EQUIVALENT ANALOG & DIGITAL MODELS

by

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Chairman, Departmental Committee on Thesis

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ABSTRACT

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Submitted to the Department of Mechanical Engineering on May 20, 1961 in partial fulfillment of the requirements for the degree of Bachelor of Science in Mechanical Engineering.

The conversion of a linear differential equation into an equivalent difference equation which yields an exact sampled solution of the differential equation is investigated. It is found that by matching the characteristic roots of the difference equation with those of the differential equations; and by suppressing the harmonic roots by modifying the excitation, the difference equation will produce an exact sampled solution of the corresponding differential equation. The procedures for such operation were developed for the first and second order equations and the corresponding results verified.

Thesis Supervisor: Prof. H. M. Paynter
Title: Associate Prof. of Mechanical Engineering

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INTRODUCTION

One of the most widely used methods of solving differential equations with the digital computer is by means of the integrater. Finite intervals are chosen and the differential equation is integrated numerically, starting from the specified initial conditions to the points whose values are required. The main disadvantage of this numerical technique is that it gives numerical values for nonlinear functions at a set of discrete points instead of the analytic expressions defined over the corresponding region. Moreover, since intervals are finite the solution obtained is only an approximation. Therefore, the accuracy of the solution depends largely on the size of the intervals chosen.

The method of converting differential equations to difference equations developed here has two distinct advantages over the method mentioned above. First, it produces exact sampled solutions of the differential equation instead of approximations. Second, its accuracy is independent of the size of the sampling intervals (provided the sampling frequency is not smaller than the frequency of the function) i.e., it is independent of the sampling frequency. This represents labor and time saved. For example, suppose it is required to compute the value of y at $x = 5$, given the boundary conditions at $x = o$. The conventional procedure is to take say, intervals of 0.01 and compute all the way from $x = 0$ to $x = 5$. This amounts to 500 steps. However, in this method, if the frequency of the wave is not greater than the sampling frequency, say less than 5, then one can choose the sampling intervals to be 1 unit apart x axis and obtain the solution in 5 steps instead of the conventional 500 or even 5,000 steps.

Since the poles and zeros of a linear system completely determine its characteristic response, it is possible to determine the coefficients of the difference equation which gives the same characteristic roots as that of the corresponding differential equation, that is to say, the poles and zeros of the difference equation and differential equation match. If the harmonic roots generated in the difference equation is now suppressed by scaling the excitation down to an optimum level such that it has only enough energy to excite the fundamental roots and not the harmonic roots, then the difference equation yields an exact sampled solution of the differential equation. In this thesis one is concerned mainly with the translation of the analog model (i.e., differential equation) to the digital model (i.e., difference equation).

The Concepts of Z - Transforms and S- Transforms

If the differential equation is Laplace Transformed and the resulting algebraic equation is in turn Z - Transformed, then a difference equation results. The Z - transform is given by

$$Z = e^{-TS} \qquad (1)$$

where Z = Z -Transform
S = Laplace Transform
T = Sampling frequency

Also $$\Delta = e^{-TS} \qquad (2)$$

where Δ = backward time difference difference-operator

This Z- Transform is the familiar Z- Transform used in Sample Data Control system analysis where a continuous input is represented at the output by trains of impulses whose amplitude correspond to the Amplitude of the input wave multiplied by a constant (or even a variable). Since the effects of a sampler is to introduce high frequency components, it is therefore desirable to suppress these harmonies to avoid distortion.

In the mapping of the poles and zeros from the S-plane to the Z-plane and back again, the same phenomena occur. Going from the S-plane to the Z-plane is a single valued translation, but the translation from Z-plane back to S-plane again is multi-valued. Thus harmonic roots are produced. These harmonic roots need be suppressed, otherwise the Z-Transformed equation (or difference equation) may not yield an exact sampled solution of the differential equation. This can be done by limiting the excitation to an appropriate fraction of its amplitude so that there is only sufficient energy, so to speak, to excite the fundamental roots and not the harmonic roots. Take Eq(3) as an example.

$$y = \frac{(S^2 - SS_1 - SS_3 + S_1 S_3).x}{(S^2 - SS_2 - SS_4 + S_2 S_4)} \qquad (3)$$

Eq(3) can be factorized and rearranged into a form given by Eq(4)

$$y = \frac{(S - S_1)(S - S_3).x}{(S - S_2)(S - S_4)} \qquad (4)$$

where: S_1, S_3 are zeros
S_2, S_4 are poles

To convert this operator into the difference operator, the poles and zeros of Eq(4) are plotted onto the S-plane. Then these poles and zeros are mapped into the Z-plane (this transformation is single valued). The poles and zeros in the Z-plane can now be represented by Eq(5).

$$y = \frac{(Z - Z_1)(Z - Z_3)}{(Z - Z_2)(Z - Z_4)} . x \qquad (5)$$

This process is shown schematically in Fig. (1). However, when the poles and zeros in the Z-plane are mapped back onto the S-plane, harmonic roots are introduced because this transformation is multi-valued as in Fig. (2). Consequently to produce an exact sampled solution of the differential equation the harmonic roots must be suppressed, as shown in Fig. (3). This suppression is done by multiplying the zeros by a scaling factor, which is a function of the poles and zeros. Take for example, the first order equation given by Eq(6).

$$y = \frac{1}{1 + TD} . x \qquad (6)$$

Direct mapping of poles and zeros yields:

$$\frac{1}{1 + TD} \longrightarrow \frac{1}{\Delta - e^{1/T}}$$

However, this does not produce the required solution. It is found that the scaling factor needed in this case is

$$(S.F.) = 1 - e^{1/T} \qquad (7)$$

Then to complete the transformation one includes the scaling factor:

$$\frac{1}{1 + TD} \longrightarrow \frac{1}{\Delta - e^{1/T}} \times (S.F.) = \frac{1 - e^{1/T}}{\Delta - e^{1/T}} \qquad (8)$$

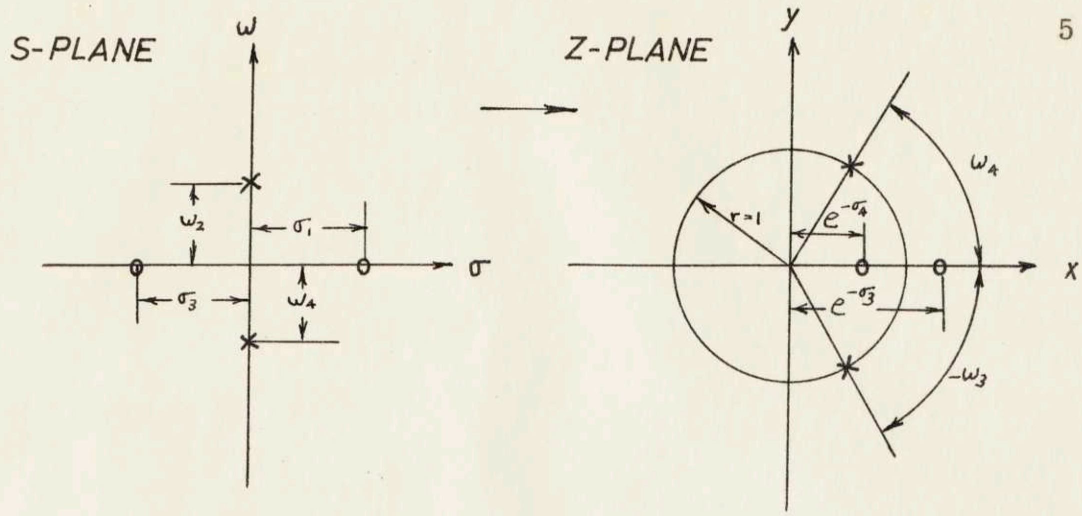


Fig. (1) Mapping of poles & zeros from S-plane to Z-plane.

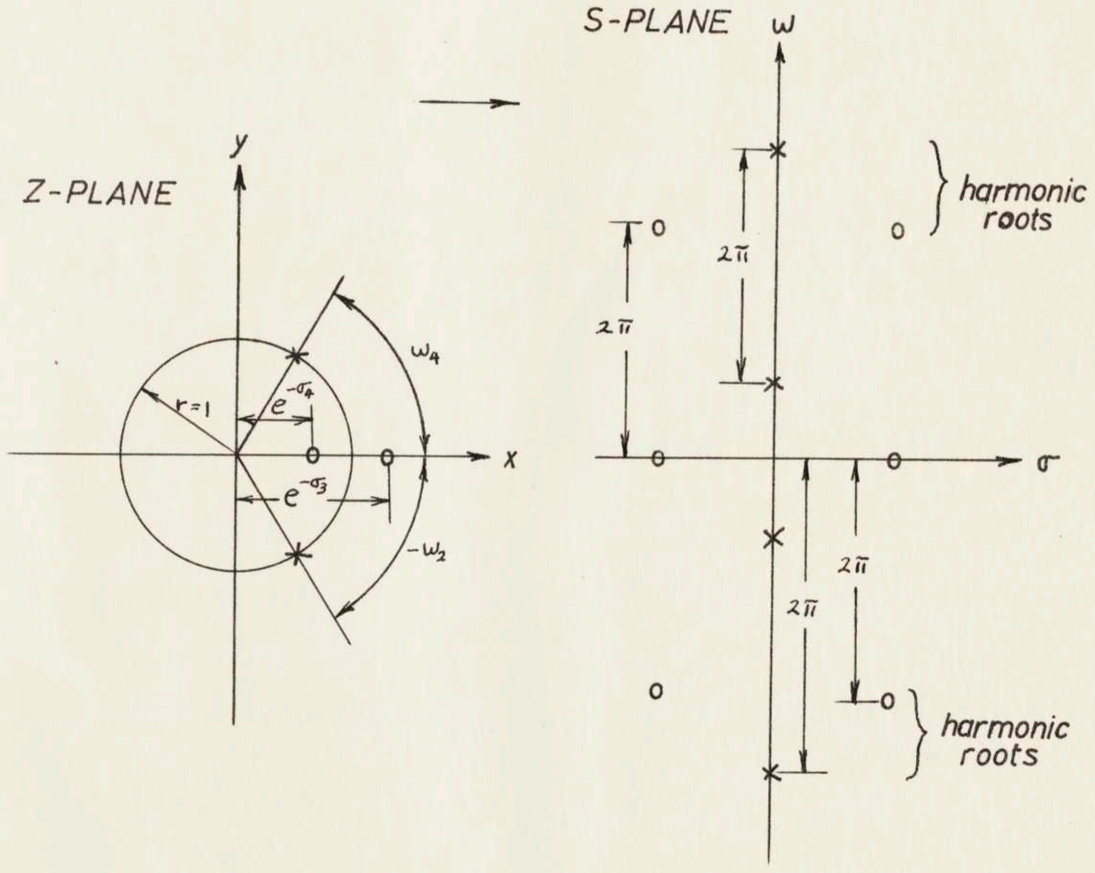


Fig. (2) Inverse mapping of poles & zeros from Z-plane back to S-plane.

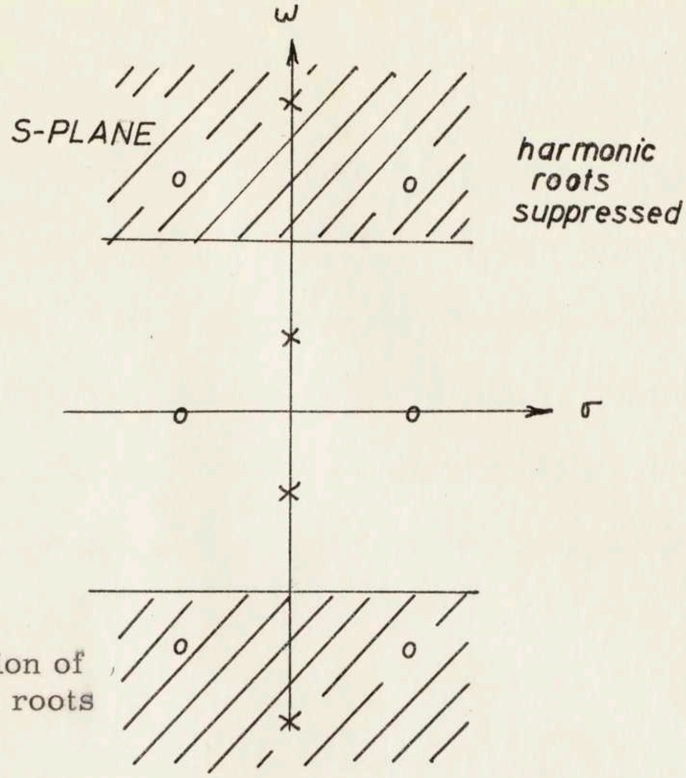


Fig. (3) Suppression of harmonic roots

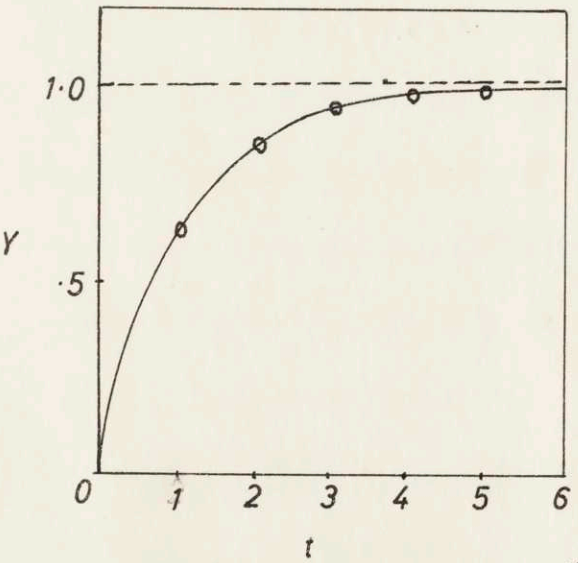


Fig. (4) Waveform of $\frac{1 - e^{1/T}}{\Delta - e^{1/T}}$

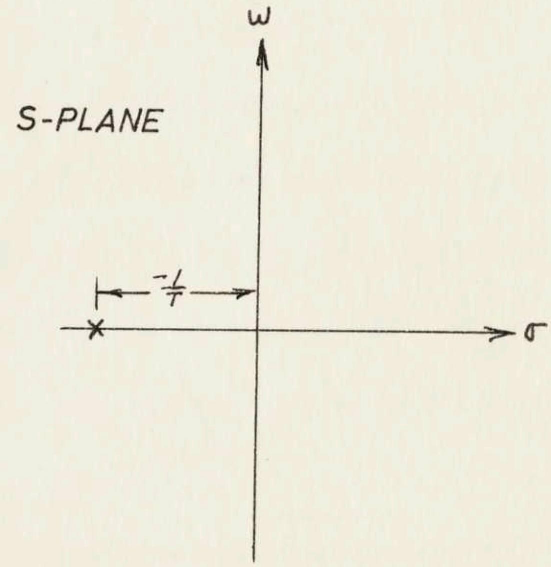


Fig. (4a) Pole configuration of curve of Fig. (4)

The waveform given by this operator is shown in Fig. (4).

