$|$.

Stokes' Theorem is $\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS$. The curve C is the **boundary** of the surface S. This is **Green's** Theorem extended to **three** dimensions. Both sides are zero when **F** is a gradient field because the **curl is zero.**

The four properties of a conservative field are **A** : $\oint \mathbf{F}\cdot d\mathbf{R} = 0$ **and B** : $\int_P^Q \mathbf{F}\cdot d\mathbf{R}$ **depends only on P and Q and C : F is the gradient of a potential function f(x, y, z) and D : curl F = 0.**
The field $y^2z^2\mathbf{i} + 2xy^2z\mathbf{k}$ fails test **D**. This field is the gradient of **no f**. The work $\int \mathbf{F}\cdot d\mathbf{R}$ from (0,0,0) to (1,1,1) is $\frac{3}{5}$ **along the straight path x = y = z = t**. For every field **F**, $\iint$ curl $\mathbf{F}\cdot \mathbf{n}dS$ is the same out through a pyramid and up through its base **because they have the same boundary, so** $\oint \mathbf{F}\cdot d\mathbf{R}$ **is the same.**

1 curl $\mathbf{F} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ **3** curl $\mathbf{F} = 0$ **5** curl $\mathbf{F} = 0$ **7** $f = \frac{1}{2}(x+y+z)^2$
9 curl $x^m\mathbf{i} = 0$; $x^n\mathbf{j}$ has zero curl if $n = 0$ **11** curl $\mathbf{F} = 2y\mathbf{i}$; $\mathbf{n} = \mathbf{j}$ on circle so $\iint \mathbf{F}\cdot\mathbf{n}dS = 0$
13 curl $\mathbf{F} = 2\mathbf{i}+2\mathbf{j}, \mathbf{n} = \mathbf{i}, \iint$ curl $\mathbf{F}\cdot\mathbf{n}dS = \iint 2\, dS = 2\pi$
15 Both integrals equal $\int \mathbf{F}\cdot d\mathbf{R}$; Divergence Theorem, V = region between S and T, always div curl $\mathbf{F} = 0$
17 Always div curl $\mathbf{F} = 0$ **19** $f = xz + y$ **21** $f = e^{x-z}$ **23** $\mathbf{F} = y\mathbf{k}$
25 curl $\mathbf{F} = (a_3b_2 - a_2b_3)\mathbf{i} + (a_1b_3 - a_3b_1)\mathbf{j} + (a_2b_1 - a_1b_2)\mathbf{k}$ **27** curl $\mathbf{F} = 2\omega\mathbf{k}$; curl $\mathbf{F} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} = 2\omega/\sqrt{3}$
29 $\mathbf{F} = x(a_3z + a_2y)\mathbf{i} + y(a_1x + a_3z)\mathbf{j} + z(a_1x + a_2y)\mathbf{k}$
31 curl $\mathbf{F} = -2\mathbf{k}, \iint -2\mathbf{k}\cdot\mathbf{R}dS = \int_0^{2\pi}\int_0^{\pi/2} -2\cos\phi(\sin\phi\, d\phi\, d\theta) = -2\pi; \int y\, dx - x\, dy =$
$\int_0^{2\pi}(-\sin^2 t - \cos^2 t)dt = -2\pi$
33 curl $\mathbf{F} = 2\mathbf{a}, 2\iint(a_1x + a_2y + a_3z)dS = 0 + 0 + 2a_3\int_0^{2\pi}\int_0^{\pi/2}\cos\phi\sin\phi\, d\phi\, d\theta = 2\pi a_3$
35 curl $\mathbf{F} = -\mathbf{i}, \mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}, \iint \mathbf{F}\cdot\mathbf{n}dS = -\frac{1}{\sqrt{3}}\pi r^2$
37 $g = \frac{y^2}{2} - \frac{x^3}{3}$ = stream function; zero divergence
39 div $\mathbf{F}$ = div $(\mathbf{V}+\mathbf{W})$ = div $\mathbf{V}$ so y = div $\mathbf{V}$ so $\mathbf{V} = \frac{y^2}{2}\mathbf{j}$ (has zero curl). Then $\mathbf{W} = \mathbf{F} - \mathbf{V} = xy\mathbf{i} - \frac{y^2}{2}\mathbf{j}$
41 curl (curl $\mathbf{F}$) = curl $(-2y\mathbf{k}) = -2\mathbf{i}$; grad (div $\mathbf{F}$) = grad $2x = 2\mathbf{i}$; $\mathbf{F}_{xx} + \mathbf{F}_{yy} + \mathbf{F}_{zz} = 4\mathbf{i}$
43 curl $\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} = \mathbf{a}\sin t$ so $\mathbf{E} = \frac{1}{2}(\mathbf{a}\times\mathbf{R})\sin t$
45 $\mathbf{n} = \mathbf{j}$ so $\int M dx + P dz = \iint(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x})dx\, dz$ **47** $M_y^* = M_y + M_z f_y + P_y f_x + P_z f_y f_x + P f_{xy}$
49 $\int \mathbf{F}\cdot d\mathbf{R} = \iint$ curl $\mathbf{F}\cdot\mathbf{n}\, dS; \iint \mathbf{F}\cdot\mathbf{n}dS = \iiint$ div $\mathbf{F}\, dV$

