Problem Set 6 Solution

17.881/882

November 5, 2004

1 Gibbons 2.4

Let us find the best-response for player 2.

If $c_1 \geq R \Longrightarrow U_2 = V \quad \forall c_2$

If $c_2 < 0 \Longrightarrow U_2 = V - c_2^2 \quad \forall c_2 \geq R - c_1$ and $U_2 = 0\ \forall c_2 < R - c_1$.

From this, we have

$BR_2(c_1) =$

0	if $c_1 \geq R$ or $c_1 < R - \sqrt{V}$
$R - c_1$	if $R - \sqrt{V} < c_1 < R$
$\epsilon\{0, \sqrt{V}\}$	if $c_1 = R - \sqrt{V}$

Anticipating this response from player 2, player 1 conjectures that his payoff is as follows.

If $c_1 \geq R \Longrightarrow U_1 = V - c_1^2$

If $R - \sqrt{V} < c_1 < R \Longrightarrow U_1 = \delta V - c_1^2$

We need to consider different cases here. If $R - \sqrt{V} < 0$, then we have characterised all possible payoffs for $c_1 \geq 0$.

If $R - \sqrt{V} = 0$, then we have that if $c_1 = 0$, $U_1 \epsilon \{\delta V, 0\}$ depending on the decision of player 2 to invest or not.

If $R - \sqrt{V} > 0$, then for $c_1 \epsilon [0, R - \sqrt{V})$, $U_1 = -c_1^2$ and for $c_1 = R - \sqrt{V}$, $U_1 \epsilon \{\delta V - c_1^2, 0\}$ depending on the decision of player 2 to invest or not.

From these observations, we can derive the Nash Equilibrium **outcomes** (I stress outcomes; I'll only specify an outcome for player 2; a Nash Equilibrium strategy would write the full best-response correspondence for player 2 as written above).

1.1 $R-\sqrt{V}<0$

Here, player 1 is choosing between $c_1=0$ and $c_1=R$, with $U_1(0,BR_2(0))=\delta V$; $U_1(R,BR_2(R))=V-R^2$

Let us write c_i^{NEO} for the outcome of player i's choice in a Nash Equilibrium. So, we have $(c_1^{NEO},c_2^{NEO})=$

$(0,R)$	if $R>[(1-\delta)V]^{1/2}$
$(R,0)$	if $R<[(1-\delta)V]^{1/2}$
$\{(0,R),(R,0)\}$	if $R=[(1-\delta)V]^{1/2}$

1.2 $R-\sqrt{V}=0$

Here, we add the possibility that player 1 plays $R-\sqrt{V}$, where his payoffs depend on player 2's strategy. Player 2 is indifferent between $c_2=\sqrt{V}$ and $c_2=0$. Let us assume that player 2 is playing the former strategy with probability p and the latter with probability $1-p$. Then it is easy to see that there is no Nash Equilibrium where $p\neq 1$. Why? Because then player 1 has no best-response. $U_1(R,BR_2(R))=V-R^2=0$, $U_1(\varepsilon,BR_2(\varepsilon))=\delta V-\varepsilon^2$ and $U_1(0,p*\sqrt{V}+(1-p)*0)=p\delta V$.

Then, if $p\neq 1$, $U_1(\varepsilon,BR_2(\varepsilon))>U_1(0,p*\sqrt{V}+(1-p)*0)\Leftrightarrow\varepsilon<\sqrt{(1-p)\delta V}$, which can always be satisfied for ε sufficiently small. But, there is no unique strategy $\varepsilon>0$ that maximises $U_1(\varepsilon,BR_2(\varepsilon))$... So, we will assume that $p=1$. Then we have $(c_1^{NEO},c_2^{NEO})=(0,R)$

1.3 $R-\sqrt{V}>0$

In addition to the previous case, we add the possibility that player 1 plays $c_1<R-\sqrt{V}$, in which case $U_1=-c_1^2$. Of course, we need only to retain the value $c_1=0$ within that interval. Yet again it is clear that we cannot have a Nash Equilibrium where player 2 is playing $c_2=0$ with some probability when $c_1=R-\sqrt{V}$.[1] So, we have $(c_1^{NEO},c_2^{NEO})=$

$(0,0)$	if $R>\left(1+\sqrt{\delta}\right)\sqrt{V}$
$(R-\sqrt{V},\sqrt{V})$	if $R<\left(1+\sqrt{\delta}\right)\sqrt{V}$
$\{(0,0),(R-\sqrt{V},\sqrt{V})\}$	if $R=\left(1+\sqrt{\delta}\right)\sqrt{V}$

[1] The choices for 1 boil down to the following possible strategies, with the corresponding payoffs: $U_1(R,BR_2(R))=V-R^2<0$, $U_1(0,BR_2(0))=0$ and $U_1(\gamma(R-\sqrt{V}),BR_2(\gamma(R-\sqrt{V})))=\delta V-(\gamma(R-\sqrt{V}))^2$ where $1<\gamma<\frac{R}{R-\sqrt{V}}$; $U_1((R-\sqrt{V}),p*\sqrt{V}+(1-p)*0)=p\delta V-(R-\sqrt{V})^2$. If $p\neq 1$, $U_1(\gamma(R-\sqrt{V}),BR_2(\gamma(R-\sqrt{V})))>U_1((R-\sqrt{V}),p*\sqrt{V}+(1-p)*0)\Leftrightarrow(\gamma^2-1)(R-\sqrt{V})^2<(1-p)\delta V$ is satisfied for γ close enough to 1. But again,

there is no unique γ that maximises $U_1(\gamma(R-\sqrt{V}),BR_2(\gamma(R-\sqrt{V})))$. (Note: I guess this depends on δ being large enough- you can always set $c_1=0$ and get 0- but I will ignore this at this point).