Determination of the Scaling Factor

Proceeding similarly a table of transforms is constructed, and since the differential equations dealt with here are linear, the principle of superposition applies. Consequently all linear differential equations can be synthesized by the superposition of operators whose zeros are at the origin of the S-plane, e.g.

$$\frac{6T^2D^2+2}{T^2D^2+2\zeta TD+1} = 6\cdot\frac{T^2D^2}{T^2D^2+2\zeta TD+1} + 2\cdot\frac{1}{T^2D^2+2\zeta TD+1}$$

The conversion of the S-operator to Z-operator involves mere algebraic substitution. However, the main bulk of the effort and time is concentrated on the determination of scaling factors which are different for different poles and zeros configurations. There are as yet no laws or rules for determining them. Consequently the only way is to use trial-and-error methods and mathematical intuition. The correct scale factor found can be verified in two ways:

1) Laplace Transform Method. The exact waveforms can be obtained by Laplace Transform operation. If the waveform obtained by actually computing the difference equation coincides, at every sampling point, with the values given by Laplace Transform method, then for all practical purposes the trial factor can be considered as the required scale factor.
2) Self-Consistent Method. If the trial factor gives exactly the same waveform, in amplitude and phase for all values of ζ and any arbitrary value

of sampling frequency T and the next value of sampling frequency then the trial factor holds for all values of sampling frequency (this is the proof by induction method). Consequently the solution is invariant with respect to sampling frequency except that there are more, or less sampling points according to whether T is increased or decreased. Therefore, even if the trial factor is not the right factor it can differ from the actual factor by only a constant multiplier, e.g., take the case of $1/(T^2 D^2 + 2 \zeta TD + 1)$, from table (1) it is found that the scaling factor is $e^{\zeta/T} \cdot 1/T^2$. If the Sampling frequency T is misplaced, e.g., $e^{\zeta/T} \cdot 1/T$ or $e^{\zeta/T}$ etc., then the error would be magnified many fold when the sampling frequency is changed. For example, the difference between $e^{\zeta/T}$ and $e^{\zeta/T} 1/T^2$ is 9 times when the frequency varies from 1 to 3. Even the factor ζ in the exponent can be verified by varying the values of ζ, especially for the value of $\zeta = 0$ where it represents a nondecaying sinusoid. (for the 2d order equation). This too is very sensitive to being misplaced, because any error would produce incorrect values of the sinusoidal envelope, however slight it may be.

PROCEDURE

Derivation of S and Z-planes conversion formulas

Once the backward difference operator is defined by $\Delta = e^{-TS}$ the relation between the S and Z-plane can be derived. The results can be represented in either cartesian or polar co-ordinates. The results given by Eq(9) and Eq(10) are expressed in Cartesian Co-ordinates, that given by Eq(11), Eq(12), Eq(13), and Eq!14) are expressed in polar co-ordinates. The S and Z-plane are shown in Fig. (5).

The Integrator

The derivation and properties of the integrator in difference form is found in Appendix F.

Linear Differential Equations

In this section the transformation of the first and second order differential equations (with either zeros at infinity or zeros at origin or both) into their corresponding digital analog, i.e., difference equations, are developed. The results are summarized in Table (1) and Table (2).

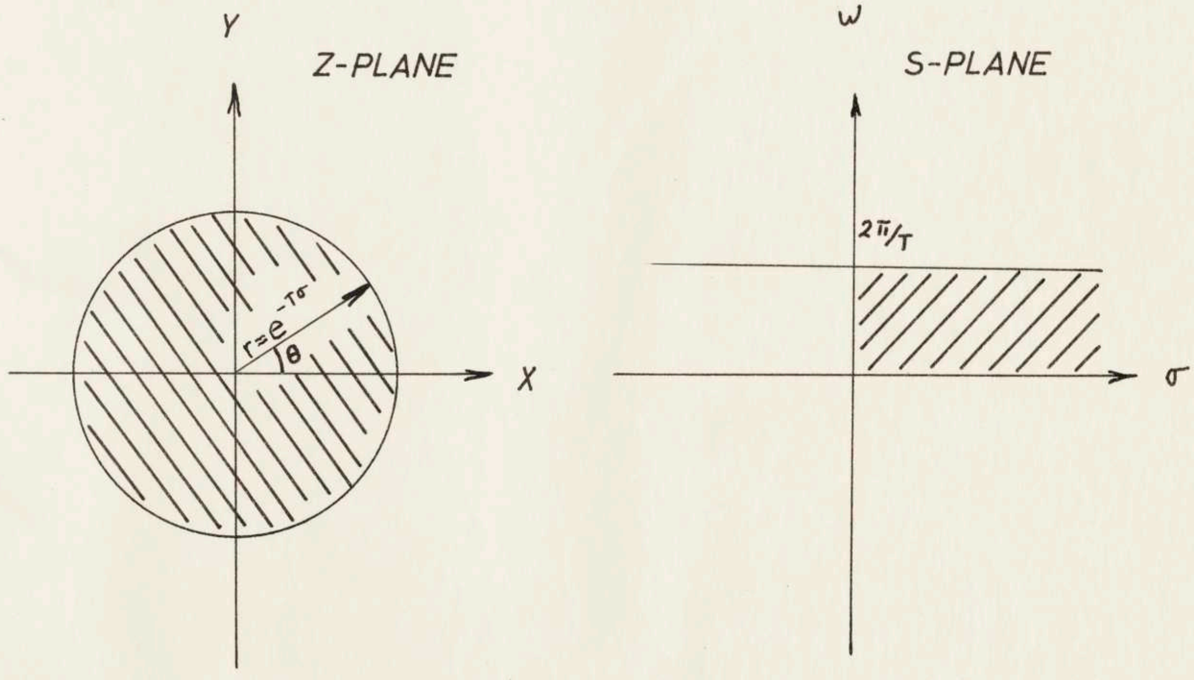


Fig. (5) the Z-plane & S-plane

S-plane in Cartesian Co-ordinates

$$Z = e^{-TS}$$

$$x + iy = e^{-T(\sigma + i w)} = e^{-T\sigma}(\cos TW - i \sin TW)$$

$$x = e^{-T\sigma} \cos TW \tag{9}$$

$$y = -e^{-T\sigma} \sin TW \tag{10}$$

$$x^2 + y^2 = e^{-2T\sigma} = r^2$$

$$\frac{y}{x} = - \tan T\sigma$$

S-plane in polar co-ordinates

$$Z = e^{-TS}$$

$$= e^{-T\sigma}(\cos TW - i \sin TW)$$

$$= re^{i\theta}$$

$$r = e^{-T\sigma} \tag{11}$$

$$\theta = -TW \tag{12}$$

also $\ln Z = - TS$

$$= - T(\sigma + i w)$$

$$= \ln r + i(\theta \pm 2K\pi)$$

$$-T\sigma = \ln r$$

$$-TW = \theta \pm 2K\pi$$

$$-\sigma = \frac{1}{T} \ln r = \frac{1}{2T} \ln (x^2 + y^2) \tag{13}$$

$$W = - \frac{\theta}{T} \mp \frac{2K\pi}{T} \tag{14}$$

The first order differential Equations

1 - a) One zero at origin of S-plane: $\frac{D}{1 + D}$ where sampling frequency equals unity. First, the poles and zeros of the operator are plotted onto the S-plane, then these are mapped onto the Z-plane or Δ -plane. Finally as a check, the poles and zeros of the Z-plane are mapped back onto the S-plane again. This process is illustrated in Fig. (9).

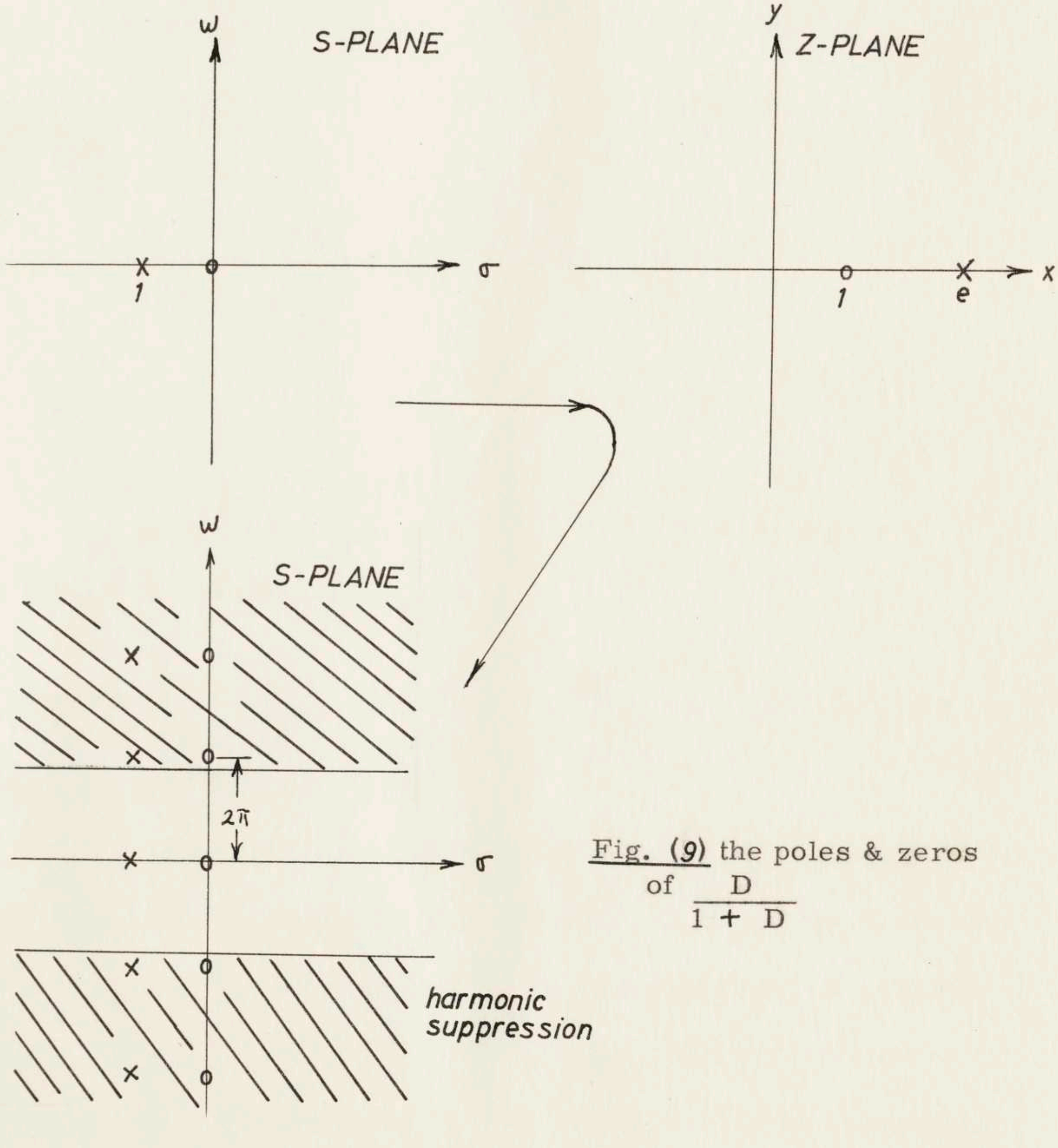


Fig. (9) the poles & zeros of $\frac{D}{1 + D}$

From the poles and zeros in the Z-plane one deduces:

$$y = \frac{\Delta - 1}{\Delta - e} . x \tag{20}$$

However, on computing and plotting the function one discovers that it is only 1/e of the true solution. Therefore, a scaling factor is needed, this gives:

$$y = \frac{\Delta - 1}{\Delta - e} . e . x \tag{21}$$

The correct and incorrect solution are shown in Fig. (10)

