ESG 8.022 Fall 2006 Final Exam

Instructor: Michael Shaw

Tuesday, December 19th, 1:30PM to 4:30PM

Instructions

Show work on all problems. Partial credit cannot be granted without adequate progress. Please explain any nonstandard notation.

Complete 4 out of 5 of the short answer questions for 10 points each. Complete 4 out of the 5 long answer questions for 40 points each. The exam is graded out of 200 points. Mark explicitly which questions are and are not to be graded if you attempt all 5 questions.

No calculators, textbooks, cheat sheets or other examination aids are permitted. A formula sheet is provided.

Good luck!

1 Short Answer Questions (10 points each)

a. The pyramid at Giz has a square base of side a and four faces which are equilateral triangles. The Scarab of Ra, buried at the very center of the base of the pyramid, has a net charge of Q. Do you know the net flux of electric field emerging from one of the triangular faces of the pyramid? If it can be determined, solve for the flux; if not, explain why.

b. A neutral particle decays into two charged particles with charge $+q$ and $-q$ respectively. They have the same mass m and fly apart in opposite directions at speed v_0. The velocities are perpendicular to a uniform magnetic field $\vec{B}$ which fills space. At what distance d from the original decay will the particles collide? Ignore any forces between the two particles.

c. Two conducting spheres of different radii are connected by a fine conducting wire. They have a net positive charge. Which sphere has more charge?

  (a) The larger sphere

  (b) The smaller sphere

  (c) They have the same charge

  (d) The answer depends on how the conductors were charged

d. Two insulated, identically charged spheres, suspended by strings from the same point are in equilibrium. An uncharged conducting plate is then placed underneath the spheres. After a new equilibrium is reached, the spheres will be:

  (a) Closer together

  (b) In the same position as before

  (c) Further Apart

  (d) Cannot be determined from the information given

e. Two spheres of linear magnetic material are placed in a uniform external magnetic field. One sphere is paramagnetic, with $\mu > \mu_0$. The other is diamagnetic, with $\mu < \mu_0$. Which of the following is correct?

  (a) The magnetic field inside the paramagnetic sphere is stronger than inside the diamagnetic sphere.

  (b) The field is stronger inside the diamagnetic sphere.

  (c) The field is the same inside the two spheres.

  (d) Because of hysterisis, the answer is history-dependant.

2   Waving with Maxwell (40 points)

Suppose that in the absence of any matter and charges, an electric field exists of the form

$$\vec{E} = E_0 \hat{x} \cos(kz - \omega t) + E_0 \hat{y} \sin(kz - \omega t)$$

In the following you are asked to show that $\vec{E}$ satisfies Maxwell's equations provided that a certain magnetic field $\vec{B}(x, y, z, t)$ also exists and a relation between k and ω is satisfied.

a. What is the appropriate relation between k and ω?

b. What is $\vec{B}(x, y, z, t)$?

c. Show directly that this $\vec{E}$ and $\vec{B}$ satisfy Maxwell's equations.

d. Describe what the electric and magnetic field look like at the origin as a function of time.

e. What is the energy flow? (Hint: be sure to find magnitude and direction)

3   A Wave in a Box (40 points)

Consider the following electromagnetic field with $k_m = m\pi/a$ and $k_n = n\pi/a$:

$$\vec{E} = E_0 \sin(k_m x) \cos(k_n y) cos(kz) sin(\omega t) \hat{y}$$
$$\vec{B} = B_z \cos(k_m x) \cos(k_n y) \cos(kz) \cos(\omega t) \hat{z} + B_x \sin(k_m x) \cos(k_n y) \sin(kz) \cos(\omega t) \hat{x}$$

a. Find the coefficients B_x and B_z and the relation between ω, k, k_m, k_n such that these fields satisfy Maxwell's equations in free space. This field can exist in a metal box with square cross section of dimension $x = y = a$ in the x and y directions, and arbitrary length in the z direction.

b. Consider the mode $m = 1$ and $n = 0$. Draw the magnetic field lines in the box.

c. What are the boundary conditions satisfied by the electric and magnetic fields on the walls of the box?

d. Apply the boundary conditions for electric and magnetic fields on a conductor to determine the surface charge densities and surface currents on the walls of the box.