2 curl $\mathbf{F} = \mathbf{0}$ because curl of gradient is always zero. **4** curl $\mathbf{F} = -\mathbf{i}+\mathbf{j}-\mathbf{k}$ from equation (1).
6 curl $\mathbf{F} = 2\mathbf{i}+2\mathbf{j}$ from Example 2: curl $(\mathbf{a}\times\mathbf{R}) = 2\mathbf{a}$.
8 $f(x,y,z) = r^{n+1}/2(n+1)$ has grad $f = \rho^n\mathbf{R}$ (so its curl is zero).
10 curl $(a_1x + a_2y + a_3z)\mathbf{k} = a_2\mathbf{i} - a_1\mathbf{j}$ which is zero when $a_1 = 0$ and $a_2 = 0$.
12 curl $(\mathbf{i}\times\mathbf{R}) = 2\mathbf{i}$ directly (or by Example 2 with $\mathbf{a} = \mathbf{i}$). Then $\oint \mathbf{F}\cdot d\mathbf{R} = \iint$ curl $\mathbf{F}\cdot\mathbf{n}dS = 0$ since $\mathbf{n} = \mathbf{j}$ is perpendicular to $\mathbf{i}$.
14 $\mathbf{F} = (x^2+y^2)\mathbf{k}$ so curl $\mathbf{F} = 2(y\mathbf{i} - x\mathbf{j})$. (Surprise that this $\mathbf{F} = \mathbf{a}\times\mathbf{R}$ has curl $\mathbf{F} = 2\mathbf{a}$ even with nonconstant $\mathbf{a}$.) Then $\oint \mathbf{F}\cdot d\mathbf{R} = \iint$ curl $\mathbf{F}\cdot\mathbf{n}dS = 0$ since $\mathbf{n} = \mathbf{k}$ is perpendicular to curl $\mathbf{F}$.
16 C is the equator (the common boundary of S and T); V is the whole ball (the earth). Note that $\mathbf{n}$ doesn't point out in the bottom half T, or the direction around C would be opposite.

For $\mathbf{F} = \mathbf{R}$ (position vector), $\iint_S \mathbf{F} \cdot \mathbf{n} dS = -\iint_T \mathbf{F} \cdot \mathbf{n} dS$.

18 If curl $\mathbf{F} = \mathbf{0}$ then $\mathbf{F}$ is the gradient of a potential: $\mathbf{F} =$ grad f. Then div $\mathbf{F} = 0$ is div grad $f = 0$ which is Laplace's equation.

20 The potential is $f = x^2 y$. **22** The potential is $f = xyz + \frac{1}{3}z^3$.

24 Start with one field that has the required curl. (Can take $\mathbf{F} = \frac{1}{2}\mathbf{i} \times \mathbf{R} = -\frac{z}{2}\mathbf{j} + \frac{y}{2}\mathbf{k}$). Then add any $\mathbf{F}$ with curl zero (particular solution plus homogeneous solution as always). The fields with curl $\mathbf{F} = \mathbf{0}$ are gradient fields $\mathbf{F} =$ grad f, since curl grad $= \mathbf{0}$. Answer: $\mathbf{F} = \frac{1}{2}\mathbf{i} \times \mathbf{R} +$ any grad f.