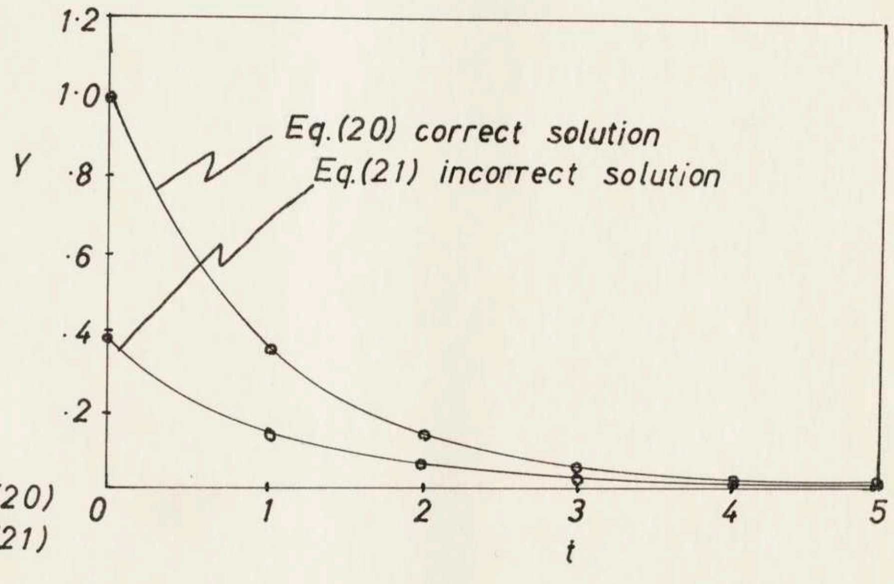


Fig. (10)
Graph Of Eq.(20)
& Eq.(21)

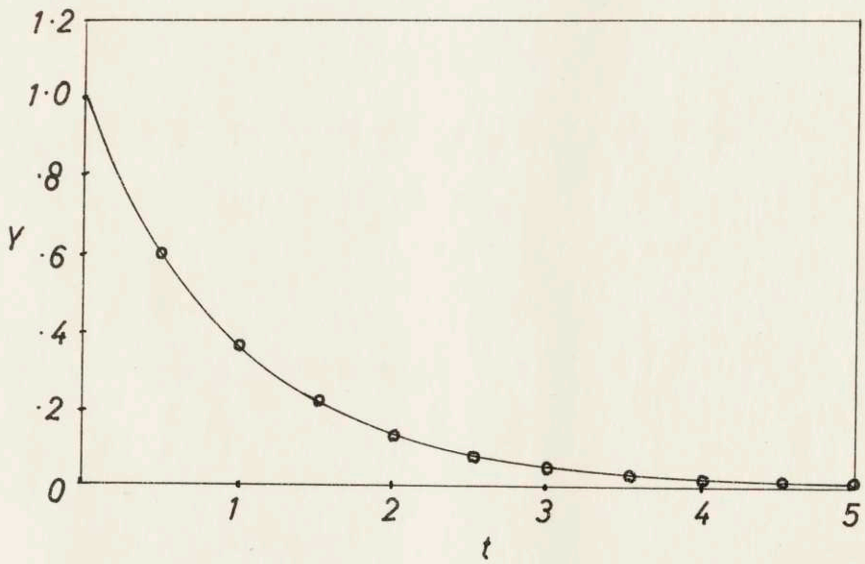
b) <u>One zero at origin of S-plane: $\frac{TD}{1 + TD}$</u> where the sampling frequency T is equal to any integer.

The foregoing analysis in section (1-a) applies only to the case where the sampling frequency equals unity. However, in actual practice, intermediate sampling points are needed for more accurate representation. Fortunately sampling points for a given interval may be increased by increasing the sampling frequency T. Proceeding in similar manner one arrives at the solution given by relation (22)

$$\frac{TD}{1 + TD} \longrightarrow \frac{\Delta - 1}{\Delta - e^{1/T}} \cdot e^{1/T} \qquad (22)$$

where $e^{1/T}$ = scaling factor

The waveform is shown in Fig. (11) which is exactly the same as that shown in Fig. (10) except that the sampling points are increased, (in this case they are doubled because T = 2).



Graph Of Eq. (22)   T = 2

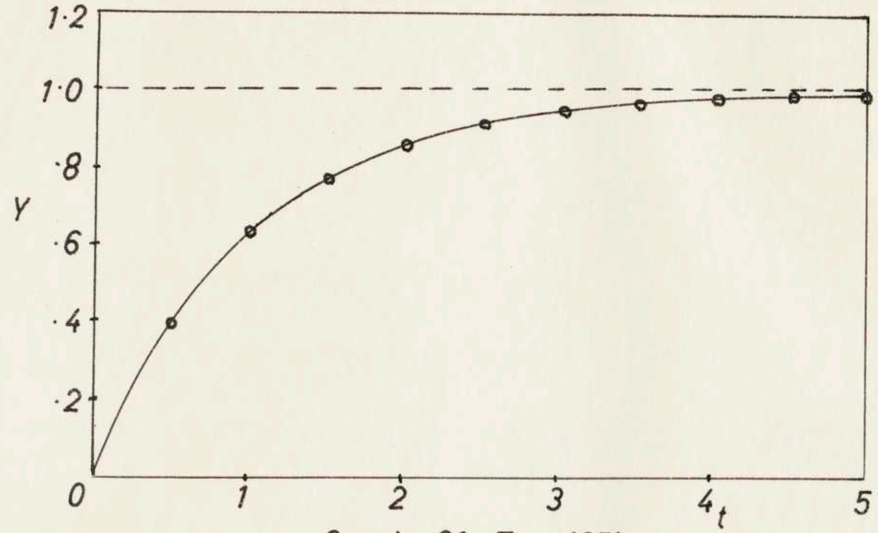
Fig. (11)

2) One zero at infinity: $\frac{1}{1 + TD}$

Proceeding in similar fashion the operator is transformed into the difference form. The scaling factor has been found, by trial-and-error, to be $(1 - e^{1/T})$.

$$\frac{1}{1 + TD} \longrightarrow \frac{1}{\Delta - e^{1/T}} \cdot (1 - e^{1/T}) \qquad (23)$$

The waveform for this operator for T = 2 is shown in Fig. (12).



Graph Of Eq. (25)

Fig. (12)

3) Verification: The above first order differential equations can be verified by using Laplace transforms.

$$y = \frac{1}{1+D} \cdot x \tag{24}$$

Taking the Laplace Transforms

$$py + y = \frac{1}{p}$$

$$y = \frac{1}{p(p+1)}$$

$$y = \frac{1}{p} - \frac{1}{p+1}$$

$$y = u(t) - e^{-t} \tag{25}$$

The curve of Eq(25) is shown in Fig. (12).

Similarly:

$$y = \frac{D}{1+D} \cdot x \tag{26}$$

Taking Laplace Transforms

$$py \quad y = p\,\frac{1}{p} = 1$$

$$y = \frac{1}{p+1}$$

$$y = e^{-t} \tag{27}$$

which is the curve shown in Fig. (10).

<u>Second Order Differential Equation</u>

1) <u>With one zero at origin and one zero at infinity of S-plane: $TD/(T^2D^2 + 2\zeta TD + 1)$</u>

To plot the poles and zeros, the denominator of the expression has to be factorized by means of the quadratic formula.

$$x = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a} \tag{28}$$

Substituting in the corresponding values of the coefficients in the denominator of the operator by letting

$$a = T^2$$

$$b = 2\zeta T$$

$$c = 1$$

then, $D = \dfrac{-2\zeta T \pm (4\zeta^2T^2 - 4T^2)^{1/2}}{2T^2}$

$$= \frac{-\zeta}{T} \pm \frac{(\zeta^2-1)^{1/2}}{T} \tag{29}$$

(for $\zeta > 1$)

$$= \frac{-\zeta}{T} \pm j\,\frac{(1-\zeta^2)^{1/2}}{T} \tag{30}$$

(for $\zeta < 1$)

Substituting for these values of D back to the operator,

$$\frac{TD}{T^2 D^2 + 2\zeta TD + 1} = \frac{TD}{\left[D - \left(-\frac{\zeta}{T} + \frac{\sqrt{\zeta^2 - 1}}{T}\right)\right]\left[D - \left(-\frac{\zeta}{T} - \frac{\sqrt{\zeta^2 - 1}}{T}\right)\right]}$$

(for $\zeta > 1$)

$$= \frac{TD}{\left[D - \left(-\frac{\zeta}{T} + \frac{j\sqrt{1 - \zeta^2}}{T}\right)\right]\left[D - \left(-\frac{\zeta}{T} - j\frac{\sqrt{1-\zeta^2}}{T}\right)\right]} \qquad (30a)$$

(for $\zeta < 1$)

The poles and zeros of Eq(30a) are plotted in Fig. (13) for $\zeta < 1$.

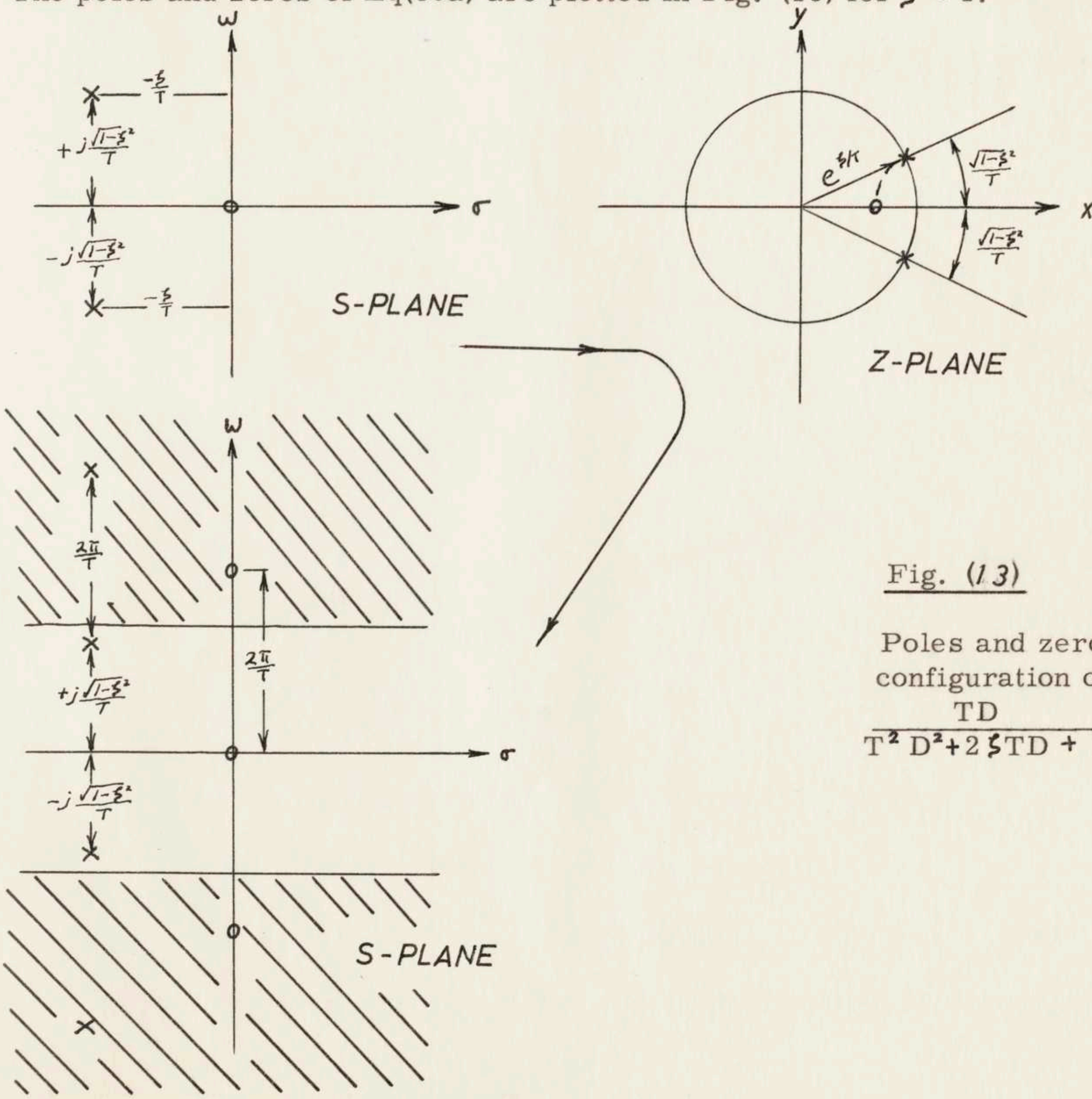


Fig. (13)

Poles and zeros configuration of $\dfrac{TD}{T^2 D^2 + 2\zeta TD + 1}$

Suppression of harmonic roots

The poles and zeros in the Z-plane can now be represented by:

$$\frac{(\Delta - 1)}{\left[\Delta - e^{\zeta/T}\left(\cos\frac{\sqrt{1-\zeta^2}}{T} + j\sin\frac{\sqrt{1-\zeta^2}}{T}\right)\right]\left[\Delta - e^{\zeta/T}\left(\cos\frac{\sqrt{1-\zeta^2}}{T} - j\sin\frac{\sqrt{1-\zeta^2}}{T}\right)\right]}$$

$$= \frac{\Delta - 1}{\Delta^2 - 2e^{\zeta/T}\cos\frac{\sqrt{1-\zeta^2}}{T} + e^{2\zeta/T}} \qquad (31)$$

$$\text{for } (\zeta < 1)$$

However, it has been found by actual computation that the difference operator in Eq(31) will not yield the correct solution . Consequently a scaling factor is needed. This can be found by trial-and-error and mathematical intuition to be: $(-e^{\zeta/T}\ 1/T)/e^{1/6T^2}$ and the translation is complete -

$$\frac{TD}{T^2D^2 + 2\zeta TD + 1} \longrightarrow \frac{\Delta - 1}{\Delta^2 - \left(2e^{\zeta/T}\cos\frac{\sqrt{1-\zeta^2}}{T}\right)\Delta + e^{2\zeta/T}} \cdot -e^{\zeta/T}\frac{1}{T}\Big/e^{1/6T^2} \qquad (34)$$

$$(\text{for } \zeta < 1)$$

$$\longrightarrow \frac{\Delta - 1}{\Delta - \left(2e^{\zeta/T}\cosh\frac{\sqrt{\zeta^2-1}}{T}\right)\Delta + e^{2\zeta/T}} \cdot -e^{\zeta/T}\frac{1}{T}\Big/e^{1/6T^2} \qquad (33)$$

$$(\text{for } \zeta > 1)$$

<u>Example (2a)</u>. Take $\frac{TD}{T^2\ D^2 + 2\ \zeta\ TD + 1}$ where

$T = 2 \qquad = 0.25$

$$y = \frac{(\Delta - 1)(-e^{\zeta/T}\ \frac{1}{T})/(e^{1/6T^2})}{\Delta^2 - \left(2e^{\zeta/T}\cos\frac{\sqrt{1-\zeta^2}}{T}\right)\Delta + e^{2\zeta/T}} \cdot x.$$

$$= \frac{-(\Delta - 1)(.567)}{\Delta^2 - 2\Delta + 1.284}$$

$$(\Delta - 2\Delta + 1.284)\ y = (.567 - .567\Delta)x$$

$$y_{k-2} - 2y_{k-1} + 284y_k = .567x_k - .567x_{k-1}$$

$$1.284y_k = -y_{k-2} + 2y_{k-1} + .567x_k - .567x_{k-1} \qquad (34)$$

The actual numerical computation of Eq(34) is shown in Appendix B and the waveform is plotted in Fig. (14).