4   A Moving Capacitor (40 points)

Two identical but oppositely charged conducting plates have been placed inside a magnet. The magnet produces a uniform field $\vec{B}_0 = B_0\hat{y}$. The plates, each of which has mass M are constrained to move sideways in the $\hat{z}$-direction (with no friction). Both plates are in the $y - z$ plane (ie: normals to their surface are in the $\hat{x}$ direction. The top plate carries charge Q_0 and the bottom plate carries charge $-Q_0$. The plates have length (in the $\hat{z}$ direction) l and width (in the $\hat{y}$ direction) w. They are separated by a distance s such that $s \ll w$ and $s \ll l$.

a. At $t = 0$, the magnet is turned down so that $\vec{B}$ decreases slowly and uniformly to zero at time T. Do the charged plates feel a force? Please explain the origin of any such force.

b. If there is a force on the plates, calculate its magnitude and direction. Feel free to assume that the motion of the plates is negative (ie: M is sufficiently large)

c. Calculate the total momentum acquired by the two plates for $t > T$.

d. The static electromagnetic fields carry momentum density. Calculate the momentum stored by the electromagnetic fields within this device for $t < 0$. Show that this momentum is identical to that which appears on the plates.

e. Now suppose instead that $\vec{B}$ is kept fixed and equal to $\vec{B}_0$ at all times, but instead that a resistance is connected across the two plates at $t = 0$ allowing them to discharge completely. Once again, the plates will feel a force. Qualitatively explain what causes this force, but do not calculate the force on the plates explicitly. Instead, explain clearly and briefly what the momentum of the plates must be when they have discharged.

5 The Yukawa Potential (40 points)

In Quantum Mechanics, it is useful to consider the Yukawa Potential, an approximation to the Coulomb potential that isn't as nasty as $r \to 0$. More physically, such a potential occurs in a field theory mediated by a massive scalar field–ie: that of the pion. We treat a Yukawa approximation to the electric potential–true if the photon had a mass m:

$$V = \frac{Q}{4\pi\epsilon_0} \frac{e^{-r/r_0}}{r}$$

where $r_0 = \frac{\hbar}{mc}$ is a constant with units of distance.

a. Find the electric field as a function of radius. (Hint: Either use proper-like spherical gradients or convert to cartesian and do it all in the $\{x, y, z\}$ basis)

b. Find the charge density as a function of radius.

c. Find the total charge enclosed in a sphere of radius $a \ll r_0$. Call this Q_0. (This should be a simple answer. Then again, many things should be simple.)

d. Find the *space charge* enclosed in a sphere of radius R, where space charge = total charge - Q_0. Explain this result physically for $R \gg r_0$.

6   Magnetic Monopoles (40 points)

You may (or may not) find the following integrals useful in this problem:

$$\int_{-\infty}^{\infty} \frac{dt}{(b^2 + v^2 t^2)^{3/2}} = \frac{2}{vb^2}; \int_{-\infty}^{\infty} \frac{v^2 t^2 dt}{(b^2 + v^2 t^2)^{3/2}} = 0$$

Consider a magnetic monopole with magnetic charge g, creating a magnetic field

$$B = \frac{\mu_0}{4\pi} g \frac{\hat{r}}{r^2}$$

a. Write down Maxwell's equations, including the effect of the magnetic monopole. Clearly define any terms you introduce.

   Suppose such a magnetic monopole is located at the origin. An electrically charged particle with charge e is located on the y-axis at $y = b$ at $t = 0$. The electric charge travels with a velocity $\vec{v} = v\hat{x}$. Make the approximation that the particle is essentially not deflected although there is a magnetic force acting on the charge.

b. Calculate the magnetic force on the electrically charged particle as a function of time t.

c. Calculate the direction and magnitude of the impulse $\Delta \vec{p} = \int_{-\infty}^{\infty} F_{mag} dt$ on the electrically charged particle.

d. Calculate the change in angular momentum $\Delta \vec{L} = \int_{-\infty}^{\infty} \vec{r_0} \times \vec{F} dt$ of the charged particle about the origin, as a result of this impulse.

e. Set the change in magnitude of the angular momentum $|\Delta \vec{L}| = n\hbar$, where n is an integer, and $\hbar = h/2\pi$ is Planck's constant. Derive a relation between the electric charge e, the magnetic charge g, the integer n, and fundamental constants that may or may not include $\hbar, c, \mu_0, \epsilon_0$. This condition is called the Dirac quantization condition for the magnetic monopole.