26 $\mathbf{F} = y\mathbf{i} - x\mathbf{k}$ has curl $\mathbf{F} = \mathbf{j} - \mathbf{k}$. (a) Angular velocity $= \frac{1}{2}$ curl $\mathbf{F} \cdot \mathbf{n} = \frac{1}{2}$ if $\mathbf{n} = \mathbf{j}$.
(b) Angular velocity $= \frac{1}{2}|\text{curl}\mathbf{F}| = \frac{\sqrt{2}}{2}$ (c) Angular velocity $= 0$.

28 One possibility: $\mathbf{F} = \frac{x^2+y^2}{2}\mathbf{k}$ has curl $\mathbf{F} =$ spin field $\mathbf{S}$. Other possibilities: $\mathbf{F} = \frac{x^2+y^2}{2}\mathbf{k}$ + any grad f.

30 **False** (curl $\mathbf{F} =$ curl $\mathbf{G}$ means curl $(\mathbf{F} - \mathbf{G}) = \mathbf{0}$ but not $\mathbf{F} - \mathbf{G} = \mathbf{0}$). **True** (curl $(\mathbf{F} - \mathbf{G}) = \mathbf{0}$ makes $\mathbf{F} - \mathbf{G}$ a gradient field). **False** ($\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{G} = \mathbf{0}$ have the same curl (zero) but div $\mathbf{F} = 3$).

32 Curl $\mathbf{R}/\rho^2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/\rho^2 & y/\rho^2 & z/\rho^2 \end{vmatrix}$ has $\mathbf{i}$ component $z\frac{\partial}{\partial y}\rho^{-2} - y\frac{\partial}{\partial z}\rho^{-2} = 0$. Similarly for $\mathbf{j}$ and $\mathbf{k}$: thus curl $\mathbf{F} = \mathbf{0}$ and $\iint$ curl $\mathbf{F} \cdot \mathbf{n} dS = 0$ and (separately) $\oint \mathbf{F} \cdot d\mathbf{R} = \oint M dx + N dy = \oint x\,dx + y\,dy = 0$.

34 Based on Problem 47 of Section 11.3, the triple vector product $(\mathbf{a} \times \mathbf{R}) \times \mathbf{R}$ is $\mathbf{F} = (\mathbf{a} \cdot \mathbf{R})\mathbf{R} - (\mathbf{R} \cdot \mathbf{R})\mathbf{a} = (ax + by + cz)\mathbf{R} - (x^2 + y^2 + z^2)\mathbf{a}$. Then by Problem 42 b of this section, or directly, the curl is grad $(ax + by + cz) \times \mathbf{R} - \text{grad}(x^2 + y^2 + z^2) \times \mathbf{a} = \mathbf{a} \times \mathbf{R} - 2\mathbf{R} \times \mathbf{a} = 3\mathbf{a} \times \mathbf{R}$. Now $\iint$ curl $\mathbf{F} \cdot \mathbf{n} dS = 0$ since $\mathbf{n} = \frac{\mathbf{R}}{|\mathbf{R}|}$ is perpendicular to the cross product curl $\mathbf{F} = 3\mathbf{a} \times \mathbf{R}$.
Also, $\oint \mathbf{F} \cdot d\mathbf{R} = \int (\mathbf{a} \cdot \mathbf{R})\mathbf{R} \cdot d\mathbf{R} - (\mathbf{R} \cdot \mathbf{R})\mathbf{a} \cdot d\mathbf{R} = 0$ because $\mathbf{R} \cdot d\mathbf{R} = 0$ on the circle and $\mathbf{R} \cdot \mathbf{R} = 1$.

36 curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & xyz \end{vmatrix} = \mathbf{i}(xz) + \mathbf{j}(1 - yz) + \mathbf{k}(1)$ and $\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. So curl $\mathbf{F} \cdot \mathbf{n} = x^2 z + y - y^2 z + z$. By symmetry $\iint x^2 z\, dS = \iint y^2 z dS$ on the half sphere and $\iint y dS = 0$.
This leaves $\iint z\, dS = \int_0^{2\pi}\int_0^{\pi/2} \cos\phi(\sin\phi\, d\phi\, d\theta) = \frac{1}{2}(2\pi) = \pi$.