<u>Verification by Laplace Transforms</u>

$$y = \frac{D}{D^2 + 2\zeta D + 1} \cdot x \tag{35}$$

Taking the Laplace Transform of Eq(35)

$$p^2 y + 2\zeta py + y = \frac{1}{p} p = 1$$

$$y = \frac{1}{p^2 + 2\zeta p + 1}$$

$$= \frac{\ }{(p + \zeta)^2 + (1 - \zeta^2)}$$

From the Laplace Transform Tables

$$y = \frac{1}{(1 - \zeta^2)^{1/2}} e^{-\zeta/T} \cdot \sin(\sqrt{1 - \zeta^2}\, t) \tag{36}$$

Substituting in the numerical value of $\zeta = .25$, $T = 2$.

$$y = (1.035)e^{-.25t} \sin(.966t)$$

The waveform of this equation coincides exactly with the values of the sampled waveform shown in Fig. (14); this proves that the conversion is correctly done and also the solution is exact. That is, its accuracy is only limited by the number of decimal places the digital computor can carry.

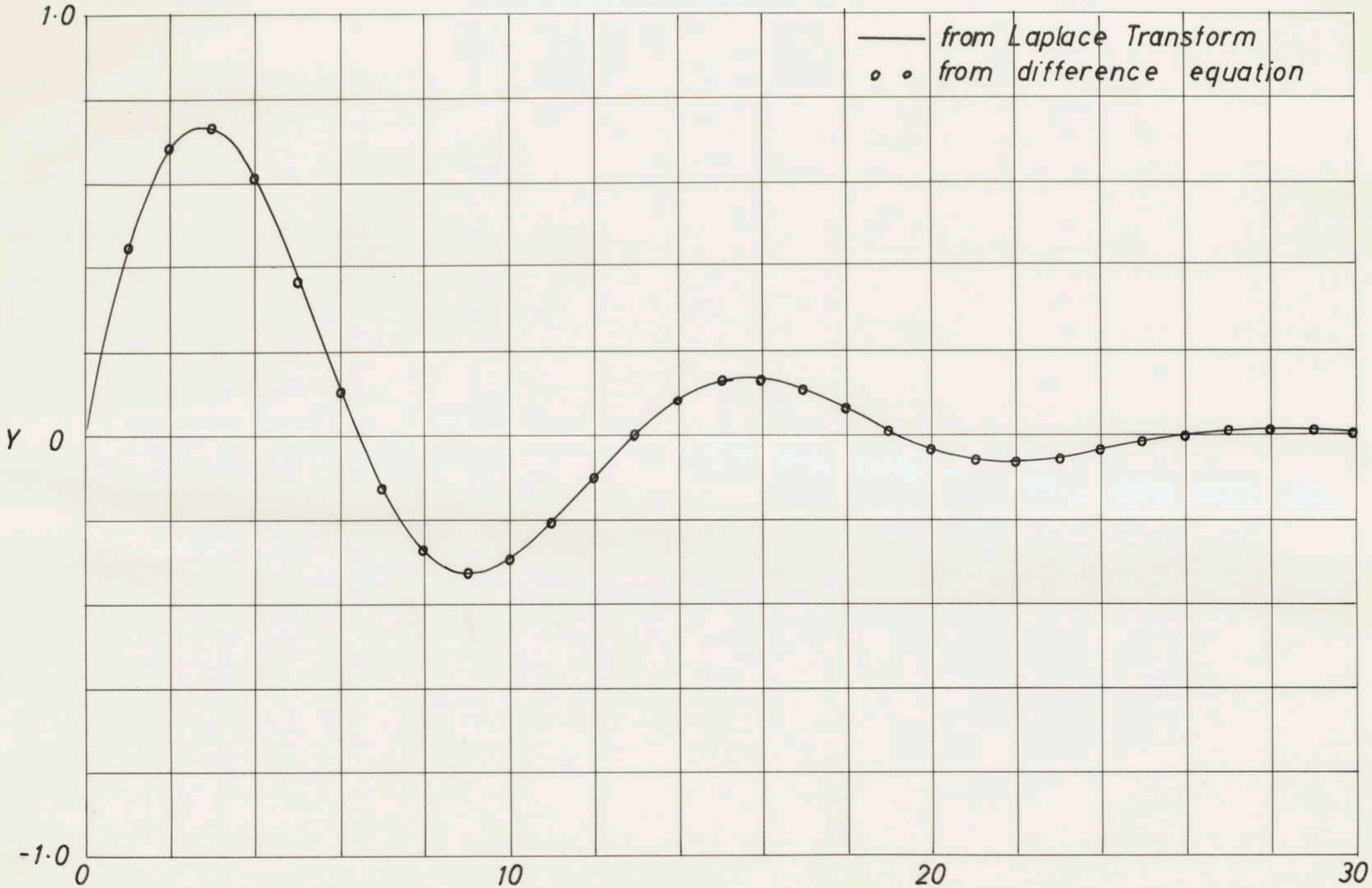
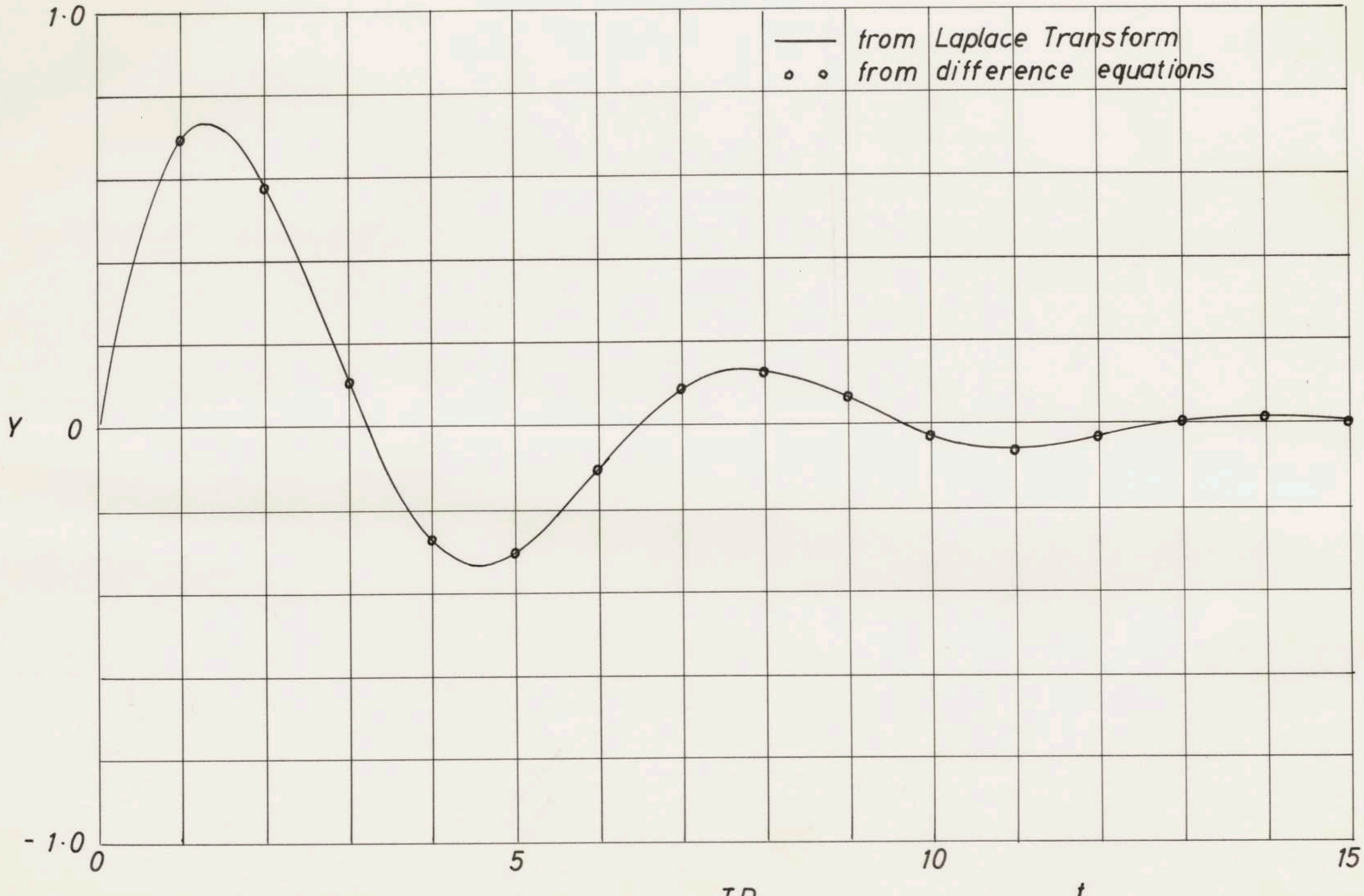


FIG. (14) Sampled Solution Of $\dfrac{TD}{T^2D^2+2\zeta TD+1}$

where $T=2$ $\zeta=\cdot25$



<u>FIG. (14a)</u> Sampled Solution Of $\dfrac{TD}{T^2D^2+2\zeta TD+1}$

where $T=1 \quad \zeta = \cdot 25$

2) With double zeros at origin of S-plane: $T^2 D^2 / (T^2 D^2 + 2\zeta TD + 1)$

Following the same scheme as in Part 1) and determining the scaling factor by trial-and-error one arrives at

$$\frac{T^2 D^2}{T^2 D^2 + 2\zeta TD + 1} \rightarrow \frac{(\Delta^2 - 2\Delta + 1)}{\Delta^2 - 2\Delta\, e^{\zeta/T} \cos \frac{\sqrt{1-\zeta^2}}{T} + e^{2\zeta/T}} \tag{35}$$

for $\zeta < 1$

$$\rightarrow \frac{(\Delta^2 - 2\Delta + 1)}{\Delta^2 - 2\Delta\, e^{\zeta/T} \cosh \frac{\sqrt{\zeta^2 - 1}}{T} + e^{2\zeta/T}} \tag{36}$$

for $\zeta > 1$

However, it is fortunate that the scaling factor turns out to be unity.

Example (2b) Take $\dfrac{T^2 D^2}{T^2 D^2 + 2\zeta TD + 1}$ where $T = 2$; $\zeta = 0.25$.

$$y = \frac{\Delta^2 - 2\Delta + 1}{\Delta^2 - 2\Delta\, e^{\zeta/T} \cos \frac{\sqrt{1-\zeta^2}}{T} + e^{2\zeta/T}} \tag{37}$$

$$= \frac{\Delta^2 - 2\Delta + 1}{\Delta^2 - 2\Delta + 1.284} .$$

$$(\Delta^2 - 2\Delta + 1.284)y = (\Delta^2 - 2\Delta + 1)x$$

$$y_{K-2} - 2y_{K-1} + 1.284\, y = x_{K-2} - 2x_{K-1} + x_K$$

$$1.284\, y_K = -y_{K-2} + 2y_{K-1} + x_{K-2} - 2x_{K-1} + x_K \tag{38}$$

The actual numerical computation of Eq(38) is shown in Appendix A, and the sampled solution is seen to coincide exactly with that given by Laplace Transform method, as shown in Fig. (15).

3) With two zeros at infinity in S-plane: $1/T^2 D^2 + 2\zeta TD + 1$.

The scaling factor for this operator has been found to be $(e^{\zeta/T} \frac{1}{T})$. Following the same method of derivation as in the foregoing examples one arrives at

$$\frac{1}{T^2 D^2 + 2\zeta TD + 1} \rightarrow \frac{e^{\zeta/T}\frac{1}{T^2}}{\Delta^2 - 2\Delta e^{\zeta/T}\cos\frac{\sqrt{1-\zeta^2}}{T} + e^{2\zeta/T}} \quad (39)$$

for $\zeta < 1$

$$\rightarrow \frac{e^{\zeta/T}\frac{1}{T^2}}{\Delta^2 - 2\Delta e^{\zeta/T}\cosh\frac{\sqrt{\zeta^2-1}}{T} + e^{2\zeta/T}} \quad (40)$$

for $\zeta > 1$

Example (2c) Take $\frac{1}{T^2 D^2 + 2\zeta TD + 1}$ with $T = 2$, $\zeta = .25$

$$y = \frac{e^{\zeta/T}\frac{1}{T^2}}{\Delta^2 - 2\Delta e^{\zeta/T}\cos\frac{\sqrt{1-\zeta^2}}{T} + e^{2\zeta/T}} \cdot X \quad (41)$$

$$= \frac{.2835}{\Delta^2 - 2\Delta + 1.284} \cdot X$$

$$(\Delta^2 - 2\Delta + 1.284)y = .2835x.$$

$$y_{K-2} - 2y_{K-1} + 1.284y_K = 0.2835x_K$$

$$1.284y_K = -y_{K-2} - 2y_{K-1} + 0.2835x_K \quad (42)$$

The actual numerical computation of EQ(42) is shown in Appendix C. The algebraic equation for the above waveform is found from Laplace Transfer to be

$$y = u(t) - \frac{e^{-\zeta t}}{1 -}\left(\sin\sqrt{1-\zeta^2}\; t + \phi\right) \quad (43)$$

where $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$

Substitute for numerical values: $\zeta = .25$.

$$y = u(t) - (1.035)e^{-.25t}\left(\sin(1.035)t + 1.31\right) \quad (44)$$

The sampled wave form obtained from the difference equation and that obtained by Eq(44) are seen to coincide, as in Fig. (16).

RESULTS

The results of the foregoing transformation of S to Z-operators are summarized in tabular form. Table (1) is a summary of translation of S-operators to Z-operators. Table (2) is a summary of translation of Z-operators back to S-operators.

It can be seen that apart from the representation given in the tables it is not possible to represent the S to Z or Z to S translations by purely algebraic relations because the scaling factor admits of no generalization. Apart from the zeros, the poles can be generalized to high order just by combining lower order equations. However, some recommendations on suggesting methods of generalizing the results of the first order and second order differential equations to higher order differential equations are found in the last section of the conclusion.

Since all these operators have either zeros at infinity and/or zeros at the origin of S-plane, the solution of operators with zeros anywhere in the S-plane other than the origin and infinity is found in the section on applications.

Perhaps one of the methods of proving qualitatively that the results are valid is by attempting to prove that as the sampling frequency T approaches ∞ , the coefficients of the difference equation approache that of the differential equation. An example of this proof on one of the second order results is found in Appendix E.

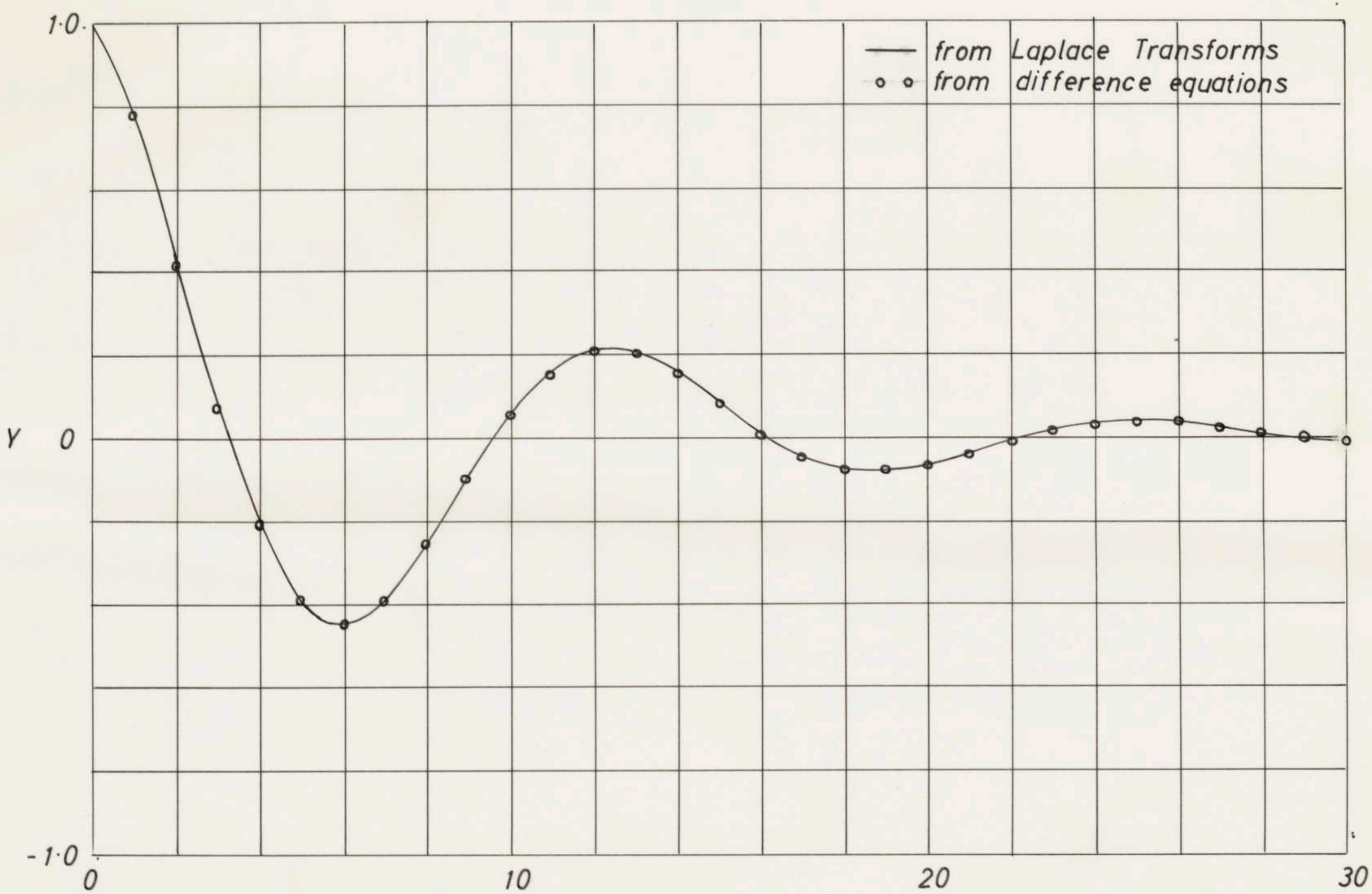


FIG. (15) Sampled Solution Of $\frac{T^2D^2}{T^2D^2+2\zeta TD+1}$

where $T=2$ $\zeta=\cdot 25$

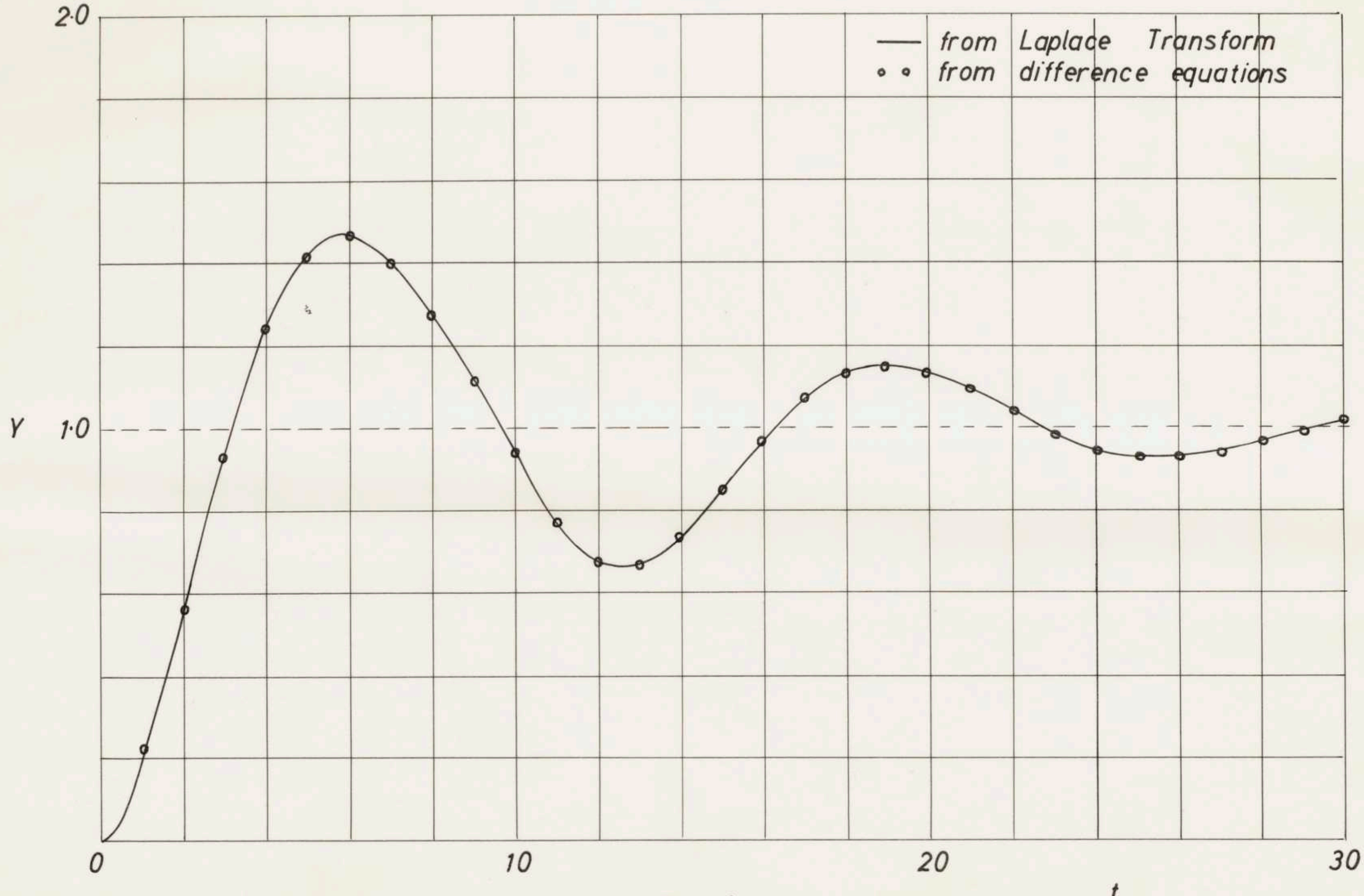


FIG. (16) Sampled Solution Of $\frac{1}{T^2D^2+2\zeta TD+1}$

where $T = 2 \quad \zeta = \cdot 25$

TABLES OF $S \longleftrightarrow Z$ TRANSFORMATIONS

TABLE (1) SUMMARY OF S→Z TRANSFORMATIONS

S-PLANE	Z-PLANE	SCALING FACTOR
Integrator		
$\frac{1}{D}$	$\frac{\Delta}{1-\Delta}$	
First Order Eq.		
$\frac{1}{TD+1}$	$\frac{1-e^{1/T}}{\Delta - e^{1/T}}$	$1-e^{1/T}$
$\frac{TD}{TD+1}$	$\frac{\Delta-1}{\Delta - e^{1/T}} \cdot e^{1/T}$	$e^{1/T}$
Second Order Eq.		
$\frac{1}{T^2D^2+2\xi TD+1}$	$\frac{e^{\xi/T}\, 1/T^2}{\Delta^2 - 2\Delta e^{\xi/T}\cos\frac{(1-\xi^2)^{1/2}}{T} + e^{2\xi/T}}$	$e^{\xi/T}\, 1/T^2$ $(-1<\xi<1)$
	$\frac{e^{\xi/T}\, 1/T^2}{\Delta^2 - 2\Delta e^{\xi/T}\cosh\frac{(\xi^2-1)^{1/2}}{T} + e^{2\xi/T}}$	$(1<\xi<\infty)$ $(-\infty<\xi<-1)$
$\frac{TD}{T^2D^2+2\xi TD+1}$	$\frac{(\Delta-1)(-e^{\xi/T}\, 1/T\ e^{-1/6T^2})}{\Delta^2 - 2\Delta e^{\xi/T}\cos\frac{(1-\xi^2)^{1/2}}{T} + e^{2\xi/T}}$	$-e^{\xi/T}\, 1/T\ e^{-1/6T^2}$ $(-1<\xi<1)$
	$\frac{(\Delta-1)(-e^{\xi/T}\, 1/T\ e^{-1/6T^2})}{\Delta^2 - 2\Delta e^{\xi/T}\cosh\frac{(\xi^2-1)^{1/2}}{T} + e^{2\xi/T}}$	$(1<\xi<\infty)$ $(-\infty<\xi<-1)$
$\frac{T^2D^2}{T^2D^2+2\xi TD+1}$	$\frac{\Delta^2-2\Delta+1}{\Delta^2 - 2\Delta e^{\xi/T}\cos\frac{(1-\xi^2)^{1/2}}{T} + e^{2\xi/T}}$	1 $(-1<\xi<1)$
	$\frac{\Delta^2-2\Delta+1}{\Delta^2 - 2\Delta e^{\xi/T}\cosh\frac{(\xi^2-1)^{1/2}}{T} + e^{2\xi/T}}$	$(1<\xi<\infty)$ $(-\infty<\xi<-1)$

TABLE (2) SUMMARY OF $Z \rightarrow S$ TRANSFORMATIONS

S-PLANE + scaling factor	Z-PLANE
Integrator	
$\frac{\Delta}{1-\Delta}$	$\frac{1}{D}$
First Order Eq.	
$\frac{n}{\Delta - x}$	$\frac{n/(1-x)}{1+(1/\ln x)D}$
$\frac{(\Delta-1)n}{\Delta - x}$	$\frac{(n/x)D}{1+(1/\ln x)D}$
Second Order Eq.	
$\frac{n}{\Delta^2 - p\Delta + q}$	$\frac{n/(q^{1/2}\,1/T^2)}{T^2D^2+2(T^2(\cos^{-1}\frac{p}{2q^{1/2}})+1)^{1/2}TD+1}$ $\quad -1<\frac{p}{2q^{1/2}}<1$ $\frac{n/(q^{1/2}\,1/T^2)}{T^2D^2+2(T^2(\cosh^{-1}\frac{p}{2q^{1/2}})+1)^{1/2}TD+1}$ $\quad 1<\frac{p}{2q^{1/2}}<\infty$
$\frac{-(\Delta-1)n}{\Delta^2 - p\Delta + q}$	$\frac{(n/(q^{1/2}\,1/T\,e^{-1/6T^2}))TD}{T^2D^2+2(T^2(\cos^{-1}\frac{p}{2q^{1/2}})+1)^{1/2}TD+1}$ $\quad -1<\frac{p}{2q^{1/2}}<1$ $\frac{(n/(q^{1/2}\,1/T\,e^{-1/6T^2}))TD}{T^2D^2+2(T^2(\cosh^{-1}\frac{p}{2q^{1/2}})+1)^{1/2}TD+1}$ $\quad 1<\frac{p}{2q^{1/2}}<\infty$
$\frac{(\Delta-2\Delta+1)n}{\Delta^2 - p\Delta + q}$	$\frac{nT^2D^2}{T^2D^2+2(T^2(\cos^{-1}\frac{p}{2q^{1/2}})+)^{1/2}TD+1}$ $\quad -1<\frac{p}{2q^{1/2}}<1$ $\frac{nT^2D^2}{T^2D^2+2(T^2(\cosh^{-1}\frac{p}{2q^{1/2}})+1)^{1/2}TD+1}$ $\quad 1<\frac{p}{2q^{1/2}}<\infty$

N.B. Use only positive root of q.