38 (The expected method is trial and error) $\mathbf{F} = 5yz\mathbf{i} + 2xy\mathbf{k} +$ any grad f.

40 Work $= \oint \mathbf{B} \cdot d\mathbf{R} = \iint$ (curl $\mathbf{B}$) $\cdot \mathbf{n} dx\, dy = \iint \mu\mathbf{J} \cdot \mathbf{n} dx\, dy$. So work is μ times current through C.

42 (a) curl $v\mathbf{i} = \frac{\partial v}{\partial z}\mathbf{j} - \frac{\partial v}{\partial y}\mathbf{k}$. Then curl (curl $\mathbf{F}$) $= (-\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 v}{\partial z^2})\mathbf{i} + \frac{\partial^2 v}{\partial x\,\partial y}\mathbf{j} + \frac{\partial^2 v}{\partial x\,\partial z}\mathbf{k}$. Also grad (div $\mathbf{F}$) $= \frac{\partial^2 v}{\partial x^2}\mathbf{i} + \frac{\partial^2 v}{\partial x\,\partial y}\mathbf{j} + \frac{\partial^2 v}{\partial x\,\partial z}\mathbf{k}$. The difference is $(v_{xx} + v_{yy} + v_{zz})\mathbf{i}$. *Note:* The same steps for the $\mathbf{j}$ and $\mathbf{k}$ components give identity (a) for any $\mathbf{F}$. My favorite is to square this matrix:

$$\begin{bmatrix} \text{curl} & \text{grad} \\ -\text{div} & 0 \end{bmatrix}\begin{bmatrix} \text{curl} & \text{grad} \\ -\text{div} & 0 \end{bmatrix} = \begin{bmatrix} \text{curl curl} - \text{grad div} & 0 \\ 0 & -\text{divgrad} \end{bmatrix} = \nabla^2 I!!$$

(b) curl $(fv\mathbf{i}) = (f_z v + f v_z)\mathbf{j} - (f v_y + f_y v)\mathbf{k}$. This is f curl $\mathbf{F} = f(v_z\mathbf{j} - v_y\mathbf{k})$ added to (grad f) $\times \mathbf{F} = f_z v\,\mathbf{j} - f_y v\mathbf{k}$. Again the identity extends to any $\mathbf{F}$.

44 $\mathbf{F} \times \mathbf{G} = (Np - Pn)\mathbf{i} + (Pm - Mp)\mathbf{j} + (Mn - Nm)\mathbf{k}$. Its divergence is the sum of x, y, and z derivatives: $[N_x p + N p_x - P_x n - P n_x] + [P_y m + P m_y - M_y p - M p_y] + [M_z n + M n_z - N_z m - N m_z]$. Note that m multiplies $P_y - N_z$, the first component of curl $\mathbf{F}$. This starts $\mathbf{G} \cdot$ curl $\mathbf{F} - \mathbf{F} \cdot$ curl $\mathbf{G}$, as we want.

46 **False.** Certainly $\mathbf{G} \times \mathbf{F}$ would be perpendicular to $\mathbf{F}$ but $\nabla \times \mathbf{F}$ is something different. For example $\mathbf{F} = \mathbf{i} + y\mathbf{k}$ has $\nabla \times \mathbf{F} = \mathbf{i}$ so $(\nabla \times \mathbf{F}) \cdot \mathbf{F} = 1$.

48 S = roof, its shadow = ground floor, C = edge of roof, shadow of C = boundary of ground floor. Similarly for spherical cap $x^2 + y^2 + z^2 = 1$ above $z = \frac{1}{2}$. Note C is on the plane $z = \frac{1}{2}$ and its shadow is a circle around the shadow of the cap, down on the plane $z = 0$.

50 curl $\mathbf{V}$ = curl $(-x\mathbf{k}) = \mathbf{j}$. A wheel in the xz plane has $\mathbf{n} = \mathbf{j}$ so it spins at full speed. A wheel perpendicular to $\mathbf{j}$ will not spin, if it is in the xy plane with $\mathbf{n} = \mathbf{k}$.