APPLICATION

Solution of Linear Differential Equations

The method of superposition can be applied to linear differential equations. It follows that the results found in table (1) can be superposed, since these are all linear. An example will illustrate the use of table (1). Take Eq(45), which is a second order linear differential equation rearranged in an explicit form, namely:

$$y = \frac{T^2 D^2 + 5\zeta TD + 2}{T^2 D^2 + 2\zeta TD + 1} \cdot x \tag{45}$$

The operator in the above form can be broken down into the basic forms found in table (1), that is

$$\frac{T^2 D^2 + 5\zeta TD + 2}{T^2 D^2 + 2\zeta TD + 1} = \frac{T^2 D^2}{T^2 D^2 + 2\zeta TD + 1} + 5\zeta \cdot \frac{TD}{T^2 D^2 + 2\zeta TD + 1} + 2 \cdot \frac{1}{T^2 D^2 + 2\zeta TD + 1} \tag{46}$$

$T = 2$

Suppose that $\zeta = .25$. Then one can make use of the results of Appendix A, B, and C. Let the sampled value of y_K be y_1, y_2, y_3 in Appendix A, Appendix B, Appendix C, respectively. Then by superposition the sampled solution of Eq(45) is given by:

$$y_K = \sum_{i=1}^{3} c_i y_i \tag{47}$$

$$y = y_1 + 5\zeta y_2 + 2y_3 \tag{48}$$

The solution of Equation (48) is found in Appendix D, and the resultant waveform is plotted in Fig. (17). As a matter of interest the small saddle or kink found at the 10th sampling point in the curve has been verified by Laplace Transform to exist.

For the convenience of using the transforms the values of $e^{\zeta/T} \cos \frac{\sqrt{1-\zeta^2}}{T}$ and $e^{\zeta/T} \cosh \frac{\sqrt{\zeta^2-1}}{T}$ vs. ζ with T as parameter are plotted in graph (1). Also the values of $e^{2\zeta/T}$ vs. ζ with T as parameter are plotted on graph (2).

CONCLUSION

From the results of the foregoing analysis one is lead to the conclusion that the initial assumption of the fact that the poles and zeros of a linear system uniquely determines its characteristics is justified. It also serves to prove that if the poles and zeros of one complex plane were mapped onto another complex plane and that if means were devised so that there is a one to one correspondence (e. g., by suppression of harmonics and multivalued roots) between the two planes then the characteristics of the linear system can be described uniquely by the poles and zeros in that other complex plane. In this thesis, the two planes in question are the S-plane and the Z-plane. By far the most significant factor that makes possible the translation of a differential equation to a corresponding difference equation which yields exact sampled solution of the former is the Z-Transform.

One of the usefulnesses of Laplace Transform is that it converts a linear differential equation into an algebraic equation to facilitate the solutioning. Consequently one is lead to speculate on the fact that if a nonlinear differential equation can be translated into an algebraic equation by some new type of transforms then it would not be too difficult to translate this fur-

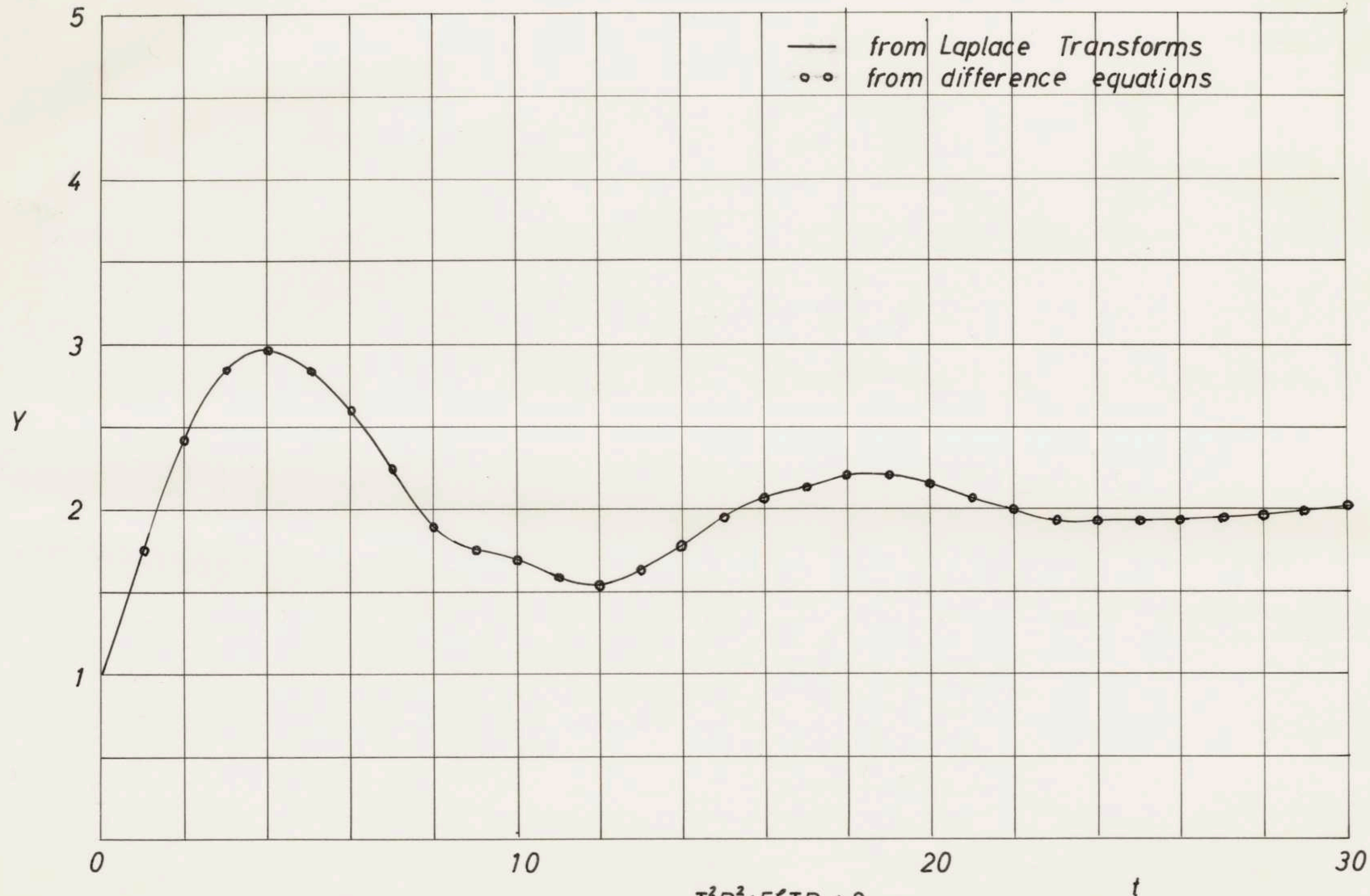
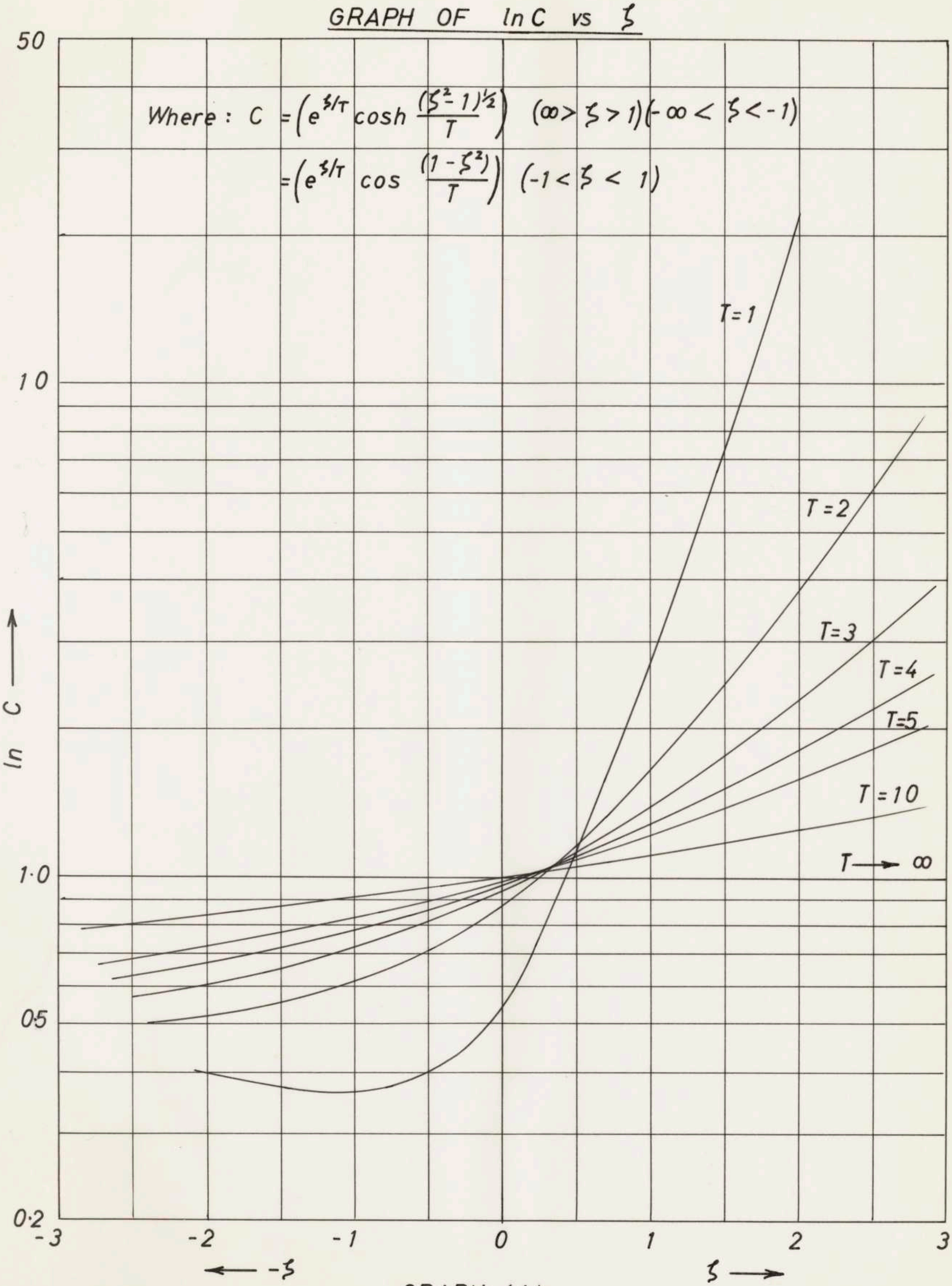
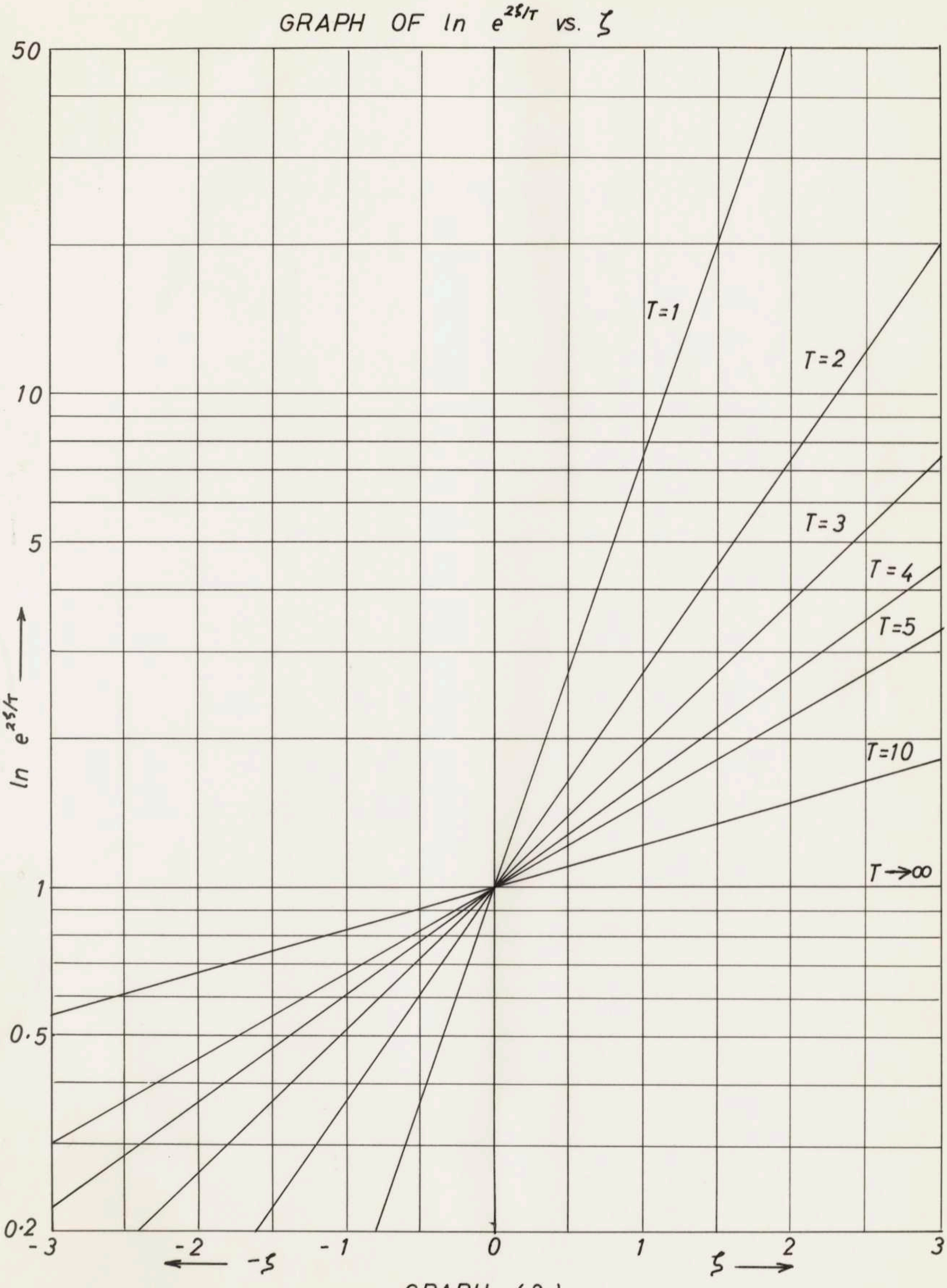


FIG. (17) Sampled Solution Of $\frac{T^2D^2+5\zeta TD+2}{T^2D^2+2\zeta TD+1}$

where $T=2 \quad \zeta = 25$


GRAPH OF ln C vs ζ
Where : C = (e^{ζ/T} cosh ((ζ² − 1)^½ / T)) (∞ > ζ > 1) (−∞ < ζ < −1)
= (e^{ζ/T} cos ((1 − ζ²) / T)) (−1 < ζ < 1)
50
10
1·0
0·5
0·2
ln C
−3
−2
−1
0
1
2
3
← −ζ
ζ →
T = 1
T = 2
T = 3
T = 4
T = 5
T = 10
T → ∞

GRAPH (1)



GRAPH OF ln $e^{2\zeta/T}$ vs. ζ
50
10
5
1
0·5
0·2
ln $e^{2\zeta/T}$
T=1
T=2
T=3
T=4
T=5
T=10
T→∞
-3
-2
-1
0
1
2
3
-ζ
ζ

GRAPH (2)

ther into a difference equation. This amounts to the exact sampled solution of a nonlinear differential equation on a digital computer, thus extending the versatility and usefulness of the digital computer. But before the problem of nonlinear differential equations is solved elegantly one can dwell on a more immediate and practical application of the foregoing approach. This is the Fourier Transform. Just as one works on the Laplace Transform with the Z-Transform it seems possible to devise analogous method to work on the Fourier Transform equations to translate it into a form suitable for digital computation. This would result in a useful method of solving a large class of physical problems using the digital computer.

However, the solutioning of the linear differential equations considered in this thesis is by no means complete because only the first and second order linear differential equations are considered. Consequently further research on third and higher order linear differential equations is needed. Perhaps it would be informative to suggest some recommendations on the approach to solving the higher order equations.

By multiplying together, say, two second order equations one obtains a fourth order equation. Since the roots are known from the second order equations the poles and zeros can be plotted on the complex plane. It can be verified that the poles and zeros transformed in this way is correct, except the scaling factor. However the product of the second order scale factors may or may not be the scale factor required by the fourth order equation. One of the ways to decide this is to compute the values of the fourth order difference equation and check these against the results produced by Laplace Transforms. From the results of the first and second order equations it is obvious that the multiplying together of two first order scale factors in no way suggests or resembles the scale factor of the second order equation. Again one may argue that since first order equations are different from all other higher order equations in that it has no complex roots, it seems more reasonable to start the scale factor generalization process from the second order equations instead of those of the first.

From the results of table (1) it can be seen that for the case of second order equations with double zeros at the origin the scale factor is unity. Consequently the scaling factor for higher order equations with numerically the same number of poles at the origin as the degree of the equation is most probably unity also, e.g.,

$$\frac{T^4 D^4}{(T^2 D^2 + 2\zeta TD + 1)(T^2 D^2 + 2\zeta TD + 1)} \longrightarrow \frac{(\Delta - 2\Delta + 1)^2 \cdot 1}{\left(\Delta^2 - 2e^{\zeta/T}\cosh\sqrt{\frac{\zeta^2 - 1}{T}} + e^{2\zeta/T}\right)}$$

$$(\text{for } \zeta > 1)$$

The analytical proof in Appendix E of the fact that as the sampling frequency T approaches infinity, the coefficients of the difference equation approaches that of the differential equation. This suggests that if the scaling factor is correct then this property is valid. Similarly by taking a higher order difference equation (without scaling factor) and reducing it into the form suitable for proving the equivalence, the factor which is needed to make the equivalence complete is thus the scaling factor. This is tedious mathematically, but it seems to be a more logical way of attacking the problem than by the method of trial-and-error even though it may take more time and effort than the latter.

APPENDIX A

Numerical Computation of Equation (38)

$$y = \frac{-y_{K-2} + 2y_{K-1} + x_{K-2} \quad -2x_{K-1} \quad x_K}{1.284} \tag{38}$$

(N. B. $S_1 = x_K - 2x_{K-1} + x_{K-2}$
$S_2 = 2y_{K-1} - y_{K-2}$)

x(k)	1.000	1.000	1.000	1..000	1.000	1.000	1.000	1.000
-2x(k-1)	0.000	-2.000	-2.000	-2.000	-2.000	-2.000	-2.000	-2.000
x(k -2)	0.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000
S_1	1.000	-1.000	0.000	0.000	0.000	0.000	0.000	0.000
2y(k -1)	0.000	1.558	0.870	0.142	-0.454	-0.816	-0.918	-0.974
$-y_{K-2}$	0.000	0.000	-0.779	-0.434	-0.071	-0.227	-0.408	0.485
S_2	0.000	1.558	0.091	-0.292	-0.525	-0.589	-0.510	-0.336
$S_1 + S_2 = S$	1.000	0.558	0.091	-0.292	-0.525	-0.589	-0.510	-0.336
y(k) = S/1.284	0.779	0.434	0.071	0.071	-0.227	-0.408	-0.458	-0.410

x(k)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-2x(k-1)	-2.000	-2.000	-2.000	-2.000	-2.000	-2.000	-2.000	-2.000
x(k -2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
S_1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2y(k -1)	-0.524	-0.197	-0.102	0.312	0.406	0.389	0.290	0.150
$-y_{K-2}$	0.397	0.262	0.098	-0.051	-0.156	-0.203	-0.194	-0.145
S_2	-0.127	0.065	0.200	0.261	0.250	0.186	0.096	0.005
$S_1 + S_2 = S$	-0.127	0.065	0.200	0.261	0.250	0.186	0.096	0.005
y(k) = S/1.284	-0.098	0.051	0.156	0.203	0.194	0.145	0.075	0.004

x(k)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-2x(k-1)	-2.000	-2.000	-2.000	-2.000	-.2000	-2.000	-2.000	-2.000
x(k -2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
S_1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2y(k -1)	0.008	-0.104	-0.156	-0.162	-0.131	-0.079	-0.022	0.026
-y(k-2)	0.075	-0.004	0.052	0.078	0.081	0.065	0.039	0.011
S_2	-0.067	-0.108	-0.104	-0.084	-0.051	-0.014	0.017	0.037
$S_1 + S_2 = S$	-0.067	-0.108	-0.104	-0.084	-0.051	-0.014	0.017	0.037
y(k) = S/1.284	-0.052	-0.078	-0.081	-0.065	-0.039	-0.011	0.013	0.029

x(K)	1.000	1.000	1.000	1.000	1.000	1.000
-2x(K-1)	-2.000	-2.000	-2.000	-2.000	-2.000	-2.000
x(k -2)	1.000	1.000	1.000	1.000	1.000	1.000
2y(k -1)	0.058	0.070	0.064	0.045	0.020	-0.005
-y(k-2)	-0.013	-0.029	-0.035	-0.032	-0.023	-0.010
S_2	0.045	0.041	0.029	0.013	-0.003	-0.015
$S_1 + S_2 = S$	0.045	0.041	0.029	0.013	-0.003	-0.015
y(k) = S/1.284	0.035	0.032	0.023	0.010	-0.002	-0.011

The values of y(k) are plotted in graphical form in Fig. (15).

APPENDIX B

Numerical Computation of Equation (34)

$$y = \frac{-y_{K-2} + 2y_{K-1} + .567x_K - .567x_{K-1}}{1.284}$$

$0.567x_K$	0.567	0.567	0.567	0.567	0.567	0.567	0.567	0.567
$-0.567x_{K-1}$	0.000	-0.567	-0.567	-0.567	-0.567	-0.567	-0.567	-0.567
$2y_{K-1}$	0.000	0.883	1.375	1.450	1.190	0.723	0.200	-0.546
$-y_{K-2}$	0.000	0.000	-0.442	-0.687	-0.726	-0.594	-0.361	0.125
S	0.567	0.883	0.933	0.763	0.464	0.129	-0.161	-0.421
$y_K = S/1.284$	0.442	0.687	0.726	0.594	0.361	0.100	-0.125	-0.328

$0.567x_K$	0.567	0.567	0.567	0.567	0.567	0.567	0.567	0.567
$-0.567x_{K-1}$	-0.567	-0.567	-0.567	-0.567	-0.567	-0.567	-0.567	-0.567
$2y_{K-1}$	-0.655	-0.595	-0.416	-0.195	0.020	0.176	0.258	0.265
$-y_{K-2}$	0.273	0.328	0.297	0.208	0.093	-0.010	-0.088	-0.129
S	-0.382	-0.267	-0.119	0.013	0.113	0.166	0.170	0.136
$y_K = S/1.284$	-0.297	-0.208	-0.093	0.010	0.088	0.129	0.133	0.106

$0.567x_K$	0.567	0.567	0.567	0.567	0.567	0.567	0.567	0.567
$-0.567x_{K-1}$	-0.567	-0.567	-0.567	-0.567	-0.567	-0.567	-0.567	-0.567
$2y_{K-1}$	0.212	0.124	1.028	-0.052	-0.104	-0.122	0.109	-0.074
$-y_{K-2}$	-0.133	-0.108	-0.062	-0.014	0.026	0.052	0.051	0.055
S	0.079	0.018	-0.034	0.066	-0.078	-0.070	-0.048	-0.019
$y_K = S/1.284$	0.062	0.014	-0.026	-0.052	-0.061	-0.055	-0.037	-0.015

$0.567x_K$	0.567	0.567	0.567	0.567	0.567	0.567	0.567
$-0.567x_{K-1}$	-0.567	-0.567	-0.567	-0.567	-0.567	-0.567	-0.567
$2y_{K-1}$	-0.030	0.010	0.039	0.053	0.053	0.041	0.022
$-y_{K-2}$	0.037	0.015	-0.005	-0.019	-0.026	-0.026	-0.026
S	0.001	0.025	0.034	0.034	0.026	0.014	-0.004
$y_K = S/1.284$	0.015	0.019	0.026	0.026	0.020	0.011	-0.003

The values of y_K are plotted in Fig. (14)

APPENDIX C

Numerical Computation of Equation (42)

$$y = \frac{-y_{K-2} + 2y_{K-1} + 0.2835x_K}{1.284} \tag{42}$$

$0.2835x_K$	0.284	0.284	0.284	0.284	0.284	0.284	0.284	0.284
$2y_{K-1}$	0.000	0.442	1.130	1.860	2.460	2.830	2.930	2.800
S_1	0.284	0.726	1.414	2.144	2.744	3.114	3.214	3.084
$-y_{K-2}$	0.000	0.000	-0.221	-0.565	-0.930	-1.230	-1.412	-1.470
$S = S_1 - y_{K-2}$	0.284	0.726	1.193	1.579	1.814	1.884	1.802	1.614
$y_K = S/1.284$	0.221	0.565	0.930	1.230	1.412	1.470	1.400	1.278

$0.2835x_K$	0.284	0.284	0.284	0.284	0.284	0.284	0.284	0.284
$2y_{K-1}$	2.550	2.230	1.820	1.540	1.340	1.330	1.470	1.700
S_1	2.834	2.514	2.104	1.824	1.624	1.614	1.754	1.984
$-y_{K-2}$	-1.400	-1.278	-1.116	-0.962	-0.770	-0.672	-0.665	-0.732
$S = S_1 - y_{K-2}$	1.434	1.236	0.988	0.862	0.854	0.942	1.089	1.252
$y_K = S/1.284$	1.116	0.962	0.770	0.672	0.665	0.732	0.850	0.975

$0.2835x_K$	0.284	0.284	0.284	0.284	0.284	0.284	0.284	0.284
$2y_{K-1}$	1.950	2.155	2.280	2.320	2.280	2.190	2.080	1.985
S_1	2.234	2.439	2.564	2.604	2.564	2.474	2.364	2.269
$-y_{K-2}$	-0.850	-0.975	-1.078	-1.140	-1.157	-1.140	-1.090	-1.040
$S = S_1 - y_{K-2}$	1.384	1.464	1.486	1.464	1.407	1.334	1.274	1.229
$y_K = S/1.284$	1.078	1.140	1.157	1.140	1.090	1.040	0.993	0.957

$0.2835x_K$	0.284	0.284	0.284	0.284	0.284	0.284
$2y_{K-1}$	1.915	1.875	1.870	1.892	1.930	1.975
S_1	2.199	2.159	2.154	2.176	2.214	2.259
$-y_{K-2}$	-0.993	-0.957	-0.938	-0.936	-0.946	-0.965
$S = S_1 - y_{K-2}$	1.206	1.202	1.216	1.240	1.268	1.294
$y_K = S/1.284$	0.938	0.936	0.946	0.965	0.987	1.008

The values of y_K are plotted in graphical form in Fig. (16)

APPENDIX D

Numerical Computation of Eq(48)

$$y = y_1 + 5\zeta y_2 + 2y \qquad (48)$$

where y_1 = values of y_K in Appendix A.
y_2 = values of y_K in Appendix B.
y_3 = values of y_K in Appendix C.
ζ = 0.25.

(N, B. in the following computation $s_1 = y_1 + 5\zeta y_2$, $y_K = s_1 + 2y_3$)

y_1	0.779	0.434	0.071	-0.227	-0.408	-0.458	-0.397	-0.262	-0.098
$5\zeta y_2$	0.553	0.859	0.907	0.743	0.451	0.125	-0.156	-0.410	-0.381
S_1	1.332	1.293	0.978	0.516	0.043	-0.333	-0.553	-0.672	-0.478
$2y_3$	0.442	1.130	1.860	2.460	2.824	2.940	2.800	2.556	2.232
y_K	1.774	2.423	2.838	2.976	2.867	2.607	2.247	1.884	1.753

y_1	0.051	0.156	0.203	0.194	0.145	0.075	0.004	-0.052	-0.078
$5\zeta y_2$	-0.260	-0.116	0.013	0.110	0.161	0.166	0.133	0.078	0.018
S_1	-0.219	0.040	0.190	0.304	0.306	0.241	0.137	-0.026	-0.060
$2y_3$	1.924	1.540	1.344	1.330	1.464	1.700	1.950	2.156	2.280
y_K	1.705	1.580	1.534	1.634	1.770	1.941	2.081	2.130	2.220

y_1	-0.081	-0.065	-0.039	-0.011	0.113	0.029	0.035	0.032	0.023
$5\zeta y_2$	-0.032	-0.065	-0.076	-0.069	-0.046	-0.019	0.006	0.024	0.032
S_1	-0.113	-0.130	-0.115	-0.080	-0.067	0.010	0.041	0.056	0.055
$2y_3$	2.314	2.280	2.180	2.080	2.080	1.986	1.914	1.876	1.892
y_K	2.201	2.150	2.065	2.000	1.919	1.924	1.917	1.928	1.947

y_1	0.010	-0.002	-0.011
$5\zeta y_2$	0.032	0.025	0.014
S_1	0.042	0.023	0.003
$2y_3$	1.930	1.974	2.016
y_K	1.972	1.997	2.019

The values of y_K are plotted in graphical form in Fig. (17).

APPENDIX E

Convergence of coefficient of differential and difference equation to the same limit as $T \rightarrow \infty$.

Require to prove: that as $T \rightarrow \infty$ the coefficients of the difference equation approaches that of the coefficients of the differential equation. Take as an example $\dfrac{\Delta^2 - 2\Delta + 1}{\Delta^2 - 2\Delta e^{\zeta/T} \cos \sqrt{\dfrac{1-\zeta^2}{T}} + e^{2\zeta/T}}$. for $\zeta < 1$.

Proof:

It is required to prove that:

$$\lim_{T\rightarrow\infty} \frac{\Delta^2 - 2\Delta + 1}{\Delta - 2\Delta e^{\zeta/T} \cos \sqrt{\dfrac{1-\zeta^2}{T}} + e^{2\zeta/T}} = \lim_{T \rightarrow \infty} \frac{T^2 D^2}{T^2 D^2 + 2\zeta T D + 1} \tag{A1}$$

Notice that on the R.H.S., as $T \rightarrow \infty$ the unit term is small compared with infinity and would drop off thus upsetting the proof. Consequently one attempt to prove that both the R.H.S. and L.H.S. approaches the same limit as $T \rightarrow \infty$ and thus constitutes an indirect proof.

Since $e^{2\zeta/T} = 1 + 2\zeta/T + 2\zeta^2/T^2 + \ldots\ldots$

$$e^{\zeta/T} = 1 + \zeta/T + \zeta^2/2T^2 + \ldots\ldots$$

$$\cos\sqrt{\frac{1-\zeta^2}{T}} = 1 + \frac{1-\zeta^2}{2T^2} + \ldots\ldots$$

Therefore $\dfrac{\Delta^2 - 2\Delta + 1}{\Delta^2 - 2\Delta e^{\zeta/T} \cos\sqrt{\dfrac{1-\zeta^2}{T}} + e^{2\zeta/T}} \cong \dfrac{\Delta^2 - 2\Delta + 1}{\Delta^2 - 2\Delta(1+\zeta/T)\left(1 + \dfrac{1-\zeta^2}{2T^2}\right) + (1+\zeta/T)}$

$$\cong \frac{\Delta^2 - 2\Delta + 1}{\Delta^2 - 2\Delta\left[1 + \dfrac{1-\zeta^2}{2T^2} + \zeta/T + \zeta/T\left(\dfrac{1-\zeta^2}{2T^2}\right)\right] + 1 + 2\zeta/T + \dfrac{2\zeta^2}{T^2}}$$

$$\lim_{T\to\infty} \frac{\Delta - 2\Delta + 1}{\Delta - 2\Delta\left\{1 + \frac{1-\zeta^2}{2T^2} + \zeta/T + \zeta/T\left(\frac{1-\zeta^2}{2T}\right)\right\} + 1 + 2\zeta/T + \frac{2\zeta^2}{T^2}}$$

$$= \frac{\Delta^2 - 2\Delta + 1}{\Delta^2 - 2\Delta\,(1+\zeta/T) + 1 + 2\zeta/T}$$

$$= \frac{\Delta^2 - 2\Delta + 1}{\Delta^2 - 2\Delta + 1 + 2\zeta/T - (2\zeta/T)\Delta} \qquad \text{(A2)}$$

Also D $\cong (1 - \Delta)$

Therefore
$$\frac{T^2 D^2}{T^2 D^2 + 2\zeta TD + 1} \cong \frac{T^2 (1-\Delta)^2}{T^2 (1-\Delta)^2 + 2\zeta T(1-\Delta) + 1}$$

$$\cong \frac{T^2 (1 - 2\Delta + \Delta^2)}{T^2 (1 - 2\Delta + \Delta^2) + 2\zeta T - 2\zeta T\Delta + 1}$$

$$\cong \frac{T^2 (1 - 2\Delta + \Delta^2)}{T^2 - 2T^2\Delta + T^2\Delta^2 + 2\zeta T - 2\zeta T\Delta + 1}$$

$$\cong \frac{\Delta^2 - 2\Delta + 1}{\Delta^2 - 2\Delta + 1 + 2\zeta/T - 2(\zeta/T)\Delta + 1/T^2}$$

$$\lim_{T\to\infty} \frac{\Delta^2 - 2\Delta + 1}{\Delta^2 - 2\Delta + 1 + \frac{2\zeta}{T} - \frac{2\zeta\Delta}{T} + \frac{1}{T^2}} = \frac{\Delta^2 - 2\Delta + 1}{\Delta^2 - 2\Delta + 1 + \frac{2\zeta}{T} + \frac{2\ \Delta}{T}} \qquad \text{(A3)}$$

Since Eq(A2) and Eq(A3) approach each other in the lim, the L. H. S. and R. H. S. of Eq(A1) approaches a definite limit as $T \to \infty$.

APPENDIX F

Some properties of the integrator

Consider the integrator which is represented in the form of difference equation, namely:

$$y_K - y_{K-2} = u_K \qquad (15)$$

1) Take the case of u =, so that the difference equation becomes:

$$y_K - y_{K-2} = x_K \qquad (16)$$

$$y_K = x_K + y_{K-2}$$

Suppose one excite the equation with a step function from 0 to . Then one obtains:

x_K	1	1	1	1	1	1	1	1	. . .
y_{K-2}	0	0	1	1	2	2	3	3	. . .
y_K	1	1	2	2	3	3	4	4	. . .

which gives the output shown in Fig. (6)

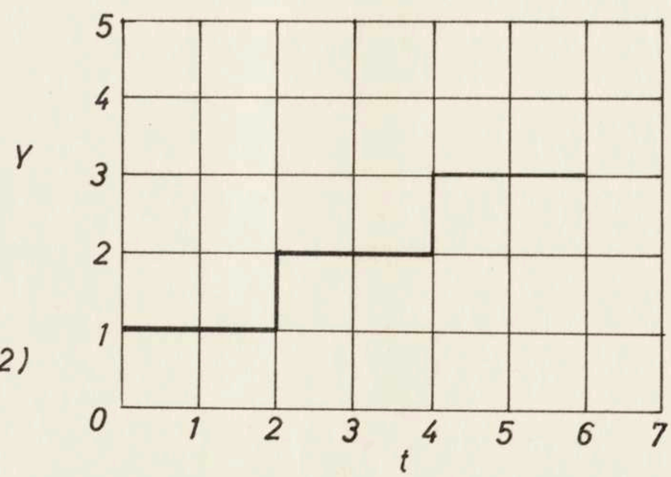


Fig. (6)
Graph Of Eq.(22)

2) Consider the case of the averaging filter where

$$u_K = \frac{x_{K-1} + x_{K-2}}{2}$$ then the excitation becomes:

x_K	1	1	1	1	1	1	. . . .
u_K	0	1/2	1	1	1	1	. . . .

which when substituted into Eq(15) gives Eq(17):

$$y_K - y_{K-2} = u_K \qquad (17)$$

where $u_K = \dfrac{x_{K-1} + x_{K-2}}{2}$

Proceed with the computation in similar manner:

u_K	0 1/2 1 1 1 1 . . .
y_{K-2}	0 0 0 1/2 1(1 1/2) . .
y_K	0 1/2, 1, 11/2, 2(2 1/2) . . .

The waveform is shown in Fig. (7) which

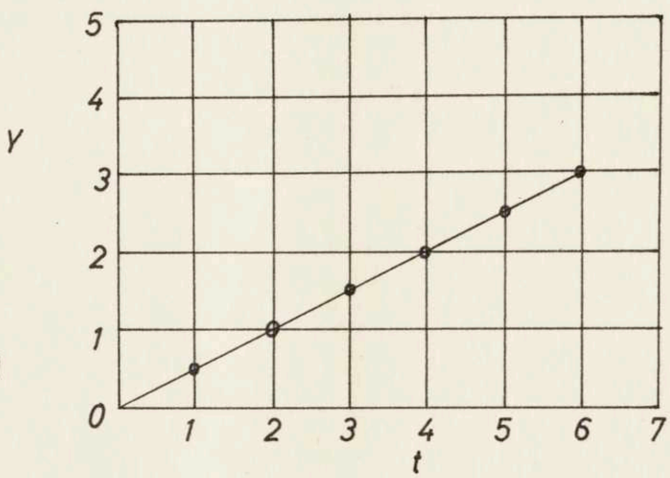


Fig (7)

Graph Of Eq (17)

shows that the solution is now exact.

3) The most comprehensive way of representing the integrator is by the Sink D integrator. The derivation is as follows:

$$\text{Sink D} = D$$

$$= \frac{e^{+D} - e^{-D}}{2}$$

$$= \frac{e^{-D}(e^{D} - e^{-D})}{2e^{-D}}$$

$$\text{but} \quad \frac{1 - e^{-2D}}{2e^{-D}} = \frac{1 - \Delta^2}{2\Delta} \tag{18}$$

$$\text{also} \quad Dy = x$$

$$\sinh \quad Dy \cong x$$

using the results of Eq(18),

$$(1 - \Delta^2)y = 2\Delta \cdot x$$

$$\text{also let} \quad \Delta x = x_{K-1} + x_{K-2}$$

$$\text{then } y_K - y_{K-2} = 2 \; \frac{x_{K-1} + x_{K-2}}{2}$$

$$y_K - y_{K-2} = x_{K-1} + x_{K-2} \tag{19}$$

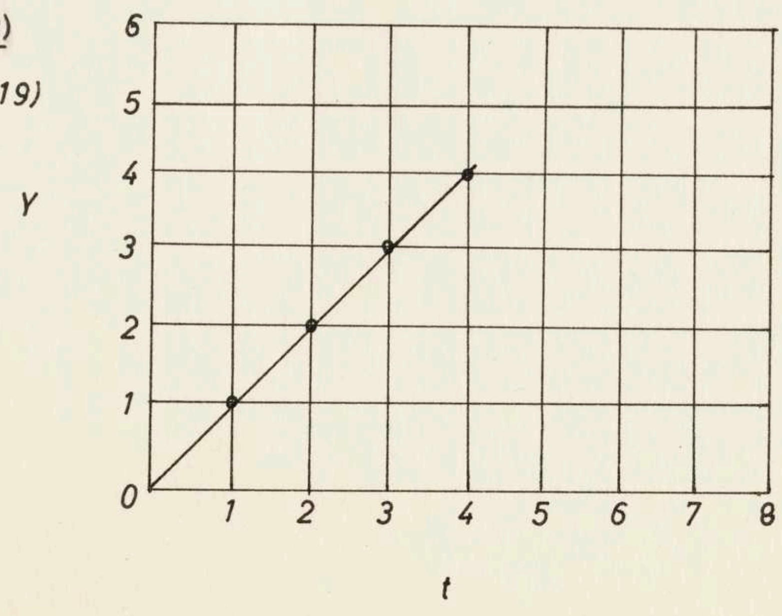
Eq(19) can be shown to be an exact integrator by the following computation:

x_{K-1}	0	1	1	1	1
x_{K-2}	0	0	1	1	1
y_{K-2}	0	0	0	1	2
y_K	0	1	2	3	4

This is shown in Fig. (8)

Fig. (8)

Graph Of Eq. (19)



APPENDIX G

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